Random Matrix Theory

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Contents

This paper is based on lectures given by Mr. Jamal Najim at Université Paris-Est Marne-la-Vallée. We begin with a short introduction to a problem of our interest.

Large Covariance Matrices

Let $X_N = (X_{ij})$ be a $N \times n$ matrix with i.i.d elements, such that

$$\mathbb{E}X_{ij} = 0, \quad \mathbb{E}|X_{ij}| = 1, \quad \mathbb{E}|X_{ij}| < \infty.$$

Let R_N be a $N \times N$ deterministic semi-positive hermitian matrix with a uniformly bounded spectral norm, i.e.

$$\mathbf{R} := \sup_N \|R_N\|_{sp} < \infty.$$

We denote $\Sigma_n = \frac{1}{\sqrt{n}} R_N^{1/2} X_N$.

Theorem 1. *Let L_N be a spectral measure of the matrix $\Sigma_n \Sigma_n^*$ and g_n its Stieltjes transformation. Let L_N^R be a spectral measure of the matrix R_N, if*

$$L_N^R \xrightarrow[N,n\to\infty]{etr} L_\infty^R,$$

therefore

(a) the following equation

$$t(z) = \int \frac{L_\infty^R(du)}{-z(1 + uct(z)) + (1-c)u}, \quad z \in \mathbb{C}^+$$

has a unique solution $z \mapsto t(z)$ which is a Stieltjes transformation of a probability measure.

(b) $\forall z \in \mathbb{C}^+$, almost surely $g_n(z) \xrightarrow[N,n\to\infty]{p.s.} t(z)$

Important example

The goal of this paper is to study the case when R_N has finite number of eigenvalues. Suppose that the spectral measure of R_N has the following from

$$L_N^R = \frac{1}{N} \sum_{l=1}^{K} n_l \delta_{\rho_l^R},$$

where K is the number of unique eigenvalues of R_N and K is independent from N, n, let

$$\frac{n_l}{N} \xrightarrow[N,n\to\infty]{} m_l > 0.$$

In this case

$$L_N^R \xrightarrow[N,n\to\infty]{etr} L_\infty^R = \sum_{l=1}^{K} m_l \delta_{\rho_l^R}$$

and we obtain the following equation

$$t(z) = \sum_{i=1}^{K} \frac{m_i}{-z(1 + c\rho_i t(z)) + (1-c)\rho_i}. \tag{1}$$

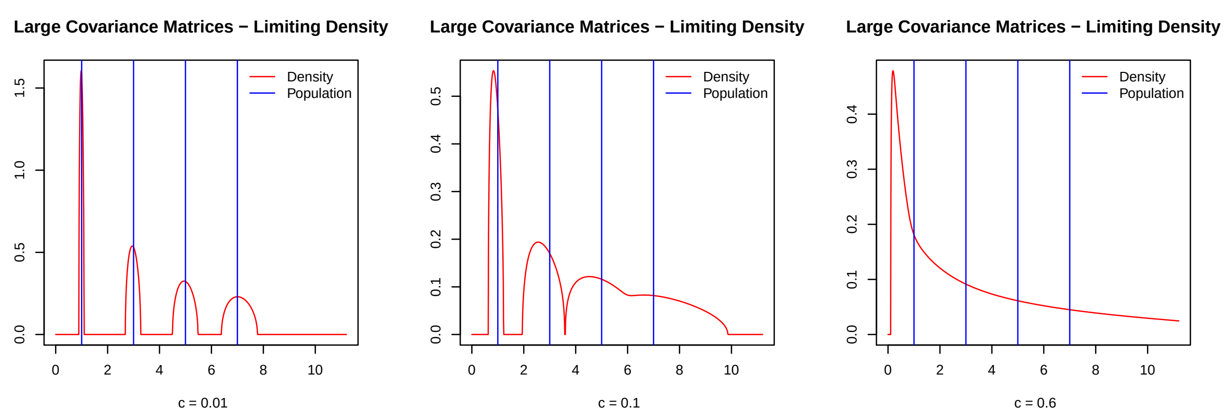
Proceeding the following algorithm one can obtain the density $f(x)$ associated with $t(z)$

- Numerically solve (1) for $z = x \in \mathbb{R}$
- Take the unique solution (if exists) $t(x)$ such that $\Im t(x) > 0$
- Density in point x is given by $f(x) = \frac{1}{\pi} \Im t(x)$
- If there is no solution with $\Im t(x) > 0$, assign $f(x) = 0$

