ELEMENTARY COMBINATORICS & PROBABILITY

Lecture 5

Pigeonhole principle, review

LAST TIME

- Review permutations and combinations
- Tuples
    - Ordered sequence with repetitions
- Combinations with repetitions
    - In how many ways can we color k balls with n colors?

TODAY

- Problem set 4
  - Permutations, combinations, tuples, …
- Pigeonhole principle
- Problem set 5

- Graded assignment 2 is out
- <u>Deadline:</u> Monday, March 22, 23:59 Barcelona time
- See Google classroom

REVIEW

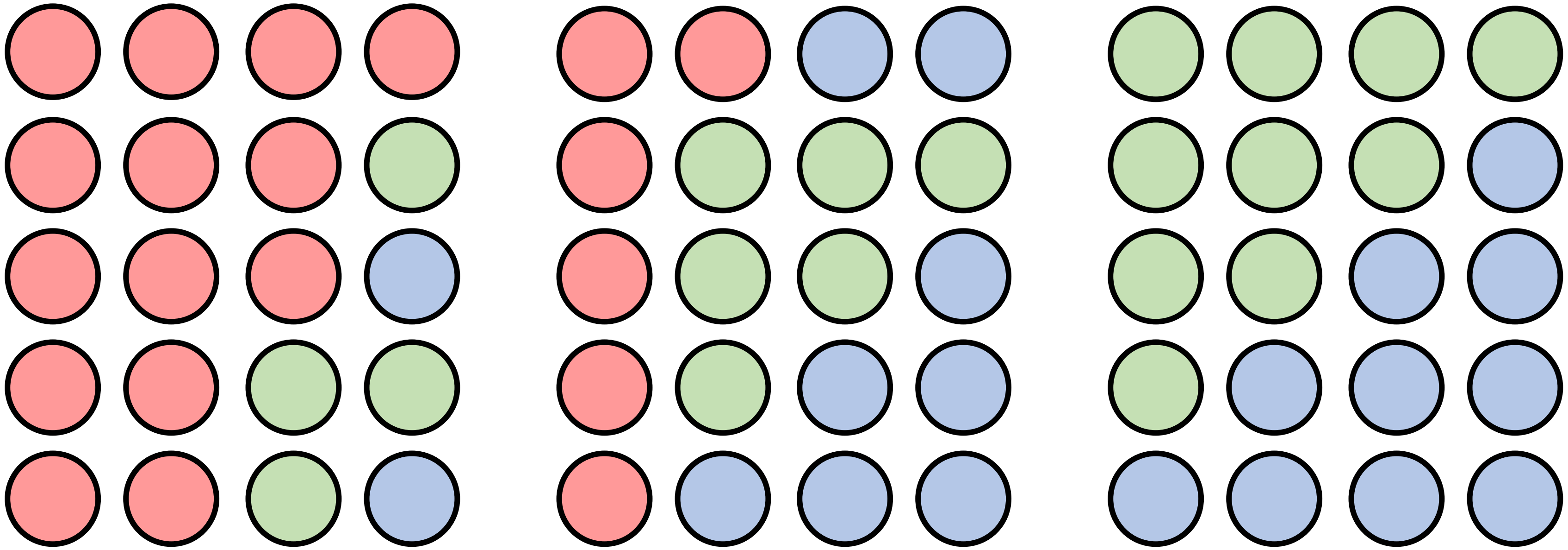
DIFFERENT ARRANGEMENTS

- Imagine you have n objects. How can you arrange them?

	WITHOUT REPETITIONS	**WITH REPETITIONS**
ORDERED	*PERMUTATIONS* Seating n people in a row $n!$	*TUPLES* Counting different n-bit strings are there? k^n
NOT ORDERED	*COMBINATIONS* Choosing k out of n different candies in a bag $C(n,k) = \frac{n!}{k!\,(n-k)!}$	*COMBINATIONS with repetitions* Distributing k identical candies among n kids $C(k+n-1, n-1)$

COMBINATIONS WITH REPETITIONS

- How many ways are there to place 4 colored balls in a bag, when each ball should be either Red, Green, or Blue?



COMBINATIONS WITH REPETITIONS

- Where to put $3 - 1 = 2$ bars?

