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PID controller design for second order nonlinear uncertain systems

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Abstract Although the classical PID (proportional-integral-derivative) controller is most widely and successfully used in engineering systems which are typically nonlinear with various uncertainties, almost all the existing investigations on PID controller focus on linear systems. The aim of this paper is to present a theory on PID controller for nonlinear uncertain systems, by giving a simple and analytic design method for the PID parameters together with a mathematic proof for the global stability and asymptotic regulation of the closed-loop control systems. To be specific, we will construct a 3-dimensional manifold within which the three PID parameters can be chosen arbitrarily to globally stabilize a wide class of second order nonlinear uncertain dynamical systems, as long as some knowledge on the upper bound of the derivatives of the nonlinear uncertain function is available. We will also try to make the feedback gains as small as possible by investigating the necessity of the manifold from which the PID parameters are chosen, and to establish some necessary and sufficient conditions for global stabilization of several special classes of nonlinear uncertain systems.

Keywords PID controller, nonlinear uncertain systems, global stability, Lyapunov function, asymptotic regulation.

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1 Introduction

It is well-known that the classical PID (proportional-integral-derivative) controller or its minor variations, is by far the most widely and successfully used controller in engineering systems, despite of the remarkable progress in modern control theory over the past half a century. For example, it was reported that more than 95% of the control loops in process control are of PID type, and most loops are actually PI control, see [1, 2].

This fact may appear to be somewhat surprising, but there are fundamental reasons behind it: the PID controller is a simple feedback structure of the form "present-past-future", which does not depend on the precise mathematical models of the dynamical systems to be controlled. In fact, the PID controller can reduce the influence of various uncertainties by feedback signals including the proportional action ("present"); it has the ability to eliminate steady state offsets via the integral action ("past"); and it can

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also anticipate the tendency through the derivative action ("future"). Moreover, the celebrated Newton's second law in mechanics still plays a fundamental role in modeling dynamical systems of the physical world, which is actually a second order differential equation of the position of a moving body, and can be well regulated by the PID controller, as will be shown rigorously in this paper. Of course, one of the most challenging tasks for the implementation of the PID controller is how to design the three parameters of the controller, which are traditionally tuned by experiments or experiences or by both, and has been investigated extensively in the literature but most for linear systems, see [1–9], among others.

One of the most eminent methods for designing the PID parameters is the Ziegler-Nichols rules (see [10]), which are based on some features of the process dynamics extracted from experiments conducted by either the step response method or the frequency response method. Many other methods including tuning and adaptation for the design of the PID parameters have also been proposed and investigated mostly for linear systems, see [1–9].

However, in order to justify the remarkable effectiveness of the PID controllers for real world systems, we have to be faced with nonlinear uncertain dynamical systems and to understand the capability of the PID controller [11]. On the other hand, better understanding of the PID control may considerably improve its widespread practice, and so contribute to better product quality [1]. These motivate the investigation of the paper.

To understand the capability of PID control, we will follow a similar theoretical framework as the investigation of the maximum capability of the feedback mechanism in [12, 13], where the maximum capability of feedback is defined by the largest possible class of nonlinear functions that can be dealt with by the feedback mechanism, where the size of the uncertain functional class is characterized by the corresponding Lipschitz constant [12], save that we will only work with continuous time PID feedback controllers here. To be specific, it will be shown that, as long as some knowledge on the upper bound of the derivatives of the nonlinear uncertain function is available, a class of uncertain functions and a simple 3-dimensional manifold can be constructed, such that the PID controller can globally stabilize the corresponding class of uncertain nonlinear dynamical systems, whenever the three PID parameters are chosen from this manifold. We will also try to make the feedback gains as small as possible by investigating the necessity of the manifold from which the PID parameters are chosen, and to establish some necessary and sufficient conditions for global stabilization of several special classes of nonlinear uncertain systems.

The rest of the paper is organized as follows. The problem formulation will be described in the next section. Section 3 will present our main results, with their proofs put in Section 4. Section 5 will conclude the paper with some remarks. For easy references, we collect several existing stability results on differential equations in Appendix A and put some auxiliary results in Appendix B.

2 Problem formulation

Let us consider a moving body in $\mathbb{R}$ which is regarded as a controlled system. Denote $p(t)$, $v(t)$, $a(t)$ as its position, velocity and acceleration at the time instant t, respectively. Assume that the external forces acting on the body consist of f and u, where $f = f(p, v, t)$ is a nonlinear function of the position p, velocity v and time t and u is the control force. There are many examples which satisfy these assumptions. Classical examples contain spring oscillator, pendulum, damped vibration, etc.

By Newton's second law, we have the equation at time t:

$$ma = f(p, v, t) + u, \tag{1}$$

where m is the mass. Our control objective is to design an output feedback controller to guarantee that for any initial position and any initial velocity, the position trajectory tracks a given constant reference value $y^* \in \mathbb{R}$ and at the same time the velocity of the body tends to 0.

In this paper, our control force is described by the classical PID controller:

$$u(t) = k_p e(t) + k_i \int_0^t e(s)\mathrm{d}s + k_d \dot{e}(t), \tag{2}$$

where e is the control error $e(t) = y^* - p(t)$, and k_p, k_i, k_d are the controller parameters to be designed.

The control variable is thus a sum of three terms: the P-term (which is proportional to the error), the I-term (which is proportional to the integral of the error) and the D-term (which is proportional to the derivative of the error). Without loss of generality, we assume that the body has the unit mass $m = 1$. Notice that $v = \dot{p}$, $a = \ddot{p}$, and then Eq. (1) can be rewritten as $\ddot{p} = f(p, \dot{p}, t) + u$. Denote $x_1 = p$ and $x_2 = \dot{p}$, and then the state space equation of this basic mechanic system under PID control is

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = f(x_1, x_2, t) + u(t), \\ u(t) = k_p e(t) + k_i \int_0^t e(s)\mathrm{d}s + k_d \dot{e}(t), \end{cases} \tag{3}$$

where $x_1(0), x_2(0) \in \mathbb{R}$ and $e(t) = y^* - x_1(t)$.

In this paper, we will show that the three controller parameters k_p, k_i, k_d can be designed explicitly such that the position of the body tracks a given constant setpoint y^* under the control law (2) for any initial position and velocity, as long as $f = f(x_1, x_2, t)$ is a continuously differentiable function with known upper bounds for its partial derivatives with respect to the variables x_1 and x_2.

3 The main results

We will provide a theory together with the design method for the PID controller under different conditions on the class of nonlinear uncertain functions.

Firstly, we define a functional space:

$$\mathscr{F}_{L_1,L_2} = \left\{ f \in C^1(\mathbb{R}^2 \times \mathbb{R}^+) \middle| \frac{\partial f}{\partial x_1} \leqslant L_1, \ \left| \frac{\partial f}{\partial x_2} \right| \leqslant L_2, \ \ \forall x_1, x_2 \in \mathbb{R}, \ \forall t \in \mathbb{R}^+ \right\},$$

where L_1 and L_2 are positive constants, and $C^1(\mathbb{R}^2 \times \mathbb{R}^+)$ denotes the space of all functions from $\mathbb{R}^2 \times \mathbb{R}^+$ to $\mathbb{R}$ which are locally Lipschitz in (x_1, x_2) uniformly in t and piecewise continuous in t, with continuous partial derivatives with respect to (x_1, x_2).

