ELEMENTARY COMBINATORICS & PROBABILITY

Lecture 3

Permutations and combinations

LAST TIME

- Basic counting principles
  - Sum rule
  - Product rule
  - Their combo
- The principle of inclusion-exclusion
  - Size of a union of several sets

Graded assignment 1 is out!
See Google classroom.

Deadline: tomorrow, end of the day.

TODAY

- Problem set 2: review.

- Counting the number of possible arrangements
  - Permutations
  - Combinations

- Problem set 3

NUMBERS

- How many integer numbers from 1 to 100 are **not** divisible neither by 2 nor by 3?

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  50 + 33 – 16 = 67 are divisible by 3 or by 2 (or by both).

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  50 + 33 – 16 = 67 are divisible by 3 or by 2 (or by both).

  100 – 67 = 33 are not divisible by either.

STUDENTS AND COURSES

- A total of 36 students plan to take at least one of Discrete Mathematics, Algebra and Calculus during the coming semester:

Discrete Mathematics	23
Algebra	19
Calculus	18
Discrete Mathematics & Algebra	7
Discrete Mathematics & Calculus	9
Algebra & Calculus	11

How many students plan to take *all three* courses?

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$36 = 23 + 19 + 18 – 7 – 9 – 11 + N$ $\quad$ $N = 3$ students

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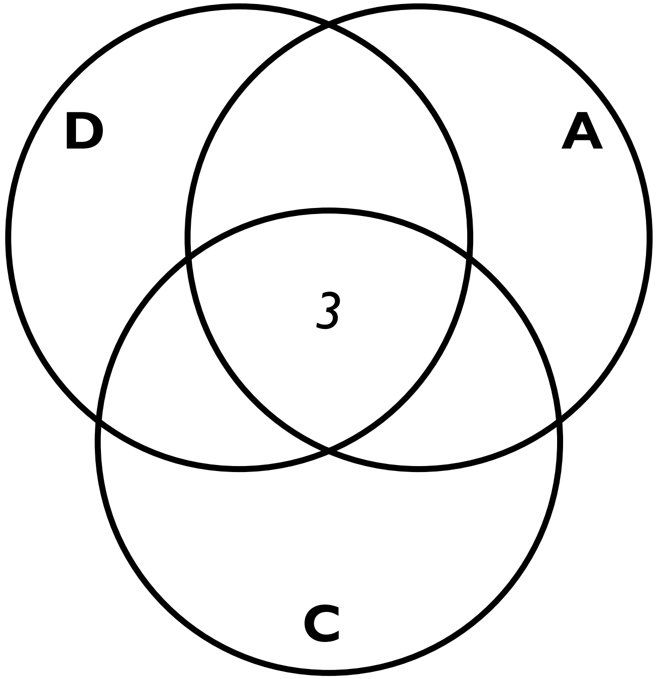

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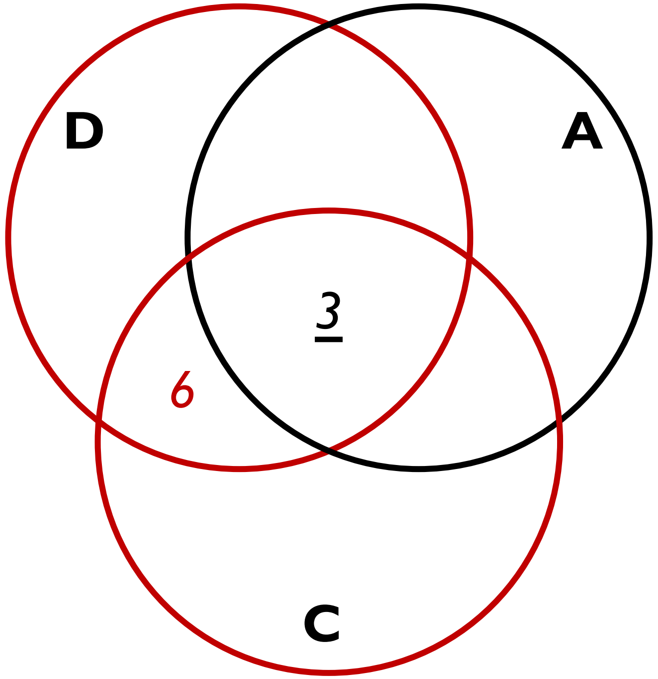

