ELEMENTARY COMBINATORICS & PROBABILITY

Lecture 1

Sets
Basic counting

TODAY

- Introduction
  - about this course;
  - about me.
- What is combinatorics (and why it's worth studying).
- Basic counting principles.
- Examples.

ABOUT ME

- EVGENIYA Korneva evgeniakorneva@gmail.com

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- Education:
  - *2015* - Bachelor of Applied Mathematics *(Moscow, Russia)*
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Leuven, Belgium

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  - Final exam (April 2)

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- Week 1: Combinatorics.
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- Evaluation:
  - 5 graded assignments *5% each*;
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LET'S START!

WHAT IS COMBINATORICS

- Fine art of counting.

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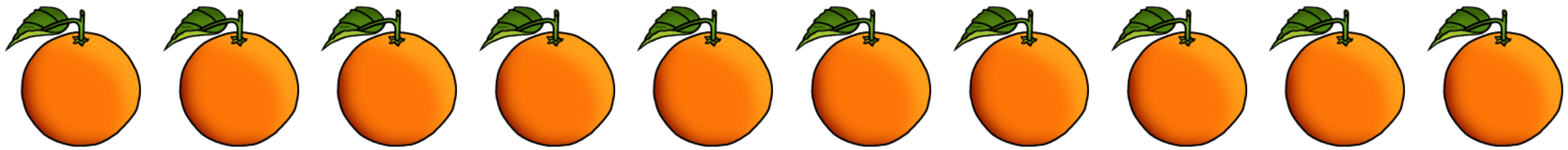

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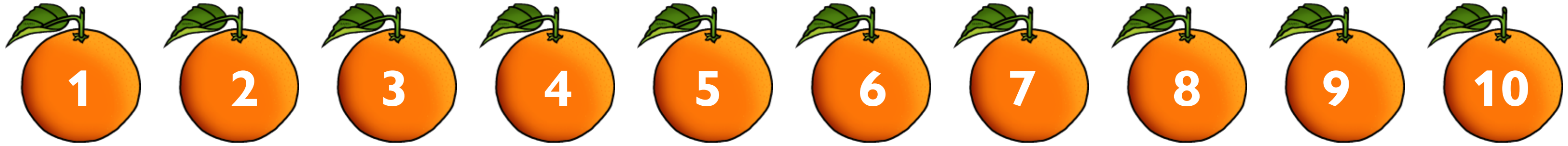

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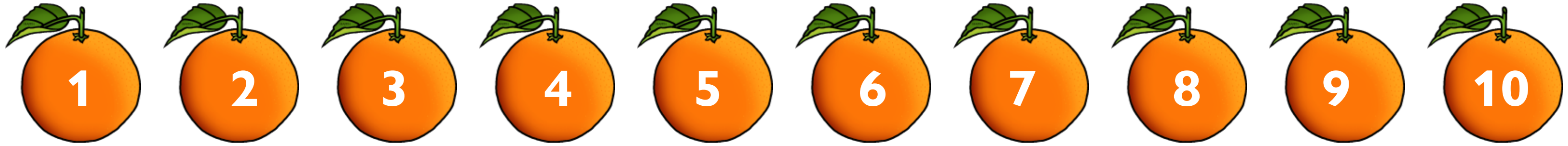

- But enumeration is not always easy.

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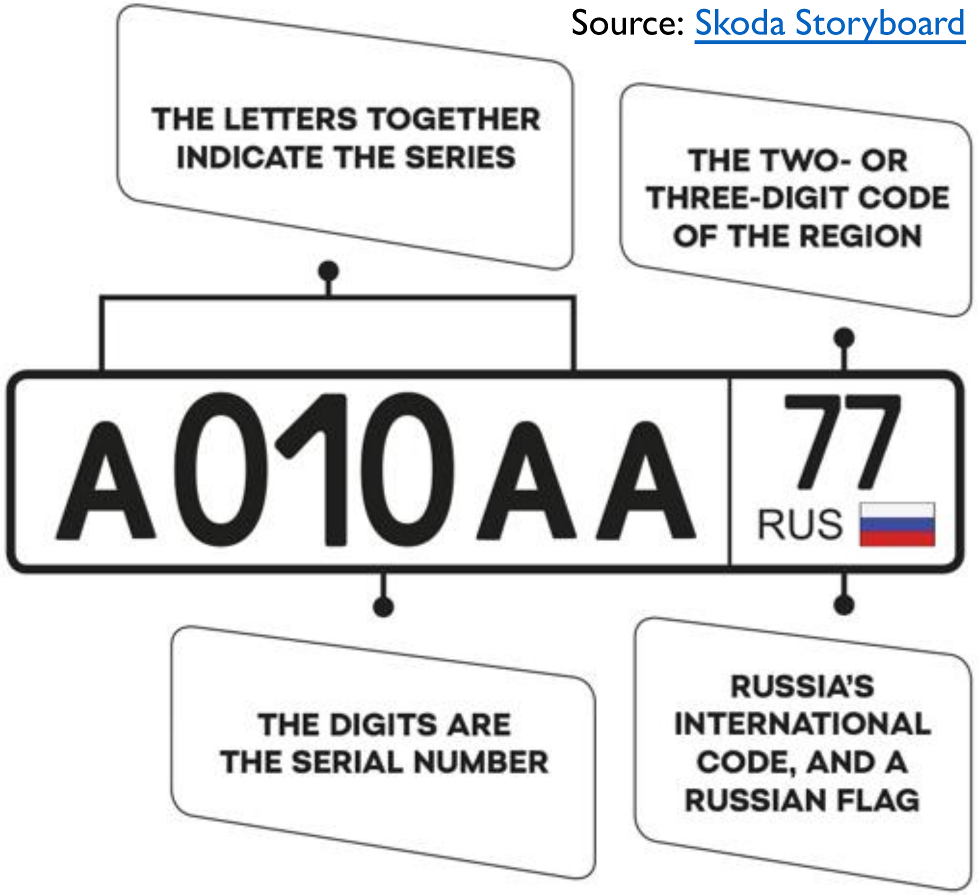

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  3 letters + 3 digits + regional code

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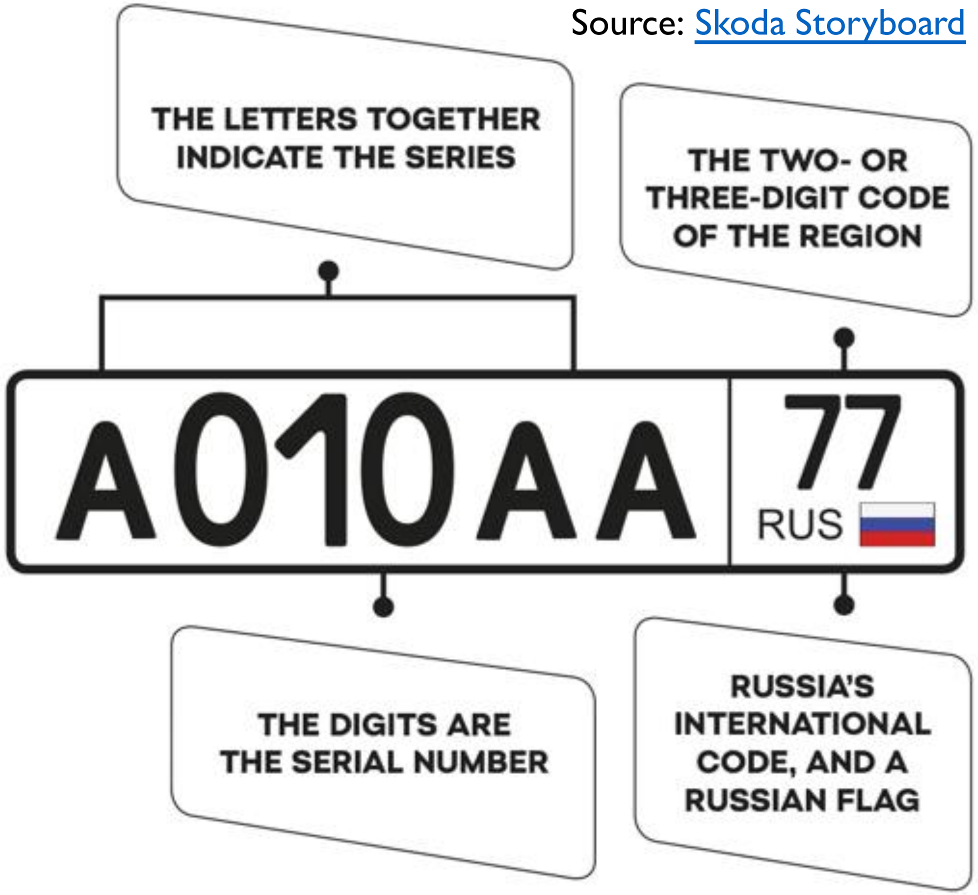

