INF5620 - Numerical methods for partial differential equations

Mandatory Exercise 1

Even Marius Nordhagen

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- For the Github repository containing programs and results, follow this link: `https://github.com/UiO-INF5620/INF5620-evenmn/tree/master/exercise_1`

Problem 1

We have the ODE problem

$$u'' + \omega^2 u = f(t) \tag{1}$$

with initial conditions

$$u(0) = I, \quad u'(0) = V \tag{2}$$

and with $t \in (0, T]$. To solve this equation numerically we need to use an approximation for the second derivative, recall the **second symmetric derivative**

$$f''(x) = \lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}. \tag{3}$$

This equation is only true when h is infinitesimal, but we can make a good approximation for small h's. We then get

$$u^{n+1} = 2u^n - u^{n-1} + (F^n - \omega^2 u^n)\Delta t^2 \tag{4}$$

by inserting equation (3) into equation (1). We now have an equation that describes the ODE at a timestep when we have the two previous, and we already know u^0. The next step is to find u^1 and for that we need **the centred scheme**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}. \tag{5}$$

By using the second initial condition, we obtain

$$u^{-1} = u^1 - 2\Delta t V \tag{6}$$

which again gives us the formula for u^1

$$u^1 = u^0 + \Delta t V + \frac{\Delta t^2}{2}(F^0 - \omega^2 u^0). \tag{7}$$

An exact solution to this problem has the form $u_e(x,t) = ct + d$ where c and d are coefficients. We can easily see that $d = I$ and $c = I$ by applying the initial conditions. Furthermore $u''(t) = 0$, which leads to

$$F(t) = \omega^2(Vt + I). \tag{8}$$

Since this is a differential equation of second order, we can also find a solution of second order (quadratic), on the form $u_e(x,t) = bt^2 + ct + d$, but a cubic solution will never fulfil the discrete equations.

Exercise 21

We now look at an elastic pendulum that is described by the ordinary differential equation

$$u'' + u = 0 \tag{9}$$

and therefore an exact solution $u = \Theta\cos(\tilde{t})$, which is good to have when we want to verify the numerical solution. The $\sim$ symbolizes a dimensionless quantity, but from now I will assume that all quantities are dimensionless and therefore skip it. The given differential equations are

$$\frac{d^2x}{dt^2} = -\frac{\beta}{1-\beta}\left(1 - \frac{\beta}{L}\right)x \quad \text{and} \tag{10}$$

$$\frac{d^2y}{dt^2} = -\frac{\beta}{1-\beta}\left(1 - \frac{\beta}{L}\right)(y-1) - \beta \tag{11}$$

where $L = \sqrt{x^2 + (y-1)^2}$ and β is a constant. A method for finding the second derivative numerical, called **second symmetric derivative**, was introduced in Problem 1, and by using that we are able to reduce the differential equations to difference equations

$$x^{n+1} = 2x^n - x^{n-1} - \Delta t^2\left(\frac{\beta}{1-\beta}\right)\left(1 - \frac{\beta}{L}\right)x^n \tag{12}$$

$$y^{n+1} = 2y^n - y^{n-1} - \Delta t^2 \left(\frac{\beta}{1-\beta}\right)\left(1 - \frac{\beta}{L}\right)(y^n - 1) - \Delta t^2 \beta \tag{13}$$

We are given the two initial conditions $x(0) = (1+\epsilon)\sin(\Theta)$ and $y(0) = 1-(1+\epsilon)\cos(\Theta)$, but to solve the difference equations we also need $x(1)$ and $y(1)$, which we again find applying the **centred scheme**. Using $\frac{dx}{dt}(0) = 0$ and $\frac{dy}{dt}(0) = 0$ we find

$$x^1 = x^0 - \frac{\Delta t^2}{2}\left(\frac{\beta}{1-\beta}\right)\left(1 - \frac{\beta}{L}\right)x^0 \tag{14}$$

$$y^1 = y^0 - \frac{\Delta t^2}{2}\left(\frac{\beta}{1-\beta}\right)\left(1 - \frac{\beta}{L}\right)(y^0 - 1) - \Delta t^2 \beta \tag{15}$$

We are now able to simulate the trajectory of the elastic pendulum, and a plot can be find in figure (1). It can also be appropriate to simulate the time evolution, especially as a function of the angle between a vertical axis and the pendulum. The angle as a function of x and y can be found graphically and is given by $\theta = \arctan(\frac{x}{1-y})$. The angle as a function of time is plotted in figure (2), with the classical solution of equation (9) as what we expect.