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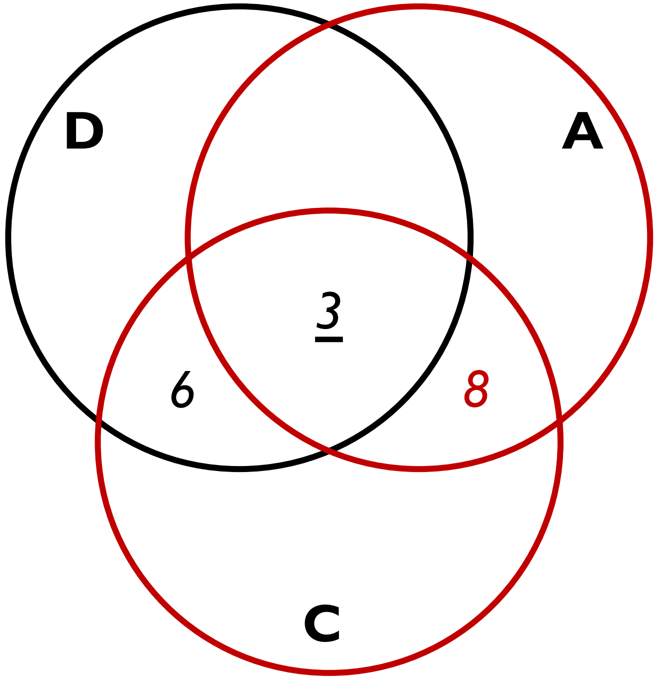

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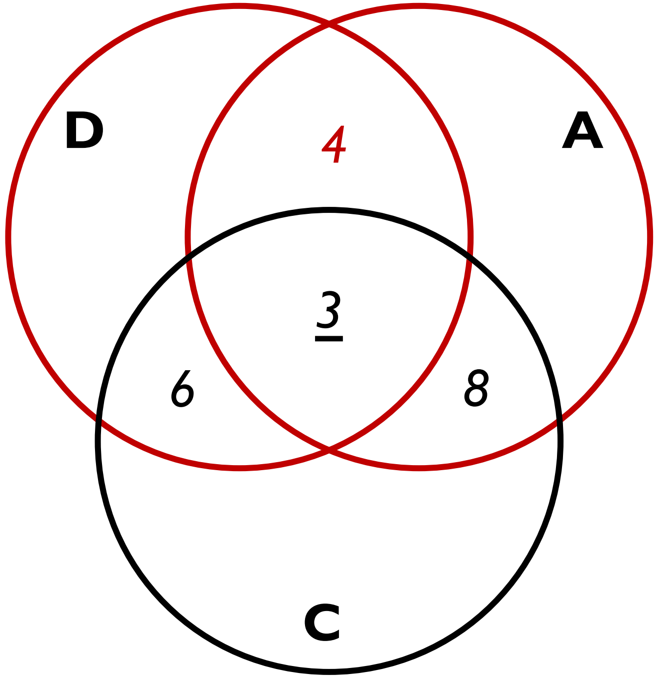