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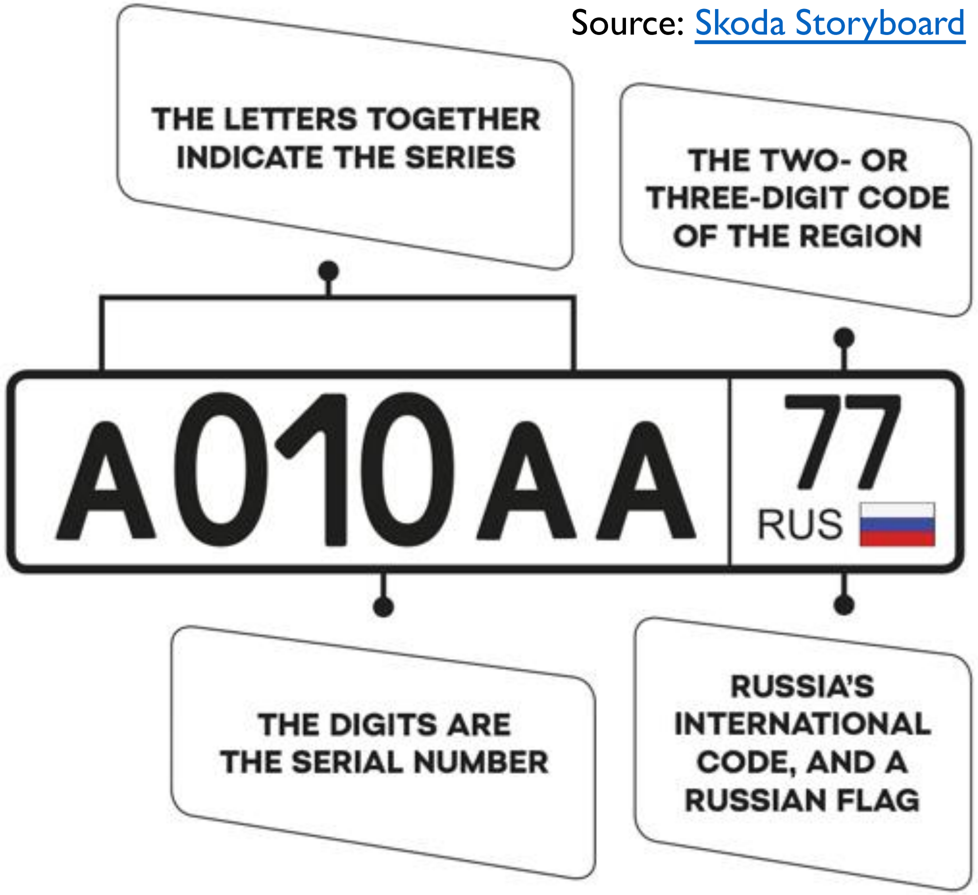

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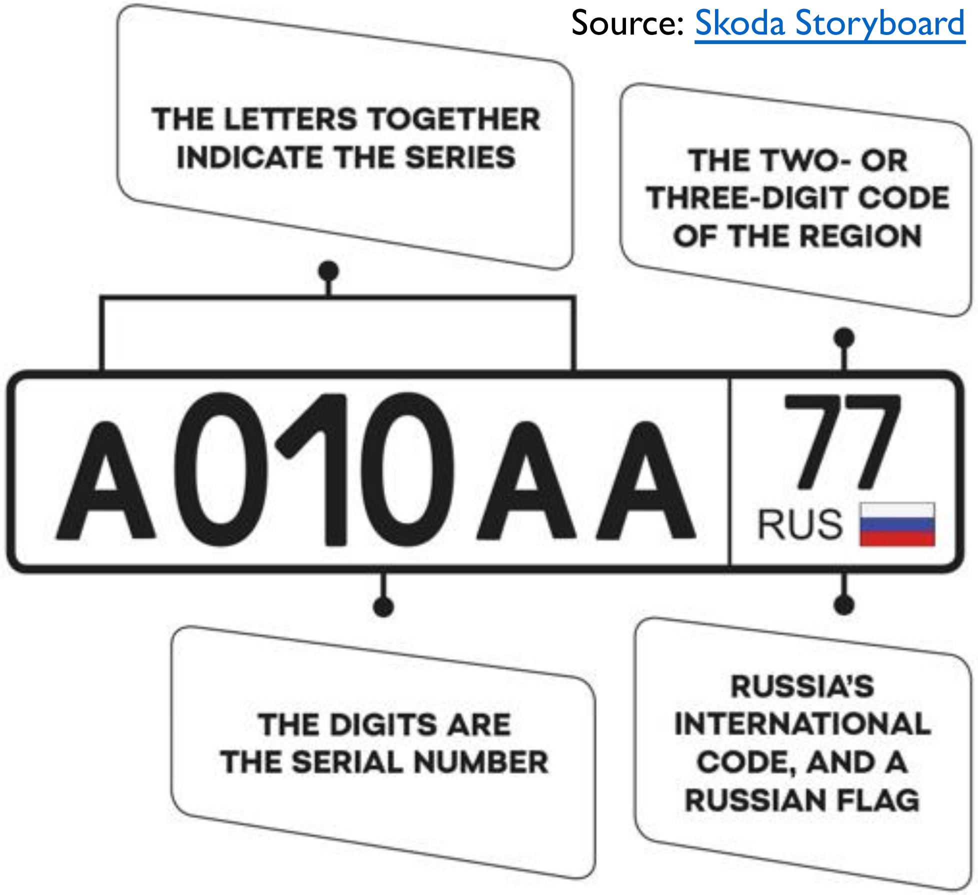

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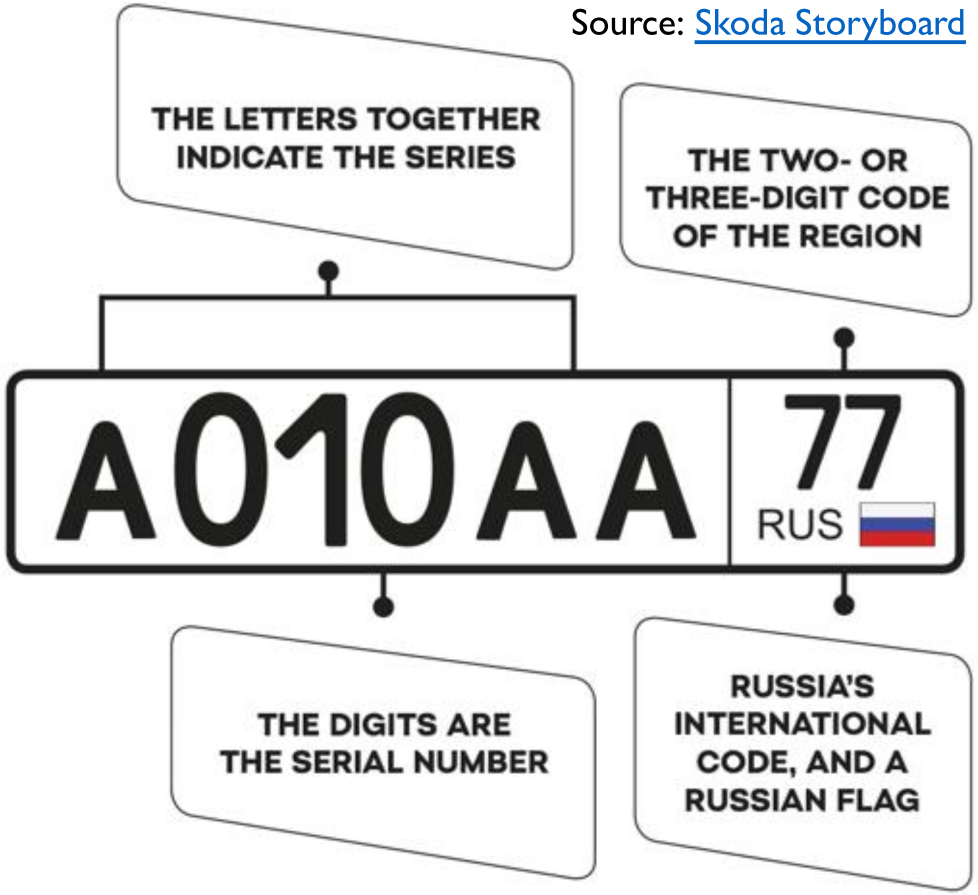

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  - ~ 1.7 million unique plates
- Moscow has **nine** regional codes.
  - ~ 1.7 million -> ~15.5 million unique plates.

