EC709 PS1

Everett Stamm

September 19, 2024

Note

I can't actually figure out how to export glmnet or locfit results to a latex table so I'm just describing them in this document. All results can be found in code.R or on `https://github.com/everettstamm1/EC709_PS1`

1 Question 1

1.1 Part 1

Omitted.

1.2 Part 2

From the slides, recall that

$$\hat{g}_h(x) = e_1'(X'WX)^{-1}X'WY$$

Where e_1 is $(r+1)\times 1$ vector with 1 in the first entry and zeros elsewhere and $W = diag(K_h(x-X_1), \dots, K_h(x-X_n))$. Add and subtract a $g_0 = (g_0(X_1), \dots, g_0(X_n))'$ to this and square to get MSE:

$$MSE = E[(\hat{g}_h(x) - g_0(x))^2|X_i] = e_1'(X'WX)^{-1}X'W\Sigma WX(X'WX)^{-1}e_1 + \left(e_1'(X'WX)^{-1}X'Wg_0 - g_0(x)\right)^2$$

Where the first term is the variance and the second is the bias squared. Also Σ is an $N\times N$ diagonal matrix of the sample variances: $\Sigma = diag(\sigma^2(X_1), \dots, \sigma^2(X_n))$

Not get taylor approximation of $g(X_i)$

$$g(X_i) \approx g_0(x) + g_0'(x)(X_i - x) + (1/2)g_0''(x)(X_i-x)^2$$

Rename $r_i = (1/2)g_0''(x)(X_i-x)^2 = g_0(X_i) - g_0(x) - g_0'(x)(X_i - x)$. Thus we can rewrite:

$$g_0(X_i) = g_0(x) - g_0'(x)(X_i - x) + r_i$$

Since $Xe_1 = (1, \dots, 1)'$ and $Xe_2 = (X_1 - x, \dots, X_n - x)'$, we can write the above in vector form:

$$g_0 = Xe_1g_0(x) - Xe_2g_0'(x) + r$$

And substitute into the equation for bias

$$Bias = e_1'(X'WX)^{-1}X'Wg_0 - g_0(x) =$$
$$e_1'(X'WX)^{-1}X'W(Xe_1g_0(x) - Xe_2g_0'(x) + r) - g_0(x) = e_1'(X'WX)^{-1}X'Wr$$

Now we need a corrolary: from the "weighted average" section of the notes we can see that

$$S_j = W((x-X^j)^j)/h^j \sum_i K_h(x-X_i)(\frac{x-X_i}{h})^j$$

Multiply, divide, do a change of variables $u = (x-x_i)/h$ and take the expectation to get that

$$E[S_j/(nh^j)] = \mu_j f_0(x) + o(1)$$

For $\mu_j = \int u^jK(u)du$. Note that the asymptotic variance of this can be shown to have

$$Var(S_j/(nh^j)) = E[(S_j/(nh^j))^2] - (E[S_j/(nh^j)])^2 \leq E[(S_j/(nh^j))^2] \leq \frac{1}{nh}\int u^{2j}K(u)^2j_0(x-hu)du$$

The integral approaches some constant as $h \to 0$ and then $\frac{1}{nh} \to 0$ as $nh \to \infty$, so the variance approaches zero. So we have that

$$S_j/(nh^j) \to \mu_j f_0(x) + o(1)$$

Now let $H = diag(1, h)$, so we have

$$n^{-1}h^{-2}H^{-1}X'Wr = n^{-1}h^{-2}H^{-1}X'S_2(1/2)g_0''(x) \to (1/2)f_0(x)\begin{pmatrix}\mu_2\\0\end{pmatrix}g_0''(x)$$

We also have that as $h \to 0$ and $nh \to \infty$

$$n^{-1}H^{-1}X'WXH^{-1} = n^{-1}\begin{bmatrix}S_0 & h^{-1}S_1\\h^{-1}S_1 & h^{-2}S_2\end{bmatrix} \to f_0(x)\begin{bmatrix}\mu_0 & \mu_1\\\mu_1 & \mu_2\end{bmatrix} = f_0(x)\begin{bmatrix}1 & 0\\0 & \mu_2\end{bmatrix}$$

Now rearrange the bias equation

$$e_1'(X'WX)^{-1}X'Wr = e_1'h^2H^{-1}(n^{-1}H^{-1}X'WXH^{-1})^{-1}n^{-1}h^{-2}H^{-1}X'Wr \to e_1'h^2H^{-1}\Big(f_0(x)\begin{bmatrix}1 & 0\\0 & \mu_2\end{bmatrix}\Big)^{-1}(1/2)f_0(x)\begin{pmatrix}\mu_2\\0\end{pmatrix}g_0''(x) =$$
$$g_0''(x)(h^2/2)e_1'H^{-1}\frac{1}{\mu_2}\begin{bmatrix}\mu_2 & 0\\0 & 1\end{bmatrix}\begin{pmatrix}\mu_2\\0\end{pmatrix} = g_0''(x)(h^2/2)\mu_2 + o(1)$$

Forgot the $o(1)$ along the way but here it is again.