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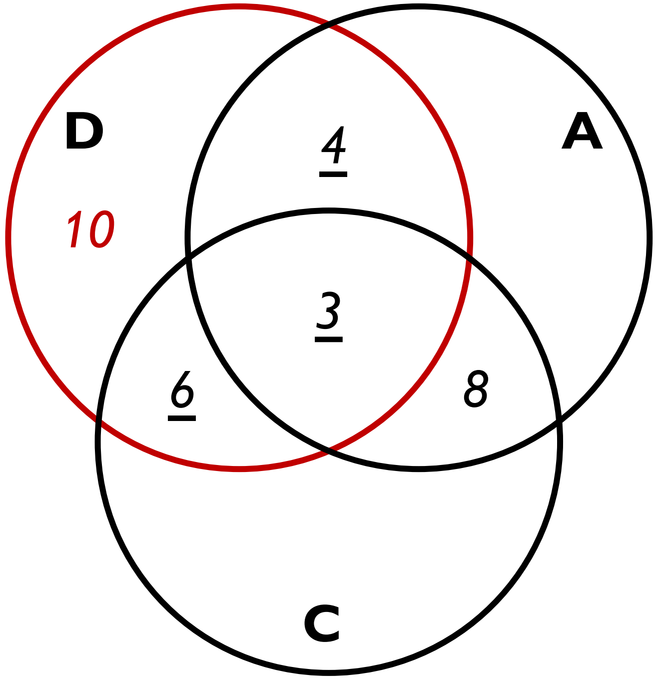

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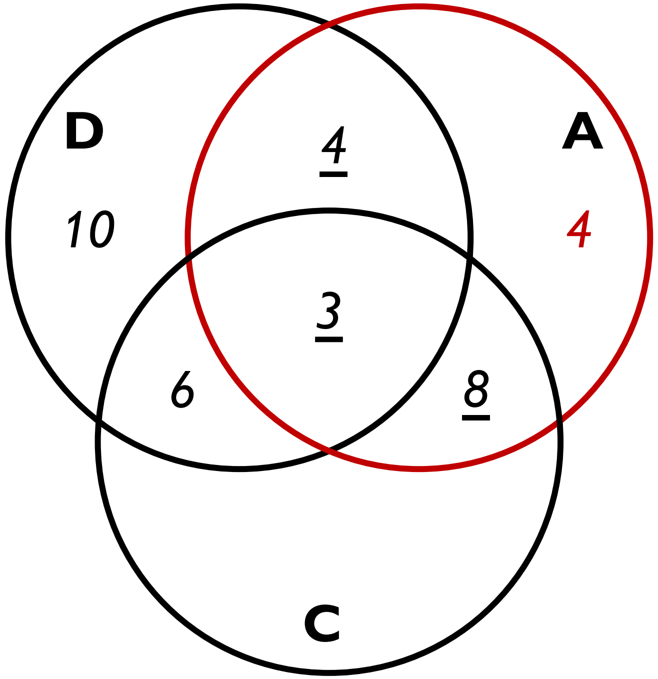

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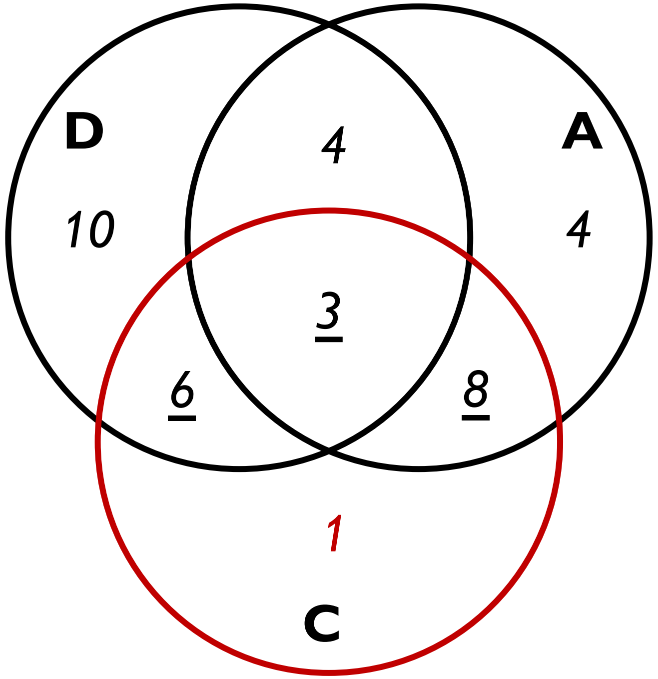