MOTIVATING EXAMPLE: PHONE NUMBERS

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Japan is running out of phone numbers, so it's making longer ones

Its current 11-digit numbers could run out by 2022

By Jon Porter | @JonPorty | May 16, 2019, 8:45am EDT

SHARE

Photo by Amelia Holowaty Krales / The Verge

Source: [The Verge]

Source: [Independent]

Local calls using area codes seen as a better alternative to adding another digit

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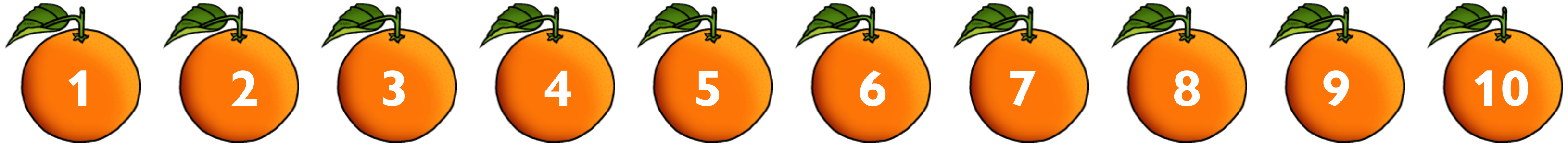

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Combinatorics = fine art of counting
(without *actually* enumerating).

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- Some proof techniques in mathematics rely on combinatorics.

SET THEORY

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  - {apple, peach, pear} is the same as {peach, apple, pear}

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- Element 5 doesn't belong to set B: $5 \notin B$

FINITE AND INFINITE SETS

- Sets can contain finite number of elements
  - digits: {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}
  - Latin characters: {A, B, C, …, X, Y, Z}
  - all possible 10-digit phone numbers

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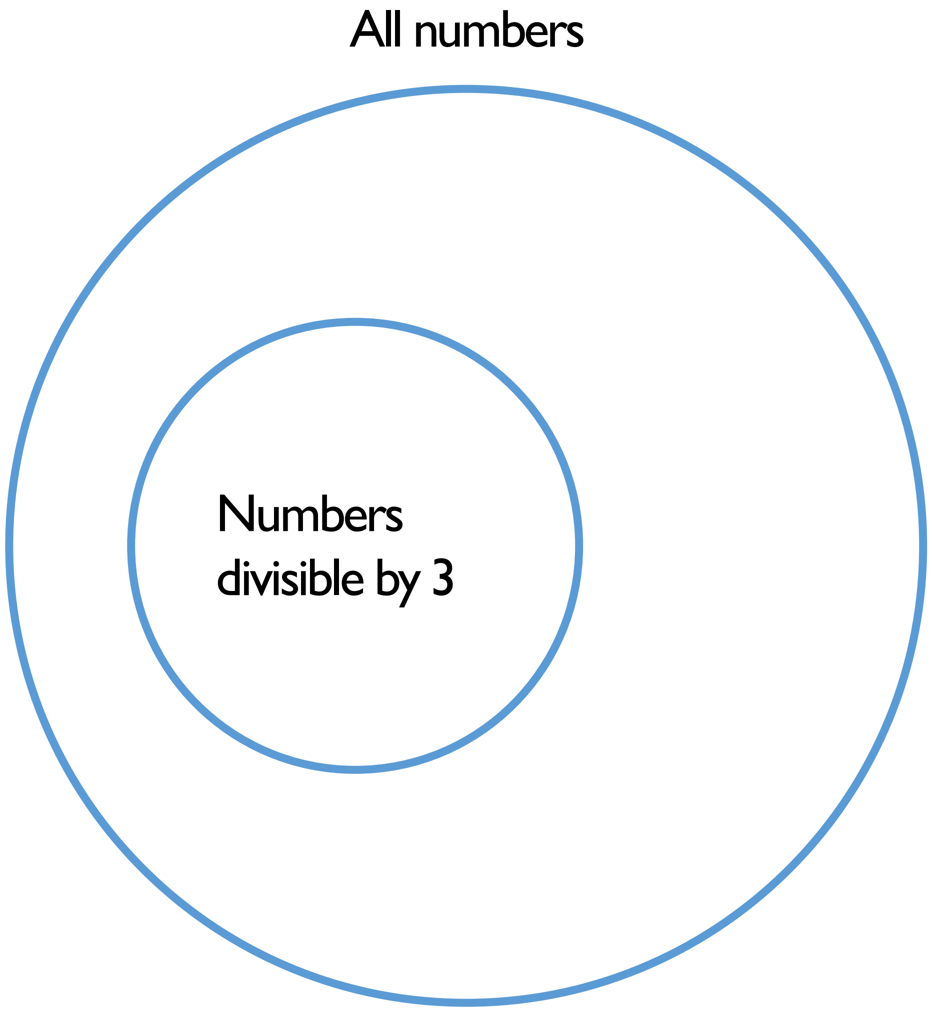
VISUALIZATION



OPERATIONS WITH SETS

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  A ∪ B = {apple, banana, orange, peach}

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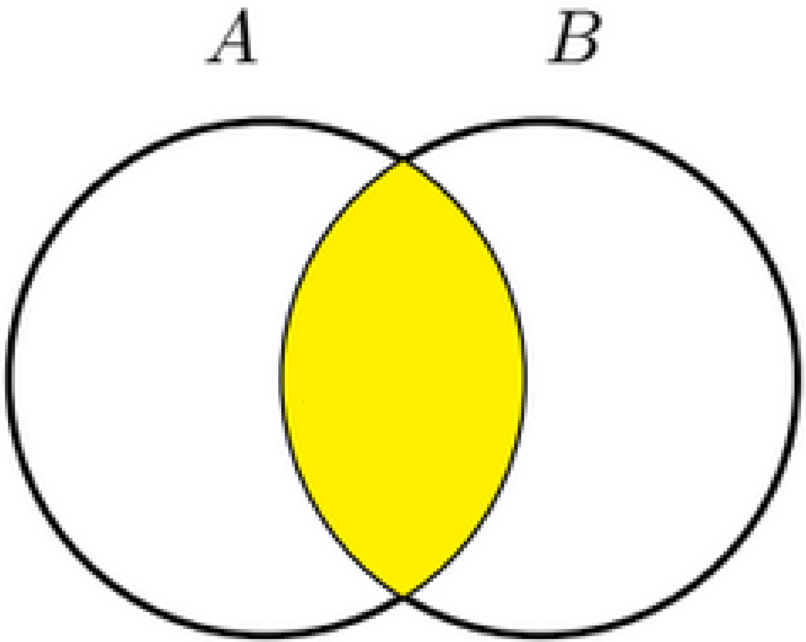

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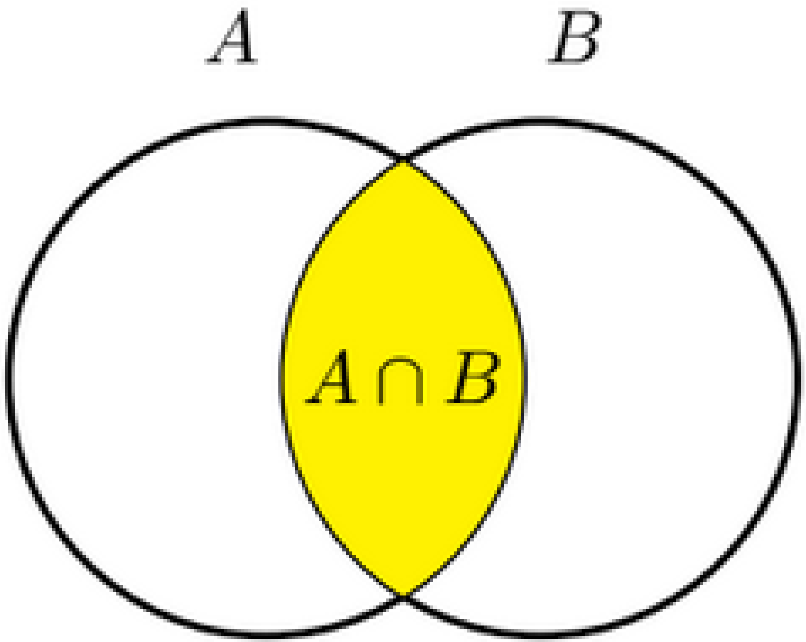

