ELEMENTARY COMBINATORICS & PROBABILITY

Lecture 12

Expectation of a random variable

LOGISTICS

- Today
  - Graded assignment 5 (pts)
- Tomorrow
  - **FINAL EXAM (25 pts)**
    - ~ 2.5 - 3h
    - Start: 09:30
  - Extra assignment (10 – 15 pts)

LAST TIME

- Probability mass function
- Cumulative distribution function
- Special distributions

TODAY

- Review special distributions
- Expectation and variance
- Review

WARM-UP

TWO EVENTS

- For two events A and B, $P(A) = 0.73$, $P(B) = 0.48$ and $P(A \cap B) = 0.29$. Are A and B independent?

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A and B aren’t independent.

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$$P(A \cup B) =$$

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- Disease A occurs in 2% of the population, while 10% suffer from a more common disease B. Assuming that these two conditions are independent, what is the probability that a random person has both?

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A and B are independent $\rightarrow$

$$P(AB) = P(A) \cdot P(B) = 0.02 \cdot 0.1 = 0.002$$

FAIR COIN

- You are tossing a fair coin 10 times. What is the probability…
- … that you will get 4 Tails in a row, followed by 6 Heads in a row?
- ... that in total you will get 4 Tails and 6 Heads?
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BERNOULLI DISTRIBUTION

- Consider a random experiment with two possible outcomes: "success" (with probability p) or "failure" (with probability $1 - p$)
  - tossing a coin: H or T;
  - a new child: a boy r a girl;
  - you take an exam: pass or fail.
- Consider a random variable $X \sim \boldsymbol{Bernoulli(p)}$

x	0	1
$P(X = x)$	$1 - p$	p

A TOURIST

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  Find the probability that the first person he encounters will be able to speak English.

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$$X \sim Bernoulli(0.3) \rightarrow$$

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$$X \sim Bernoulli(0.3) \rightarrow P(X = 1) = p = 0.3$$

BINOMIAL DISTRIBUTION

- A random variable Y is said to follow Binomial distribution with parameters n and p if its PMF is given by

$$P_Y(k) = P(Y = k) = \begin{cases} C(n,k) \cdot p^k (1-p)^{n-k}, & k = 0, 1, \dots, n \\ 0, & \text{otherwise} \end{cases}$$

- Represents the number of successes in a series of n independent Bernoulli trials, each of which results in success with probability p.

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  $P(Y \geq 1) =$

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  $$P(Y \geq 1) = P(Y = 1) + P(Y = 2) + P(Y = 3) + P(Y = 4) =$$

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  $$= 1 - P(Y = 0) = 1 - C(4,0) \cdot 0.3^0 \cdot 0.7^4 =$$

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$$= 1 - P(Y = 0) = 1 - C(4{,}0) \cdot 0.3^0 \cdot 0.7^4 = 1 - 0.7^4$$

FLOWERS

- Cross-fertilizing a red and a white flower produces red flowers 25% of the time. You cross-fertilize five pairs of red and white flowers.

  Find the probability that there will be no red-flowered plants in the produced five offspring.

  Random variable Y $-$ the number of red-flowered offspring.

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$$P(Y = 0) = C(5, 0) \cdot 0.25^0 \cdot 0.75^5 =$$

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$$Y \sim Binomial(n = 5, p = 0.25)$$

$$P(Y = 0) = C(5, 0) \cdot 0.25^0 \cdot 0.75^5 = 0.75^5 \sim 0.237$$

EXPECTED VALUE

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x	-1	0	1
$P(x)$	$1/3$	$1/3$	$1/3$

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- Which value would you expect to get *on average*?
- And now?

x	-1	0	1
$P(x)$	0.1	0.1	0.8

EXPECTED VALUE

- Let X be a discrete random variable with a finite range $R_X = \{x_1, \dots, x_n\}$. The expected value of X is defined as:

$$EX = \sum_{x_k \in R_X} x_k \cdot P(X = x_k)$$

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- Different notations: $EX, E[X], \mu_X, \dots$

