Tina Memo No. 2003-003
Internal Report

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Last updated
14 / 8 / 2014

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Products and Convolutions of Gaussian Probability Density Functions

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Abstract

It is well known that the product and the convolution of Gaussian probability density functions (PDFs) are also Gaussian functions. This document provides proofs of this for several cases; the product of two univariate Gaussian PDFs, the product of an arbitrary number of univariate Gaussian PDFs, the product of an arbitrary number of multivariate Gaussian PDFs, and the convolution of two univariate Gaussian PDFs. These results are useful in calculating the effects of smoothing applied as an intermediate step in various algorithms.

1 The Product of Two Univariate Gaussian PDFs

Let $f(x)$ and $g(x)$ be Gaussian PDFs with arbitrary means μ_f and μ_g and standard deviations σ_f and σ_g

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_f} e^{-\frac{(x-\mu_f)^2}{2\sigma_f^2}} \quad \text{and} \quad g(x) = \frac{1}{\sqrt{2\pi}\sigma_g} e^{-\frac{(x-\mu_g)^2}{2\sigma_g^2}}$$

Their product is

$$f(x)g(x) = \frac{1}{2\pi\sigma_f\sigma_g} e^{-\left(\frac{(x-\mu_f)^2}{2\sigma_f^2} + \frac{(x-\mu_g)^2}{2\sigma_g^2}\right)}$$

Examine the term in the exponent

$$\beta = \frac{(x-\mu_f)^2}{2\sigma_f^2} + \frac{(x-\mu_g)^2}{2\sigma_g^2}$$

Expanding the two quadratics and collecting terms in powers of x gives

$$\beta = \frac{(\sigma_f^2 + \sigma_g^2)x^2 - 2(\mu_f\sigma_g^2 + \mu_g\sigma_f^2)x + \mu_f^2\sigma_g^2 + \mu_g^2\sigma_f^2}{2\sigma_f^2\sigma_g^2}$$

Dividing through by the coefficient of x^2 gives

$$\beta = \frac{x^2 - 2\frac{\mu_f\sigma_g^2 + \mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2}x + \frac{\mu_f^2\sigma_g^2 + \mu_g^2\sigma_f^2}{\sigma_f^2+\sigma_g^2}}{2\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2+\sigma_g^2}}$$

This is again a quadratic in x, and so Eq. 2 is a Gaussian function. Compare the terms in Eq. 5 to a the usual Gaussian form

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x^2-2\mu x+\mu^2)}{2\sigma^2}}$$

Since a term ϵ that is independent of x can be added to complete the square in β, this is sufficent to complete the proof in cases where the normalisation can be ignored. The product of two Gaussian PDFs is proportional to a Gaussian PDF with a mean that is half the coefficient of x in Eq. 5 and a standard deviation that is the square root of half of the denominator i.e.

$$\sigma_{fg} = \sqrt{\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2+\sigma_g^2}} \quad \text{and} \quad \mu_{fg} = \frac{\mu_f\sigma_g^2 + \mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2}$$

i.e. the variance σ_{fg}^2 is twice the harmonic mean of the individual variances σ_f^2 and σ_g^2, and the mean μ_{fg} is the sum of the individual means μ_f and μ_g weighted by their variances. In general, the product is not itself a PDF as, due to the presence of the scaling factor, it will not have the correct normalisation.

The product $f(x)g(x)$ can now be written in the usual Gaussian form directly, with an unknown scaling constant (this may be sufficient in cases where renormalisation can be applied). Alternatively, proceeding from Eq. 5, suppose that ϵ is the term required to complete the square in β i.e.

$$\epsilon = \frac{\left(\frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2}\right)^2 - \left(\frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2}\right)^2}{\frac{2\sigma_f^2\sigma_g^2}{(\sigma_f^2+\sigma_g^2)}} = 0$$

Adding this term to β gives

$$\beta = \frac{x^2 - 2x\frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2} + \left(\frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2}\right)^2}{\frac{2\sigma_f^2\sigma_g^2}{(\sigma_f^2+\sigma_g^2)}} + \frac{\frac{\mu_f^2\sigma_g^2+\mu_g^2\sigma_f^2}{\sigma_f^2+\sigma_g^2} - \left(\frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2}\right)^2}{\frac{2\sigma_f^2\sigma_g^2}{(\sigma_f^2+\sigma_g^2)}}$$

After some manipulation, this reduces to

$$\beta = \frac{\left(x - \frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2}\right)^2}{2\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2+\sigma_g^2}} + \frac{(\mu_f-\mu_g)^2}{2(\sigma_f^2+\sigma_g^2)} = \frac{(x-\mu_{fg})^2}{2\sigma_{fg}^2} + \frac{(\mu_f-\mu_g)^2}{2(\sigma_f^2+\sigma_g^2)}$$

Substituting back into Eq. 2 gives

$$f(x)g(x) = \frac{1}{2\pi\sigma_f\sigma_g}\exp\left[-\frac{(x-\mu_{fg})^2}{2\sigma_{fg}^2}\right]\exp\left[-\frac{(\mu_f-\mu_g)^2}{2(\sigma_f^2+\sigma_g^2)}\right]$$

Multiplying by σ_{fg}/σ_{fg} and rearranging gives

$$= \frac{1}{\sqrt{2\pi}\sigma_{fg}}\exp\left[-\frac{(x-\mu_{fg})^2}{2\sigma_{fg}^2}\right]\frac{1}{\sqrt{2\pi(\sigma_f^2+\sigma_g^2)}}\exp\left[-\frac{(\mu_f-\mu_g)^2}{2(\sigma_f^2+\sigma_g^2)}\right]$$

Therefore, the product of two Gaussians PDFs $f(x)$ and $g(x)$ is a scaled Gaussian PDF

$$f(x)g(x) = \frac{S_{fg}}{\sqrt{2\pi}\sigma_{fg}}\exp\left[-\frac{(x-\mu_{fg})^2}{2\sigma_{fg}^2}\right]$$

where

$$\sigma_{fg} = \sqrt{\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2+\sigma_g^2}} \quad \text{and} \quad \mu_{fg} = \frac{\mu_f\sigma_g^2+\mu_g\sigma_f^2}{\sigma_f^2+\sigma_g^2} \tag{1}$$

and the scaling factor S is itself a Gaussian PDF on both μ_f and μ_g with standard deviation $\sqrt{\sigma_f^2+\sigma_g^2}$

$$S_{fg} = \frac{1}{\sqrt{2\pi(\sigma_f^2+\sigma_g^2)}}\exp\left[-\frac{(\mu_f-\mu_g)^2}{2(\sigma_f^2+\sigma_g^2)}\right]$$

These can be written more conveniently as

$$\frac{1}{\sigma_{fg}^2} = \frac{1}{\sigma_f^2} + \frac{1}{\sigma_g^2} \quad , \quad \mu_{fg} = \left(\frac{\mu_f}{\sigma_f^2} + \frac{\mu_g}{\sigma_g^2}\right)\sigma_{fg}^2 \quad \text{and} \quad S_{fg} = \frac{1}{\sqrt{2\pi\frac{\sigma_f^2\sigma_g^2}{\sigma_{fg}^2}}}\exp\left[-\frac{1}{2}\frac{(\mu_f-\mu_g)^2}{\sigma_f^2\sigma_g^2}\sigma_{fg}^2\right] \tag{2}$$