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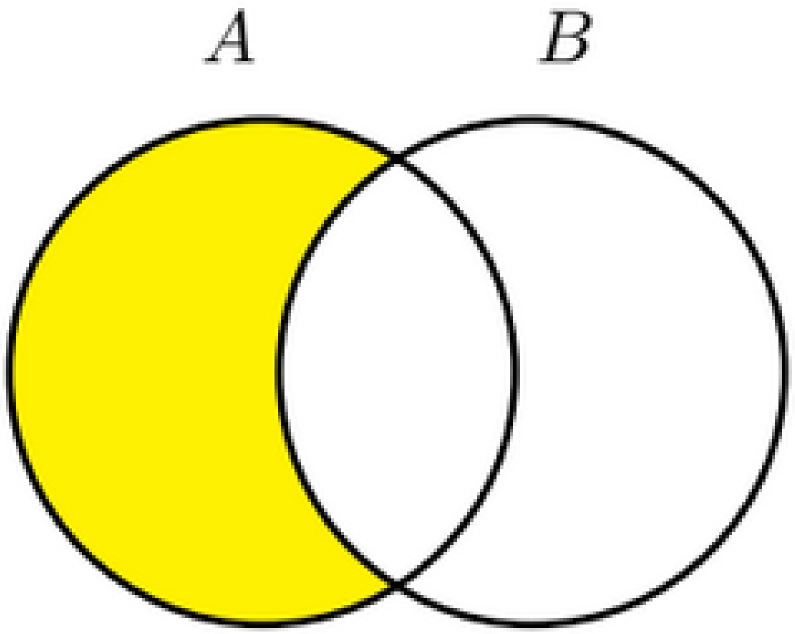

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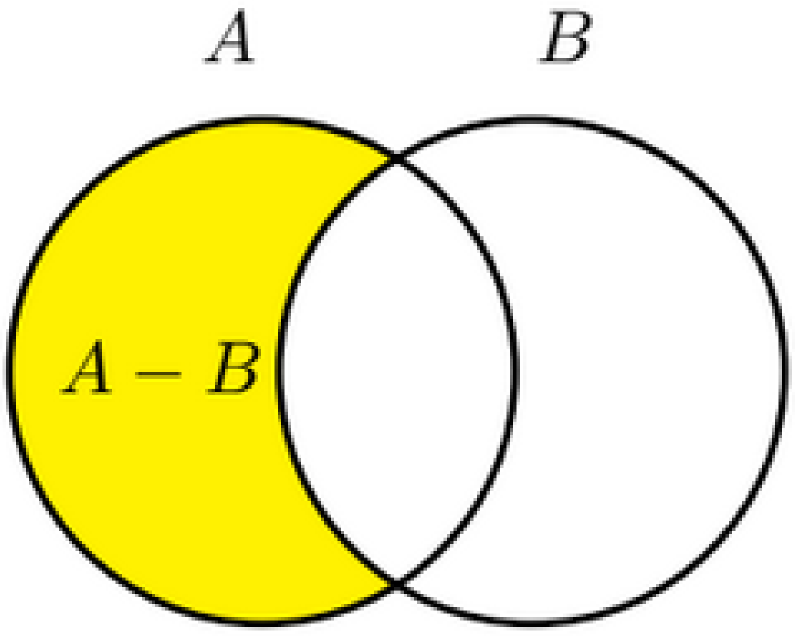

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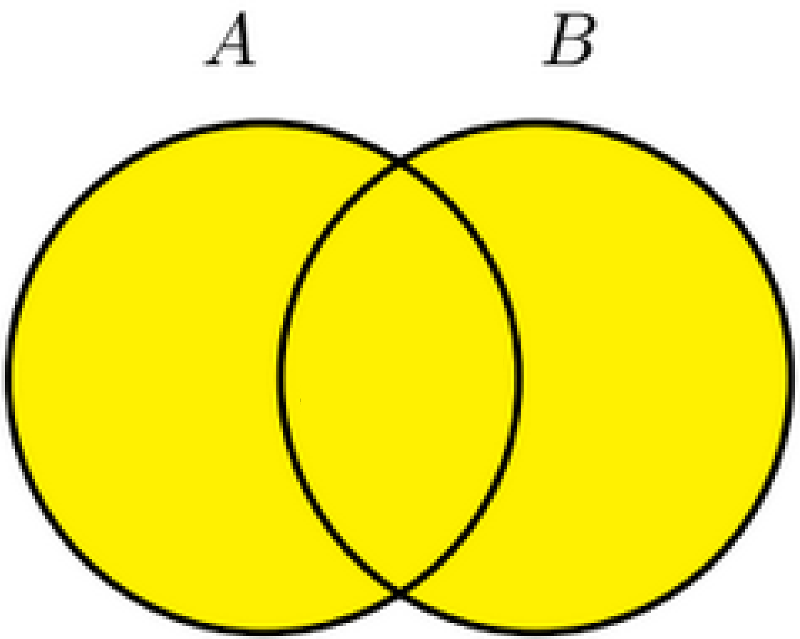

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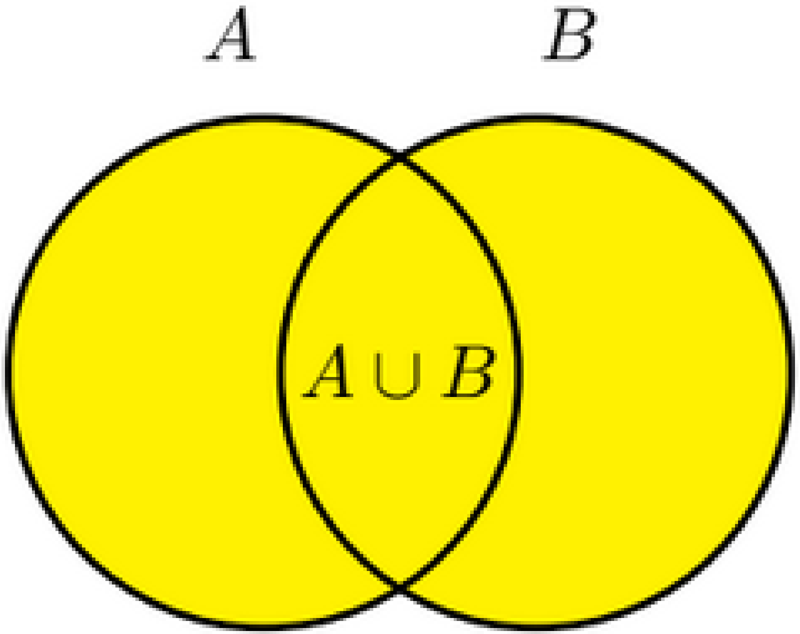

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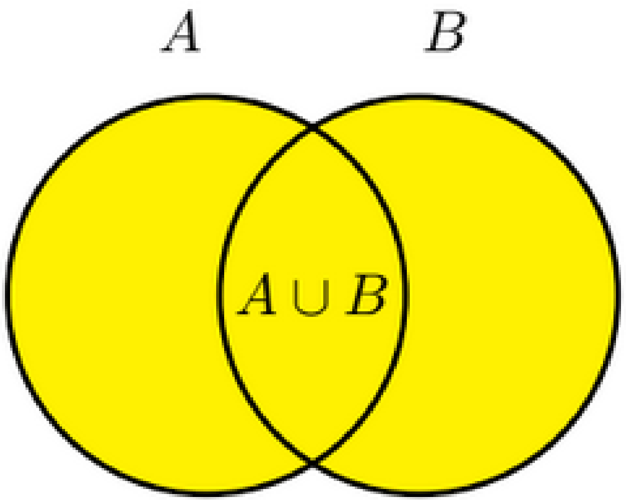

