$$L = 2$$

$$\begin{array}{cccc}
E_1 = 0 & E_2 = 0 & E_3 = 0 & E_4 = 0 \\
\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} & \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} & \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \\
E_5 = 0 & E_6 = 0 & E_7 = 0 & E_8 = 0 \\
\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\
E_9 = 0 & E_{10} = 0 & E_{11} = 0 & E_{12} = 0 \\
\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} & \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \\
E_{13} = 8J & E_{14} = 8J & E_{15} = -8J & E_{16} = -8J \\
\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} & \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\end{array}$$

T_c

$$E = -J \sum_{<kl>}^{N} s_k s_l$$

$s_k = \pm 1 N J$

$< kl > J > 0 L N = L^2 M = 2^N$

$$Z = \sum_{i=1}^{M} e^{-\beta E_i}$$

$L = 2 L = 2 \Rightarrow N = 2 \cdot 2 = 4 \Rightarrow M = 2^4 = 16 L = 2$

$$Z = e^{-\beta E_1} + ... e^{-\beta E_{16}}$$

$\begin{bmatrix} s(0,0) & s(0,1) \\ s(1,0) & s(1,1) \end{bmatrix}$

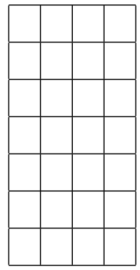
$s(1,1) s(0,1) s(1,0) s_l s_k s_l s_k$

$$E = -J\Big(s(0,0) \cdot \big[s(1,0) + s(0,1)\big] + s(0,1) \cdot \big[s(1,1) + s(0,0)\big]$$
$$+ s(1,0) \cdot \big[s(0,0) + s(1,1)\big] + s(1,1) \cdot \big[s(0,1) + s(1,0)\big]\Big)$$

$E \in \{-8J, 0, 8J\} \{2, 12, 2\}$

$$Z = 2e^{-\beta(-8J)} + 12e^{-\beta \cdot 0} + 2e^{-\beta \cdot 8J} = 2(e^{\beta \cdot 8J} + e^{-\beta \cdot 8J}) + 12$$
$$= 4\cosh(8J\beta) + 12$$

$$L = 2$$



$\cosh(x) = \frac{1}{2}(e^{-x} + e^{x})$

$$\langle E \rangle = \frac{1}{Z}\sum_{i=1}^{M} E_i e^{-\beta E_i} = \frac{1}{Z}\big[2\cdot(-8J)e^{8J\beta} + 0 + 2\cdot 8Je^{-8J\beta}\big] = \frac{1}{Z}\big[-16Je^{8J\beta} + 16Je^{-8J\beta}\big]$$
$$= -\frac{16J}{Z}\big[-e{-}8J\beta + e^{8J\beta}\big] = -\frac{32J}{Z}\sinh(8J\beta) = -\frac{8J\sinh(8J\beta)}{\cosh(8J\beta)+3}$$

$\sinh(x) = \frac{1}{2}(-e^{-x} + e^{x})$

$$C_v = \frac{1}{k_b T}\sigma_E^2 = \frac{1}{k_b T}\big(\langle E^2\rangle - \langle E\rangle^2\big)$$

$$\langle E^2 \rangle = \frac{1}{Z}\sum_{i=1}^{M} E_i^2 e^{-\beta E_i} = \frac{1}{Z}\big[2\cdot(-8J)^2 e^{8J\beta} + 0 + 2\cdot(8J)^2 e^{-8J\beta}\big] = \frac{128J^2}{Z}\big[e^{8J\beta} + e^{-8J\beta}\big]$$
$$= \frac{128J^2\cdot 2\cosh(8J\beta)}{4\cosh(8J\beta)+12} = \frac{64J^2\cosh(8J\beta)}{\cosh(8J\beta)+3}$$