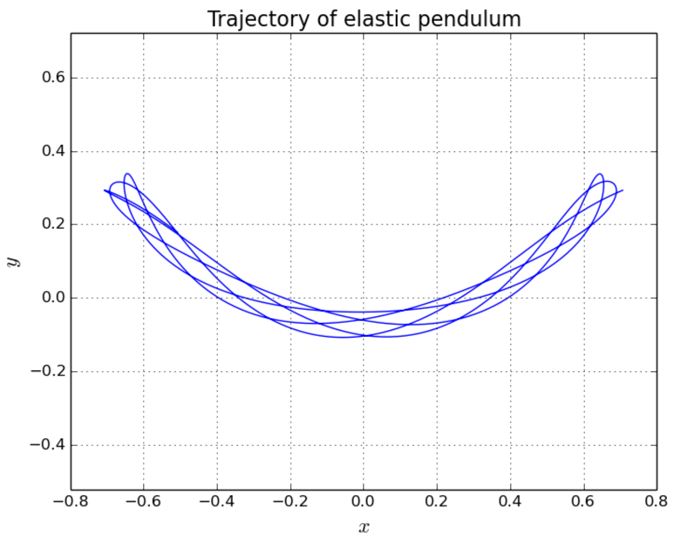

Figure 1: Trajectory of the pendulum in xy-direction with initial angle $\Theta = 45°$ and $\beta = 0.9$ simulated over 3 periods with 600 timesteps per period.

A classical non-elastic pendulum with angular frequency ω is described by the differential equation

$$u'' - \omega^2 u = 0 \tag{16}$$

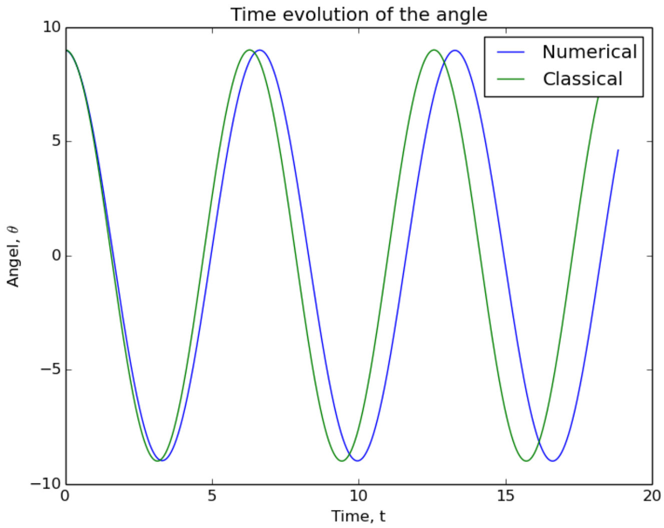

Figure 2: Time evolution of the angle between pendulum and the y-axis where the numerical wave is calculated and the classical is the expected. The plot is made with an initial angle $\Theta = 9°$ and $\beta = 0.9$, plotted over 3 periods with 600 timesteps per period.

with exact solution $u = \Theta \cos(\omega t)$. In our case the angular frequency is given by the formula $\omega = \sqrt{\frac{\beta}{1-\beta}}$, and we can use this solution to compare our numerical solution in y-direction, see figure (3).

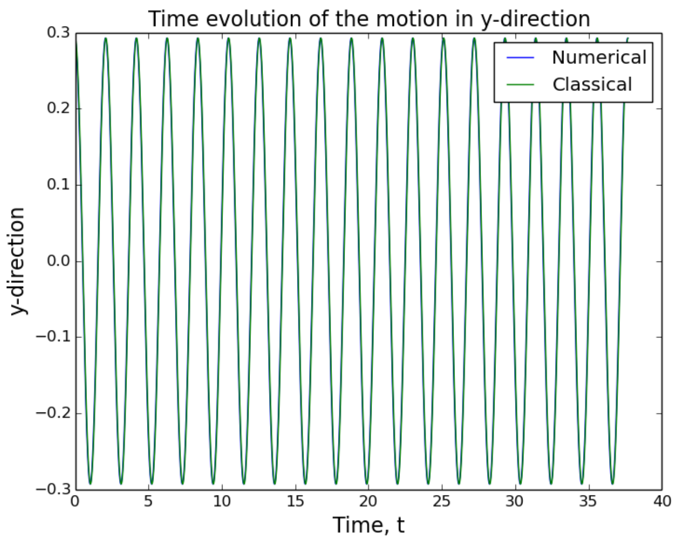

Figure 3: Time evolution of the motion in y-direction, where both the numerical and classical solution are plotted. Plotted over 6 periods and 600 timesteps per period.