Exercises Week 2

Econometrics

1. Consider the projections matrices P_X and $P_{X,Z}$, where the second one extends the regressors to include the set Z. Show that the matrix $P_{X,Z} - P_X$ is an orthogonal projection matrix. That is, show that it is symmetric and idempotent.

2. **Ex. 3.14 in ETM**: In this exercise you are going to prove that $E(\hat{U}'\hat{U}) = (n-k)\sigma^2$.

i. Use the *maker of residuals* matrix to write the residuals in terms of the errors.

ii. Use the fact that the sum of squares is a scalar to write this product using the trace operator.

iii. Use the fact that the trace of a product of matrices is invariant to cyclic permutations, $Tr(ABC) = Tr(CAB)$, to show that $E(\hat{U}'\hat{U}) = \sigma Tr(M_X)$, where σ is the variance of the error.

iv. Use the cyclic permutation property again to show that $Tr(M_X) = n-k$, where n is the sample size and k is the number of regressors.

v. Use the above to construct an unbiased estimator for the variance of the residuals.

3. In the following exercises you are going to show directly, without referring to the Gauss-Markov theorem, that the estimator from the correct specification is more efficient than the one from a specification augmented with irrelevant regressors.

i. **Ex. 3.8 in ETM**: If A is a symmetric positive definite $k \times k$ matrix, then $I - A$ is positive definite if and only if $A^{-1} - I$ is positive definite, where I is the $k \times k$ identity matrix. Prove this result by considering the quadratic form $x'(I - A)x$ and expressing x as $R^{-1}z$, where R is a symmetric matrix such that $A = R^2$.

ii. Extend the above result to show that, if A and B are symmetric positive definite matrices of the same dimensions, then $A - B$ is positive definite if and only if $B^{-1} - A^{-1}$ is positive definite.

iii. Show that the difference from the inverse of the variance of the estimator of the correctly specified model and the inverse of the variance of the estimator for the augmented model can be written as a quadratic form; that is, show that $Var^{-1}(\hat{\beta}) - Var^{-1}(\tilde{\beta}) = \sigma^{-2}(P_Z X)'(P_Z X)$.

iv. Use the above to show that the estimator from the correct specification is more efficient than the one from a specification augmented with irrelevant regressors.

4. **Ex. 3.1 in ETM**: Generate a sample of size 25 from the model

$$y_t = \beta_1 + \beta_2 y_{t-1} + u_t,$$

with $\beta_1 = 1$ and $\beta_2 = 0.8$. For simplicity, assume that $y_0 = 0$ and that the u_t are $NID(0, 1)$. Use this sample to compute the OLS estimates $\hat{\beta}_1$ and $\hat{\beta}_2$. Repeat at least 100 times, and find the averages of the $\hat{\beta}_1$ and $\hat{\beta}_2$. Use these averages to estimate the bias of the OLS estimators of β_1 and β_2.

Repeat this exercise for sample sizes of 50, 100, and 200. What happens to the bias of $\hat{\beta}_1$ and $\hat{\beta}_2$ as the sample size is increased?

5. **Goodness of fit for irrelevant regressors**: In this exercise, you are going to conduct a Monte Carlo simulation to show graphically that the R-squared increases as we add more regressors regardless of whether they are part of the model or not.

Hence, you will need to follow the steps below:

- Set the sample size $n = 100$.
- Then, for $R = 1000$ repetitions, do the following:

  i. Generate a regressor x from a normal distribution with mean 0 and variance 1.
  ii. Generate an error term u from a normal distribution with mean 0 and variance σ^2 of your choosing.
  iii. Generate a dependent variable y from the following model:

  $$y = \beta_1 x + u,$$

  for $\beta_1 = 1$.
  iv. Generate a matrix of $m = 1$ irrelevant regressors drawing independent normal distributions with mean 0 and variance 1.
  v. Estimate the model with the additional regressors, compute the R-square, and store it in a matrix with R rows and 20 columns.
  vi. Repeat steps iv and v adding 1 to m up to 20.

- Plot the average R-squared for each size model.