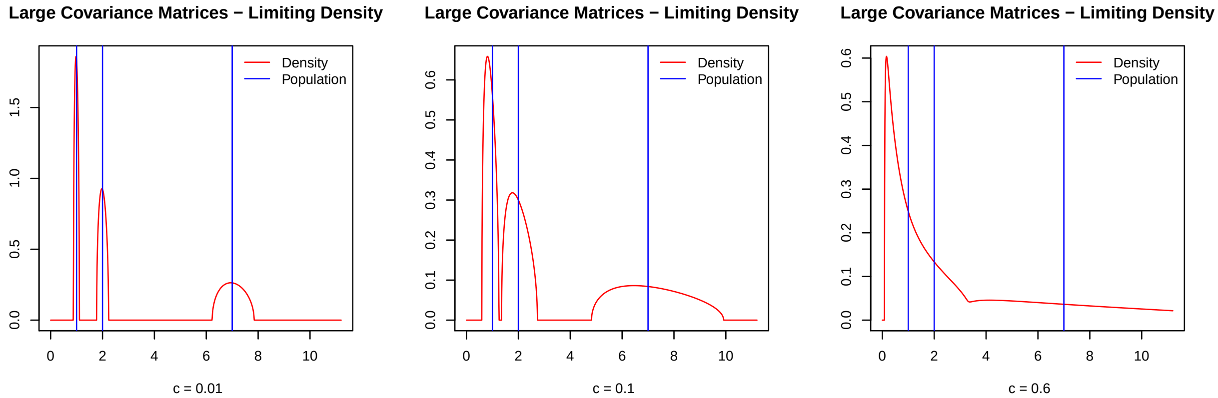
Numerical experiments

We want to make numerical experiments of this case and study the dependence of density on parameter $c = \frac{N}{n}$. To proceed these numerical experiment we have implemented a function which recieves 3 parameters, namely a vector of "weights" $\mathbf{m} = (m_1, ..., m_k)$ such that $\sum_{i=1}^{K} m_i = 1$, a vector of eigenvalues $\rho = (\rho_1, ..., \rho_k)$ and $c = \frac{N}{n}$, and gives a plot of density. This parameters give us an explicit information about matrix R_N and a matrix X_n therefore about matrix Σ_N.

Assume that R_N is such that $K = 4$ and $\rho_1 = 1, \rho_2 = 3, \rho_3 = 5, \rho_4 = 7$, $m_1 = m_2 = m_3 = m_4 = \frac{1}{4}$



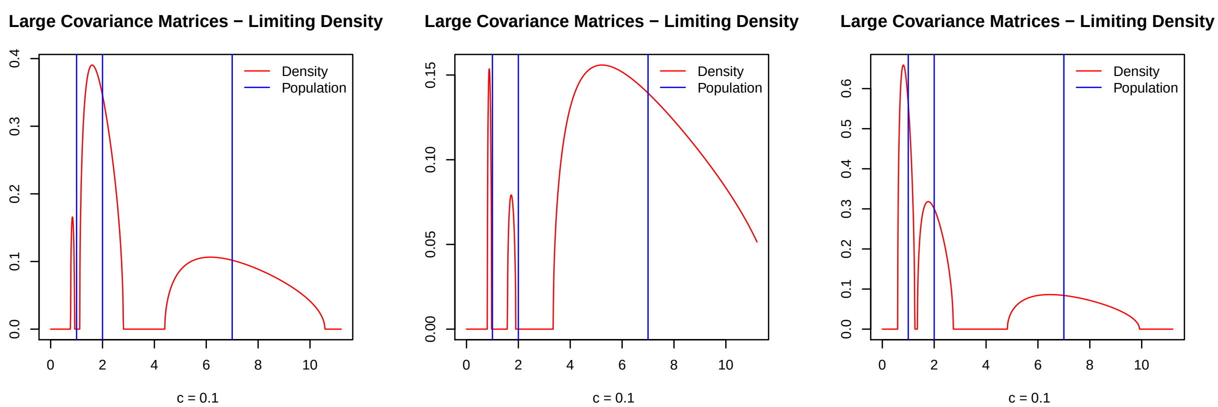
Assume that R_N is such that $K = 3$ and $\rho_1 = 1, \rho_2 = 2, \rho_3 = 7$, $m_1 = m_2 = m_3 = \frac{1}{3}$



One may notice that with an increase of c the density becomes less separated (harder to make an assumption about eigenvalues), a statistical explanation of this effect is following. The ratio $c = \frac{N}{n}$ is a relation between a dimension of feature space and a number of observations, when the amount of observation is less then the dimension of the feature space (case of big ratio c) we can't gather any relevant information from this matrix (in fact in this case we are not in high dimensional statistics). In contrast, if we are able to obtain "a lot of" observation compared to the dim. of the feature space (ratio c is small) we can gather a lot more information (we are able to localize eigenvalues).

We would also like to study the effect of "weights", we take fix $c = 0.1, K = 3$ and take three types of "weight" vectors,

1. left plot: $m_1 << m_2 \approx m_3$
2. middle plot: $m_1 \approx m_2 << m_3$
3. right plot: $m_1 \approx m_2 \approx m_3$



One may notice the following effects

1. left plot: density gives a "great" approximation of ρ_1 and "normal" for ρ_2, ρ_3
2. middle plot: density gives a "great" approximation of ρ_1, ρ_2 and "normal" for ρ_3
3. right plot: density gives a "normal" approximation of ρ_1, ρ_2, ρ_3

Our statistical explanation is following. When one of eigenvalues occurs extremely less times then the others the corresponding eigenspace has a “small” dimension therefore it allows us to have less observations to gather same amount of information. It is also obvious that the absolute value of eigenvalue has an effect on the density.

Corollary

We proceeded a numerical experiment to study an effect of different parameters in the described model. Our conclusion is that the ratio c has an important and natural statistical influence on the density, moreover we have tried to show and explain the influence of “weights” of eigenvalues. For numerical experiments we used R language and implemented a function which can be used in further study of the phenomenon. Our function is mainly based on **polynom** package which provides an easy and intuitive way of working with polynoms.