EXPECTED VALUE

- $EX =$

x	-1	0	1
$P(x)$	$1/3$	$1/3$	$1/3$

- $EY =$

x	-1	0	1
$P(x)$	0.1	0.1	0.8

EXPECTED VALUE

- $EX = -1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = 0$

x	-1	0	1
$P(x)$	$1/3$	$1/3$	$1/3$

- $EY =$

x	-1	0	1
$P(x)$	0.1	0.1	0.8

EXPECTED VALUE

- $EX = -1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = 0$

x	-1	0	1
$P(x)$	$1/3$	$1/3$	$1/3$

- $EY = -1 \cdot 0.1 + 0 \cdot 0.1 + 1 \cdot 0.8 = 0.7$

x	-1	0	1
$P(x)$	0.1	0.1	0.8

BERNOULLI

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x	0	1
$P(X = x)$	$1 - p$	p

$$EX =$$

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x	0	1
$P(X = x)$	$1 - p$	p

$$EX = 0 \cdot (1 - p) + 1 \cdot p = p$$

PROPERTIES OF EXPECTED VALUE

- Let X be a random variable with $EX = \mu_X$ and Y be a random variable such that $Y = aX + b$. Then

$$EY = E(aX + b) = aEX + b = a\mu_X + b$$

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$$EY = E(aX + b) = aEX + b = a\mu_X + b$$

- For any set of random variables $X_1, \dots, X_n$

$$E(X_1 + \cdots + X_n) = EX_1 \dots + E\_X_n$$

EXAMPLE

- Let X be a random variable with $EX\ = 10.$
- Let $Y = 5X + 1$

$$E(Y) =$$

- Let $Z = 3X - 2$

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EXAMPLE

- Let X be a random variable with $EX = 10$.
- Let $Y = 5X + 1$

$$E(Y) = 5 \cdot 10 + 1 = 51$$

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EXAMPLE

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- Let $Y = 5X + 1$

$$E(Y) = 5 \cdot 10 + 1 = 51$$

- Let $Z = 3X - 2$

$$E(Z) = 3 \cdot 10 - 3 = 28$$

EXAMPLE

- Let X be a random variable with $EX = 10$ and Y be a random variable with $EX = 100$.

  What is the expected value of $5X - Y + 50$?

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  What is the expected value of $5X - Y + 50$?

$$E(5X - Y + 50) = 5EX - EY + 50 = 50 - 100 + 50 = 0$$

TOSSING A COIN

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  $\mathrm{E}(X_1 + \cdots + X_n) = EX_1 + \cdots + EX_n = np$

SO,…

- The expected value of a Bernoulli random variable with parameter p is p:

$$X \sim Bernoulli(p) \quad \rightarrow \quad EX = p$$

- The expected value of a Binomial random variable with parameters (n, p) is np:

$$Y \sim Binomial(n, p) \quad \rightarrow \quad EY = np$$

FLOWERS

- Cross-fertilizing a red and a white flower produces red flowers 25% of the time. You cross-fertilize five pairs of red and white flowers.

  Of the five cross-fertilized offspring, how many red-flowered plants do you expect?

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$$EY = np = 5 \cdot 0.25 = 1.25$$

EXPECTATION OF X^2

- Consider the following random variable X:

x	-1	1
$P(x)$	$1/3$	$2/3$

$$EX =$$

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PROPERTIES OF EXPECTED VALUE 2

- Let X be a random variable with range $R_X = \{x_1, \ldots, x_n\}$ and $Y = g(X)$. Then

$$\mathrm{E}Y = E\big(g(X)\big) = \sum_{x_k \in R_X} g(x_k) \cdot P(X = x_k)$$

VARIANCE

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- Consider the following two random variables X and Y:

x	-100	0	100
$P(X = x)$	$1/3$	$1/3$	$1/3$

y	-1	1
$P(Y = y)$	$1/2$	$1/2$

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$EX =$, $EY =$

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$$EX = \frac{1}{3} \cdot (-100) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 100 = 0, \qquad EY =$$

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But X and Y are very different....

VARIANCE

- The **variance** of a random variable X with $EX = \mu_X$ is defined as

$$Var(X) = E(X - EX)^2 = (EX)^2 - E(X^2)$$

VARIANCE

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$$Var(X) = E(X - EX)^2 = (EX)^2 - E(X^2)$$

- “How often does X take values far from its mean?”