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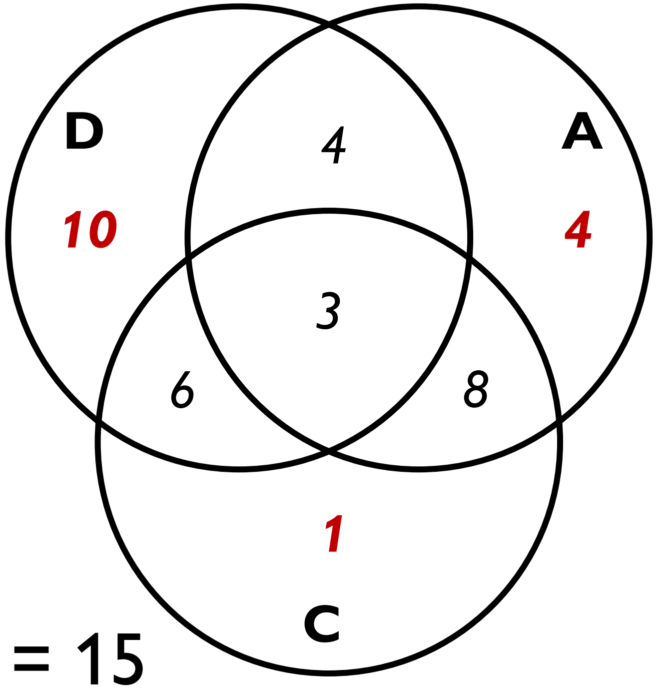

How many students plan to take *exactly one* of the courses? 10 + 4 + 1 = 15

BIT STRINGS

- How many different 8-bit strings begin with 10, end with 01 or have 00 as its two middle bits?

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  End begin with 10:

1	**0**	*	*	*	*	*	*

  End with 01:

*	*	*	*	*	*	**0**	**1**

  Have 00 in the middle:

*	*	*	**0**	**0**	*	*	*

BIT STRINGS

- How many different 8-bit strings begin with 10, end with 01 or have 00 as its two middle bits?

End begin with 10: 2^6

1	**0**	*	*	*	*	*	*

End with 01: 2^6

*	*	*	*	*	*	**0**	**1**

Have 00 in the middle: 2^6

*	*	*	**0**	**0**	*	*	*

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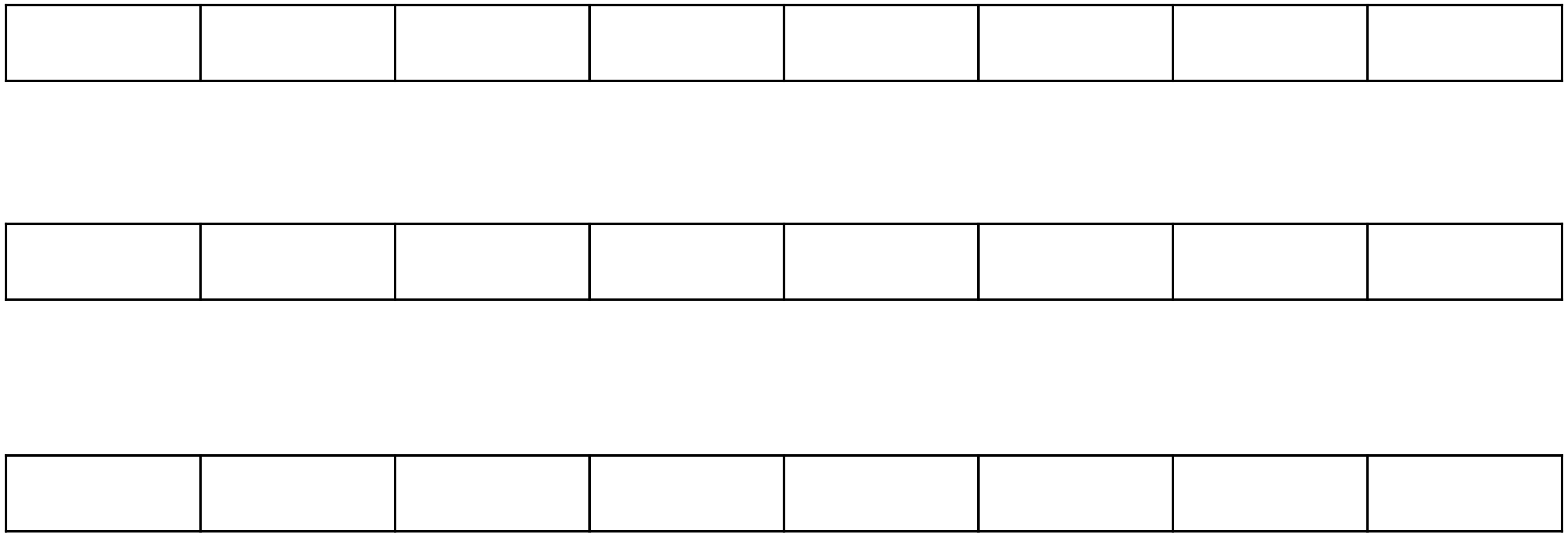