○|○|○○ |○|○○○   ○○○○||

* * * * * *

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○|○|○○ |○|○○○   ○○○○||

$$* * * * * *$$

$$C(4 + 3 - 1, 3 - 1) = C(6,2) = \frac{6!}{2!\,4!} = 15$$

COMBINATIONS WITH REPETITIONS

- How many ways are there to place k colored balls in a bag, when each ball can be of one of the n colours?

$$C(k+n-1, n-1)$$

BAGELS

- Chris is ordering bagels for three friends he's studying with, as well as one for himself. The bagel shop sells eight varieties of bagel. In how many ways can he choose the bagels to give to Jan, Tom, Olive, and himself?

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$$8 \cdot 8 \cdot 8 \cdot 8 = 8^4$$

ways to choose the bagels

DOUGHNUTS, PART I

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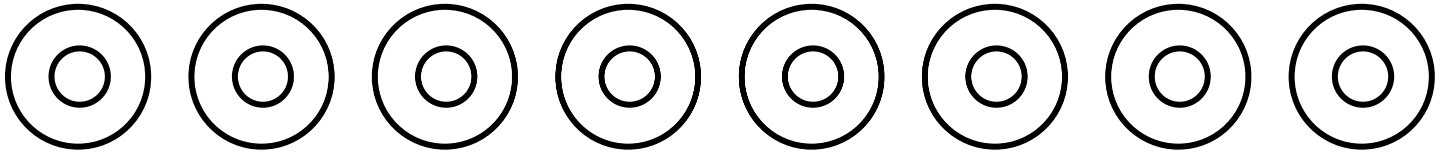

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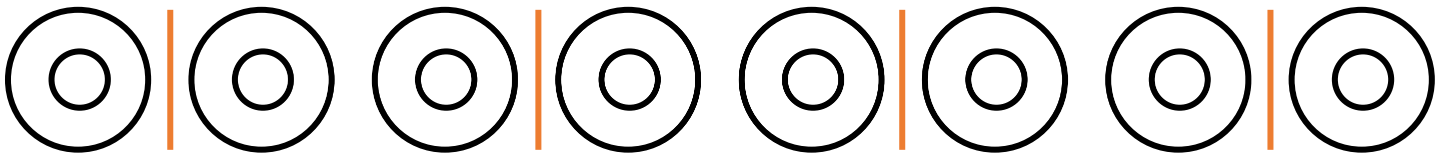

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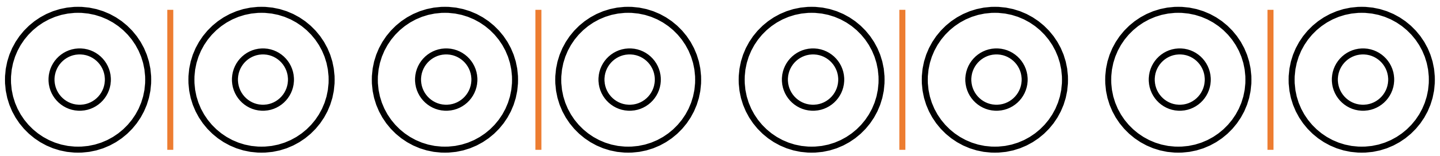

$$C(8+5-1,5-1)=$$

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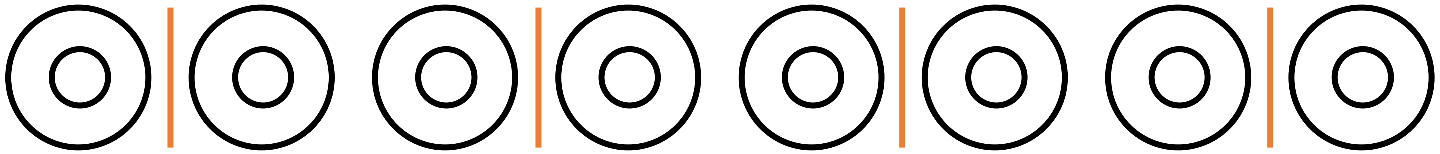