It is much easier to generate a proof by induction for the scaling factor of products of larger numbers of Gaussians if it is written in the form of a sum of terms, each of which involves a single subscript i.e. the parameters of a single Gaussian PDF. Appendix A provides the necessary proof, giving

$$S_{fg} = \frac{1}{\sqrt{2\pi\frac{\sigma_f^2\sigma_g^2}{\sigma_{fg}^2}}}\exp\left[-\frac{1}{2}\left(\frac{\mu_f^2}{\sigma_f^2} + \frac{\mu_g^2}{\sigma_g^2} - \frac{\mu_{fg}^2}{\sigma_{fg}^2}\right)\right] \tag{3}$$

2 The Product of n Univariate Gaussian PDFs

Let $N(\mu, \sigma)$ represent a Gaussian PDF with mean μ and standard deviation σ. Let subscript i refer to an individual Gaussian PDF in a product of n univariate Gaussian PDFs. Furthermore, let the subscript $i = 1...n$ refer to the parameters of the distribution that is the product n individual Gaussian PDFs and subscripts of the form $i = (1...n-1)n$ refer to the parameters of a distribution that is the product of two Gaussian PDFs, one of which is itself the product of $n-1$ Gaussian PDFs. Therefore, the results from Section 1 can be applied to the first two Gaussian PDFs in the product of n Gaussian PDFs to produce a Gaussian PDF and a scaling factor. The remaining $n-2$ PDFs can then be introduced iteratively using the same expressions i.e.

$$\prod_{i=1}^{n} N(\mu_i, \sigma_i) = S_{i=1...2} N(\mu_{i=1...2}, \sigma_{i=1...2}) \prod_{i=3}^{n} N(\mu_i, \sigma_i)$$

$$= S_{i=1...2} S_{(i=1...2)3} N(\mu_{(i=1...2)3}, \sigma_{(i=1...2)3}) \prod_{i=4}^{n} N(\mu_i, \sigma_i) = ...$$

$$= S_{i=1...2} ... S_{(...((i=1...2)3)...n)} N(\mu_{(...((i=1...2)3)...n)}, \sigma_{(...((i=1...2)3)...n)}) = S_{i=1...n} N(\mu_{i=1...n}, \sigma_{i=1...n})$$

Applying the expression for the standard deviation from Eq. 2 iteratively gives

$$\frac{1}{\sigma^2_{i=1...n}} = \frac{1}{\sigma^2_{i=1...n-1}} + \frac{1}{\sigma^2_n} = \frac{1}{\sigma^2_{i=1...n-2}} + \frac{1}{\sigma^2_{n-1}} + \frac{1}{\sigma^2_n} = ... = \sum_{i=1}^{n} \frac{1}{\sigma^2_i} \tag{4}$$

Similarly, the mean is given by

$$\mu_{i=1...n} = \left[\frac{\mu_{i=1...n-1}}{\sigma^2_{i=1...n-1}} + \frac{\mu_n}{\sigma^2_n}\right] \sigma^2_{i=1...n} = \left[\left(\frac{\mu_{i=1...n-2}}{\sigma^2_{i=1...n-2}} + \frac{\mu_{n-1}}{\sigma^2_{n-1}}\right) \frac{\sigma^2_{i=1...n-1}}{\sigma^2_{i=1...n-1}} + \frac{\mu_n}{\sigma^2_n}\right] \sigma^2_{i=1...n}$$

$$= \left[\frac{\mu_{i=1...n-2}}{\sigma^2_{i=1...n-2}} + \frac{\mu_{n-1}}{\sigma^2_{n-1}} + \frac{\mu_n}{\sigma^2_n}\right] \sigma^2_{i=1...n} = ... = \left[\sum_{i=1}^{n} \frac{\mu_i}{\sigma^2_i}\right] \sigma^2_{i=1...n} \tag{5}$$

By inspection of Eq. 3, state the form

$$S_{i=1...n} = \frac{1}{(2\pi)^{(n-1)/2}} \sqrt{\frac{\sigma^2_{i=1...n}}{\prod_{i=1}^{n} \sigma^2_i}} \exp\left[-\frac{1}{2}\left(\sum_{i=1}^{n} \frac{\mu^2_i}{\sigma^2_i} - \frac{\mu^2_{i=1...n}}{\sigma^2_{i=1...n}}\right)\right] \tag{6}$$

for the scaling factor. Similarly, using Eq. 4 to manipulate some of the standard deviation terms,

$$S_{(i=1...n)(n+1)} = \frac{1}{(2\pi)^{(1/2)}} \sqrt{\frac{\sigma^2_{i=1...n+1}}{\sigma^2_{i=1...n}\sigma^2_{n+1}}} \exp\left[-\frac{1}{2}\left(\frac{\mu^2_{i=1...n}}{\sigma^2_{i=1...n}} + \frac{\mu^2_{n+1}}{\sigma^2_{n+1}} - \frac{\mu^2_{(i=1...n)(n+1)}}{\sigma^2_{(i=1...n)(n+1)}}\right)\right]$$

The scaling factor is the product of individual scaling factors for each pairwise multiplication, so

$$S_{i=1...n+1} = S_{i=1...n} S_{(i=1...n)(n+1)} =$$

$$\frac{1}{(2\pi)^{n/2}} \sqrt{\frac{\sigma^2_{i=1...n}}{\prod_{i=1}^{n} \sigma^2_i} \frac{\sigma^2_{i=1...n+1}}{\sigma^2_{i=1...n}\sigma^2_{n+1}}} \exp\left[-\frac{1}{2}\left(\sum_{i=1}^{n} \frac{\mu^2_i}{\sigma^2_i} + \frac{\mu^2_{n+1}}{\sigma^2_{n+1}} - \frac{\mu^2_{(i=1...n)(n+1)}}{\sigma^2_{(i=1...n)(n+1)}}\right)\right]$$

This gives two terms to deal with: First, the standard devation term

$$\frac{\prod_{i=1}^{n} \sigma^2_i}{\sigma^2_{i=1...n}} \frac{\sigma^2_{i=1...n}\sigma^2_{n+1}}{\sigma^2_{i=1...n+1}} = \frac{\sigma^2_{n+1} \prod_{i=1}^{n} \sigma^2_i}{\sigma^2_{i=1...n+1}} = \frac{\prod_{i=1}^{n+1} \sigma^2_i}{\sigma^2_{i=1...n+1}}$$

Second, the term in the exponent; using Eq. 5 gives

$$\frac{\mu^2_{(i=1...n)(n+1)}}{\sigma^2_{(i=1...n)(n+1)}} = \frac{\mu^2_{i=1...n}}{\sigma^2_{i=1...n}} + \frac{\mu^2_{n+1}}{\sigma^2_{n+1}} = \sum_{i=1}^{n} \frac{\mu^2_i}{\sigma^2_i} + \frac{\mu^2_{n+1}}{\sigma^2_{n+1}} = \sum_{i=1}^{n+1} \frac{\mu^2_i}{\sigma^2_i} = \frac{\mu^2_{i=1...n+1}}{\sigma^2_{i=1...n+1}}$$

Therefore

$$\sum_{i=1}^{n} \frac{\mu^2_i}{\sigma^2_i} + \frac{\mu^2_{n+1}}{\sigma^2_{n+1}} - \frac{\mu^2_{(i=1...n)(n+1)}}{\sigma^2_{(i=1...n)(n+1)}} = \sum_{i=1}^{n+1} \frac{\mu^2_i}{\sigma^2_i} - \frac{\mu^2_{(i=1...n+1)}}{\sigma^2_{(i=1...n+1)}}$$

So

$$S_{i=1...n+1} = \frac{1}{(2\pi)^{n/2}} \sqrt{\frac{\sigma^2_{i=1...n+1}}{\prod_{i=1}^{n+1} \sigma_i^2}} \exp\left[-\frac{1}{2}\left(\sum_{i=1}^{n+1} \frac{\mu_i^2}{\sigma_i^2} - \frac{\mu^2_{i=1...n+1}}{\sigma^2_{i=1...n+1}}\right)\right]$$

which, together with Eq. 3, constitutes a proof by induction of Eq. 6. As with the product of two univariate Gaussian PDFs, the scaling factor is a Gaussian function. However, it is not a PDF, as it does not have the correct normalisation.