$$C_v = \frac{1}{k_b T}\left[\frac{64J^2\cosh(8J\beta)}{\cosh(8J\beta)+3} - \left(-\frac{8J\sinh(8J\beta)}{\cosh(8J\beta)+3}\right)^2\right] = \frac{1}{k_b T}\left[\frac{64J^2\cosh(8J\beta)}{\cosh(8J\beta)+3} - \frac{64J^2\sinh^2(8J\beta)}{(\cosh(8J\beta)+3)^2}\right]$$
$$= \frac{64J^2}{k_b T}\left[\frac{\cosh(8J\beta)(\cosh(8J\beta)+3) - \sinh^2(8J\beta)}{(\cosh(8J\beta)+3)^2}\right] = \frac{64J^2}{k_b T}\left[\frac{\cosh^2(8J\beta) + 3\cosh^2(8J\beta) - \sinh^2(8J\beta)}{(\cosh(8J\beta)+3)^2}\right]$$
$$= \frac{64J^2\beta}{T}\left[\frac{1+3\cosh(8J\beta)}{(\cosh(8J\beta)+3)^2}\right]$$

$\mathcal{M}s2\times 2\mathcal{M} \in \{-4,-2,0,2,4\}$

$$\langle \mathcal{M} \rangle = \frac{1}{Z}\sum_{i=1}^{M}\mathcal{M}e^{-\beta E_i}$$

$$\langle \mathcal{M}^2 \rangle = \frac{1}{Z}\sum_{i=1}^{M}\mathcal{M}^2 e^{-\beta E_i}$$

$$\chi = \frac{1}{k_b T}\sigma_{\mathcal{M}} = \frac{1}{k_b T}\big(\langle\mathcal{M}^2\rangle - \langle\mathcal{M}\rangle^2\big).$$

$$L = 2$$

$L\langle\mathcal{M}\rangle$

$|\mathcal{M}| \in \{0, 2, 4\}$

$$\begin{aligned}
\langle|\mathcal{M}|\rangle &= \frac{1}{Z}\sum_{i=1}^{M}|\mathcal{M}|e^{-\beta E_i} = \frac{1}{Z}\Big(2\cdot 4e^{-\beta(-8J)} + 8\cdot 2e^{0} + 0\cdot 4e^{0} + 0\cdot 2e^{-\beta 8J}\Big) \\
&= \frac{8}{Z}\big(e^{8J\beta}+2\big) = \frac{8(e^{8J\beta}+2)}{4\cosh(8j\beta)+12} = \frac{2(e^{8J\beta}+2)}{\cosh(8j\beta)+3}.
\end{aligned}$$

$$\begin{aligned}
\langle\mathcal{M}^2\rangle &= \frac{1}{Z}\sum_{i=1}^{M}\mathcal{M}^2e^{-\beta E_i} = \frac{1}{Z}\big(2\cdot 4^2e^{-\beta(-8J)} + 8\cdot 2^2e^{0} + 0 + 0\big) = \frac{32}{Z}\big(e^{8J\beta}+1\big) \\
&= \frac{8\big(e^{8J\beta}+1\big)}{\cosh(8J\beta)+3}.
\end{aligned}$$

$$\begin{aligned}
\chi &= \frac{1}{k_bT}\big(\langle\mathcal{M}^2\rangle - \langle|\mathcal{M}|\rangle^2\big) = \frac{1}{k_bT}\left[\frac{8(e^{8J\beta}+1)}{\cosh(8J\beta)+3} - \left(\frac{2(e^{8J\beta}+2)}{\cosh(8j\beta)+3}\right)^2\right] \\
&= \frac{1}{k_bT}\left[\frac{8(e^{8J\beta}+1\big)(\cosh(8j\beta)+3) - 4(e^{8J\beta}+2)^2}{(\cosh(8j\beta)+3)^2}\right] \\
&= 4\beta\left[\frac{2(e^{8J\beta}+1\big)(\cosh(8j\beta)+3) - (e^{8J\beta}+2)^2}{(\cosh(8j\beta)+3)^2}\right]
\end{aligned}$$

$$\begin{aligned}
T' &= T\frac{k_b}{J} \\
\Rightarrow T &= T'\frac{J}{k_b} \\
\Rightarrow \beta &= \frac{1}{k_bT} = \frac{1}{k_b\cdot T'\frac{J}{k_b}} = \frac{1}{T'J}
\end{aligned}$$

$J = 1$

$$\beta = \frac{1}{T'}$$

T'

$$C_v' = \frac{C_v}{k_b} = \frac{64}{T'^2}\frac{1+3\cosh(8/T')}{(\cosh(8/T')+3)^2}$$