$$C(8+5-1,5-1) = C(12,4) = \frac{12!}{4!8!} = 495$$

ways to fill a box of 8 doughnuts

DOUGHNUTS, PART 1

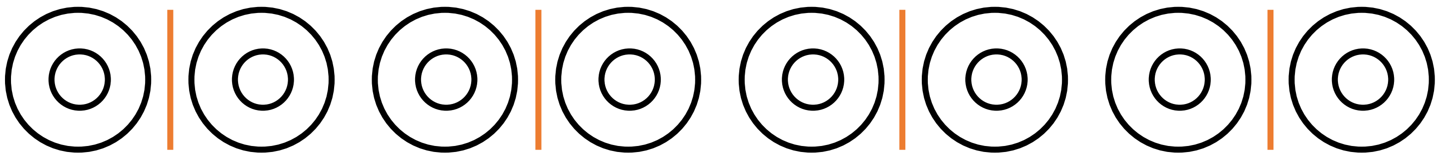
This corresponds to the number of non-negative integer solutions of which equation?

A. $x_1 + x_2 + x_3 + x_4 = 12$

B. $x_1 + x_2 + x_3 + x_4 + x_5 = 8$

C. $x_1 + x_2 + \cdots + x_{11} + x_{12} = 4$

D. $x_1 + x_2 + \cdots + x_7 + x_8 = 5$



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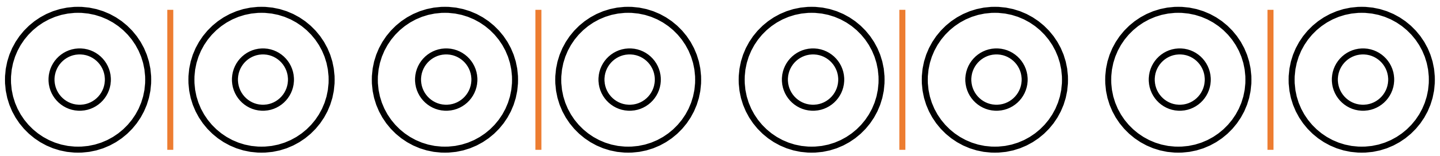

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$$\frac{8!}{2! \cdot 3! \cdot 3!}$$

unique ways to "order" repeating doughnuts

PROBLEM SET 4

Permutations and combinations

A PARTY – PART 1

- Five couples need to hold a meeting dedicated to the planning of the party. The meeting should consist of five people, one from each couple. How many possible ways do they have to pick people for the meeting?

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We need a representative from every couple:

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5$$

A PARTY – PART 2

- Five couples need to pick three couples who will be responsible for bringing food for the party. How many ways are there to do this?

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$$C(5,3) = \frac{5!}{3!\,2!} = 10$$

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Now, let’s chose one of the two people in each couple:

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Putting it all together:

$$C(5,3) \cdot 2 \cdot 2 \cdot 2 = 80$$

BOOK CLUB

- Alice has 7 textbooks and Bob has 5 textbooks. All textbooks are different. Alice gives Bob three of her books and Bob gives Alice three of his books. How many ways do they have to do it?

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$$C(10+5-1,5-1)=C(14,4)=\frac{14!}{10!4!}=\frac{14\cdot 13\cdot 12\cdot 11}{4!}=1001$$

ways of eating 10 slices of pizza of 5 types

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Alternative representation:

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$$x_1 + x_2 + x_3 + x_4 + x_5 = 10$$

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$$C(50+6-1,6-1) = C(55,5) = \frac{55!}{5!50!} = \frac{55\cdot 54\cdot 53\cdot 52\cdot 51}{5\cdot 4\cdot 3\cdot 2}$$

ways of distributing the gold without constraints

PIRATES

- In how many ways can 6 pirates split 50 gold pieces if the first pirate needs at least 10 pieces, the second needs at least 8, the third needs at least 6, the fourth needs at least 4, and the fifth needs at least 2, and the sixth can get any amount?

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$$y_1 = x_1 - 10, \qquad y_2 = x_2 - 8, \qquad y_3 = x_3 - 4, \qquad y_4 = x_4 - 2,$$
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20 "balls", 6 "colours"

$$C(20+6-1, 6-1) = C(26,5) = \frac{25!}{5!20!} = \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21}{5!}$$

ways of distributing the gold with constraints above

STUDENTS AND ASSIGNMENTS

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Different assignments for different students $\rightarrow$
order matters, no repetitions

$$9 \cdot 8 \cdot 7 \cdot 6 = \frac{9!}{(9-4)!}$$

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Different assignments for different students $\rightarrow$ order matters.