3 The Product of n Multivariate Gaussian PDFs

The multivariate Gaussian PDF can be written as

$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{d/2}\sqrt{|\boldsymbol{V}|}} exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{V}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right]$$

where d is the dimensionality of $\boldsymbol{x}$, $\boldsymbol{\mu}$ is the d-dimensional mean vector, and $\boldsymbol{V}$ is the d-by-d dimensional covariance matrix; this document adopts the standard notation of using bold face symbols to represent vectors and matrices. The Gaussian PDF can also be written in canonical notation as

$$p(\boldsymbol{x}) = exp\left[\boldsymbol{\zeta} + \boldsymbol{\eta}^T\boldsymbol{x} - \frac{1}{2}\boldsymbol{x}^T\boldsymbol{\Lambda}\boldsymbol{x}\right] \tag{7}$$

where

$$\boldsymbol{\Lambda} = \boldsymbol{V}^{-1} \quad , \quad \boldsymbol{\eta} = \boldsymbol{V}^{-1}\boldsymbol{\mu} \quad \text{and} \quad \boldsymbol{\zeta} = -\frac{1}{2}\left(d\log 2\pi - log|\boldsymbol{\Lambda}| + \boldsymbol{\eta}^T\boldsymbol{\Lambda}^{-1}\boldsymbol{\eta}\right)$$

So the product of n Gaussian PDFs $i = 1...n$ is

$$\prod_{i=1}^{n} p_i(\boldsymbol{x}) = exp\left[\boldsymbol{\zeta}_{i=1...n} + \left(\sum_{i=1}^{n}\boldsymbol{\eta}_i\right)^T \boldsymbol{x} - \frac{1}{2}\boldsymbol{x}^T\left(\sum_{i=1}^{n}\boldsymbol{\Lambda}_i\right)\boldsymbol{x}\right]$$

where

$$\boldsymbol{\zeta}_{i=1...n} = \sum_{i=1}^{n}\boldsymbol{\zeta}_i = -\frac{1}{2}\left(nd\log 2\pi - \sum_{i=1}^{n}\log|\boldsymbol{\Lambda}_i| + \sum_{i=1}^{n}\boldsymbol{\eta}_i^T\boldsymbol{\Lambda}_i^{-1}\boldsymbol{\eta}_i\right)$$

So

$$\begin{aligned}\prod_{i=1}^{n} p_i(\boldsymbol{x}) &= exp\left[\boldsymbol{\zeta}_{i=1...n} + \boldsymbol{\zeta}_n - \boldsymbol{\zeta}_n + \left(\sum_{i=1}^{n}\boldsymbol{\eta}_i\right)^T \boldsymbol{x} - \frac{1}{2}\boldsymbol{x}^T\left(\sum_{i=1}^{n}\boldsymbol{\Lambda}_i\right)\boldsymbol{x}\right] \\ &= exp(\boldsymbol{\zeta}_{i=1...n} - \boldsymbol{\zeta}_n) exp\left[\boldsymbol{\zeta}_n + \boldsymbol{\eta}_n^T\boldsymbol{x} - \frac{1}{2}\boldsymbol{x}^T\boldsymbol{\Lambda}_n\boldsymbol{x}\right] \end{aligned} \tag{8}$$

where

$$\boldsymbol{\Lambda}_n = \sum_{i=1}^{n}\boldsymbol{\Lambda}_i \quad , \quad \boldsymbol{\eta}_n = \sum_{i=1}^{n}\boldsymbol{\eta}_i$$

and

$$\boldsymbol{\zeta}_n = -\frac{1}{2}\left(d\log 2\pi - log|\boldsymbol{\Lambda}_n| + \boldsymbol{\eta}_n^T\boldsymbol{\Lambda}_n^{-1}\boldsymbol{\eta}_n\right) \tag{9}$$

Comparing Eqs. 7, 9 and 8 shows that the result is, as in the previous sections, a scaled Gaussian PDF over $\boldsymbol{x}$ with a mean vector and covariance matrix given by

$$\boldsymbol{V}_n^{-1} = \sum_{i=1}^{n}\boldsymbol{V}_i^{-1} \quad \text{and} \quad \boldsymbol{V}_n^{-1}\boldsymbol{\mu_n} = \sum_{i=1}^{n}\boldsymbol{V}_i^{-1}\boldsymbol{\mu}_i$$

The scaling factor is again a Gaussian function.

4 The Convolution of Two Univariate Gaussian PDFs

We wish to find the convolution of two Gaussian PDFs

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_f} e^{-\frac{(x-\mu_f)^2}{2\sigma_f^2}} \quad \text{and} \quad g(x) = \frac{1}{\sqrt{2\pi}\sigma_g} e^{-\frac{(x-\mu_g)^2}{2\sigma_g^2}}$$

in the most general case i.e. non-identical means. The convolution of two functions $f(t)$ and $g(t)$ over a finite range[1] is defined as

$$\int_0^x f(x-\tau)g(\tau)d\tau = f \otimes g$$

However, the usual approach is to use the convolution theorem [2],

$$F^{-1}[F(f(x))F(g(x))] = f(x) \otimes g(x)$$

where F is the Fourier transform

$$F(f(x)) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ikx}\ dx$$

and F^{-1} is the inverse Fourier transform

$$F^{-1}(F(k)) = \int_{-\infty}^{\infty} F(k) e^{2\pi ikx}\ dk$$

Using the transformation

$$x' = x - \mu_f$$

the Fourier transform of $f(x)$ is given by

$$F(f(x)) = \frac{1}{\sqrt{2\pi}\sigma_f} \int_{-\infty}^{\infty} e^{-\frac{x'^2}{2\sigma_f^2}} e^{-2\pi ik(x'-\mu_f)}\ dx' = \frac{e^{-2\pi ik\mu_f}}{\sqrt{2\pi}\sigma_f} \int_{-\infty}^{\infty} e^{-\frac{x'^2}{2\sigma_f^2}} e^{-2\pi ikx'}\ dx'$$

Using Euler's formula [2],

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

we can split the term in $e^{x'}$ to give

$$F(f(x)) = \frac{e^{-2\pi ik\mu_f}}{\sqrt{2\pi}\sigma_f} \int_{-\infty}^{\infty} e^{-\frac{x'^2}{2\sigma_f^2}} [\cos(2\pi kx') - i\sin(2\pi kx')]\ dx'$$

The term in $\sin(x')$ is odd and so its integral over all space will be zero, leaving

$$F(f(x)) = \frac{e^{-2\pi ik\mu_f}}{\sqrt{2\pi}\sigma_f} \int_{-\infty}^{\infty} e^{-\frac{x'^2}{2\sigma_f^2}} \cos(2\pi kx')\ dx'$$

