FYS3110
Quantum Mechanics
Oblig 01

Even Marius Nordhagen

September 2, 2016

1 Introduction

The purpose of this problem set is to repeat what we have learned in the linear algebra courses, and to be familiar with the Dirac notation.

2 Exercise 1

We start with this ket:

$$|\psi\rangle = c(\sqrt{5}\,|0\rangle) + i\,|1\rangle \tag{1}$$

a)

$$\begin{aligned}\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^* &= c^*(\sqrt{5}\,\langle\phi|0\rangle^* - i\,\langle\phi|1\rangle^*)\\ &= c^*(\sqrt{5}\,\langle 0|\phi\rangle - i\,\langle 1|\phi\rangle)\\ &= c^*(\sqrt{5}\,\langle 0| - i\,\langle 1|)\,|\phi\rangle = \langle\psi|\phi\rangle\end{aligned}$$

So $\langle\psi|$ has to be

$$\langle\psi| = c^*(\sqrt{5}\,\langle 0| - i\,\langle 1|) \tag{2}$$

I want to find the value of c when the hypotenuse is fixed to 1. This is found by Pythagoras:

$$\begin{aligned}1 = \sqrt{c^2\sqrt{5}^2 + c^2} &= c\sqrt{6}\\ c &= \frac{1}{\sqrt{6}}\end{aligned} \tag{3}$$

b) We have that

$$|0\rangle \simeq \begin{pmatrix}1\\0\end{pmatrix}, \quad |1\rangle \simeq \begin{pmatrix}0\\1\end{pmatrix}$$

I want to write $|\psi\rangle$ as a matrix, and from Equation (1) I see that

$$|\psi\rangle = c(\sqrt{5}\begin{pmatrix}1\\0\end{pmatrix} + i\begin{pmatrix}0\\1\end{pmatrix}) = c\begin{pmatrix}\sqrt{5}\\ i\end{pmatrix} \tag{4}$$

I also want to find the $\hat{A}$-matrix, and for that I need the equations

$$\hat{A}\,|0\rangle = -i\,|1\rangle\,, \quad \hat{A}\,|1\rangle = i\,|0\rangle$$

On matrix form I then have two equations

$$\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\-i\end{pmatrix}, \quad \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}i\\0\end{pmatrix}$$

Which leads to $a_{11} = 0,\ a_{12} = i,\ a_{21} = -i,\ a_{22} = 0$, and

$$\hat{A} = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix} \tag{5}$$

c) First I want to compute $\langle\psi|\hat{A}|\psi\rangle$ by using the representation in the previous sub exercise, but then I need to find $\langle\psi|$ on matrix form. The question is, what are $\langle 0|$ and $\langle 1|$ represented by?

This is not so hard, on the basis of normalization we can write

$$\langle 0|0\rangle = \langle 0| \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

$$\langle 1|1\rangle = \langle 1| \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1$$

which tell us that

$$\langle 0| \simeq (1,0), \quad \langle 1| \simeq (0,1)$$

If we insert these in Equation (2), we easily find that

$$\langle\psi| = c^*(\sqrt{5}, -i)$$

and finally we can begin computing $\langle\psi|\hat{A}|\psi\rangle$:

$$\langle\psi|\hat{A}|\psi\rangle = c^*(\sqrt{5}, -i) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} c \begin{pmatrix} \sqrt{5} \\ i \end{pmatrix} = c^* c(-\sqrt{5} - \sqrt{5})$$

$$\langle\psi|\hat{A}|\psi\rangle = \underline{-2\sqrt{5}c^2}$$

Now I want to compute $\langle\psi|\hat{A}|\psi\rangle$ directly from the definitions:

$$\begin{aligned} \langle\psi|\hat{A}|\psi\rangle &= \langle\psi|\, \hat{A}|c(\sqrt{5}\,|0\rangle + i\,|1\rangle) \\ &= \langle\psi|\, c(\sqrt{5}\hat{A}\,|0\rangle + i\hat{A}\,|1\rangle) \\ &= \langle\psi|\, c(-\sqrt{5}i\,|1\rangle + i^2\,|0\rangle) \end{aligned}$$

I also need to use my expression of $\langle\psi|$ from Equation 2:

$$\Rightarrow c^*(\sqrt{5}\,\langle 0| - i\,\langle 1|)c(-\sqrt{5}i\,|1\rangle - |0\rangle)$$

$$= c^* c(-5i\,\langle 0|1\rangle - \sqrt{5}\,\langle 0|0\rangle - \sqrt{5}\,\langle 1|1\rangle + i\,\langle 1|0\rangle)$$

$$\langle\psi|\hat{A}|\psi\rangle = \underline{-2\sqrt{5}c^2}$$

Mark: The inner products $\langle i|j\rangle$ are valued by

$$\langle i|j\rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{6}$$

and is called the Kronecker delta δ_{ij}.

