Gain Patterns Described by Spherical Harmonics

Definitions of the Gain:

For CalPoly, I believe this describes the gain integral:

$$TotalGain_{CalPoly} = \frac{1}{\max[G(\theta,\phi)]}\int_0^{\pi}\int_0^{2\pi}[a_0 Y_0^0(\theta,\phi) + a_1 Y_1^0(\theta,\phi) + \cdots + a_{12} Y_{12}^0(\theta,\phi)]\sin\theta\, d\theta d\phi$$

Note: If I am not mistaken, $\max[G(\theta,\phi)]$ is not constant; it depends on the value of a_0. This is because the first Spherical Harmonic is the only one that integrates over all angles to a nonzero value.

OSU's method is taking the square of the linear sum:

$$TotalGain_{OSU} = \int_0^{2\pi}\int_0^{\pi}[a_0 Y_0^0(\theta,\phi) + a_1 Y_1^0(\theta,\phi) + \cdots + a_{12} Y_{12}^0(\theta,\phi)]^2\sin\theta\, d\theta d\phi$$

To calculate a fitness score, change the bounds of integration to the region of interest. For example, if trying to maximize a ring at theta of 90 degrees with a spread of 20 degrees, make the bounds of the theta integral 80 to 100 degrees (the min and max acceptable theta values) instead of 0 to 180 degrees (the bounds of an integration overall angles).

Normalizing the Gain Pattern:

The OSU definition of gain can be further simplified using the orthogonality of Spherical Harmonics:

$$= \int_0^{\pi}\int_0^{2\pi}([a_0 Y_0^0(\theta,\phi)]^2 + [a_1 Y_1^0(\theta,\phi)]^2 + \cdots + [a_{12} Y_{12}^0(\theta,\phi)]^2)\sin\theta\, d\theta d\phi$$

Using the normality of the Spherical Harmonics over all angles, the integral can be evaluated:

$$TotalGain_{OSU} = [a_0^2 + a_1^2 + \cdots + a_{12}^2] = 1$$

The OSU Total Gain is set to be 1. This is why the best fitness score achievable is 1. Such a score corresponds to the entire volume of the gain pattern fitting between the region of interest (e.g. between theta of 80 to 100 degrees).

The use of the square of the linear sum of spherical harmonics makes normalizing the coefficients convenient. No integration necessary! We can calculate the normalization factor n as follows:

$$\left[\left(\frac{a_0}{n}\right)^2 + \left(\frac{a_1}{n}\right)^2 + \cdots + \left(\frac{a_n}{n}\right)^2\right] = 1 \qquad n = \sqrt{[a_0^2 + a_1^2 + \cdots + a_{12}^2]}$$

So, for any arbitrary set of 13 coefficient numbers, we can calculate n and divide each coefficient by n to set the total gain to 1.

Code Integration:

In the code, integration is slightly more complex. The code calculates the value of the integrand at 1440 different theta values, from 0 to 180 degrees in steps of 0.125 degrees (e.g. at theta = 0.125, 0.25, 0.375, … 179.875, 180). It sums these values and outputs the sum. One might already see a problem with this definition: integration with half the stepsize would output twice the integral's value because it sums the same region twice as many times. Therefore, the integral for the gain needs to be redefined as:

$$TotalGain_{OSU} = 2\pi\, N_\theta * IntegrateCode\{[a_0 Y_0^0(\theta,\phi) + a_1 Y_1^0(\theta,\phi) + \cdots + a_{12} Y_{12}^0(\theta,\phi)]^2 \sin(\theta)\, , N_\theta\}$$

Where N_θ is the stepsize in theta (in radians) the integrating code takes. In our code, N_θ is 1/8 of a degree (0.002182 radians) and so the integral samples the function 1440 times from 0 to pi.

Benefits of OSU's Gain Definition:

The benefit of OSU's definition of Gain is that there are *no* stray constants. The only nonintuitive thing is the multiplication by the integration stepsize. Therefore, all fitness scores are calculated as:

$$FScore_{OSU} = 2\pi\, N_\theta * IntegrateCode\{\, [a_0 Y_0^0(\theta,\phi) + a_1 Y_1^0(\theta,\phi) + \cdots + a_{12} Y_{12}^0(\theta,\phi)]^2 \sin(\theta)\, , ThetaMIN, ThetaMAX\}$$

Where: $N_\theta = 0.125 * \frac{\pi}{180}$

And all normalization constant for a set of coefficients can be calculated without any integration and implemented as:

$$a_n' = \frac{a_n}{n} \qquad n = \sqrt{[a_0^2 + a_1^2 + \cdots + a_{12}^2]}$$

Where the prime coefficient is normalized. This normalization method saves a lot of time in the code. It about halved the time it takes for my code to run when compared to the normalization by integration method, and it is more accurate.

Finally, all gain patterns are always positive with no adjusting necessary.