This integral is given in standard form in [1]

$$\int_0^{\infty} e^{-at^2} \cos(2xt)\ dt = \frac{1}{2}\sqrt{\frac{\pi}{a}} e^{-\frac{x^2}{a}}$$

and so

$$F(f(x)) = e^{-2\pi ik\mu_f} e^{-2\pi^2\sigma_f^2 k^2}$$

The second term in this expression is a Gaussian PDF in k: the Fourier transform of a Gaussian PDF is another Gaussian PDF. The first term is a phase term accounting for the mean of $f(x)$ i.e. its offset from zero. The Fourier transform of $g(x)$ will give a similar expression, and so

$$F(f(x))F(g(x)) = e^{-2\pi ik\mu_f} e^{-2\pi^2\sigma_f^2k^2} e^{-2\pi ik\mu_g} e^{-2\pi^2\sigma_g^2k^2} = e^{-2\pi ik(\mu_f+\mu_g)} e^{-2\pi^2(\sigma_f^2+\sigma_g^2)k^2}$$

[1] In practice, convolutions are more often performed over an infinite range

$$\int_{-\infty}^{\infty} f(x-\tau)g(\tau)d\tau = f \otimes g$$

Comparing Eq. 25 to Eq. 24, we can see that it is the Fourier transform of a Gaussian PDF with mean and standard deviation

$$\mu_{f\otimes g} = \mu_f + \mu_g \quad \text{and} \quad \sigma_{f\otimes g} = \sqrt{\sigma_f^2 + \sigma_g^2}$$

and therefore, since the Fourier transform is invertible,

$$P_{f\otimes g}(x) = F^{-1}[F(f(x))F(g(x))] = \frac{1}{\sqrt{2\pi(\sigma_f^2 + \sigma_g^2)}} e^{-\frac{(x-(\mu_f+\mu_g))^2}{2(\sigma_f^2+\sigma_g^2)}}$$

It may be worth noting a general result at this point; the area under a convolution is equal to the product of the areas under the factors

$$\begin{aligned}\int_{-\infty}^{\infty}(f\otimes g)dt &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)g(t-u)du\right]dt \\ &= \int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} g(t-u)dt\right]du \\ &\left[\int_{-\infty}^{\infty} f(u)du\right]\left[\int_{-\infty}^{\infty} g(t)dt\right]\end{aligned}$$

Therefore, the preservation of the normalisation when convolving PDFs i.e. the fact that the convolution is also a PDF, normalised such that the area under the function is equal to unity, is a special case rather than being true in general.

5 Summary

It is well known that the product and the convolution of a pair of Gaussian PDFs are also Gaussian. In the case of the product of two univariate Gaussian PDFs $N(\mu_f, \sigma_f)$ and $N(\mu_g, \sigma_g)$, the result is a scaled Gaussian PDF where the scaling factor is itself a Gaussian PDF on both μ_f and μ_g

$$N(\mu_f,\sigma_f)N(\mu_g,\sigma_g) = \frac{S_{fg}}{\sqrt{2\pi}\sigma_{fg}}\exp\left[-\frac{(x-\mu_{fg})^2}{2\sigma_{fg}^2}\right] \quad \text{where} \quad \frac{1}{\sigma_{fg}^2} = \frac{1}{\sigma_f^2} + \frac{1}{\sigma_g^2} \quad , \quad \mu_{fg} = \left(\frac{\mu_f}{\sigma_f^2} + \frac{\mu_g}{\sigma_g^2}\right)\sigma_{fg}^2$$

$$\text{and} \quad S_{fg} = \frac{1}{\sqrt{2\pi\frac{\sigma_f^2\sigma_g^2}{\sigma_{fg}^2}}}\exp\left[-\frac{1}{2}\frac{(\mu_f-\mu_g)^2}{\sigma_f^2\sigma_g^2}\sigma_{fg}^2\right]$$

It should be noted that this result is not the PDF of the product of two Gaussian random variates; in that case, the product normal distribution applies.

The product of n univariate Gaussian PDFs is given by

$$\prod_{i=1}^{n} N(\mu_i,\sigma_i) = \frac{S_{i=1\ldots n}}{\sqrt{2\pi\sigma_{i=1\ldots n}^2}} exp\left[-\frac{(x-\mu_{i=1\ldots n})^2}{2\sigma_{i=1\ldots n}^2}\right] \quad \text{where} \quad \frac{1}{\sigma_{i=1\ldots n}^2} = \sum_{i=1}^{n}\frac{1}{\sigma_i^2} \quad , \quad \mu_{i=1\ldots n} = \left[\sum_{i=1}^{n}\frac{\mu_i}{\sigma_i^2}\right]\sigma_{i=1\ldots n}^2$$

$$\text{and} \quad S_{i=1\ldots n} = \frac{1}{(2\pi)^{(n-1)/2}}\sqrt{\frac{\sigma_{i=1\ldots n}^2}{\prod_{i=1}^{n}\sigma_i^2}}\exp\left[-\frac{1}{2}\left(\sum_{i=1}^{n}\frac{\mu_i^2}{\sigma_i^2} - \frac{\mu_{i=1\ldots n}^2}{\sigma_{i=1\ldots n}^2}\right)\right]$$

i.e. is a Gaussian PDF scaled by a Gaussian function.

The product of n multivariate Gaussian PDFs is given by

$$\prod_{i=1}^{n} N(\boldsymbol{\mu}_i, \boldsymbol{V}_i^{-1}) = exp(\boldsymbol{\zeta}_{i=1\ldots n} - \boldsymbol{\zeta}_n)exp\left[\boldsymbol{\zeta}_n + \boldsymbol{\eta}_n^T\boldsymbol{x} - \frac{1}{2}\boldsymbol{x}^T\boldsymbol{\Lambda}_n\boldsymbol{x}\right]$$

where

$$\boldsymbol{\Lambda}_i = \boldsymbol{V}_i^{-1} \quad , \quad \boldsymbol{\eta}_i = \boldsymbol{V}_i^{-1}\boldsymbol{\mu}_i \quad , \quad \boldsymbol{\Lambda}_n = \sum_{i=1}^{n}\boldsymbol{\Lambda}_i \quad , \quad \boldsymbol{\eta}_n = \sum_{i=1}^{n}\boldsymbol{\eta}_i \quad ,$$

$$\boldsymbol{\zeta}_n = -\frac{1}{2}\left(d\log 2\pi - log|\boldsymbol{\Lambda}_n| + \boldsymbol{\eta}_n^T\boldsymbol{\Lambda}_n^{-1}\boldsymbol{\eta}_n\right) \quad \text{and}$$

$$\boldsymbol{\zeta}_{i=1\dots n} = \sum_{i=1}^{n} \boldsymbol{\zeta}_i = -\frac{1}{2}\left(nd\log 2\pi - \sum_{i=1}^{n}\log|\boldsymbol{\eta}_i| + \sum_{i=1}^{n}\boldsymbol{\eta}_i^T\boldsymbol{\Lambda}_i^{-1}\boldsymbol{\eta}_i\right)$$

i.e. a Gaussian PDF scaled by a Gaussian function.

The convolution of two Gaussian PDFs is a Gaussian PDF with mean and standard deviation

$$\mu_{f\otimes g} = \mu_f + \mu_g \quad \text{and} \quad \sigma_{f\otimes g} = \sqrt{\sigma_f^2 + \sigma_g^2}$$

These results can be useful in a number of applications; for example, the convolution of Gaussian distributions freqently occurs in smoothing applied as an intermadiate step in various machine vision algorithms. Products of Gaussian PDFs may occur during the application of Bayes theorem, and in some problems related to Gaussian processes.