Theorem 1. Consider the PID controlled system (3) with any unknown $f \in \mathscr{F}_{L_1,L_2}$. Assume that for all $t \in \mathbb{R}^+$ and $y \in \mathbb{R}$, $f(y, 0, t) = f(y, 0, 0)$. Then for any $L_1, L_2 > 0$, there exists a three dimensional manifold $\Omega_{\mathrm{pid}} \subset \mathbb{R}^3$, defined by

$$\Omega_{\mathrm{pid}} = \left\{ (k_p, k_i, k_d) \in \mathbb{R}^3 \middle| k_p > L_1, k_d > L_2, k_i > 0, (k_p - L_1)(k_d - L_2) - k_i > L_2 \sqrt{k_i (k_d + L_2)} \right\}, \tag{4}$$

such that whenever the controller parameters (k_p, k_i, k_d) are taken from Ω_{pid}, the closed-loop system (3) will satisfy $\lim_{t\to\infty} x_1(t) = y^*$ and $\lim_{t\to\infty} x_2(t) = 0$ for any initial value $(x_1(0), x_2(0)) \in \mathbb{R}^2$ and any constant setpoint $y^* \in \mathbb{R}$.

The proof is given in Section 4.

Remark 1. The selection of the three controller parameters has wide flexibility and is robust to some extent, due to the open property of the parameter manifold and the precise expression of Ω_{pid}. Theorem 1 gives a global convergence result, for which the upper bounds of the partial derivatives play a key role, and we remark that without such bounds it would not be possible for PID control to achieve global stabilization in general (see [14]). We also note that the selection of the three parameters does not rely on the initial values (position and velocity) or the setpoint y^*.

Remark 2. From the above theorem, we find that the integral parameter k_i of the PID controller can be taken arbitrarily small, but cannot be zero, since otherwise there will be no integral action. At the same time, for any fixed $k_p > L_1$ and $k_d > L_2$, we have $(k_p, k_i, k_d) \in \Omega_{\text{pid}}$ for all sufficiently small $k_i > 0$. Moreover, from the proof of Theorem 1, one may find that the uncertain function class $\mathscr{F}_{L_1,L_2}$ may also be expanded by replacing the conditions on the partial derivatives with certain global Lipschitz-like properties.

If we have more constrains on the unknown function $f(x_1, x_2, t)$, for example f is independent of t and is linear in the second variable x_2, then we can find a larger and necessary parameter manifold to stabilize the system. The next proposition rigorously proves this fact.

Let us introduce the following functional space,

$$\mathscr{G}_{L_1,L_2} = \left\{ f \in C^2(\mathbb{R}^2) \middle| \frac{\partial f}{\partial x_1} \leqslant L_1,\ \frac{\partial f}{\partial x_2} \leqslant L_2,\ \frac{\partial^2 f}{\partial x_2^2} = 0,\ \forall x_1, x_2 \in \mathbb{R} \right\},$$

where $L_1 > 0,\ L_2 > 0$ are constants and $C^2(\mathbb{R}^2)$ is the space of twice continuously differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}$.

Proposition 1. Consider the PID controlled system (3) where the unknown function $f \in \mathscr{G}_{L_1,L_2}$ does not depend on time t. Then for any $f \in \mathscr{G}_{L_1,L_2}$ and any setpoint $y^* \in \mathbb{R}$, the closed-loop system (3) satisfies $\lim_{t\to\infty} x_1(t) = y^*$ and $\lim_{t\to\infty} x_2(t) = 0$ for any initial value $(x_1(0), x_2(0)) \in \mathbb{R}^2$ if and only if the PID parameters (k_p, k_i, k_d) lie in the following 3-dimensional manifold:

$$\Omega_{\text{pid}}^{'} = \left\{ (k_p, k_i, k_d) \in \mathbb{R}^3 \middle| k_p > L_1, k_d > L_2, k_i > 0, (k_p - L_1)(k_d - L_2) - k_i > 0 \right\}. \tag{5}$$

The proof is given in Section 4.

We remark that when $(y^*, 0)$ is an equilibrium point of the uncontrolled systems, the I-term is not necessary for regulation. The next theorem gives a necessary and sufficient condition for tracking a given setpoint (equilibrium point) with PD control for a class of two-dimensional affine nonlinear uncertain systems.

Define a functional space $\mathscr{F}_{L_1,L_2,y^*} \subset C^1(\mathbb{R}^2 \to \mathbb{R}^2)$ as follows:

$$\left\{ f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \middle| \frac{\partial f_1}{\partial x_2} > 0, -\left(\frac{\partial f_1}{\partial x_2}\right)^{-1} \det(Df) \leqslant L_1,\ \left(\frac{\partial f_1}{\partial x_2}\right)^{-1} \operatorname{tr}(Df) \leqslant L_2,\ \forall x_1, x_2 \in \mathbb{R},\ f(y^*, 0) = 0 \right\},$$

where $\det(Df)$ is the determinant of the Jacobian matrix of f defined by

$$Df = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{pmatrix}$$

and $\operatorname{tr}(Df)$ is the trace of Df defined by $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$.

Theorem 2. Consider the following affine nonlinear system with unknown $f \in \mathscr{F}_{L_1,L_2,y^*}$,

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \dot{x}_2 = f_2(x_1, x_2) + u, \end{cases} \tag{6}$$

where u is the PD control: $u = k_p e(t) + k_d \dot{e}(t)$ and $e(t) = y^* - x_1(t)$. Then for any $f \in \mathscr{F}_{L_1,L_2,y^*}$, the closed-loop system (6) satisfies $\lim_{t\to\infty} x_1(t) = y^*$ and $\lim_{t\to\infty} x_2(t) = 0$ for any initial value $(x_1(0), x_2(0)) \in \mathbb{R}^2$ if and only if the PD parameters lie in the 2-dimensional manifold $\Omega_{pd} = \{(k_p, k_d) \in \mathbb{R}^2 | k_p > L_1,\ k_d > L_2\}$.

The proof of Theorem 2 is also given in Section 4. We remark that if $\frac{\partial f_1}{\partial x_2} < 0,\ \forall x_1, x_2 \in \mathbb{R}$, then Theorem 2 still holds provided that the inequalities "$>$" and "$\leqslant$" are replaced by "$<$" and "$\geqslant$" respectively in the definitions of $\mathscr{F}_{L_1,L_2,y^*}$ and Ω_{pd}.