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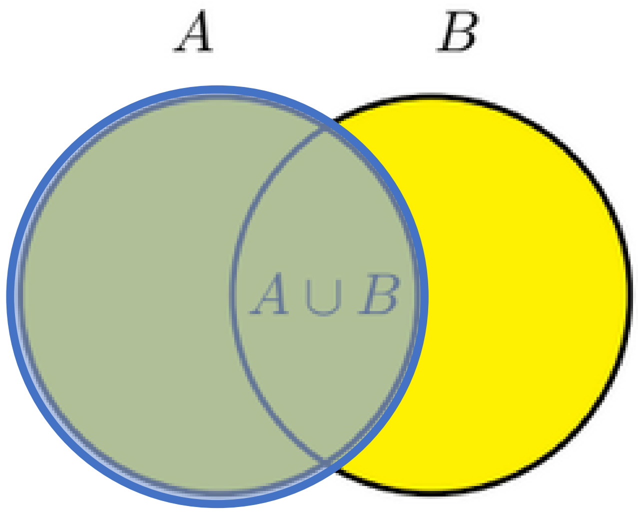

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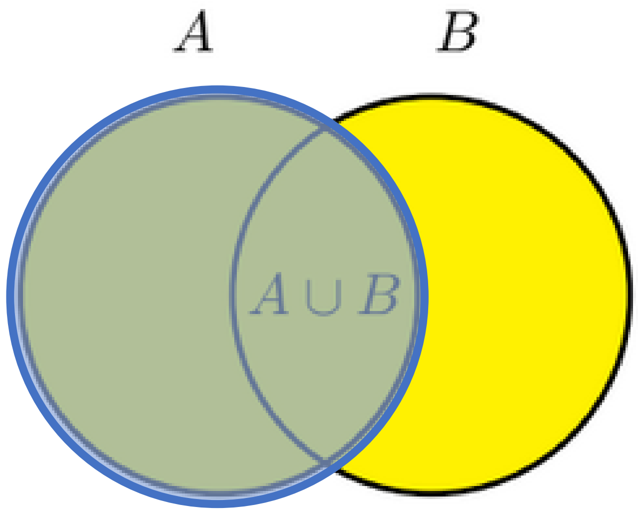

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- **Complement** of A is

$$A^c = A = U \setminus A$$

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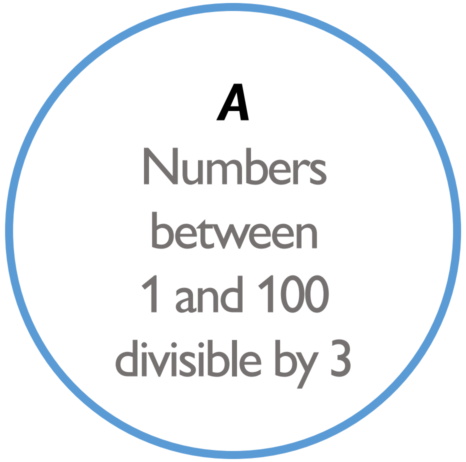

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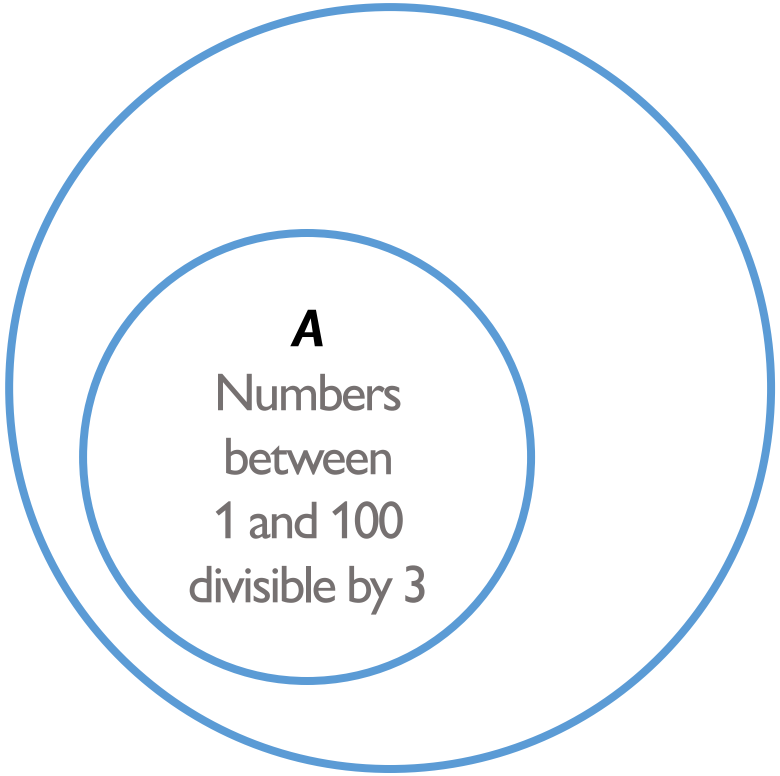

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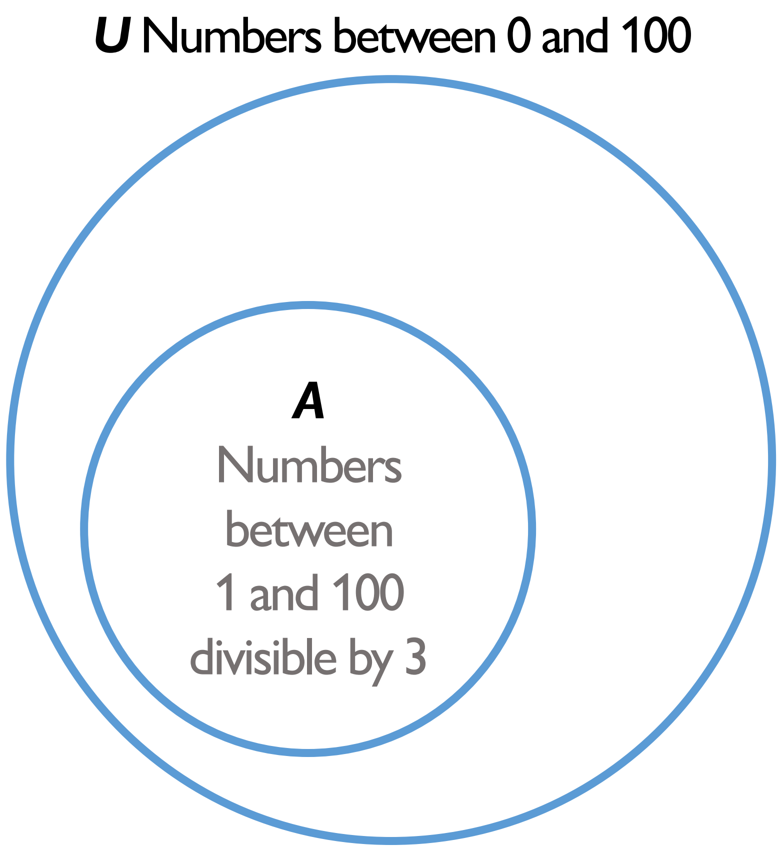

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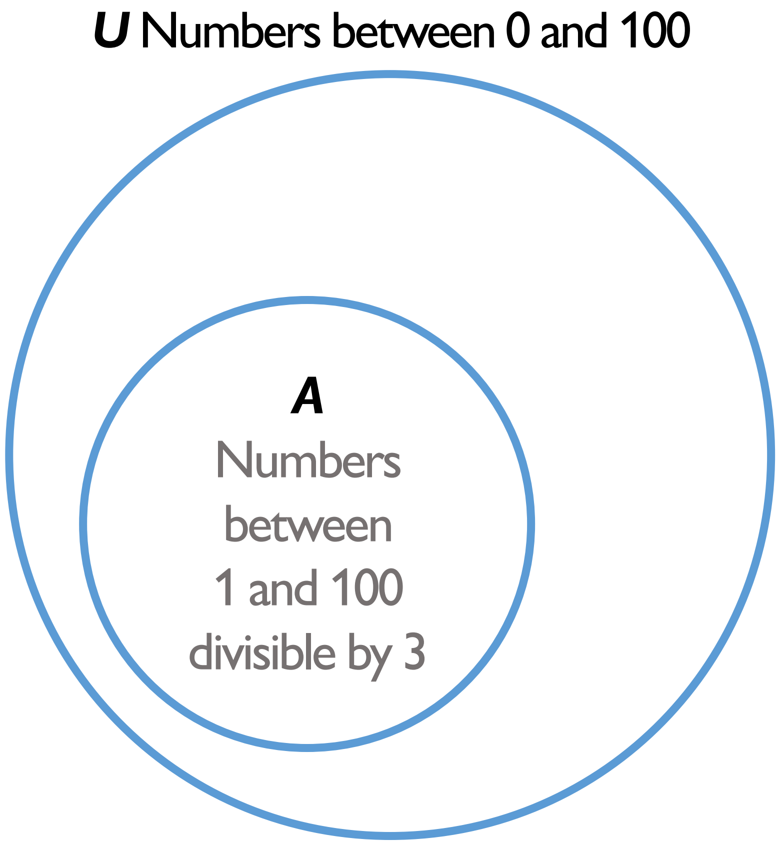