BIT STRINGS

- How many different 8-bit strings begin with 10, end with 01 or have 00 as its two middle bits?

  End begin with 10 and end with 01:

1	**0**	*	*	*	*	**0**	**1**

  Begin with 10 and have 00 in the middle:

1	**0**	*	**0**	**0**	*	*	*

  Have 00 in the middle and end with 01:

*	*	*	**0**	**0**	*	**0**	**1**

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  End begin with 10 and end with 01: 2^4

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  Begin with 10 and have 00 in the middle: 2^4

1	**0**	*	**0**	**0**	*	*	*

  Have 00 in the middle and end with 01: 2^4

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BIT STRINGS

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Begin with 10, have 00 in the middle and end with 01: 2^2

1	**0**	*	**0**	**0**	*	**0**	**1**

BIT STRINGS

- How many different 8-bit strings begin with 10, end with 01 or have 00 as its two middle bits?

Begin with 10, have 00 in the middle and end with 01: 2^2

1	**0**	*	**0**	**0**	*	**0**	**1**

$3 \cdot 2^6 - 3 \cdot 2^4 + 2^2 = 3 \cdot 64 - 3 \cdot 16 + 4 = 148$
8-bit strings begin with 10, end with 01 or have 00 in the middle

COUNTING ARRANGEMENTS

ARRANGEMENTS OF OBJECTS

- Imagine you have n objects. How can you arrange them?

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ORDERED		
NOT ORDERED		

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PERMUTATIONS

https://youtu.be/uNS1QvDzCVw

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  Example:

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TRY TO LIST ALL POSSIBILITIES

EXAMPLE: RUNNERS

- In how many ways can three runners, a, b and c, finish the race, if no ties are allowed?

abc

acb

EXAMPLE: RUNNERS

- In how many ways can three runners, a, b and c, finish the race, if no ties are allowed?

abc	bac
acb	bca

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acb	bca	cba

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There are **6** permutations.

PERMUTATIONS OF n ELEMENTS

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$$n \cdot (n-1)$$

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  Example:

  In how many orders can we put n books on the shelf?

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3$$

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  Example:

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$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2$$

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  In how many orders can we put n books on the shelf?

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

FACTORIAL

- Many counting problems involve multiplying together long strings of numbers.

- Factorial notation:

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Examples:

$3! = 3 \cdot 2 \cdot 1 = 6$

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Examples:

$3! = 3 \cdot 2 \cdot 1 = 6$ $\qquad$ $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

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Examples:

$3! = 3 \cdot 2 \cdot 1 = 6$ $\quad$ $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ $\quad$ $\mathbf{0! = 1}$

FACTORIAL: EXERCISE

$$\frac{7!}{5!} =$$

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$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

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$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 6 = 42$$

FACTORIAL: EXERCISE

$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 6 = 42$$

$$\frac{100!}{98!} =$$

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$$\frac{100!}{98!} = \frac{100 \cdot 99 \cdot 98!}{98!}$$

FACTORIAL: EXERCISE

$$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 6 = 42$$

$$\frac{100!}{98!} = \frac{100 \cdot 99 \cdot 98!}{98!} = 9900$$

PERMUTATIONS OF n ELEMENTS

- In how many ways can n distinct objects be ordered?