6 Acknoweldgements

Thanks are due to David Kirchheimer, University of Bristol, for pointing out the importance of the product of an arbitrary number of Gaussian PDFs and providing Matlab code for numerical testing of the results. Thanks are also due to Thomas Dent, Institute for Gravitational Physics (Albert Einstein Institute), Hannover, Germany, for pointing out several typographical errors.

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A Rewriting the Scaling Factor

Using Eq. 4 and 5

$$\frac{\mu_{i=1\dots n}^2}{\sigma_{i=1\dots n}^4} = \left(\sum_{i=1}^{n}\frac{\mu_i}{\sigma_i^2}\right)^2 = \sum_{i=1}^{n}\frac{\mu_i^2}{\sigma_i^4} + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\mu_i\mu_j}{\sigma_i^2\sigma_j^2}$$

So

$$2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\mu_i\mu_j}{\sigma_i^2\sigma_j^2} = \frac{\mu_{i=1\dots n}^2}{\sigma_{i=1\dots n}^4} - \sum_{i=1}^{n}\frac{\mu_i^2}{\sigma_i^4}$$

The terms in the exponent of the scaling factor for the product of univariate Gaussians take the form

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{(\mu_i-\mu_j)^2}{\sigma_i^2\sigma_j^2}\sigma_{i=1\dots n}^2 = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\frac{\mu_i^2}{\sigma_i^2\sigma_j^2} - \frac{2\mu_i\mu_j}{\sigma_i^2\sigma_j^2} + \frac{\mu_j^2}{\sigma_i^2\sigma_j^2}\right)\sigma_{i=1\dots n}^2$$

which, substituting the above expression for the cross term,

$$= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\frac{\mu_i^2}{\sigma_i^2\sigma_j^2}\sigma_{i=1\dots n}^2 + \frac{\mu_j^2}{\sigma_i^2\sigma_j^2}\sigma_{i=1\dots n}^2\right) + \sum_{i=1}^{n}\frac{\mu_i^2}{\sigma_i^4}\sigma_{i=1\dots n}^2 - \frac{\mu_{i=1\dots n}^2}{\sigma_{i=1\dots n}^4}\sigma_{i=1\dots n}^2$$

$$= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\mu_i^2\left(\frac{\sigma_{i=1\dots n}^2}{\sigma_i^2\sigma_j^2} + \frac{\sigma_{i=1\dots n}^2}{\sigma_i^4}\right) - \frac{\mu_{i=1\dots n}^2}{\sigma_{i=1\dots n}^2} = \sum_{i=1}^{n}\left(\frac{\mu_i^2}{\sigma_i^2}\right) - \frac{\mu_{i=1\dots n}^2}{\sigma_{i=1\dots n}^2}$$

Note: Appendices B and C are an older version of the derivation for the product of n univariate Gaussian PDFs; they do not use the manipulation given in Appendix A and so are considerably more complicated. The aim here was to illustrate the derivation using the proof for three Gaussian PDFs, and then to replicate each step with n Gaussian PDFs. However, these derivations are made redundant by the simpler versions given in the main document.

B The Product of Three Univariate Gaussian PDFs

Since the product of two Gaussian PDFs is a scaled Gaussian PDF, the above proof can be extended to give the product of larger numbers of Gaussian PDFs. We adopt the following notation: $N(\mu, \sigma)$ denotes a Gausian PDF with mean μ and standard deviation σ; subscripts f, g, h etc. indicate the parameters of individual Gaussian PDFs in the product; subscripts e.g. fg indicate the parameters of the products of those distributions; subscripts e.g. $(fg)h$ indicate the parameters of the product of the distribution h with a distribution that is itself the product of the distributions f and g. Therefore, the product of three Gaussian PDFs is

$$
\begin{aligned}
&N(\mu_f, \sigma_f)N(\mu_g, \sigma_g)N(\mu_h, \sigma_h) \\
&= S_{fg}N(\mu_{fg}, \sigma_{fg})N(\mu_h, \sigma_h) \\
&= S_{fg}S_{(fg)h}N(\mu_{(fg)h}, \sigma_{(fg)h})
\end{aligned}
$$

Defining

$$
S_{fgh}N(\mu_{fgh}, \sigma_{fgh}) = S_{fg}S_{(fg)h}N(\mu_{(fg)h}, \sigma_{(fg)h}) \tag{10}
$$

we have

$$
S_{fgh} = S_{fg}S_{(fg)h} \quad , \quad \mu_{fgh} = \mu_{(fg)h} \quad \text{and} \quad \sigma_{fgh} = \sigma_{(fg)h}
$$

Since the expressions for the mean and standard deviation in Equation 2 are expressed as the sums over individual terms that feature only the parameters of a single distribution f, g, they can be extended to multiple distributions easily

$$
\frac{1}{\sigma^2_{fgh}} = \frac{1}{\sigma^2_{(fg)h}} = \frac{1}{\sigma^2_{fg}} + \frac{1}{\sigma^2_h} = \frac{1}{\sigma^2_f} + \frac{1}{\sigma^2_g} + \frac{1}{\sigma^2_h} \tag{11}
$$

and

$$
\mu_{fgh} = \mu_{(fg)h} = \left(\frac{\mu_{fg}}{\sigma^2_{fg}} + \frac{\mu_h}{\sigma^2_h}\right)\sigma^2_{(fg)h} = \left(\frac{\mu_f}{\sigma^2_f} + \frac{\mu_g}{\sigma^2_g} + \frac{\mu_h}{\sigma^2_h}\right)\sigma^2_{fgh} \tag{12}
$$

The scaling factor is given by

$$
\begin{aligned}
S_{fgh} = S_{fg}S_{(fg)h} &= \frac{1}{\sqrt{2\pi(\sigma^2_f + \sigma^2_g)}} \exp\left[-\frac{(\mu_f - \mu_g)^2}{2(\sigma^2_f + \sigma^2_g)}\right] \frac{1}{\sqrt{2\pi(\sigma^2_{fg} + \sigma^2_h)}} \exp\left[-\frac{(\mu_{fg} - \mu_h)^2}{2(\sigma^2_{fg} + \sigma^2_h)}\right] \\
&= \frac{1}{2\pi\sqrt{(\sigma^2_f + \sigma^2_g)(\sigma^2_{fg} + \sigma^2_h)}} \exp\left[-\frac{1}{2}\left(\frac{(\mu_f - \mu_g)^2}{\sigma^2_f + \sigma^2_g} + \frac{(\mu_{fg} - \mu_h)^2}{\sigma^2_{fg} + \sigma^2_h}\right)\right]
\end{aligned}
$$

This can be dealt with as two separate terms; first

$$
(\sigma^2_f + \sigma^2_g)(\sigma^2_{fg} + \sigma^2_h) = (\sigma^2_f + \sigma^2_g)(\frac{\sigma^2_f\sigma^2_g}{\sigma^2_f + \sigma^2_g} + \sigma^2_h) = \sigma^2_f\sigma^2_g + \sigma^2_f\sigma^2_h + \sigma^2_g\sigma^2_h
$$

Second

$$
\frac{(\mu_f - \mu_g)^2}{\sigma^2_f + \sigma^2_g} + \frac{(\mu_{fg} - \mu_h)^2}{\sigma^2_{fg} + \sigma^2_h} = \frac{(\mu_f - \mu_g)^2(\sigma^2_{fg} + \sigma^2_h) + (\mu_{fg} - \mu_h)^2(\sigma^2_f + \sigma^2_g)}{(\sigma^2_f + \sigma^2_g)(\sigma^2_{fg} + \sigma^2_h)}
$$