$A^c = ?$

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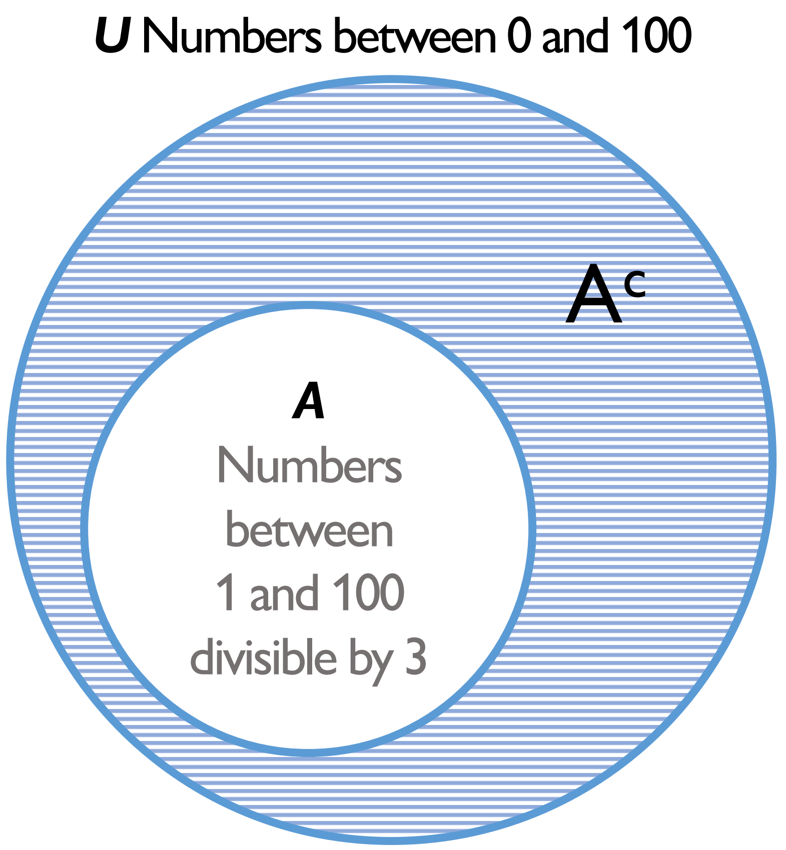

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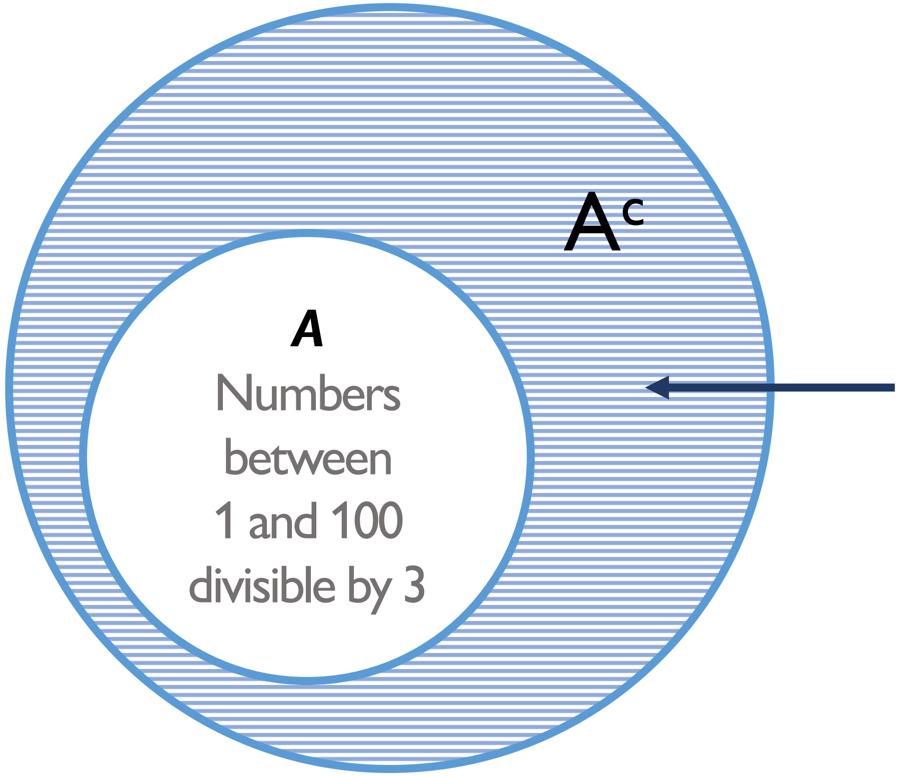

Numbers between 0 and 100
not divisible by 3

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OTHER LAWS

Name	Rule
Commutative laws	$A \cap B = B \cap A, \quad A \cup B = B \cup A$
Associative laws	$A \cap (B \cap C) = (A \cap B) \cap C$
	$A \cup (B \cup C) = (A \cup B) \cup C$
Distributive laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
De Morgan's laws	$(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c$
Complement laws	$A \cap A^c = \varnothing, \quad A \cup A^c = U$
Double complement law	$(A^c)^c = A$
Idempotent laws	$A \cap A = A, \quad A \cup A = A$
Absorption laws	$A \cap (A \cup B) = A, \quad A \cup (A \cap B) = A$
Dominance laws	$A \cap \varnothing = \varnothing, \quad A \cup U = U$
Identity laws	$A \cup \varnothing = A, \quad A \cap U = A$

CARTESIAN PRODUCT

- Let A and B be two sets.
- **Cartesian product** of two sets A and B is a set, containing all sequences, where the first element is from set A and the second is from set B:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

CARTESIAN PRODUCT: EXAMPLE

- D = {wine, beer, water}
- F = {spaghetti, burgers}

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CARTESIAN PRODUCT: EXAMPLE

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D x F = {(wine, spaghetti),
(wine, burgers),
(beer, spaghetti),
(beer, burgers),
(water, spaghetti),
(water, burgers)}

BACK TO COUNTING...

ADDITION PRINCIPLE

(aka Rule of sum)

RULE OF SUM: EXAMPLE

- We want to order dinner for tonight:
  - Pizza:
  - Sushi:

RULE OF SUM: EXAMPLE

- We want to order dinner for tonight:
  - Pizza:

    Dominos, Pizza Hut or King Slice (3 restaurants)
  - Sushi:

RULE OF SUM: EXAMPLE

- We want to order dinner for tonight:
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    Wabi Sabi or Sakura (2 restaurants)

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- How many options are there in total?

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  3 pizza places **+** **2** sushi places =

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  - Sushi:

    Wabi Sabi or Sakura (2 restaurants)

- How many options are there in total?

  3 pizza places **+** **2** sushi places = **3** **+** **2** = 5 options in total

RULE OF SUM

- If there are **a** ways of doing something and **b** number of ways of doing another thing and we can not do both at the same time, then there are **a + b** ways to choose one of the actions.

RULE OF SUM (SET THEORY)

- If A and B are <u>disjoint sets</u> ($A \cap B = \emptyset$), then

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$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|$$

MULTIPLICATION PRINCIPLE

(aka Rule of product)

RULE OF PRODUCT INFORMALLY

- If there are **a** ways of doing something and **b** ways of doing another thing, then there are **a** · **b** ways of performing both actions.

RULE OF PRODUCT: EXAMPLE 1

- You want to order pizza:
  - first, choose the type of crust:
  - second, choose one topping:

RULE OF PRODUCT: EXAMPLE 1

- You want to order pizza:
  - first, choose the type of crust:
    thin or thick (2 choices);
  - second, choose one topping:

RULE OF PRODUCT: EXAMPLE 1

- You want to order pizza:
  - first, choose the type of crust:
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    cheese, pepperoni, or sausage (3 choices).