Corollary 1. If $f_1(x_1, x_2) = x_2$, then the function space $\mathscr{F}_{L_1,L_2,y^*}$ in Theorem 2 reduces to

$$\mathscr{F}_{L_1,L_2,y^*}^{'} = \left\{ f_2 \in C^1(\mathbb{R}^2) \middle| \frac{\partial f_2}{\partial x_1} \leqslant L_1,\ \frac{\partial f_2}{\partial x_2} \leqslant L_2,\ \forall x_1, x_2 \in \mathbb{R},\ f_2(y^*, 0) = 0 \right\}.$$

Finally, it is worth mentioning that for first order systems, PI control is sufficient. The next proposition gives a rigorous description. Define

$$\mathscr{F}_L = \{f \in H(\mathbb{R}\times\mathbb{R}^+) : |f(x,t) - f(y,t)| \leqslant L|x-y|, \forall x, y \in \mathbb{R},\ \forall t \in \mathbb{R}^+\},$$

where $L > 0$ is a constant and $H(\mathbb{R}\times\mathbb{R}^+)$ is the space of functions from $\mathbb{R}\times\mathbb{R}^+$ to $\mathbb{R}$, which are piecewise continuous in the second variable t.

Proposition 2. Consider the following first order nonlinear system $\dot{x} = f(x,t) + u$, where the unknown $f \in \mathscr{F}_L$ and the PI control is defined by $u(t) = k_p e(t) + k_i \int_0^t e(s)\mathrm{d}s$. Then for any $f \in \mathscr{F}_L$ and any setpoint y^* satisfying $f(y^*,t) = f(y^*,0)$ for all $t \in \mathbb{R}^+$, the closed-loop system is globally stable and satisfies $\lim_{t\to\infty} x(t) = y^*$ if and only if the PI parameters lie in the following 2-dimensional manifold: $\Omega_{pi} = \{(k_p,k_i) \in \mathbb{R}^2 | k_p > L, k_i > 0\}$. In fact, if the PI parameters (k_p,k_i) are taken from Ω_{pi}, then the convergence $\lim_{t\to\infty} x(t) = y^*$ has an exponential rate for any initial value $x(0) \in \mathbb{R}$.

The proof is given in Section 4.

Remark 3. If the control channel contains an unknown parameter, say b (i.e. $u(t)$ is replaced by $bu(t)$ in the control systems), where b is an unknown positive constant with a known lower bound $\underline{b} > 0$, then all the results in the above theorems, propositions and corollary remain true, provided that $(\underline{b}k_p, \underline{b}k_i, \underline{b}k_d)$ are chosen from the respective manifolds.

This assertion is quite obvious from the proofs given in the next section, where more explanations will be provided.

4 Proofs of the main results

Proof of Theorem 1. First, we introduce some notations. Denote $x(t) = \int_0^t e(s)\mathrm{d}s + \frac{f(y^*,0,0)}{k_i}$, $y(t) = e(t)$, $z(t) = \dot{e}(t)$, $g(y,z,t) = -f(y^*-y,-z,t) + f(y^*,0,t)$, then (3) turns into

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = g(y,z,t) - k_i x - k_p y - k_d z. \end{cases} \tag{7}$$

By $f \in \mathscr{F}_{L_1,L_2}$, it is easy to see that $g \in \mathscr{F}_{L_1,L_2}$ and $g(0,0,t) = 0$ for all $t \geqslant 0$. Hence $(0,0,0)$ is an equilibrium of (7).

Note that $g(y,z,t)$ can be expressed as $g(y,z,t) = b(y,t)y + a(y,z,t)z$, where the functions $b(y,t)$ and $a(y,z,t)$ are defined by

$$b(y,t) = \begin{cases} \dfrac{g(y,0,t)}{y}, & y \neq 0, \\ \dfrac{\partial g}{\partial y}(0,0,t), & y = 0, \end{cases} \quad \text{and} \quad a(y,z,t) = \begin{cases} \dfrac{g(y,z,t) - g(y,0,t)}{z}, & z \neq 0, \\ \dfrac{\partial g}{\partial z}(y,0,t), & z = 0. \end{cases}$$

Obviously, we have $b(y,t) \leqslant L_1$, $|a(y,z,t)| \leqslant L_2$ for all y,z,t by the mean value theorem and the definition of $\mathscr{F}_{L_1,L_2}$. Since for all $t \geqslant 0$ and $y \in \mathbb{R}$, $f(y,0,t) = f(y,0,0)$, it is easy to see that $b(y,t) = \frac{g(y,0,0)}{y}$ is merely a function of y, denoted henceforth by $b(y)$. Obviously, $b(\cdot)$ is continuous.

Hence, the closed-loop equation (7) can be rewritten as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = A(x,y,z,t) \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{8}$$

where

$$A(x,y,z,t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_i & -k_p + b(y) & -k_d + a(y,z,t) \end{bmatrix}.$$

To construct a Lyapunov function, we denote $\psi = \frac{\psi_0+\psi_1}{2}$, where $\psi_0 = \inf\limits_{y,z,t}\{-a(y,z,t)+k_d\}$, $\psi_1 = \sup\limits_{y,z,t}\{-a(y,z,t)+k_d\}$, and $\phi_0 = \inf\limits_{y} \phi(y)$ where $\phi(y) = -b(y)+k_p$. It is easy to see that $\phi_0 \geqslant k_p - L_1 > 0$ and $\psi_0 \geqslant k_d - L_2 > 0$ by the fact that $k_p > L_1$ and $k_d > L_2$.

We now adopt a similar method as that used for autonomous differential equations (see [15]), and proceed to show that the following quadratic form plus an integral term is indeed a Lyapunov function,

$$V(x,y,z) = [x,y,z]P[x,y,z]^\tau + \int_0^y (\phi(s)-\phi_0)s\mathrm{d}s,$$

where the constant matrix P is

$$P = \frac{1}{2}\begin{bmatrix} \mu k_i & k_i & 0 \\ k_i & \phi_0+\mu\psi & \mu \\ 0 & \mu & 1 \end{bmatrix}, \tag{9}$$

and $\mu > 0$ is a constant defined by $\mu = \frac{2(\phi_0\psi_0+k_i)}{4\phi_0+L_2^2}$.

It is not difficult to verify that P is a positive definite matrix (see Lemma B1 in Appendix B), hence $V(x,y,z)$ is a positive definite function which is radically unbounded in x,y,z.

Next, by simple calculations based on the definitions of P and $A(x,y,z,t)$, it follows that the time derivative of $V(x,y,z)$ along the trajectories of (8), is given by

$$\begin{aligned}\dot{V}(x,y,z) &= [x,y,z](PA(x,y,z,t)+A(x,y,z,t)^\tau P)[x,y,z]^\tau + (\phi(y)-\phi_0)yz \\ &= [x,y,z]\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_i-\mu\phi(y) & \frac{\mu(\psi+a(y,z,t)-k_d)+\phi_0-\phi(y)}{2} \\ 0 & \frac{\mu(\psi+a(y,z,t)-k_d)+\phi_0-\phi(y)}{2} & \mu+a(y,z,t)-k_d \end{bmatrix}\begin{bmatrix} x \\ y \\ z\end{bmatrix} + (\phi(y)-\phi_0)yz \\ &= [x,y,z]\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_i-\mu\phi(y) & \dfrac{\mu(\psi+a(y,z,t)-k_d)}{2} \\ 0 & \dfrac{\mu(\psi+a(y,z,t)-k_d)}{2} & \mu+a(y,z,t)-k_d \end{bmatrix}\begin{bmatrix} x \\ y \\ z\end{bmatrix} \\ &= -[y,z]Q(y,z,t)\begin{bmatrix} y \\ z\end{bmatrix},\end{aligned}$$

where $Q(y,z,t)$ is a symmetric matrix, expressed by

$$Q(y,z,t) = \begin{bmatrix} -k_i+\mu\phi(y) & -\dfrac{\mu}{2}[\psi+a(y,z,t)-k_d] \\ -\dfrac{\mu}{2}[\psi+a(y,z,t)-k_d] & -\mu-a(y,z,t)+k_d \end{bmatrix}.$$

We now prove that $Q(y,z,t)$ is actually positive definite for all $y,z\in\mathbb{R}$ and $t\in\mathbb{R}^+$.