The denominator is the same as the first term, above, so only the numerator need be dealt with

$$
M = (\mu_f - \mu_g)^2(\sigma^2_{fg} + \sigma^2_h) + (\mu_{fg} - \mu_h)^2(\sigma^2_f + \sigma^2_g)
$$

Substituting the expressions for μ_{fg} and σ_{fg} from Eq. 1,

$$
M = (\mu_f - \mu_g)^2\sigma^2_h + (\mu_f - \mu_g)^2\frac{\sigma^2_f\sigma^2_g}{\sigma^2_f + \sigma^2_g} + \left[\frac{\mu_f\sigma^2_g + \mu_g\sigma^2_f}{\sigma^2_f + \sigma^2_g} - \mu_h\right]^2(\sigma^2_f + \sigma^2_g)
$$

One approach is expand the expression fully and then pair all of the terms to give an overall factor of $\sigma_f^2 + \sigma_g^2$. However, this is impractical as a route to a proof for the product of arbitrary numbers of Gaussians. Instead,

$$M = (\mu_f - \mu_g)^2\sigma_h^2 + (\mu_f - \mu_g)^2 \frac{\sigma_f^2\sigma_g^2}{\sigma_f^2 + \sigma_g^2} + \frac{\left[\mu_f\sigma_g^2 + \mu_g\sigma_f^2 - \mu_h(\sigma_f^2 + \sigma_g^2)\right]^2}{(\sigma_f^2 + \sigma_g^2)}$$

$$= (\mu_f - \mu_g)^2\sigma_h^2 + (\mu_f - \mu_g)^2 \frac{\sigma_f^2\sigma_g^2}{\sigma_f^2 + \sigma_g^2} + \frac{\left[(\mu_f - \mu_h)\sigma_g^2 + (\mu_g - \mu_h)\sigma_f^2\right]^2}{\sigma_f^2 + \sigma_g^2}$$

$$= (\mu_f - \mu_g)^2\sigma_h^2 + (\mu_f - \mu_g)^2 \frac{\sigma_f^2\sigma_g^2}{\sigma_f^2 + \sigma_g^2} + \frac{(\mu_f - \mu_h)^2\sigma_g^4}{\sigma_f^2 + \sigma_g^2} + 2(\mu_f - \mu_h)(\mu_g - \mu_h)\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2 + \sigma_g^2} + \frac{(\mu_g - \mu_h)^2\sigma_f^4}{\sigma_f^2 + \sigma_g^2}$$

Now, observe that

$$(A - B)^2 + 2(A - C)(B - C) = A^2 - 2AB + B^2 + 2AB - 2AC - 2BC + 2C^2$$
$$= A^2 - 2AC + C^2 + B^2 - 2BC + C^2 = (A - C)^2 + (B - C)^2 \tag{13}$$

Therefore

$$(\mu_f - \mu_g)^2 \frac{\sigma_f^2\sigma_g^2}{\sigma_f^2 + \sigma_g^2} + 2(\mu_f - \mu_h)(\mu_g - \mu_h)\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2 + \sigma_g^2} = (\mu_f - \mu_h)^2\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2 + \sigma_g^2} + (\mu_g - \mu_h)^2\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2 + \sigma_g^2}$$

and so

$$M = (\mu_f - \mu_g)^2\sigma_h^2 + \frac{(\mu_f - \mu_h)^2\sigma_g^4}{\sigma_f^2 + \sigma_g^2} + \frac{(\mu_g - \mu_h)^2\sigma_f^4}{\sigma_f^2 + \sigma_g^2} + (\mu_f - \mu_h)^2\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2 + \sigma_g^2} + (\mu_g - \mu_h)^2\frac{\sigma_f^2\sigma_g^2}{\sigma_f^2 + \sigma_g^2}$$

$$= (\mu_f - \mu_g)^2\sigma_h^2 + (\mu_f - \mu_h)^2\frac{\sigma_g^4 + \sigma_f^2\sigma_g^2}{\sigma_f^2 + \sigma_g^2} + (\mu_g - \mu_h)^2\frac{\sigma_f^4 + \sigma_f^2\sigma_g^2}{\sigma_f^2 + \sigma_g^2}$$

$$= (\mu_f - \mu_g)^2\sigma_h^2 + (\mu_f - \mu_h)^2\sigma_g^2 + (\mu_g - \mu_h)^2\sigma_f^2$$

Collecting terms, this gives

$$S_{fgh} = \frac{1}{2\pi\sqrt{\sigma_f^2\sigma_g^2 + \sigma_f^2\sigma_h^2 + \sigma_g^2\sigma_h^2}} \exp\left[-\frac{1}{2}\frac{(\mu_f - \mu_g)^2\sigma_h^2 + (\mu_f - \mu_h)^2\sigma_g^2 + (\mu_g - \mu_h)^2\sigma_f^2}{\sigma_f^2\sigma_g^2 + \sigma_f^2\sigma_h^2 + \sigma_g^2\sigma_h^2}\right]$$

which can be written more conveniently as

$$S_{fgh} = \frac{1}{2\pi\sqrt{\frac{\sigma_f^2\sigma_g^2\sigma_h^2}{\sigma_{fgh}^2}}} \exp\left[-\frac{1}{2}\left(\frac{(\mu_f - \mu_g)^2}{\sigma_f^2\sigma_g^2} + \frac{(\mu_f - \mu_h)^2}{\sigma_f^2\sigma_h^2} + \frac{(\mu_g - \mu_h)^2}{\sigma_g^2\sigma_h^2}\right)\sigma_{fgh}^2\right] \tag{14}$$

Therefore, the product of three Gaussian PDFs is a scaled Gaussian PDF

$$f(x)g(x)h(x) = \frac{S_{fgh}}{\sqrt{2\pi}\sigma_{fgh}} \exp\left[-\frac{(x - \mu_{fgh})^2}{2\sigma_{fgh}^2}\right]$$

where σ_{fgh}, μ_{fgh} and S_{fgh} are given by Eqs. 11, 12 and 14 respectively.

As in Section 1, the scaling factor can be rewritten using Appendix A to give

$$S_{fgh} = \frac{1}{2\pi\sqrt{\frac{\sigma_f^2\sigma_g^2\sigma_h^2}{\sigma_{fgh}^2}}} \exp\left[-\frac{1}{2}\left(\frac{\mu_f^2}{\sigma_f^2} + \frac{\mu_g^2}{\sigma_g^2} + \frac{\mu_h^2}{\sigma_h^2} - \frac{\mu_{fgh}^2}{\sigma_{fgh}^2}\right)\right]$$

C The Product of n Univariate Gaussian PDFs

Let subscript i refer to an individual Gaussian PDF in a product of n univariate Gaussian PDFs. Based on the derivations in Sections 1 and B, it is clear that the product is also a Gaussian PDF, multiplied by a scaling factor. The notation used in Section B is extended, so that the subscript $i = 1...n$ refers to the parameters of the distribution that is the product n individual Gaussian PDFs and subscript $i = (1...n-1)n$ refers to the parameters of a distribution that is the product of two Gaussian PDFs, one of which is itself the product of $n-1$ Gaussian PDFs. In addition, define

$$\alpha_n = \sum_{i=1}^{n} \left[\prod_{\substack{j=1 \\ j \neq i}}^{n} \sigma_i^2 \right] \quad \text{and} \quad \gamma_n = \prod_{i=1}^{n} \sigma_i^2$$