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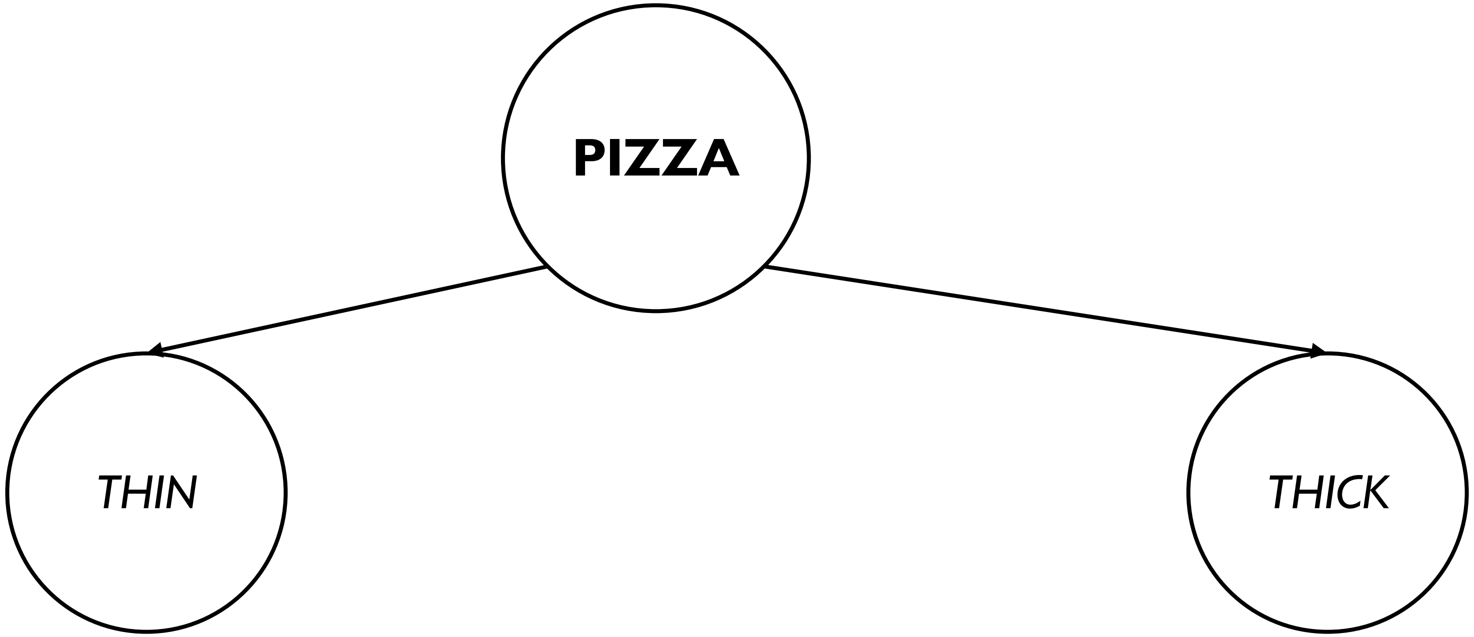
- How many different pizzas are there?

2 ways of choosing crust **x** **3** ways of choosing topping =
= **2 x 3** = 6 different pizzas