Let's look at this problem from the position of the assignments:

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Different assignments for different students → order matters.

Let's look at this problem from the position of the assignments:

$$4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^9$$

ways of distributing different assignments between different students

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What would make it a "combinations with repetitions" problem?

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What would make it a "combinations with repetitions" problem?

- indistinguishable students (no ordering)
- indistinguishable assignments (repetitions)

CANDIES

- There are 15 identical candies. In how many ways can you distribute them among 7 kids?

Identical candies →

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Identical candies →
repetitions allowed, order doesn't matter →

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combinations with repetitions

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Identical candies $\rightarrow$
repetitions allowed, order doesn't matter $\rightarrow$
combinations with repetitions

15 candies = 15 balls,   7 kids = 7 colours, categories

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$$C(15+7-1, 7-1) = C(21,6) = \frac{21!}{6!\,15!} = 54264$$

ways of distributing the candies among the kids with no constraints

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$$C(8+7-1,7-1)=C(14,6)=\frac{14!}{6!8!}=3003$$

ways of distributing the candies among the kid so that everyone gets at least one candy

WORKING GROUPS

- There are 12 students in the class. How many ways are there to split them into working groups of size 2 to work on the same assignment?

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$\frac{12!}{6!(2!)^6} = 10395$ possible splits

THE PIGEONHOLE PRINCIPLE

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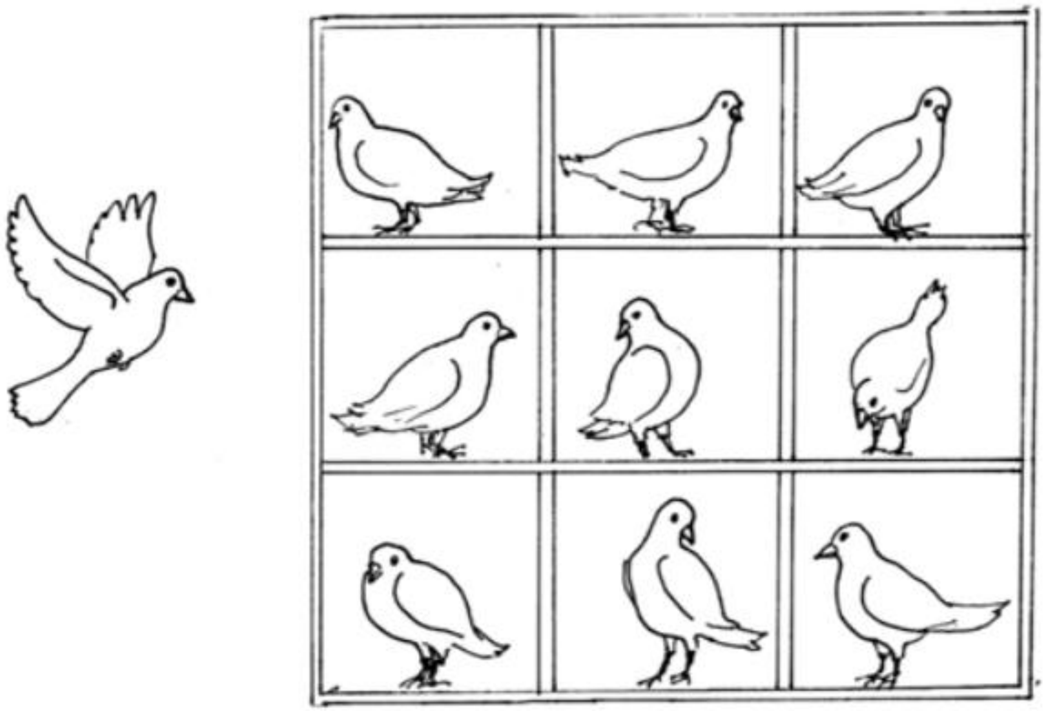
If n or more pigeons are distributed among k pigeonholes and $n > k$, then at least one pigeonhole contains two or more pigeons.



BIRTHDAYS 1

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There are $k = 12$ months.

If there are $n > k$ students,
at least two of them are born in the same month.