VARIANCE

$$\boldsymbol{Var(X) \geq 0}$$

VARIANCE

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because by definition it's the expected value of

$$(X - \mu_X)^2 \geq 0$$

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$$Var(X) =$$

$$Var(Y) =$$

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$$EX=\frac{1}{3}\cdot(-100)+\frac{1}{3}\cdot 0+\frac{1}{3}\cdot 100=0, \qquad EY=\frac{1}{2}\cdot(-1)+\frac{1}{2}\cdot 1=0$$

$$Var(X)=E(X-EX)^2=EX^2=\frac{2}{3}\cdot 10^4$$

$$Var(Y)=$$

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$$EX = \frac{1}{3} \cdot (-100) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 100 = 0, \qquad EY = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$$

$$Var(X) = E(X - EX)^2 = EX^2 = \frac{2}{3} \cdot 10^4$$

$$Var(Y) = E(X - EY)^2 = E(Y^2) = 1$$

EXAMPLE

- You roll a fair dice once, random variable X – the outcome.

$$EX =$$

$$Var(X) =$$

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- You roll a fair dice once, random variable X – the outcome.

$$EX = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) =$$

$$Var(X) =$$

EXAMPLE

- You roll a fair dice once, random variable X – the outcome.

$$EX = \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = \frac{7}{3}$$

$$Var(X) =$$

EXAMPLE

- You roll a fair dice once, random variable X – the outcome.

$$EX = \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = \frac{7}{3}$$

$$Var(X) = (EX)^2 - E(X^2) =$$

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$$EX^2 =$$

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$$Var(X) = (EX)^2 - E(X^2) =$$

$$EX^2 = \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}$$

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$$EX = \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = \frac{7}{3}$$

$$Var(X) = (EX)^2 - E(X^2) = \frac{91}{6} - \frac{49}{9} \sim 2.92$$

$$EX^2 = \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}$$

VARIANCE OF A LINEAR COMBINATION

$$Var(aX + b) =$$

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$$= E\big(aX + b - E(aX + b)\big)^2 =$$

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$$= E(aX + b - aEX - b)^2 = E(aX - aEX)^2 =$$

$$= a^2E(X - EX)^2 =$$

VARIANCE OF A LINEAR COMBINATION

$$Var(aX + b) =$$

$$= E\left(aX + b - E(aX + b)\right)^2 =$$

$$= E(aX + b - aEX - b)^2 = E(aX - aEX)^2 =$$

$$= a^2 E(X - EX)^2 =$$

$$= a^2 Var(X)$$

EXAMPLE

- Let X be a random variable with $\mathrm{Var}(X) \ = 10.$
- Let $Y = 5X + 1$

$$Var(Y) =$$

- Let $Z = -3X - 2$

$$Var(Z) =$$

EXAMPLE

- Let X be a random variable with $\mathrm{Var}(X) \ = 10.$
- Let $Y = 5X + 1$

$$Var(Y) = Var\ (5X + 1) =$$

- Let $Z = -3X - 2$

$$Var(Z) =$$

EXAMPLE

- Let X be a random variable with $\mathrm{Var}(X) = 10.$
- Let $Y = 5X + 1$

$$Var(Y) = Var\ (5X + 1) = 25Var(X) = 250$$

- Let $Z = -3X - 2$

$$Var(Z) =$$

EXAMPLE

- Let X be a random variable with $\mathrm{Var}(X) \ = 10.$
- Let $Y = 5X + 1$

$$Var(Y) = Var\ (5X + 1) = 25Var(X) = 250$$

- Let $Z = -3X - 2$

$$Var(Z) = Var(-3X - 2) =$$

EXAMPLE

- Let X be a random variable with $\mathrm{Var}(X) = 10$.
- Let $Y = 5X + 1$

$$Var(Y) = Var\ (5X + 1) = 25Var(X) = 250$$

- Let $Z = -3X - 2$

$$Var(Z) = Var(-3X - 2) = 9Var(X) = 90$$

STANDARD DEVIATION

- What are the measurement units of EX? Same as those of X.
- What are the measurement units of $\mathrm{Var}(X)$? $[units\ of\ X]^2$
  - Difficult to interpret.

- Standard deviation:

$$std\ (X) = \sqrt{Var(X)}$$