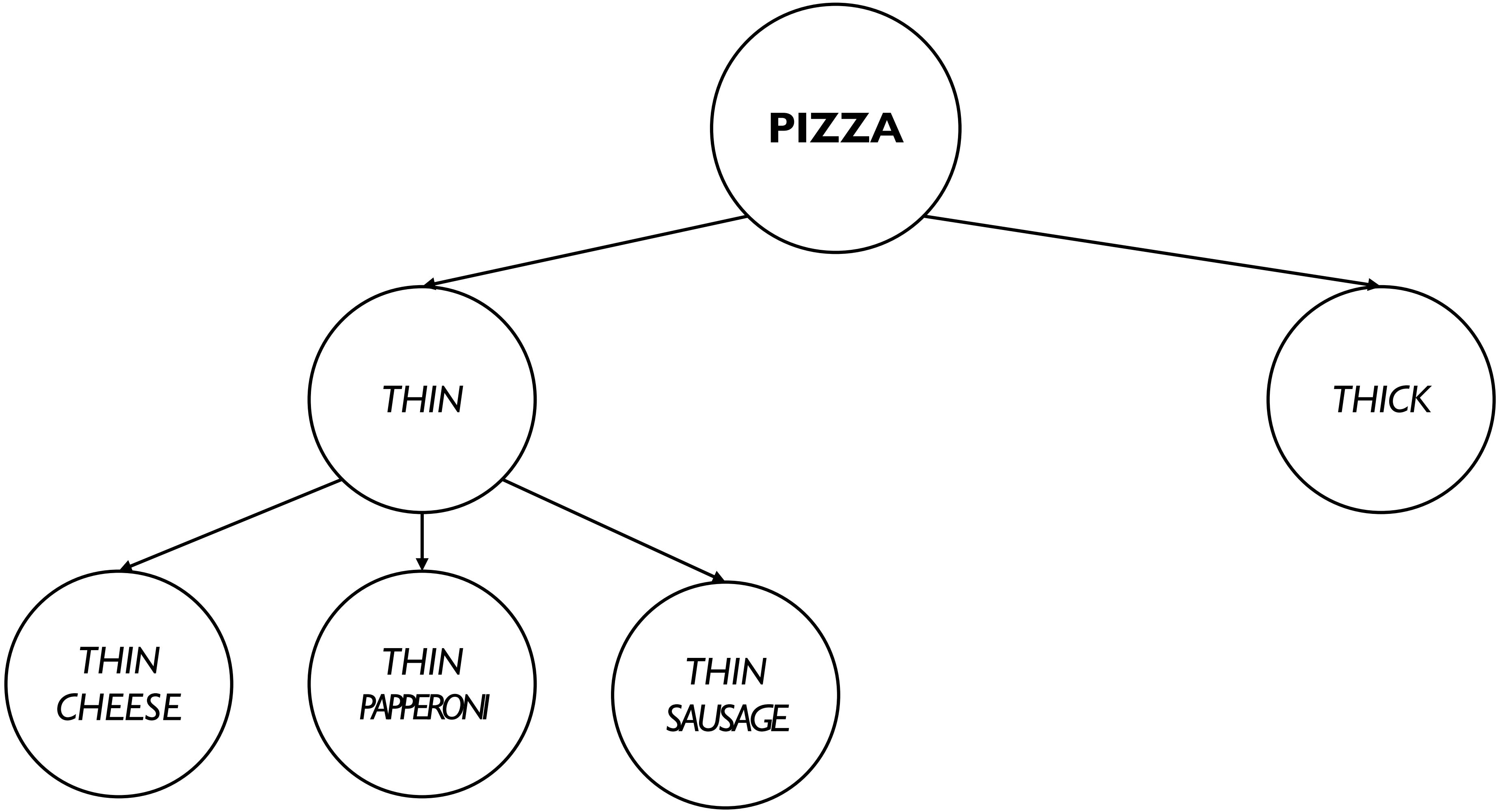
RULE OF PRODUCT: EXAMPLE 1

PIZZA

RULE OF PRODUCT: EXAMPLE 1



RULE OF PRODUCT: EXAMPLE 1



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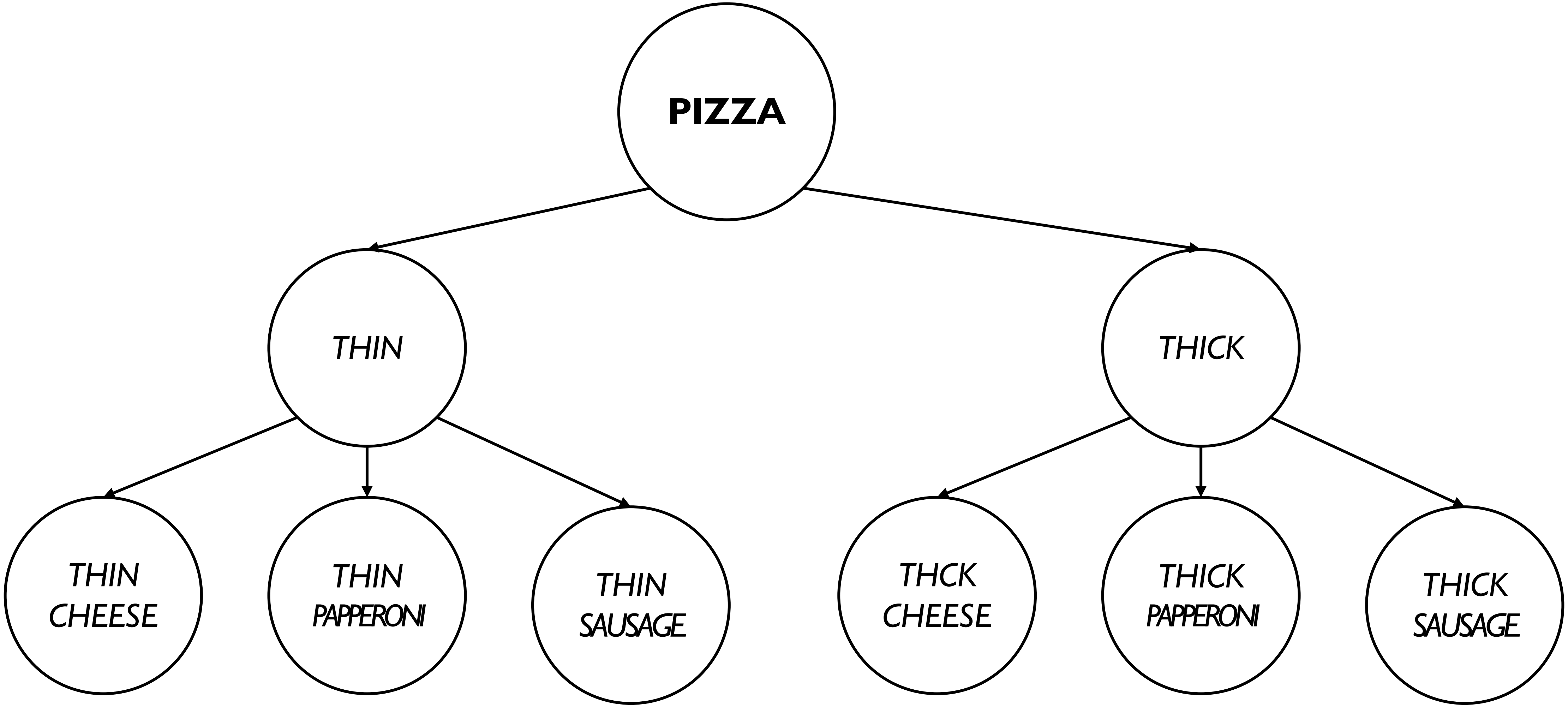

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RULE OF PRODUCT (SET THEORY)

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$$|A \times B| = |A| \cdot |B|$$

- **Generalized product rule:**

  For any sets $A_1, A_2, \ldots, A_n$

$$|A_1 \times A_2 \times \ldots \times A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|$$

RULE OF PRODUCT: EXAMPLE 2

- One can chose from the following meals during the day:
  - **B**reakfast = {pancakes, eggs, bagel}
  - **L**unch = {salad, soup, bagel}
  - **D**inner = {soup, pasta, burgers}

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How many different day menus can one have?

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How many different day menus can one have?

$$|B| \cdot |L| \cdot |D| = 3 \cdot 3 \cdot 3 = 27$$

RULE OF PRODUCT: EXAMPLE 3

- License plate:

3 letters + 3 digits + regional code

- How many unique plates are there per regional code?

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$$12 \cdot 12 \cdot 12 \cdot 10 \cdot 10 \cdot 10$$

RULE OF PRODUCT: EXAMPLE 3

- License plate:

3 letters + 3 digits + regional code

- How many unique plates are there per regional code?

- 12 letters and 10 digits can be used:

$$12 \cdot 12 \cdot 12 \cdot 10 \cdot 10 \cdot 10 =$$

1 728 000 unique plates

MORE EXAMPLES

Sum and product rules in action

SUM & PRODUCT RULES IN PROGRAMMING

```
for i in range(n):
    print('hello')

for j in range(m):
    print('hello')
```

```
for i in range(n):
    for j in range(m):
        print('hello')
```

SUM & PRODUCT RULES IN PROGRAMMING

RULE OF SUM

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hello will be printed **m + n** times

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RULE OF PRODUCT

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for i in range(n):
    for j in range(m):
        print('hello')
```

hello will be printed **m*n** times

NUMBER OF 6-BIT STRINGS

- Example of a 6-bit string:

0	**1**	**1**	**1**	**0**	**1**

- How many such strings are there?

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*	*	*	*	*	*

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Each position:

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Each position: 0 or 1 (2 choices)

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- Solution:

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Each position: 0 or 1 (2 choices)
Product rule:

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 = 64 \text{ different 6-bit strings.}$$

NUMBER OF 6-BIT STRINGS

- How many such strings are there that start with *10*?

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- Solution:

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Positions 1 & 2: 10 Other positions: 0 or 1 (2 choices)

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Product rule:

$$1 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^4 = 16$$

different 6-bit strings that start with **10**.

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not 10

*	*	*	*	*	*

NUMBER OF 6-BIT STRINGS

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SOLUTION 1 |strings that **don’t** start with *10*| =

NUMBER OF 6-BIT STRINGS

- How many such strings are there that **don’t** start with *10*?

- Solution: **not 10**

*	*	*	*	*	*

SOLUTION 1 |strings that **don’t** start with *10*| =
|strings that start with *00*| + |strings that start with *01*| +
+ |strings that start with *11*|

NUMBER OF 6-BIT STRINGS

- How many such strings are there that **don’t** start with *10*?

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SOLUTION 1 |strings that **don’t** start with *10*| =
|strings that start with *00*| + |strings that start with *01*| +
+ |strings that start with *11*|

$2^4 + 2^4 + 2^4 = 16 + 16 + 16 = 48$

NUMBER OF 6-BIT STRINGS

- How many such strings are there that **don’t** start with *10*?

- Solution:

not 10

*	*	*	*	*	*

SOLUTION 2 |strings that **don’t** start with *10*| =

NUMBER OF 6-BIT STRINGS

- How many such strings are there that **don’t** start with *10*?

- Solution:

not 10

*	*	*	*	*	*

SOLUTION 2

|strings that **don’t** start with *10*| =
|all strings| - |strings that **do** start with *10*|

NUMBER OF 6-BIT STRINGS

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$2^6 - 2^4 = 64 – 16 = 48$

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$$N_i = 2^i$$

$$N = 2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^1 = 126$$

SUBSETS OF A SET

- Consider a set {a, b, c}. How many subsets does it have? List them all.

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  2 options per each of the *n* element ***(just like bit-strings!)***.

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  2 options per each of the *n* element ***(just like bit-strings!)***.

  Product rule: $2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$ subsets

EXAMPLE: NUMBER OF PASSWORDS

- A valid password contains 6 to 8 symbols
  - first symbol is a letter (upper- or lowercase);
  - other symbols are letters (upper- or lowercase) or digits.
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  Sum rule:

$$N = P_6 + P_7 + P_8 \qquad P_i \text{ – \# passwords of length i}$$

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EXAMPLE: NUMBER OF PASSWORDS

- Solution:

Sum rule: $N = P_6 + P_7 + P_8$ P_i – # passwords of length i

First symbol:

upper- or lowercase letter – $2 \cdot 26 = 52$ options

Other symbols:

upper- or lowercase letter or digit

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Sum rule: $N = P_6 + P_7 + P_8$ P_i – # passwords of length i

First symbol:

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Other symbols:

upper- or lowercase letter or digit – $52 + 10 = 62$ options

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$P_6 =$ $P_7 =$ $P_8 =$

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Product rule: $P_6 = 52 \cdot 62^5$ $P_7 =$ $P_8 =$

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$$N = 52 \cdot 62^5 + 52 \cdot 62^6 + 52 \cdot 62^7$$

OTHER USEFUL TRICKS

COUNTING ONE THING BY COUNTING ANOTHER

- How to count the number of people in the class?

COUNTING ONE THING BY COUNTING ANOTHER

- How to count the number of people in the class?
  - Count heads

COUNTING ONE THING BY COUNTING ANOTHER

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  - Count their laptops

COUNTING ONE THING BY COUNTING ANOTHER

- How to count the number of people in the class?
  - Count heads
  - Count their laptops
  - Count ears and divide by 2 ☺
    (assuming that everyone has exactly two ears)

NUMBER OF HANDSHAKES

- There are n people in a room. Everyone shakes hands with everyone. How many handshakes are there?

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LET'S PRACTICE

https://docs.google.com/document/d/1XM0qoghJNkjnrVL3QVb_cSYRnfz1qdf7CLK6yFezF18/edit?usp=sharing

TO SUM UP

- The basics of set theory.
- Basic counting principles:
  - sum rule;
  - product rule;
  - combining the rules.
- Other tricks
  - using complement of a set;
  - overcounting.