# Summary of Gradient descent and Newton's method from coursera course

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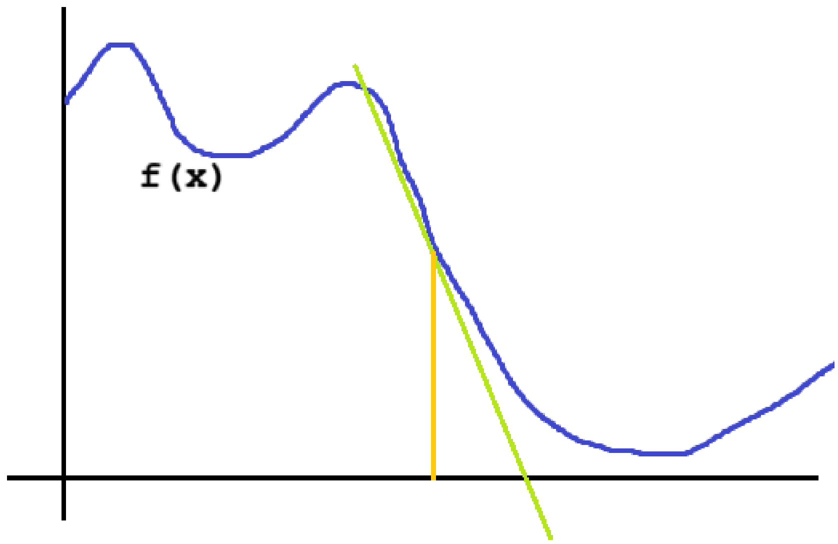
In this document we provide brief summary of how coursera explains

- Gradient descent with one variable
- Gradient descent with two or more variables
- Newton's method with one variable
- Newton's method with two or more variables

## 1 Gradient descent with one variable

Assume that we have a continuous function f defined on $\mathbb{R}$:

*Inspired by coursera mathematics for ML specialization.




Assume that f is also differentiable with derivative $f'(x)$.
We also have a starting point $x_0$.
Then we get $x_1$ by subtracting $f'(x_0) \cdot \alpha$ from x, where $\alpha$ is called the learning rate, which we can choose before doing this procedure.
Usual values for $\alpha$ are $0,01$ or $0,05$.
We iterate this, so that we get an array which is recursively defined as:

$$x_{k+1} = x_k - f'(x_k) \cdot \alpha$$

This array will converge to the minimum of f. The pitchfalls here are, that the procedure may end in a local minimum, while f has a stronger minimum elsewhere.
Or with a less than optimal choice for the learning rate, the array could even diverge.

## 2 Gradient descent with two or more variables

With a function $f(x, y)$ of more variables, we can determine the gradient:

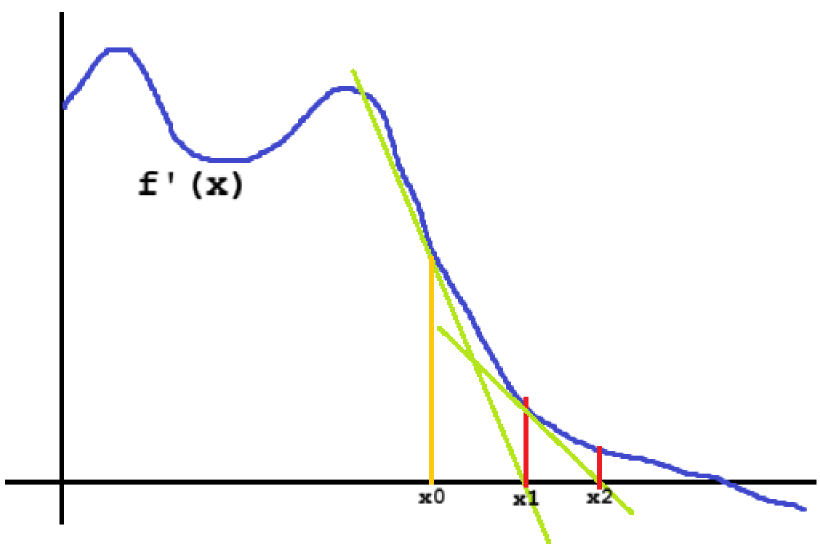
$$\nabla f(x, y) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

The method is the same, but where we took the derivative for one variable, we will now take the gradient, and the recursive definition of our array $(x_k, y_k)$ becomes:

$$(x_{k+1}, y_{k+1}) = (x_k, y_k) - \nabla f(x, y) \cdot \alpha$$

# 3 Newton's method with one variable

Newton's method finds the zeroes of a function f. Because we're interested in finding a minimum of f, Newton's method will help us find the zero of it's derivative $f'$.




Geometrically, when you have the graph of f, and you have a starting point $x_0$, we start by drawing the tangent line.
Then we see where this tangent line intersects with the x-axis. That will be $x_1$.
By iterating this procedure we get an array $(x_k)_{k=0,1,...}$.

From the geometrical aspect of the procedure, we can give a formula between $x_{k+1}$ and $x_k$:

$$x_{k+1} = x_k - (f'(x_k)/f''(x_k))$$

(that is for finding the zero of $f'$) The idea is that the array $(x_k)$ converges to the value x where $f'(x) = 0$.

In order to know if $f'(x)$ points to a minimum of f, we need to look at the second derivative $f''(x)$:
$f''(x) > 0 \Rightarrow$ f has a minimum at x
$f''(x) < 0 \Rightarrow$ f has a maximum at x
$f''(x) = 0 \Rightarrow$ inconclusive, perhaps an inflection point

# 4 Newton's method with two or more variables

Say we have function $f(x, y)$ of 2 variables.

Then here we have it's Hessian matrix:

$$H_f(x, y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}.$$

Now at a given point (x,y) we'll calculate it's eigenvalues $\lambda_1, \lambda_2, \ldots$
(A 2 by 2 matrix would have at most two eigenvalues)

If the gradient has value (0,0) at point (x,y) then:

- If all the eigenvalues of $H_f$ at (x,y) are positive, it's a minimum
- If all the eigenvalues of $H_f$ at (x,y) are negative, it's a maximum
- In other cases, it's inconclusive

In Newton's method generalized to more than one variables, the formula for the next point is:

$$(x_{k+1}, y_{k+1}) = (x_k, y_k) - (H_f^{-1}(x_k, y_k) \cdot \nabla f(x_k, y_k))$$