Denote $\alpha = -\frac{\mu}{2}[\psi+a(y,z,t)-k_d]$, and $\beta = -\mu-a(y,z,t)+k_d$, note that by the definitions of ϕ_0, ψ_0, ψ_1 and (B1), (B3) in the Appendix B, we have

$$-k_i+\mu\phi(y) \geqslant -k_i+\mu\phi_0 > 0, \tag{10}$$

$$-\mu+\psi_1 \geqslant \beta \geqslant -\mu+\psi_0 > 0, \tag{11}$$

$$|\psi+a(y,z,t)-k_d| \leqslant L_2, \tag{12}$$

where we have used the fact that $\psi = \frac{\psi_0+\psi_1}{2}$ and $|a(y,z,t)| \leqslant L_2$.

Therefore, by (B2), (10)–(12), we know that $(-k_i+\mu\phi(y))\beta \geqslant (-k_i+\mu\phi_0)(-\mu+\psi_0) > \frac{\mu^2}{4}L_2^2 \geqslant \alpha^2$. By this and (10), (11), it is easy to see that the matrix $Q(y,z,t)$ is positive definite for all $y,\ z,\ t$.

By some standard calculations, we can get the minimum eigenvalue of $Q(y,z,t)$ as

$$\lambda_{\min}\{Q(y,z,t)\}=h(y,\alpha,\beta)\triangleq\frac{1}{2}\Big\{-k_i+\mu\phi(y)+\beta-\sqrt{(-k_i+\mu\phi(y)-\beta)^2+4\alpha^2}\Big\}.$$

Define $\lambda(y)=\inf_{\alpha,\ \beta}h(y,\alpha,\beta)$, where the infimum is taken for all $|\alpha|\leqslant\frac{\mu L_2}{2}$ and $\beta\in[-\mu+\psi_0,-\mu+\psi_1]$.

It is easy to see that $\lambda(\cdot)$ is a positive function of y. We also know that $\lambda(\cdot)$ is a continuous function by Lemma B2 in Appendix B.

Finally, we apply the Theorem A1 in Appendix A to conclude the proof of this theorem. If we take $W(x,y,z)=\lambda(y)(y^2+z^2)$, then all the conditions in Theorem A1 are satisfied. Hence all the solutions of (7) are bounded and satisfy $\lim_{t\to\infty}\lambda(y(t))(y(t)^2+z(t)^2)=0$. Since $y(t)$ is bounded, there exists a positive $\delta>0$ such that $\lambda(y(t))>\delta$. Therefore $\lim_{t\to\infty}(y(t)^2+z(t)^2)=0$, which gives $\lim_{t\to\infty}x_1(t)=y^*$ and $\lim_{t\to\infty}x_2(t)=0$.

Hence, the proof of Theorem 1 is complete.

Proof of Proposition 1. Similarly, we introduce the following notations. Denote $x(t)=\int_0^t e(s)\mathrm{d}s+\frac{f(y^*,0)}{k_i}$, $y(t)=e(t)$, $z(t)=\dot e(t)$, $g(y,z)=-f(y^*-y,-z)+f(y^*,0)$, then Eq. (3) turns into

$$\begin{cases}\dot x=y,\\ \dot y=z,\\ \dot z=g(y,z)-k_ix-k_py-k_dz.\end{cases}\tag{13}$$

By $f\in\mathscr{G}_{L_1,L_2}$, it is easy to see that $g\in\mathscr{G}_{L_1,L_2}$ and $g(0,0)=0$. Hence $(0,0,0)$ is an equilibrium of (13).

Sufficiency. Note that $g(y,z)$ can be expressed as $g(y,z)=b(y)y+a(y,z)z$, where $b(y)$ and $a(y,z)$ are defined by

$$b(y)=\begin{cases}\dfrac{g(y,0)}{y}, & y\neq0,\\ \dfrac{\partial g}{\partial y}(0,0), & y=0,\end{cases}\quad\text{and}\quad a(y,z)=\begin{cases}\dfrac{g(y,z)-g(y,0)}{z}, & z\neq0,\\ \dfrac{\partial g}{\partial z}(y,0), & z=0.\end{cases}$$

Obviously, $b(\cdot)$ and $a(\cdot,\cdot)$ are continuous functions and $b(y)\leqslant L_1,\ a(y,z)\leqslant L_2$ for all y,z by the mean value theorem.

Since $\frac{\partial^2 f}{\partial x_2^2}=0$, it is easy to see that $a(y,z)$ is independent of z, i.e. merely a function of y, denoted by $a(y)$. Denote $\psi(y)=-a(y)+k_d$ and $\phi(y)=-b(y)+k_p$. Then Eq. (13) turns to be

$$\begin{cases}\dot x=y,\\ \dot y=z,\\ \dot z=-k_ix-\phi(y)y-\psi(y)z.\end{cases}$$

By Theorem A2 in Appendix A, the above system is globally asymptotically stable if $\phi_0>0$, $\psi_0>0$ and $\phi_0\psi_0>k_i$. This is easy to verify because by the definition of ϕ_0,ψ_0 and $(k_p,k_i,k_d)\in\Omega^{'}_{\rm pid}$, we have $\phi_0\geqslant k_p-L_1,\psi_0\geqslant k_d-L_2$ and hence $\phi_0\psi_0\geqslant(k_p-L_1)(k_d-L_2)>k_i$. Thus $(0,0,0)$ is a globally asymptotically stable equilibrium point of the above system for any $f\in\mathscr{G}_{L_1,L_2}$, which gives $\lim_{t\to\infty}x_1(t)=y^*$ and $\lim_{t\to\infty}x_2(t)=0$.

Necessity. Next, we proceed to show that if for any $f\in\mathscr{G}_{L_1,L_2}$ and any setpoint y^*, the closed-loop system under the PID controller (2) satisfies $\lim_{t\to\infty}e(t)=0$ for any initial values $(x_1(0),x_2(0))$, then we must have $(k_p,k_i,k_d)\in\Omega^{'}_{\rm pid}$. We now prove this claim by the contradiction argument.

If $(k_p,k_i,k_d)\notin\Omega^{'}_{\rm pid}$, we consider two cases separately.