  Example:

  In how many orders can we put n books on the shelf?

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 = \boldsymbol{n}!$$

A PHOTOGRAPH

- In how many ways can Ann, John, David, Mary and Sam line up for a photograph?

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$$5! = 120 \text{ ways}$$

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$$2 \times 4! = 48 \text{ ways}$$

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REPEATED ELEMENTS

If among n elements $k \leq n$ elements are not unique, with $n_1, n_2, \dots, n_k$ repetitions respectively, then the number of possible permutations is

$$\frac{n!}{n_1! \cdot n_2! \cdot \dots \cdot n_k!}$$

SPOT A PROBLEM IN THE TEXT

TEDEd

Source: [TEDEd]

Discover Create Support

Anagrams:

An ANAGRAM is a kind of wordplay where the letters in a word, phrase or sentence are rearranged to make a new word, phrase or sentence. For example, the word ANAGRAM has 7 letters and can be rearranged 7! = 5040 ways. One of these arrangements spells the word ANAGRAM itself, another spells MARGANA, and so on. It is believed that Shakespeare played with this idea when naming the protagonist in his play *Hamlet*. Hamlet's name is thought to have been an anagram of AMALETH, the name of a Danish Prince. Another famous anagram comes from J.K. Rowling's book *Harry Potter and the Chamber of Secrets*. The name "Tom Marvolo Riddle" has 17 letters. These 17 letters can be rearranged approximately 355-thousand billion ways. One of these arrangements spells "I am Lord Voldemort."

History:

The first person to ever use ! to symbolize a factorial was a French mathematician named

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$$\frac{7!}{3!} = 7 \cdot 6 \cdot 5 \cdot 4 = 840$$

THAT'S NOT ENTIRELY TRUE EITHER

TEDEd

Source: [TEDEd]

Discover Create Support

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TOM MARVOLO RIDDLE

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- TOM MARVOLO RIDDLE has 16 characters:

T	V
O (3 times)	L (2 times)
M (2 times)	I
A	D (2 times)
R (2 times)	E

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- Therefore, the # on arrangements is

$$\frac{16!}{3! \cdot 2! \cdot 2! \cdot 2! \cdot 2!} = 15 \cdot 14 \cdot \ldots \cdot 4$$

R-PERMUTATIONS

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  How many ways are there to choose a president, vice-president, and secretary from a group of 17 people?

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$$17 \cdot 16 \cdot 15 = 4080$$

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Note that:

$$17 \cdot 16 \cdot 15 = \frac{17!}{14!} = \frac{17!}{(17-3)!}$$

R-PERMUTATIONS: GENERAL CASE

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Chan, Harris	Jones, Vello	Harris, Chan	Vello, Jones
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$$\frac{4!}{\mathbf{2}!(4-3)!} \text{ different committees}$$

COMBINATIONS: GENERAL CASE

- Suppose we want to choose k elements from a set with n elements, in no specific order.

  In other words: select a subset of size k from a set of size n.

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$$C(n,k) = \frac{n!}{k!\,(n-k)!}$$

NOTATION

$$C(n,k) = \binom{n}{k} = C_n^k = \frac{n!}{k!\,(n-k)!}$$

SPOT A PROBLEM HERE

Example 3. *In how many ways can a set of two positive integers less than 100 be chosen?*

Solution. $99 \times 98 = 9702$ ways. □

Source: [UC Berkeley]

COMBINATIONS

- Consider the formulas:

$$(a+b)^2 = a^2 + 2ab + b^2$$

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$$(a+b)(a+b)(a+b) =$$
$$= a^3 + a^2b + aba + ab^2 + ba^2 + b^2a + bab + b^3$$