Using what we learned for the bias, let's identify a new quantity:

$$Q_j = \frac{h}{n}\sum_i K(x - X_i)^2(\frac{x - X_i}{h})^j\sigma^2(x_i)$$

Thus with change of variables $u = (x - X_i)/h$

$$E[Q_j] = E[hK_h(x - X_i)^2(\frac{x - X_i}{h})^j\sigma^2(x_i)] = f_0(x)\sigma^2(x)\int u^jK(u)^2du = f_0(x)\sigma^2(x)v_j$$

Turns out that

$$n^{-1}hH^{-1}X'W\Sigma WXH^{-1} = \begin{bmatrix}Q_0 & 0\\0 & Q_2\end{bmatrix}$$

So then the variance equation becomes

$$e_1'(X'WX)^{-1}X'W\Sigma WX(X'WX)^{-1}e_1 = n^{-1}h^{-1}e_1'H^{-1}\Big(\frac{H^{-1}X'WXH^{-1}}{n}\Big)\Big(\frac{hH^{-1}X'W\Sigma WXH^{-1}}{n}\Big)\Big(\frac{H^{-1}X'WXH^{-1}}{n}\Big)H^{-1}e_1 \to$$
$$n^{-1}h^{-1}e_1'(f_0(x)\begin{bmatrix}1 & 0\\0 & \mu_2\end{bmatrix})^{-1}\begin{bmatrix}Q_0 & 0\\0 & Q_2\end{bmatrix}(f_0(x)\begin{bmatrix}1 & 0\\0 & \mu_2\end{bmatrix})^{-1}e_1 = n^{-1}h^{-1}\frac{v_0\sigma^2(x)}{f_0(x)} + o(1)$$

1.3 Part 3

Below is the table of simulated values.

Table 1

	h	kern_bias	kern_var	local_bias	local_var
1	0.050	0.013	0.142	0.018	0.150
2	0.100	0.015	0.120	0.011	0.143
3	0.150	0.014	0.100	0.012	0.137
4	0.200	0.010	0.079	0.019	0.136