Case A. $k_i=0$. Let us take $f=L_1x_1+L_2x_2$ and $y^*\neq0$, and then $f\in\mathscr{G}_{L_1,L_2}$. Denote $e_1=y^*-x_1$, $e_2=-x_2$, and then Eq. (3) turns to be

$$\begin{cases}\dot e_1=e_2,\\ \dot e_2=(L_1-k_p)e_1+(L_2-k_d)e_2-L_1y^*.\end{cases}$$

If $k_p \neq L_1$, the equilibrium point of this system is $(\frac{L_1 y^*}{L_1 - k_p}, 0)$, thus $\lim_{t\to\infty} e_1(t) = \frac{L_1 y^*}{L_1 - k_p} \neq 0$ for initial value $(e_1(0), e_2(0)) = (\frac{L_1 y^*}{L_1 - k_p}, 0)$.

If $k_p = L_1$, then $\dot{e}_2 = (L_2 - k_d)e_2 - L_1 y^*$, and it is easy to see that $\lim_{t\to\infty} |e_1(t)| = \infty$ for any $k_d \in \mathbb{R}$ and any initial value $(e_1(0), e_2(0))$.

Case B. $k_i \neq 0$. Let us take $f = L_1 x_1 + L_2 x_2 + c$, where $c \in \mathbb{R}$ is to be determined. Then Eq. (13) turns to be

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -k_i x + (-k_p + L_1)y + (-k_d + L_2)z. \end{cases} \tag{14}$$

The sufficient and necessary condition for (14) to be asymptotically stable is $k_i > 0,\ k_p - L_1 > 0,\ k_d - L_2 > 0,\ (k_p - L_1)(k_d - L_2) > k_i$ by the Routh-Hurwitz stability criterion. Hence if $(k_p, k_i, k_d) \notin \Omega'_{\text{pid}}$, then there exists an eigenvalue λ of A whose real part $\Re(\lambda)$ will be $\geqslant 0$, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_i & -k_p + L_1 & -k_d + L_2 \end{bmatrix}.$$

We first consider the case where the three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of A are distinct . Since $-k_i = \lambda_1\lambda_2\lambda_3 \neq 0$, then $\lambda_1, \lambda_2, \lambda_3$ are all nonzero. Denote

$$P = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}.$$

By simple calculations, we have

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \mathrm{e}^{At} \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = P \begin{bmatrix} \mathrm{e}^{\lambda_1 t} & 0 & 0 \\ 0 & \mathrm{e}^{\lambda_2 t} & 0 \\ 0 & 0 & \mathrm{e}^{\lambda_3 t} \end{bmatrix} P^{-1} \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{\lambda_1 t} & \mathrm{e}^{\lambda_1 t} & \mathrm{e}^{\lambda_1 t} \\ \lambda_1 \mathrm{e}^{\lambda_1 t} & \lambda_2 \mathrm{e}^{\lambda_2 t} & \lambda_3 \mathrm{e}^{\lambda_3 t} \\ \lambda_1^2 \mathrm{e}^{\lambda_1 t} & \lambda_2^2 \mathrm{e}^{\lambda_2 t} & \lambda_3^2 \mathrm{e}^{\lambda_3 t} \end{bmatrix} P^{-1} \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix}.$$

Without loss of generality, we assume that $\Re(\lambda_1) \geqslant 0$. If λ_1 is a real number, we choose $c = k_i - L_1 y^*$ and $(x_1(0), x_2(0)) = (y^* - \lambda_1, -\lambda_1^2)$ such that

$$\begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda_1 \\ \lambda_1^2 \end{bmatrix} = P \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

then $y(t) = \lambda_1 \mathrm{e}^{\lambda_1 t}$, which does not tend to 0 as $t \to \infty$. If λ_1 is not real, it is easy to see that either $(x(0), y(0), z(0)) = (1, \Re(\lambda_1), \Re(\lambda_1^2))$ or $(x(0), y(0), z(0)) = (1, \Im(\lambda_1), \Im(\lambda_1^2))$ would satisfy $\lim_{t\to\infty} e(t) \neq 0$, where $\Re(\cdot)$ and $\Im(\cdot)$ denote the real part and the imaginary part of a complex number, respectively.

If A has multiple eigenvalues λ, then all the eigenvalues of A is real since A is 3×3 real coefficient matrix.

(a): A is similar to $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$. The similar matrix can be chosen as $P = \begin{bmatrix} 1 & 1 & 1 \\ \lambda & \lambda+1 & \lambda+1 \\ \lambda^2 & \lambda^2+2\lambda & (\lambda+1)^2 \end{bmatrix}$, therefore

$$\mathrm{e}^{At} = \begin{bmatrix} 1 & 1 & 1 \\ \lambda & \lambda+1 & \lambda+1 \\ \lambda^2 & \lambda^2+2\lambda & (\lambda+1)^2 \end{bmatrix} \begin{bmatrix} \mathrm{e}^{\lambda t} & t\mathrm{e}^{\lambda t} & \frac{t^2}{2}\mathrm{e}^{\lambda t} \\ 0 & \mathrm{e}^{\lambda t} & t\mathrm{e}^{\lambda t} \\ 0 & 0 & \mathrm{e}^{\lambda t} \end{bmatrix} P^{-1}.$$

(b): A is similar to $\begin{bmatrix}\lambda & 1 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda_3\end{bmatrix}$. The similar matrix can be chosen as $P=\begin{bmatrix}1 & 1 & 1\\ \lambda & 1+\lambda & \lambda_3\\ \lambda^2 & \lambda^2+2\lambda & \lambda_3^2\end{bmatrix}$, then

$$\mathrm{e}^{At}=\begin{bmatrix}1 & 1 & 1\\ \lambda & 1+\lambda & \lambda_3\\ \lambda^2 & \lambda^2+2\lambda & \lambda_3^2\end{bmatrix}\begin{bmatrix}\mathrm{e}^{\lambda t} & t\mathrm{e}^{\lambda t} & 0\\ 0 & \mathrm{e}^{\lambda t} & 0\\ 0 & 0 & \mathrm{e}^{\lambda_3 t}\end{bmatrix}P^{-1}.$$

We choose $c=k_i-L_1y^*$ and $(x_1(0),x_2(0))=(y^*-\lambda_1,-\lambda_1^2)$ both in case (a) and (b) such that

$$\begin{bmatrix}x(0)\\ y(0)\\ z(0)\end{bmatrix}=\begin{bmatrix}1\\ \lambda_1\\ \lambda_1^2\end{bmatrix}=P\begin{bmatrix}1\\ 0\\ 0\end{bmatrix},$$

then $y(t)=\lambda\mathrm{e}^{\lambda t}$, which does not tend to zero, which contradicts with $e(t)\to 0$ since $y(t)=e(t)$.

Proof of Theorem 2.