Therefore, there must be at least **13** students.

BIRTHDAYS 2

- How many people there should be in the room to guarantee that two people have birthday on the same day?

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People = pigeons, days = holes

There are $k = 366$ days.

If there are $n > k$ people,
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Therefore, there must be at least **367** people.

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- Prove that if seven distinct numbers are selected from $\{1,2,\ldots,11\}$, then some two of these numbers sum to 12.

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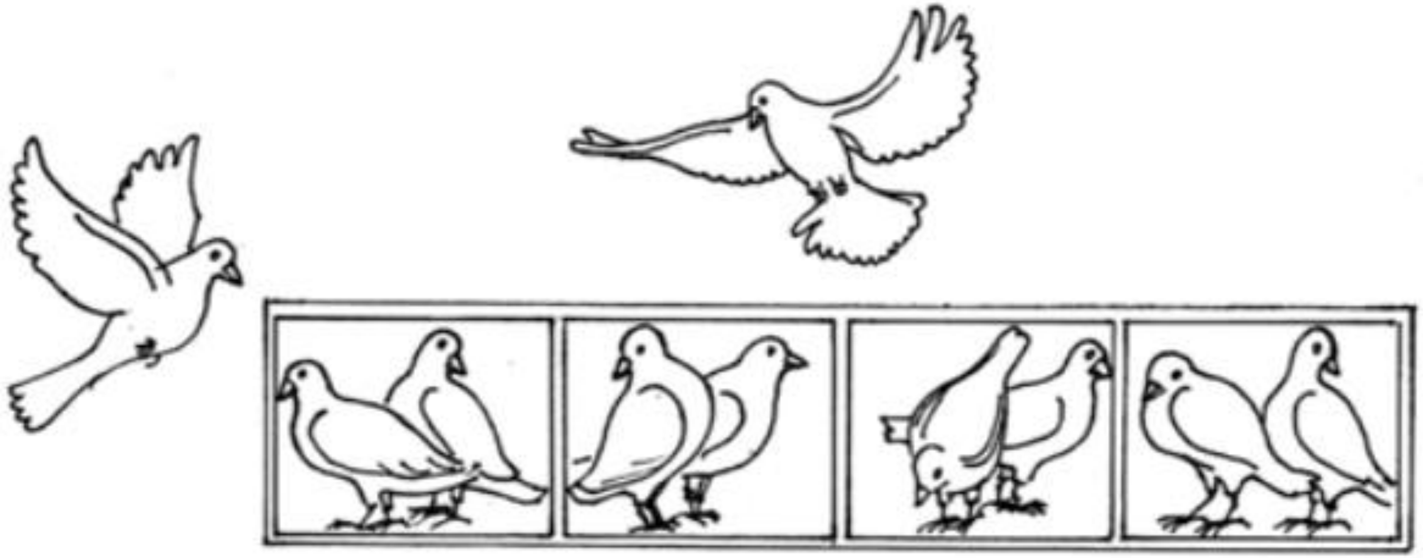
Those are our $k=6$ holes.

$n=7>k$ numbers are selected.

At least two of them will belong to the same set, meaning that at least two of them will sum up to 12.

GENERALIZED PIGEONHOLE PRINCIPLE

- Given n items that fall into k different categories, if $n > m \cdot k$ for some positive integer m, then at least $m + 1$ of the items must fall into the same category.



BREAKOUT ROOMS

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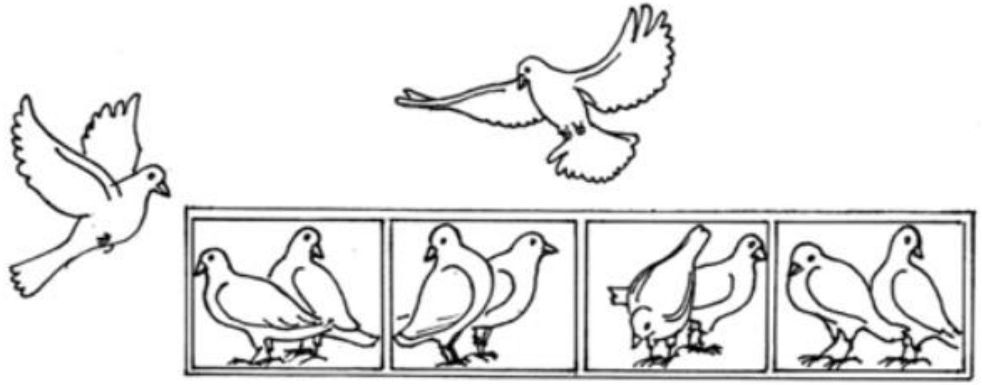
  Example:



There are $n = 5$ people in this class.