$T'2 \times 2$

$$\langle E \rangle = -\frac{8\sinh(8/T')}{\cosh(8/T')+3}$$
$$\langle E^2 \rangle = \frac{64J^2\cosh(8/T')}{\cosh(8/T')+3}$$
$$\langle |\mathcal{M}| \rangle = \frac{2(e^{8/T'}+2)}{\cosh(8/T')+3}$$
$$\langle \mathcal{M}^2 \rangle = \frac{8\big(e^{8/T'}+1\big)}{\cosh(8/T')+3}$$
$$\chi = \frac{4}{T'}\frac{2(e^{8/T'}+1\big)(\cosh(8/T')+3)-(e^{8/T'}+2)^2}{(\cosh(8/T')+3)^2}$$

$L = 2$
T_C

$$\langle \mathcal{M}(T) \rangle \sim (T - T_C)^{\beta} \,,$$

$\beta = 1/8$

$$C_V(T) \sim |T_C - T|^{\alpha} \,,$$

$$\chi(T) \sim |T_C - T|^{\gamma} \,,$$

$\alpha = 0 \gamma = 7/4 T >> T_C T T_C \xi T_C$

$$\xi(T) \sim |T_C - T|^{-\nu} \,.$$

ξ

$$T_C(L) - T_C(L=\infty) = aL^{-1/\nu},$$

$a\nu = 1 T_c \approx 2.269$
$T_C(L=\infty) x L$

$$T_C(L) - x = a/L$$

$$T_C(L^*) - x = a/L^*$$

$$x = \frac{T_c(L^*)L^* - T_c(L)L}{L^* - L}$$

E_t

$\Delta E = E_t - E$

- $\Delta E \leq 0$

- $\Delta E > 0 w = e^{-\beta \Delta E} r$

  $r \leq w$

$L \times L$

-

  $L = 2$

- T_c

-

$\Delta E \Delta \mathcal{M} 10^5 \Delta E N$

$$\Delta E = E_2 - E_1 = -J \sum_{<kl>}^{N} s_{k,2} s_{l,2} + J \sum_{<kl>}^{N} s_{k,1} s_{l,1}$$

$lkl s_l s_{k,2} = s_{k,1} = s_k$

$$\Delta E = J \sum_{<kl>}^{N} s_k s_{l,1} - J \sum_{<kl>}^{N} s_k s_{l,2} = J \sum_{<kl>}^{N} s_k (s_{l,1} - s_{l,2})$$

$s_{l,1} s_{l,1} = 1 s_{l,2} = -1 s_{l,1} - s_{l,2} = 1 - (-1) = 2 s_{l,1} = -1 s_{l,2} = 1 s_{l,1} - s_{l,2} = -1 - 1 = -2$
$s_{l,1} - s_{l,2} = 2 s_{l,1}$

$$\Delta E = J \sum_{<kl>}^{N} s_k \cdot 2 s_{l,1} = 2 J s_{l,1} \sum_{<k>}^{N} s_k$$

$$\Delta \mathcal{M} = \mathcal{M}_2 - \mathcal{M}_1 = \sum_{i}^{N} s_{i,2} - \sum_{i}^{N} s_{i,1} = \sum_{i}^{N} (s_{i,2} - s_{i,1}) = s_{l,2} - s_{l,1} = -2 s_{l,1}$$

$$\Rightarrow \mathcal{M}_2 = \mathcal{M}_1 - 2 s_{l,1}$$

$\Delta E w = e^{-\beta \Delta E} \Delta E \Delta E \in \{-8, -4, 0, 4, 8\} w w \Delta E > 0 \Rightarrow \Delta E \in \{4, 8\} w \Delta E$

$L = 2 N = L^2$
$5 \cdot 10^5$
$T = 1$
$L = 2$
$2 \cdot 10^5 0.002$
$2 \cdot 10^5$
$L = 20 T \in \{1.0, 2.4\}$