3 Exercise 2

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

a)

$$U^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad U^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

b) A hermitian matrix is equal to its own conjugate transposed. With other words, if U is hermitian, it has to satisfy

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

This give us this set of equations:

i . $a = a^*$

ii . $d = d^*$

iii . $b = c^*$

iv . $c = b^*$

Equation 1 and 2 tell us that a and d have to be real numbers, and equation 3 and 4 tell us that b and c have to be in the same number category (both have to be either real or complex).

c) I compute the the eigenvalues by the standard formula

$$det(U - \lambda I) \tag{7}$$

$$= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

To find the roots for this characteristic polynomial, I have to use the famous ABC-formula:

$$\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{8}$$

In our case $A = 1$, $B = -(a+d)$ and $C = (ad - bc)$. By inserting this, we will get that

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} = \frac{a + d \pm \sqrt{(a-d)^2 + 4bc}}{2} \tag{9}$$

I can not take this longer, but this is good enough for proving that the eigenvalues always are real when U is hermitian. There are two ways that λ can be complex. A possibility is when one or more of the constants are complex. In the previous sub exercise we saw that a and d must be real, and since $4bc = 4c^*c \in \text{Re}$, we should not worry about this case.

An other way that λ may could be complex, is if the square root is negative. This is neither possible, because both of the terms must me positive. So λ has to be a real number.

d) We call a matrix unitary when it has the properties

$$U^*U = UU^* = I, \quad U^\dagger U = UU^\dagger = I$$

and we have already seen that a matrix

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is hermitian when a and d are real numbers, and b and c are in the same category. A hermitian unitary matrix therefore has to satisfy the matrix equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} aa + bc & ab + bd \\ ac + cd & bc + dd \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

From this we can find a set with four independent equations:

i . $a^2 + bc = 1$

ii . $ab + bd = 0$

iii . $ac + cd = 0$

iv . $bc + d^2 = 1$

Of course I could solve this set of equation by row operations on the matrix, but I think it goes just as fast to solve the equations on the "primary school way". Anyway we get two different solutions that depends on how we choose b:

i . $b = 0$: We can see that this is a solution that satisfies equation ii. and iii., and we have

$$a^2 = d^2 = 1$$

ii . $b \neq 0$: This is of course also a possible solution which gives us that

$$a = \pm\sqrt{1 - b^2}, \quad d = \mp\sqrt{1 - b^2}, \quad a = -d$$

Mark: Since a and d have to be positive, we have to choose a b in the interval $b \in [-1, 1]$.

e) After we have choose b, we can easily find the eigenvalues by inserting into Equation (9). I'll solve it for the two solutions, and I start with $b = 0$:

I

$$a = \pm 1 \quad b = 0 \quad c = 0 \quad d = \pm 1$$

$$\lambda = \frac{\pm 1 \pm 1 \pm (\pm 1 \pm 1)}{2} = \pm 1$$

II

$$a = \pm\sqrt(1 - b^2) \quad b \neq 0 \quad c = b \quad d = -a$$

$$\lambda = \pm\frac{\sqrt{4(1 - b^2) + 4b^2}}{2} = \pm 1$$

So the conclusion is that the eigenvalues have to be $\pm$!

4 Exercise 3

We start with the following statements:

$$\hat{H}\left|\psi\right\rangle = g\left|\phi\right\rangle, \quad \hat{H}\left|\phi\right\rangle = g^*\left|\psi\right\rangle, \quad \hat{H}\left|\gamma_n\right\rangle = 0$$

If the operator $\hat{H}$ is hermitian, it has to satisfy $\hat{H}^\dagger = \hat{H}$. I will try to find which conditions $\left|\phi\right\rangle$ and $\left|\psi\right\rangle$ must satisfy if $\hat{H}$ is hermitian:

$$\langle\phi|\hat{H}|\psi\rangle = \langle\phi|g|\phi\rangle = g\left\langle\phi|\psi\right\rangle = g$$

$$\langle\phi|\hat{H}^\dagger|\psi\rangle = \langle\psi|\hat{H}|\phi\rangle = g\left\langle\psi|\psi\right\rangle^* = g\left\langle\psi|\psi\right\rangle = g$$

So $\hat{H}$ is hermitian if ψ and ϕ are normalized.

5 Comment

I decided try to write this delivery in English just to improve my English skills. Please tell me if something is terrible, I'm sure I have a lot to learn about writing physic reports in English.