$k = 2$ breakout rooms are created.

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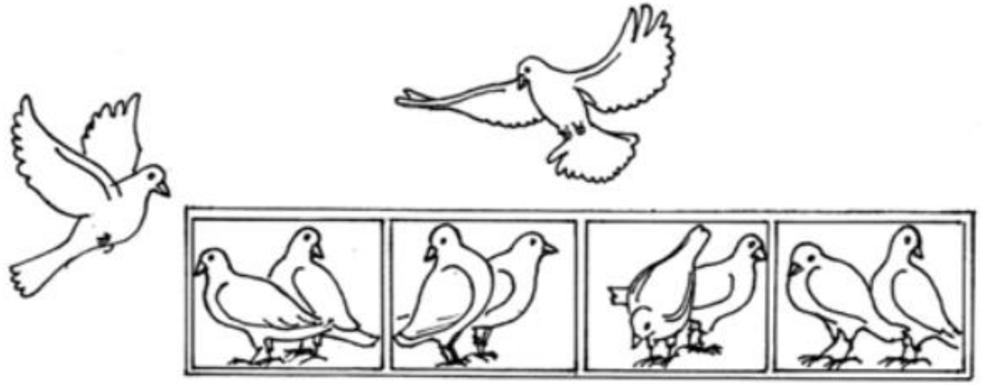

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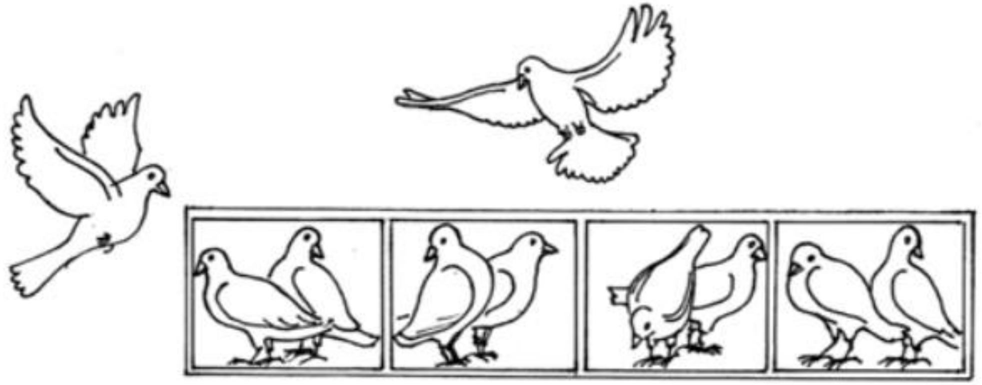

There are $n = 5$ people in this class.

$k = 2$ breakout rooms are created.

$$n > 2 \cdot k = 4 \implies$$

at least $2 + 1 = 3$ students will be put in the same breakout room.

TEST

- 30 students participated in an exam. The worst student in class got 13 answers wrong, while others made fewer mistakes. Show that there are at least 3 students who all made the same number of mistakes.

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Students = pigeons, number of mistakes = pigeonholes

$$30 > 2 \cdot 13$$

According to the generalized pigeonhole principle, at least $2 + 1 = 3$ students must have received the same score.

FLOOR AND CEILING

- Useful functions:
  - Floor $\lfloor x \rfloor -$ the largest integer smaller than x.