By inspection of the results for the products of two and three Gaussian PDFs, state

$$\prod_{i=1}^{n} N(\mu_i, \sigma_i) = \frac{S_{i=1...n}}{\sqrt{2\pi\sigma_{i=1...n}^2}} e^{-\frac{(x-\mu_{i=1...n})^2}{2\sigma_{i=1...n}^2}}$$

where

$$\frac{1}{\sigma_{i=1...n}^2} = \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \quad \text{or} \quad \sigma_{i=1...n}^2 = \frac{\gamma_n}{\alpha_n} \quad , \quad \mu_{i=1...n} = \left[\sum_{i=1}^{n} \frac{\mu_i}{\sigma_i^2} \right] \sigma_{i=1...n}^2$$

and

$$S_{i=1...n} = \frac{1}{\sqrt{(2\pi)^{n-1}\alpha_n}} \exp\left[-\frac{\sigma_{i=1...n}^2}{2} \left(\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{(\mu_i - \mu_j)^2}{\sigma_i^2 \sigma_j^2} \right) \right] \tag{15}$$

The above expressions can be proved by observing that, following Eq. 10,

$$\begin{aligned} S_{i=1...n} N(\mu_{i=1...n}, \sigma_{i=1...n}) &= S_{i=1...n-1} N(\mu_{i=1...n-1}, \sigma_{i=1...n-1}) N(\mu_n, \sigma_n) \\ &= S_{i=1...n-1} S_{(i=1...n-1)n} N(\mu_{i=1...n}, \sigma_{i=1...n}) \end{aligned} \tag{16}$$

Therefore, using Eq. 2

$$\frac{1}{\sigma_{i=1...n}^2} = \frac{1}{\sigma_{i=1...n-1}^2} + \frac{1}{\sigma_n^2} \quad \text{and} \quad \mu_{i=1...n} = \left(\frac{\mu_{i=1...n-1}}{\sigma_{i=1...n-1}^2} + \frac{\mu_n}{\sigma_n^2} \right) \sigma_{i=1...n}^2$$

Eq. 2 can then be substituted to expand $\sigma_{i=1...n-1}^2$ into $\sigma_{i=1...n-2}^2$ and σ_{n-1}^2, and $\mu_{i=1...n-1}$ into $\mu_{i=1...n-2}$ and μ_{n-1}; repeating this gives

$$\frac{1}{\sigma_{i=1...n}^2} = \frac{1}{\sigma_{i=1...n-1}^2} + \frac{1}{\sigma_n^2} = \frac{1}{\sigma_{i=1...n-2}^2} + \frac{1}{\sigma_{n-1}^2} + \frac{1}{\sigma_n^2} = ... = \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \quad \text{Q.E.D.}$$

$$\begin{aligned} \mu_{i=1...n} &= \left[\frac{\mu_{i=1...n-1}}{\sigma_{i=1...n-1}^2} + \frac{\mu_n}{\sigma_n^2} \right] \sigma_{i=1...n}^2 = \left[\left(\frac{\mu_{i=1...n-2}}{\sigma_{i=1...n-2}^2} + \frac{\mu_{n-1}}{\sigma_{n-1}^2} \right) \frac{\sigma_{i=1...n-1}^2}{\sigma_{i=1...n-1}^2} + \frac{\mu_n}{\sigma_n^2} \right] \sigma_{i=1...n}^2 \\ &= \left[\frac{\mu_{i=1...n-2}}{\sigma_{i=1...n-2}^2} + \frac{\mu_{n-1}}{\sigma_{n-1}^2} + \frac{\mu_n}{\sigma_n^2} \right] \sigma_{i=1...n}^2 = ... = \left[\sum_{i=1}^{n} \frac{\mu_i}{\sigma_i^2} \right] \sigma_{i=1...n}^2 \quad \text{Q.E.D.} \end{aligned}$$

Eq. 15 can be written using α as

$$S_{i=1...n} = \frac{1}{\sqrt{(2\pi)^{n-1}\alpha_n}} \exp\left[-\frac{1}{2} \left(\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left[(\mu_i - \mu_j)^2 \prod_{\substack{k=1 \\ k \neq i,j}}^{n} \sigma_k^2 \right] \right) \frac{1}{\alpha_n} \right]$$

Similarly, Eq. 2 gives the scaling factor for the product of $N(\mu_{i=1...n}, \sigma_{i=1...n})$ and $N(\mu_n, \sigma_n)$ as

$$S_{(i=1...n)n+1} = \frac{1}{\sqrt{2\pi(\sigma_{i-1...n}^2 + \sigma_{n+1}^2)}} \exp\left[-\frac{1}{2} \frac{(\mu_{i=1...n} - \mu_{n+1})^2}{\sigma_{i=1...n}^2 + \sigma_{n+1}^2} \right]$$

Therefore, the aim here is to show that

$$S_{i=1\ldots n+1} = \frac{1}{\sqrt{(2\pi)^n \alpha_n(\sigma^2_{i-1\ldots n} + \sigma^2_{n+1})}} \exp -\frac{1}{2}\left[\left(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[(\mu_i-\mu_j)^2\prod_{\substack{k=1\\k\neq i,j}}^{n}\sigma_k^2\right]\right)\frac{1}{\alpha_n} + \frac{(\mu_{i=1\ldots n}-\mu_{n+1})^2}{\sigma^2_{i=1\ldots n}+\sigma^2_{n+1}}\right]$$

The standard deviation term is

$$\alpha_n(\sigma^2_{i-1\ldots n} + \sigma^2_{n+1}) = \alpha_n\sigma^2_{i-1\ldots n} + \alpha_n\sigma^2_{n+1} = \gamma_n + \alpha_n\sigma^2_{n+1} = \alpha_{n+1}$$

The exponential term, ignoring the $-1/2$, is

$$\left(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[(\mu_i-\mu_j)^2\prod_{\substack{k=1\\k\neq i,j}}^{n}\sigma_k^2\right]\right)\frac{1}{\alpha_n} + \frac{(\mu_{i=1\ldots n}-\mu_{n+1})^2}{\sigma^2_{i=1\ldots n}+\sigma^2_{n+1}}$$

$$= \frac{\left(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[(\mu_i-\mu_j)^2\prod_{\substack{k=1\\k\neq i,j}}^{n}\sigma_k^2\right]\right)(\sigma^2_{i=1\ldots n}+\sigma^2_{n+1}) + (\mu_{i=1\ldots n}-\mu_{n+1})^2\alpha_n}{(\sigma^2_{i=1\ldots n}+\sigma^2_{n+1})\alpha_n}$$

The denominator is, as expected, the standard deviation term that was dealt with above; ignore this, and let the numerator be called M

$$M = \left(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[(\mu_i-\mu_j)^2\prod_{\substack{k=1\\k\neq i,j}}^{n}\sigma_k^2\right]\right)(\sigma^2_{i=1\ldots n}+\sigma^2_{n+1}) + (\mu_{i=1\ldots n}-\mu_{n+1})^2\alpha_n$$

$$= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left((\mu_i-\mu_j)^2\prod_{\substack{k=1\\k\neq i,j}}^{n}\sigma_k^2\right)\sigma^2_{i=1\ldots n} + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left((\mu_i-\mu_j)^2\prod_{\substack{k=1\\k\neq i,j}}^{n}\sigma_k^2\right)\sigma^2_{n+1} + \left[\left(\sum_{i=1}^{n}\frac{\mu_i}{\sigma_i^2}\right)\sigma^2_{i=1\ldots n} - \mu_{n+1}\right]^2\alpha_n$$