$T = 2.4$
$T = 1 T = 1$
$T = 2.4$
$L = 20 L$
$10^5 t_{eq} = 2 \cdot 10^5$
$500000 \sigma_E$
$L = 20 T \in \{1.0, 2.0, 1.4\}$
$T \in \{1.0, 2.0, 1.4\} \sigma_E \in \{0.15, 1.71, 2.8\} \sigma_E = 0.15 T = 2.0 T = 2.4 \sigma_E = 2.8$

$T_c L \in \{20, 40, 60, 80\}$
$L = 20 |\mathcal{M}| = 1$
$L \in \{20, 40, 60, 80\} L$

$T_c L$

$5 \cdot 10^5 L = 2$

$T_c L$

L	$T_C(C_V)$	$T_c(\chi)$

$5 \cdot 10^5 L = 2$

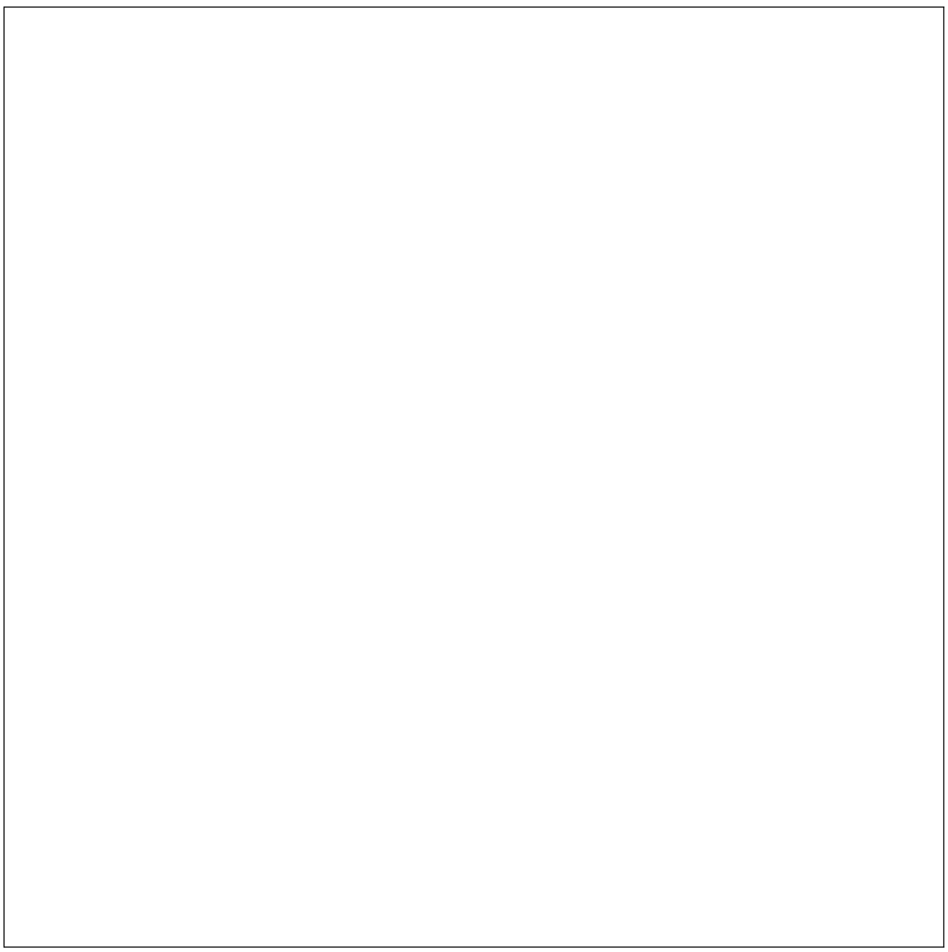
$L = 2T = 1$

$L = 2T = 1$

$L = 20T = 1$

$L = 20T = 2.4$

$T = 1$



$T = 2.4$

$L = 20$

$T \in \{1.0, 2.0, 1.4\}\sigma_E \in \{0.15, 1.71, 2.8\}$

$L = 20$

$L \in \{20, 40, 60, 80\}$

$L \in \{20, 40, 60, 80\}$

$T_cT_c = 2.269$

$\Delta E \Delta E \leq 0 e^{-\Delta E/T} > 1 \Delta E \leq 0 e^{-\Delta E/T} > r$
L