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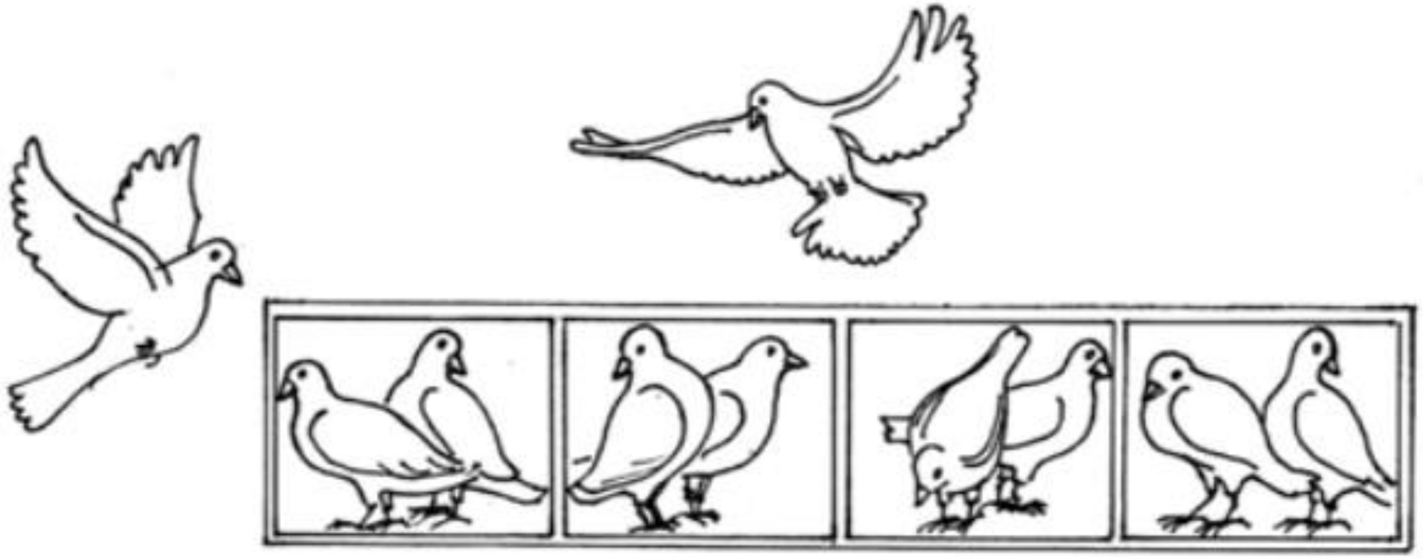
$$\lfloor 5 \rfloor = 5, \qquad \lfloor 2.5 \rfloor = 2, \qquad \lfloor -3.3 \rfloor = -4$$

  - Ceiling $\lceil x \rceil -$the smallest integer larger than x.

$$\lceil 5 \rceil = 5, \qquad \lceil 2.5 \rceil = 3, \qquad \lceil -3.3 \rceil = -3$$

GENERALIZED PIGEONHOLE PRINCIPLE

- Given n items that fall into k different categories, then at least $\left\lceil \frac{n}{k} \right\rceil$ of the items must fall into the same category.



CONFERENCE ROOMS

- A conference room contains 8 tables and 107 chairs. What is the smallest possible number of chairs at the table having the most chairs?

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Pigeons: 107 chairs

Pigeonholes: 8 tables

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$$\left\lceil \frac{107}{8} \right\rceil = 14$$

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Pigeons: 107 chairs

Pigeonholes: 8 tables

$$\left\lceil \frac{107}{8} \right\rceil = 14$$

Therefore, according to the generalized pigeonhole principle, at least 14 chairs are at the same table.

ODD OR EVEN

- Show that in any finite set of n numbers, at least half of the numbers are odd or at least half of them are even.

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  There are $k = 2$ "holes": even and odd numbers.

  If n numbers are distributed between $k = 2$ holes, at least $\left\lceil \frac{n}{2} \right\rceil$ of them belong to the same hole $\Leftrightarrow$

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  At least $\left\lceil \frac{n}{2} \right\rceil$ are odd or $\left\lceil \frac{n}{2} \right\rceil$ are even.

PROBLEM SET 5

https://docs.google.com/document/d/1Kf4NYeMPyABbbiWnm5sYKGgZOr3oK3XDznbb1Sf0oyY/edit?usp=sharing

LOGISTICS INTERIM EXAM

- Monday, March 22
  - 09:00 – 10:00 (roughly): review
  - 10:20 – 12:20: exam

- Topics included:
  - Basic counting
  - Inclusion-exclusion
  - Permutations and combinations
  - Pigeonhole principle