Sufficiency. Let $f\in\mathscr{F}_{L_1,L_2,y^*}$ and $(k_p,k_d)\in\Omega_{pd}$. Denote $e_1=y^*-x_1=e, e_2=-x_2$, $g_1(e_1,e_2)=-f_1(y^*-e_1,-e_2)$ and $g_2(e_1,e_2)=-f_2(y^*-e_1,-e_2)$. Then the closed-loop system (6) turns to be

$$\begin{cases}\dot{e_1}=g_1(e_1,e_2),\\ \dot{e_2}=g_2(e_1,e_2)-k_pe_1-k_dg_1(e_1,e_2).\end{cases}\tag{15}$$

Since $f(y^*,0)=0$, then $(0,0)$ is an equilibrium of (15). Denote the vector field of (15) by $F(e_1,e_2)$, i.e.,

$$F(e_1,e_2)=\begin{bmatrix}g_1(e_1,e_2)\\ g_2(e_1,e_2)-k_pe_1-k_dg_1(e_1,e_2)\end{bmatrix}.$$

Then the Jacobian matrix $DF(e_1,e_2)$ of F is

$$\begin{bmatrix}\dfrac{\partial g_1}{\partial e_1}(e_1,e_2) & \dfrac{\partial g_1}{\partial e_2}(e_1,e_2)\\ \dfrac{\partial g_2}{\partial e_1}(e_1,e_2)-k_d\dfrac{\partial g_1}{\partial e_1}(e_1,e_2)-k_p & \dfrac{\partial g_2}{\partial e_2}(e_1,e_2)-k_d\dfrac{\partial g_1}{\partial e_2}(e_1,e_2)\end{bmatrix}.$$

Note that the trace of $DF(e_1,e_2)$ satisfies

$$\frac{\partial g_1}{\partial e_1}+\frac{\partial g_2}{\partial e_2}-k_d\frac{\partial g_1}{\partial e_2}\leqslant(L_2-k_d)\frac{\partial g_1}{\partial e_2}<0,$$

and the determinant of $DF(e_1,e_2)$ satisfies

$$\frac{\partial g_1}{\partial e_1}\left(\frac{\partial g_2}{\partial e_2}-k_d\frac{\partial g_1}{\partial e_2}\right)-\frac{\partial g_1}{\partial e_2}\left(\frac{\partial g_2}{\partial e_1}-k_d\frac{\partial g_1}{\partial e_1}-k_p\right)=\frac{\partial g_1}{\partial e_1}\frac{\partial g_2}{\partial e_2}-\frac{\partial g_1}{\partial e_2}\frac{\partial g_2}{\partial e_1}+k_p\frac{\partial g_1}{\partial e_2}\geqslant(k_p-L_1)\frac{\partial g_1}{\partial e_2}>0$$

for any $(e_1,e_2)\in\mathbb{R}^2$, which implies that the two eigenvalues of $DF(e_1,e_2)$ have negative real parts for any (e_1,e_2). By the Markus-Yamabe's theorem (Theorem A3 in Appendix A), we know that $(0,0)$ is globally asymptotically stable, thus we have $\lim_{t\to\infty}x_1(t)=y^*$ and $\lim_{t\to\infty}x_2(t)=0$ for any initial value $(x_1(0),x_2(0))\in\mathbb{R}^2$.

Necessity. Next, we prove a result slightly stronger than the necessity of this theorem. In fact, we will prove that if Eq. (6) under the PD control satisfies $\lim_{t\to\infty}x_1(t)=y^*$ for all the $f\in\mathscr{F}_{L_1,L_2,y^*}$ with any initial $(x_1(0),x_2(0))$, then we must have $(k_p,k_d)\in\Omega_{pd}$.

Let $f_1(x_1,x_2)=x_2$ and $f_2(x_1,x_2)=L_1(x_1-y^*)+L_2x_2$, and then $f=(f_1,f_2)\in\mathscr{F}_{L_1,L_2,y^*}$ and Eq. (6) turns to be

$$\begin{cases}\dot{e_1}=e_2,\\ \dot{e_2}=(L_1-k_p)e_1+(L_2-k_d)e_2,\end{cases}$$

where $e_1 = y^* - x_1$. If $(k_p, k_d) \notin \Omega_{pd}$, then either $L_1 - k_p \geqslant 0$ or $L_2 - k_d \geqslant 0$. Therefore the matrix $A = \begin{bmatrix} 0 & 1 \\ L_1 - k_p & L_2 - k_d \end{bmatrix}$ is not a Hurwitz matrix by the Routh-Hurwitz criteria.

Case 1. A has two distinct eigenvalues λ_1 and λ_2. It is easy to verify that

$$\mathrm{e}^{At} = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 \mathrm{e}^{\lambda_1 t} - \lambda_1 \mathrm{e}^{\lambda_2 t} & \mathrm{e}^{\lambda_2 t} - \mathrm{e}^{\lambda_1 t} \\ \lambda_1 \lambda_2 (\mathrm{e}^{\lambda_1 t} - \mathrm{e}^{\lambda_2 t}) & \lambda_2 \mathrm{e}^{\lambda_2 t} - \lambda_1 \mathrm{e}^{\lambda_1 t} \end{bmatrix}.$$

Without loss of generality, we assume $\Re(\lambda_1) \geqslant 0$. Then for the initial value $(e_1(0), e_2(0)) = (1, 0)$, we have

$$e_1(t) = \mathrm{e}^{At} \begin{bmatrix} e_1(0) \\ e_2(0) \end{bmatrix} = \frac{\lambda_2 \mathrm{e}^{\lambda_1 t} - \lambda_1 \mathrm{e}^{\lambda_2 t}}{\lambda_2 - \lambda_1},$$

which does not tend to zero as $t \to \infty$.

Case 2. A has a multiple eigenvalue λ.

$$\mathrm{e}^{At} = \begin{bmatrix} 1 & 1 \\ \lambda & 1 + \lambda \end{bmatrix} \begin{bmatrix} \mathrm{e}^{\lambda t} & t\mathrm{e}^{\lambda t} \\ 0 & \mathrm{e}^{\lambda t} \end{bmatrix} \begin{bmatrix} 1 + \lambda & -1 \\ -\lambda & 1 \end{bmatrix} = \begin{bmatrix} (1 - \lambda t)\mathrm{e}^{\lambda t} & * \\ -\lambda^2 t \mathrm{e}^{\lambda t} & * \end{bmatrix},$$

where those $*$ mean that we don't care about what they are in the proof of our theorem. Then for the initial value $(e_1(0), e_2(0)) = (1, 0)$, we have $e_1(t) = (1 - \lambda t)\mathrm{e}^{\lambda t}$. Obviously, $\lim_{t \to \infty} e_1(t) \neq 0$ since λ is a real number and $\lambda \geqslant 0$, which contradicts with $e_1(t) \to 0$, and hence we must have $(k_p, k_d) \in \Omega_{pd}$. The proof of Theorem 2 is complete.

Proof of Proposition 2.

Sufficiency. Let $f \in \mathscr{F}_L$ and $f(y^*, t) = f(y^*, 0)$, $\forall t \in \mathbb{R}^+$ for some y^*. Suppose $(k_p, k_i) \in \Omega_{pi}$ be given. Denote $e_0(t) = \int_0^t e(s)\mathrm{d}s + \frac{f(y^*, 0)}{k_i}$ and $h(e, t) = -f(y^* - e, t) + f(y^*, t)$.

Then we have

$$\begin{cases} \dot{e_0} = e, \\ \dot{e} = h(e, t) - k_i e_0 - k_p e, \end{cases} \tag{16}$$

where $h(0, t) = 0$ for any $t \in \mathbb{R}^+$ and $h \in \mathscr{F}_L$.