Focussing on the last of these three terms

$$\alpha_n\left[\left(\sum_{i=1}^{n}\frac{\mu_i}{\sigma_i^2}\right)\sigma^2_{i=1\ldots n} - \mu_{n+1}\right]^2 = \alpha_n\left[\left(\sum_{i=1}^{n}\frac{\mu_i}{\sigma_i^2}\right)\frac{\gamma^2_{i=1\ldots n}}{\alpha^2_{i=1\ldots n}} - \mu_{n+1}\right]^2 = \frac{1}{\alpha_n}\left[\sum_{i=1}^{n}\left(\mu_i\prod_{\substack{j=1\\j\neq i}}^{n}\sigma_j^2\right) - \alpha^2_{i=1\ldots n}\mu_{n+1}\right]^2$$

$$= \frac{1}{\alpha_n}\left[\sum_{i=1}^{n}\left(\mu_i\prod_{\substack{j=1\\j\neq i}}^{n}\sigma_j^2\right) - \sum_{i=1}^{n}\left(\mu_{n+1}\prod_{\substack{j=1\\j\neq i}}^{n}\sigma_j^2\right)\right]^2 = \frac{1}{\alpha_n}\left[\sum_{i=1}^{n}\left((\mu_i-\mu_{n+1})\prod_{\substack{j=1\\j\neq i}}^{n}\sigma_j^2\right)\right]^2$$

$$= \frac{1}{\alpha_n}\left[\sum_{i=1}^{n}\left((\mu_i-\mu_{n+1})^2\prod_{\substack{j=1\\j\neq i}}^{n}\sigma_j^4\right) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left((\mu_i-\mu_{n+1})(\mu_j-\mu_{n+1})\prod_{k=1}^{n}\sigma_k^2\prod_{\substack{l=1\\l\neq i,j}}^{n}\sigma_l^2\right)\right]$$

However,

$$\frac{\prod_{k=1}^{n}\sigma_k^2}{\alpha_n} = \sigma^2_{i=1\ldots n}$$

so, recombining this with the first two terms of M gives

$$M = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left((\mu_i-\mu_j)^2\prod_{\substack{k=1\\k\neq i,j}}^{n}\sigma_k^2\sigma_{n+1}^2\right) + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left((\mu_i-\mu_j)^2\prod_{\substack{k=1\\k\neq i,j}}^{n}\sigma_k^2\right)\sigma^2_{i=1\ldots n} +$$

$$\frac{1}{\alpha_n}\sum_{i=1}^{n}\left((\mu_i-\mu_{n+1})^2\prod_{\substack{j=1\\j\neq i}}^{n}\sigma_j^4\right) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left((\mu_i-\mu_{n+1})(\mu_j-\mu_{n+1})\prod_{\substack{k=1\\k\neq i,j}}^{n}\sigma_k^2\right)\sigma^2_{i=1\ldots n}$$

Applying Eq. 13 to the second and fourth terms gives

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left((\mu_i-\mu_j)^2\prod_{\substack{k=1\\k\neq i,j}}^{n}\sigma_k^2\sigma_{n+1}^2\right)+\frac{1}{\alpha_n}\sum_{i=1}^{n}\left((\mu_i-\mu_{n+1})^2\prod_{\substack{j=1\\j\neq i}}^{n}\sigma_j^4\right)$$

$$+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[\left((\mu_i-\mu_{n+1})^2+(\mu_j-\mu_{n+1})^2\right)\prod_{\substack{k=1\\k\neq i,j}}^{n}\sigma_k^2\right]\sigma_{i=1\ldots n}^2$$

$$=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left((\mu_i-\mu_j)^2\prod_{\substack{k=1\\k\neq i,j}}^{n}\sigma_k^2\sigma_{n+1}^2\right)+\frac{1}{\alpha_n}\sum_{i=1}^{n}\left((\mu_i-\mu_{n+1})^2\frac{\gamma_n^2}{\sigma_i^4}\right)$$

$$+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left[\left((\mu_i-\mu_{n+1})^2+(\mu_j-\mu_{n+1})^2\right)\sum_{\substack{j=1\\j\neq i}}^{n}\frac{\gamma_n}{\sigma_i^2\sigma_j^2}\right]\sigma_{i=1\ldots n}^2$$

$$=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left((\mu_i-\mu_j)^2\prod_{\substack{k=1\\k\neq i,j}}^{n}\sigma_k^2\sigma_{n+1}^2\right)+\frac{1}{\alpha_n}\sum_{i=1}^{n}\left((\mu_i-\mu_{n+1})^2\frac{\gamma_n^2}{\sigma_i^4}\right)+\sum_{i=1}^{n}\left[(\mu_i-\mu_{n+1})^2\sum_{\substack{j=1\\j\neq i}}^{n}\frac{\gamma_n}{\sigma_i^2\sigma_j^2}\right]\sigma_{i=1\ldots n}^2$$

$$=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left((\mu_i-\mu_j)^2\prod_{\substack{k=1\\k\neq i,j}}^{n}\sigma_k^2\sigma_{n+1}^2\right)+\sum_{i=1}^{n}\left((\mu_i-\mu_{n+1})^2\left[\frac{\gamma_n^2}{\alpha_n\sigma_i^4}+\sum_{\substack{j=1\\j\neq i}}^{n}\frac{\gamma_n\sigma_{i=1\ldots n}^2}{\sigma_i^2\sigma_j^2}\right]\right)$$

Now, examine

$$\frac{\gamma_n^2}{\alpha_n\sigma_i^4}+\sum_{\substack{j=1\\j\neq i}}^{n}\frac{\gamma_n\sigma_{i=1\ldots n}^2}{\sigma_i^2\sigma_j^2}=\frac{\gamma_n^2}{\alpha_n}\left(\frac{1}{\sigma_i^4}+\sum_{\substack{j=1\\j\neq i}}^{n}\frac{1}{\sigma_i^2\sigma_j^2}\right)=\frac{\gamma_n^2}{\alpha_n}\sum_{j=1}^{n}\frac{1}{\sigma_i^2\sigma_j^2}=\sigma_{i=1\ldots n}^2\gamma_n\frac{1}{\sigma_i^2}\sum_{j=1}^{n}\frac{1}{\sigma_j^2}=\frac{\sigma_{i=1\ldots n}}{\sigma_i^2}=\prod_{\substack{j=1\\j\neq i}}^{n}\sigma_j^2$$

So,

$$M=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left((\mu_i-\mu_j)^2\prod_{\substack{k=1\\k\neq i,j}}^{n}\sigma_k^2\sigma_{n+1}^2\right)+\sum_{i=1}^{n}\left((\mu_i-\mu_{n+1})^2\prod_{\substack{j=1\\j\neq i}}^{n}\sigma_j^2\right)=\sum_{i=1}^{n}\sum_{j=i+1}^{n+1}\left((\mu_i-\mu_j)^2\prod_{\substack{k=1\\k\neq i,j}}^{n+1}\sigma_k^2\right)$$

Collecting terms

$$S_{i=1\ldots n}S_{(i=1\ldots n)n+1}=\frac{1}{\sqrt{(2\pi)^n\alpha_{n+1}}}\exp\left[-\frac{\sigma_{i=1\ldots n+1}^2}{2}\left(\sum_{i=1}^{n}\sum_{j=i+1}^{n+1}\frac{(\mu_i-\mu_j)^2}{\sigma_i^2\sigma_j^2}\right)\right]=S_{i=1\ldots n+1}\quad\text{Q.E.D.}$$