1.4 Part 4

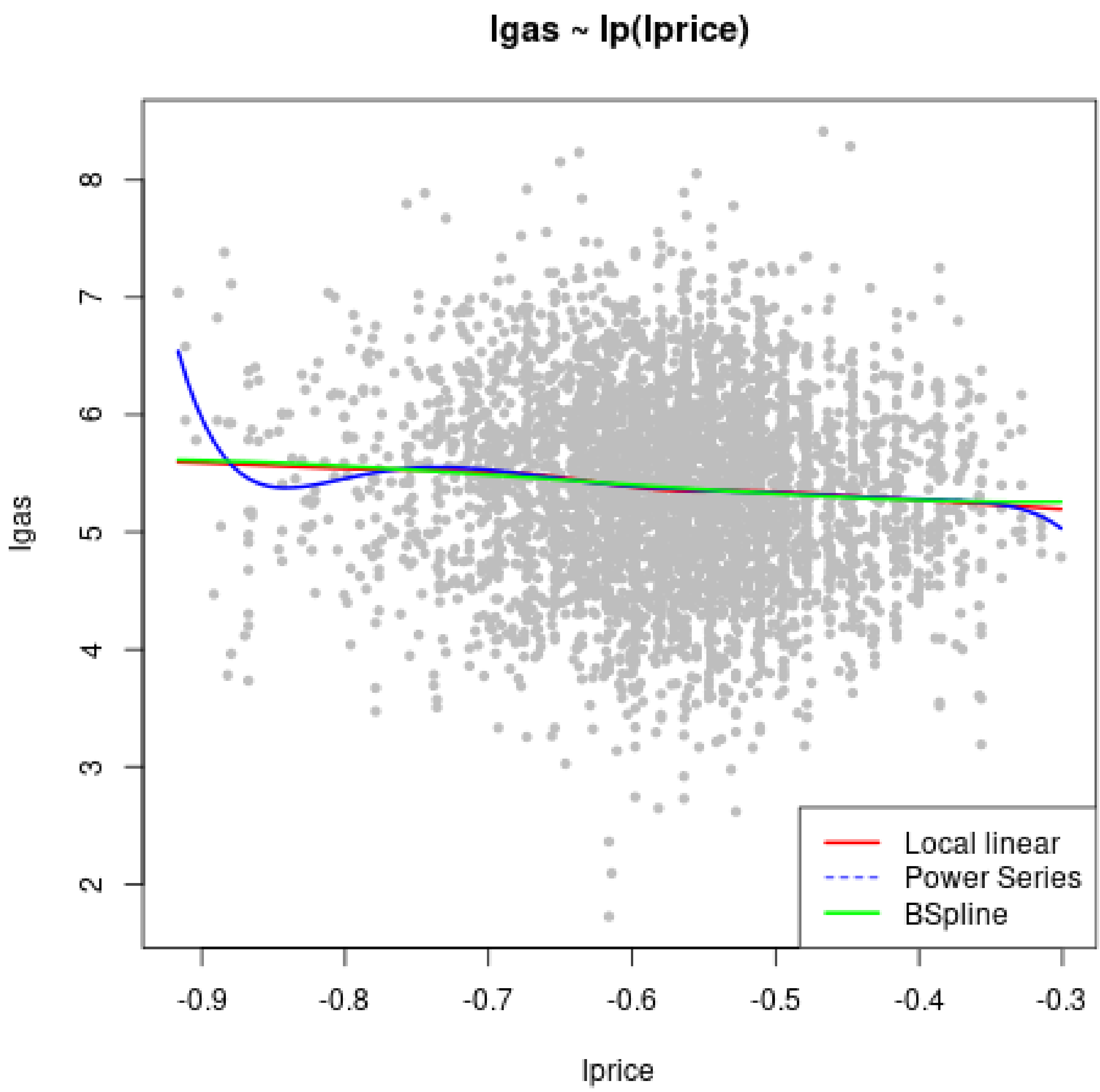
Asymptotic theory would predict the following for local linear regression:

$$E[Bias] = E[g_0''(x)(h^2/2)\mu_2] = E[\exp(X)(h^2/2)(1/5)] = (h^2/2)(1/5)(\exp(1) - \exp(0))$$

Thus for $h = (0.05, 0.1, 0.15, 0.2)$ we'd expect $Bias = (0.0004, 0.0017, 0.0039, 0.0067)$. Kinda makes sense that since we're holding n constant while $h \to 0$, the values get further from their true value. Similarly:

$$E[Variance] = E[\frac{1}{1000h}\frac{(3/5)\sigma^2(x)}{f_0(x)}] = \frac{1}{1000h}\frac{(3/5)0.5^2/12}{1}$$

Thus for $h = (0.05, 0.1, 0.15, 0.2)$ we'd expect $Variance = (0.00025, 0.000125, 0.000083, 0.0000625)$. Clearly we're very far off!



2 Question 2

2.1 Part 1

In code.R, I optimized the local linear regression over bandwidth values of $(0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8)$ and the power and bspline regressions over 1-7 knots. The optimal values were 0.1 for local linear, 7 for power, and 1 for bspline.

2.2 Part 2

With the price set to 0.57. I get betas of 5.34, 5.36, and 5.37 with standard errors of 0.02 0.016, and 0.013 for local linear, power, and bspline regressions, respectively.

2.3 Part 3

Again in code.R, I optimized the local linear regression over bandwidth values of $(0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8)$ and the power and bspline regressions over 1-7 knots. The optimal values were 0.1 for local linear, 7 for power, and 1 for bspline.

2.4 Part 4

In the standard lasso, the coefficient on log price is -0.554, while in the two-way one it's fixed at zero. That said, there are coefficients of -0.253, -0.057, and 0.125 on the interaction of log price and driver, hhsize, and urban, respectively.

3 Question 3

Below is the prediction error for the 5 estimators. Clearly, there are benefits to cross validation, though surprisingly post lasso CV is higher than just the lasso CV.

Table 2

Statistic	N	Mean	St. Dev.	Min	Max
lasso_pe	500	5.917	0.829	3.657	9.089
post_lasso_pe	500	2.871	0.394	1.696	4.140
lassoCV_pe	500	0.258	0.130	0.039	0.934
post_lassoCV_pe	500	0.562	0.246	0.027	1.300
oracle_pe	500	0.061	0.036	0.006	0.278