Now, we construct the Lyapunov function $V(e_0, e) = \frac{1}{2}(e_0^2 + 2\epsilon e_0 e + \frac{1}{k_i} e^2)$, where $\epsilon > 0$ is a number to be determined. Then

$$\begin{aligned} \dot{V}(e_0, e) &= e_0 e + \epsilon e^2 + \left(\epsilon e_0 + \frac{1}{k_i} e\right)(h(e, t) - k_i e_0 - k_p e) \\ &= -k_i \epsilon e_0^2 - k_p \epsilon e_0 e + \epsilon e_0 h(e, t) + \frac{1}{k_i} e h(e, t) + \left(\epsilon - \frac{k_p}{k_i}\right) e^2 \\ &\leqslant -k_i \epsilon e_0^2 + (k_p + L)\epsilon |e_0 e| + \left(\epsilon + \frac{L - k_p}{k_i}\right) e^2. \end{aligned}$$

It is easy to see that for sufficiently small $\epsilon > 0$, the following three inequalities hold,

$$\epsilon^2 < \frac{1}{k_i}, \tag{17}$$

$$\epsilon < \frac{k_p - L}{k_i}, \tag{18}$$

$$4 k_i \epsilon \left(\frac{k_p - L}{k_i} - \epsilon\right) > (k_p + L)^2 \epsilon^2. \tag{19}$$

It is not difficult to see that, V is a positive definite quadratic form if (17) holds and $\dot{V}$ is a negative definite function of e_0, e if both (18) and (19) hold. Therefore, the closed-loop system (16) is globally exponentially stable, which in turn gives $\lim_{t \to \infty} x(t) = y^*$ with an exponential convergence rate for any initial value $x(0) \in \mathbb{R}$.

Necessity. We use contradiction argument and assume that $e(t)\to 0$ for all $f\in\mathscr{F}_L$ satisfying $f(y^*,t)=f(y^*,0)$ and for all initial value but $(k_p,k_i)\notin\Omega_{pi}$. We consider two cases separately.

(i) If $k_i=0$, let $f(x)=Lx$, then $\dot e=(L-k_p)e-Ly^*$. Obviously, $e(t)$ does not tend to 0 as long as $y^*\neq 0$ for any initial value $e(0)\in\mathbb{R}$.

(ii) If $k_i\neq 0$, we take $f(x)=Lx$. Then Eq. (16) becomes

$$\begin{cases}\dot{e_0}=e,\\ \dot e=-k_ie_0+(L-k_p)e.\end{cases}\tag{20}$$

Since $(k_p,k_i)\notin\Omega_{pi}$, then there exists at least one eigenvalue λ of $A=\begin{bmatrix}0 & 1\\ -k_i & L-k_d\end{bmatrix}$ whose real part satisfies $\Re(\lambda)\geqslant 0$. Similar to the necessity proof of Theorem 2, we can show that $e(t)\nrightarrow 0$ as $t\to\infty$ for the initial value $(e_0(0),e(0))=(1,0)$, which contradicts with our assumption $e(t)\to 0$.

This completes the proof of Proposition 2.

Proof of Remark 3. We only consider the case in Theorem 1, since the other conclusions are obvious.

It suffices to show that for any $b\geqslant\underline{b}$, we have $(bk_p,bk_i,bk_d)\in\Omega_{\rm pid}$ as long as $(\underline{b}k_p,\underline{b}k_i,\underline{b}k_d)\in\Omega_{\rm pid}$. The first three inequalities in $\Omega_{\rm pid}$ are easy to verify, and we only need to prove the fourth inequality holds for any $b\geqslant\underline{b}$.

Define $k(b)=(bk_p-L_1)(bk_d-L_2)-bk_i-L_2\sqrt{bk_i(bk_d+L_2)}$. By some simple calculations, using the assumption that $(\underline{b}k_p,\underline{b}k_i,\underline{b}k_d)\in\Omega_{\rm pid}$, we have

$$\begin{aligned}
k'(b)&=2bk_pk_d-(L_1k_d+L_2k_p)-k_i-L_2\frac{2bk_ik_d+k_iL_2}{2\sqrt{bk_i(bk_d+L_2)}}\\
&\geqslant bk_pk_d+\underline{b}k_pk_d-(L_1k_d+L_2k_p)-k_i-L_2\frac{2bk_ik_d+k_iL_2}{2\sqrt{bk_i(bk_d+L_2)}}\\
&=\left(bk_pk_d-\frac{L_1L_2}{\underline{b}}\right)+\frac{1}{\underline{b}}\left\{(\underline{b}k_p-L_1)(\underline{b}k_d-L_2)-\underline{b}k_i\right\}-L_2\frac{2bk_ik_d+k_iL_2}{2\sqrt{bk_i\,(bk_d+L_2)}}\\
&\geqslant\left(bk_pk_d-\frac{L_1L_2}{\underline{b}}\right)+L_2\left(\frac{\sqrt{\underline{b}k_i\,(\underline{b}k_d+L_2)}}{\underline{b}}-\frac{2bk_ik_d+k_iL_2}{2\sqrt{bk_i\,(bk_d+L_2)}}\right)\\
&\geqslant\sqrt{k_i}L_2\left(\frac{\sqrt{\underline{b}\,(\underline{b}k_d+L_2)}}{\underline{b}}-\frac{2bk_d+L_2}{2\sqrt{b\,(bk_d+L_2)}}\right)\\
&\geqslant\sqrt{k_i}L_2\left(\frac{\sqrt{\underline{b}\,(\underline{b}k_d+L_2)}}{\underline{b}}-\frac{bk_d+L_2}{\sqrt{b\,(bk_d+L_2)}}\right)\\
&\geqslant\sqrt{k_i}L_2\left(\sqrt{k_d+L_2/\underline{b}}-\sqrt{k_d+L_2/b}\right)\geqslant 0.
\end{aligned}$$

Therefore, $k(\cdot)$ is an increasing function of b, i.e. $k(b)\geqslant k(\underline{b})>0$ and hence $(bk_p,bk_i,bk_d)\in\Omega_{\rm pid}$ for any $b\geqslant\underline{b}$.

5 Conclusion

In this paper, we have presented a mathematical theory together with a design method for the well-known PID control of a basic class of second order nonlinear uncertain dynamical systems, and investigated several related issues including global stabilization, asymptotic regulation, and minimality of the feedback gains. We remark that the PID design rules given in this paper is quite simple and is almost necessary for global stabilization. We also remark that both our theory and design methods demonstrate that the PID controller is indeed quite robust with respect to both the design parameters and the nonlinear uncertainties. For future investigation, it is desirable to extend the results and methods to more general nonlinear uncertain systems, albeit we know that the standard PID controller can neither be used to globally stabilize nonlinear uncertain systems with nonlinear growth rate faster than linear, nor be used to globally stabilize third-order (or higher) nonlinear uncertain systems in general (see [14]). Moreover, it would be interesting to consider more complicated situations such as time-delayed inputs and sampled-data PID controllers under a prescribed sampling rate, and to connect the related boundaries established for the maximum capability of the general feedback mechanism (cf. e.g. [12, 16]).

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Conflict of interest The authors declare that they have no conflict of interest.

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Appendix A

Theorem A1 (LaSalle-Yoshizawa). Consider the following time-varying nonlinear system

$$\dot{x} = f(x, t),$$

where $f : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is a locally Lipschitz in x uniformly in t and piecewise continuous in t. Let $x = 0$ be an equilibrium point of this system, i.e. $f(0, t) = 0, \forall t \geqslant 0$. Let $V(x) \in C^1(\mathbb{R}^n \to \mathbb{R}^+)$ be continuously differentiable, positive definite and radially unbounded function such that

$$\dot{V} = \frac{\partial V}{\partial x} f(x, t) \leqslant -W(x) \leqslant 0, \quad \forall t \geqslant 0, \quad \forall x \in \mathbb{R}^n,$$

where W is a continuous function. Then, all solutions of this system are globally uniformly bounded and satisfy

$$\lim_{t\to\infty} W(x(t)) = 0.$$

See [A1,A2] for detailed discussion.

Theorem A2. Consider the following third order autonomous equation,

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -cx - \phi(y)y - \psi(y)z, \end{cases}$$

where $\phi(y)$ and $\psi(y)$ are continuous functions of y and c is a constant, and then the orign is globally asymptotically stable if $\inf\limits_{y\in\mathbb{R}} \phi(y) > 0$, $\inf\limits_{y\in\mathbb{R}} \psi(y) > 0$ and $\inf\limits_{y\in\mathbb{R}} \phi(y) \cdot \inf\limits_{y\in\mathbb{R}} \psi(y) > c$.

See Theorem 4.9 in [15] for detailed discussion.

The result in the following theorem was a two-dimensional version of a general conjecture on globally asymptotical stability, called Markus-Yamabe conjecture (or Jacobian conjecture), which was proved to be true for the two-dimensional case in [A3], see also [A3–A5].

Theorem A3 (Markus-Yamabe). Let $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, $f(0) = 0$. Consider the plane differential equation,

$$\dot{x} = f(x).$$

If for any $x \in \mathbb{R}^2$, the eigenvalues of the Jacobian matrix $Df(x)$ of f at x have negative real parts, then the zero solution of the differential equation is globally asymptotically stable.

Appendix B

Lemma B1. The matrix P defined by (9) is positive definite.

Proof. To show that the matrix P is positive definite, we first show that the following three inequalities hold,

$$\mu < \psi_0, \tag{B1}$$

$$4(-k_i + \mu\phi_0)(-\mu + \psi_0) > \mu^2 L_2^2, \tag{B2}$$

$$-k_i + \mu\phi_0 > 0. \tag{B3}$$

Note that, by the definition of $\Omega_{\rm pid}$, we have $(k_p - L_1)(k_d - L_2) > k_i + L_2\sqrt{k_i(k_d + L_2)}$, thus $\phi_0\psi_0 > k_i + L_2\sqrt{k_i(k_d + L_2)}$, hence by $0 < \psi_0 \leqslant L_2 + k_d$, we have $\phi_0\psi_0 - k_i > L_2\sqrt{k_i\psi_0} > 0$, i.e.

$$(\phi_0\psi_0 - k_i)^2 > L_2^2 k_i\psi_0. \tag{B4}$$

Hence Eq. (B1) is true since

$$\mu - \psi_0 = \frac{-(2\phi_0\psi_0 - 2k_i + L_2^2\psi_0)}{4\phi_0 + L_2^2} < 0.$$

Furthermore, by rewriting (B4) as

$$(\phi_0\psi_0 + k_i)^2 - (4\phi_0 + L_2^2)k_i\psi_0 > 0, \tag{B5}$$

we know that

$$\begin{aligned} 4(-k_i + \mu\phi_0)(-\mu + \psi_0) - \mu^2 L_2^2 &= -(4\phi_0 + L_2^2)\mu^2 + 4(\phi_0\psi_0 + k_i)\mu - 4k_i\psi_0 \\ &= \frac{4[-(4\phi_0 + L_2^2)k_i\psi_0 + (\phi_0\psi_0 + k_i)^2]}{4\phi_0 + L_2^2} > 0. \end{aligned}$$

Hence Eq. (B2) is also valid, and consequently (B3) follows from (B1) and (B2).

Next, by (B1) and (B3), it is easy to verify that P is positive definite, since the following three inequalities hold,

$$\mu k_i > 0,$$

$$\det\begin{bmatrix} \mu k_i & k_i \\ k_i & \phi_0 + \mu\psi \end{bmatrix} = \mu k_i(\phi_0 + \mu\psi) - k_i^2 \geqslant k_i(\mu\phi_0 - k_i + \mu^2\psi_0) > k_i\mu^2\psi_0 > 0,$$

and

$$\det\begin{bmatrix} \mu k_i & k_i & 0 \\ k_i & \phi_0 + \mu\psi & \mu \\ 0 & \mu & 1 \end{bmatrix} = k_i(\mu\phi_0 + \mu^2\psi - k_i - \mu^3) > k_i(\mu^2\psi_0 - \mu^3) = k_i\mu^2(\psi_0 - \mu) > 0.$$

Lemma B2. Let $f(\cdot,\cdot)$ be a continuous function defined in $\mathbb{R} \times K$, where $K \subset \mathbb{R}^n$ is a compact set. Define $h(y) = \inf\limits_{w\in K} f(y, w)$, and then $h(\cdot)$ is a continuous function of y.

Proof. Firstly, we prove that $A_c = \{y \in \mathbb{R} : h(y) > c\}$ is open for any $c \in \mathbb{R}$. Assume that $h(y_0) > c$ for some y_0, and then $f(y_0, w) \geqslant h(y_0) > c$ for any $w \in K$ by the definition of $h(\cdot)$. Therefore, there exists a $\delta > 0$, such that $f(y, w) \geqslant \frac{h(y_0)+c}{2}$ for any y which satisfies $|y - y_0| < \delta$ and for any $w \in K$, since f is uniformly continuous in $[y_0 - 1, y_0 + 1] \times K$. Hence $(y_0 - \delta, y_0 + \delta) \subset A_c$, which gives the open property of A_c.

Next, we prove $B_c = \{y \in \mathbb{R} : h(y) < c\}$ is also open for any c. Assume that $h(y_0) < c$ for some y_0. Since K is compact, there exists a $w_0 \in K$ such that $h(y_0) = f(y_0, w_0)$. By the continuous property of f at the point (y_0, w_0), we conclude that there exists a $\eta > 0$ such that $f(y, w_0) \leqslant \frac{h(y_0)+c}{2}$ for any y which satisfies $|y - y_0| < \eta$. Hence $h(y) \leqslant f(y, w_0) \leqslant \frac{h(y_0)+c}{2} < c$ for $y \in (y_0 - \eta, y_0 + \eta)$. We conclude that B_c is also open for any $c \in \mathbb{R}$.

In conclusion, $h(\cdot)$ is a continuous function of y.

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