BINOMIAL THEOREM

$$(a+b)^n = \sum_{i=0}^{n} C(n,i) \cdot a^{n-i} b^i$$

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$$= a^4 + 4a^3 b + 3a^2 b^2 + 4ab^3 + b^4$$

SUM OF BINOMIAL COEFFICIENTS

- Can you guess the value of the following sum

$$C(n,0) + C(n,1) + C(n,2) + \cdots + C(n,n-1) + C(n,n) = ?$$

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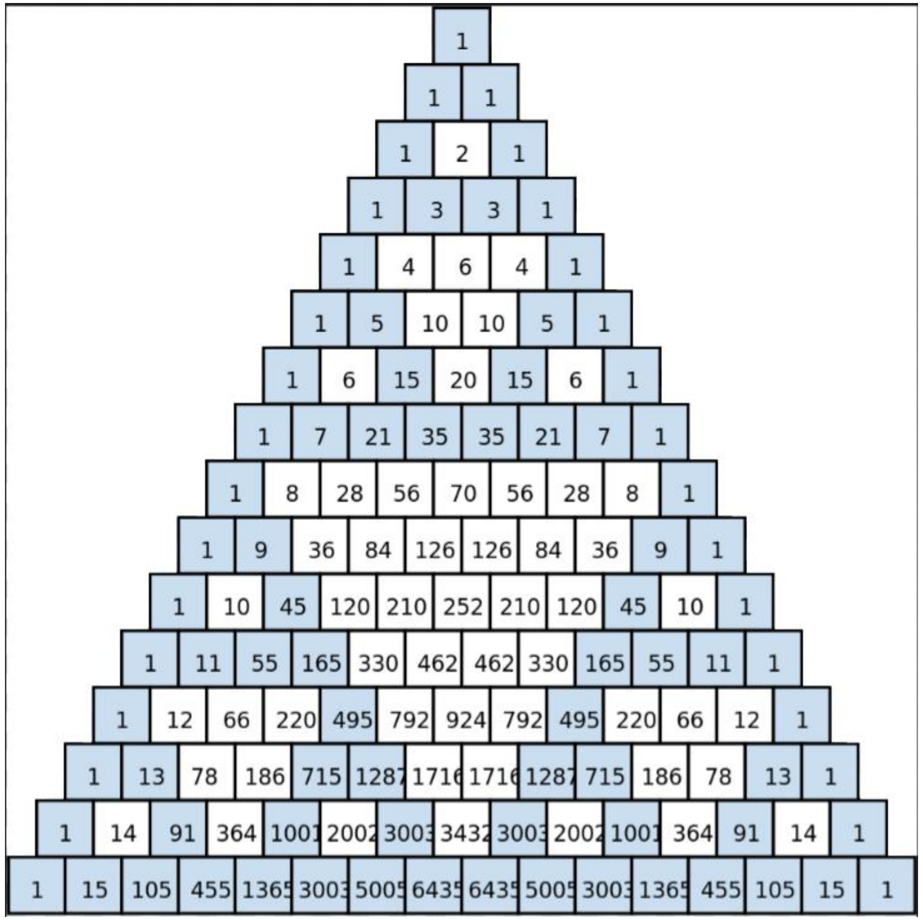
Thus, the sum above = number of subsets of a set of size n.

How many subsets does a set of n elements have? $\mathbf{2^n}$.

PASCAL'S TRIANGLE



PASCAL'S TRIANGLE

https://youtu.be/XMriWTvPXHI

SALADS

- Is this true?

- Assume that
  - there are exactly 50 ingredients;
  - letuce (salad base) is one of them, so there are 49 ingredients left;
  - you can still choose 5 ingredients in addition to the base.

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$$= \frac{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 44!} =$$

$$= \frac{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45}{120} =$$

$$= 1{,}906{,}884$$

PERMUTATIONS VS COMBINATIONS

COMMITTEE 1

- A committee should consist of 3 faculty members and 2 students:
  - faculty members can be chosen from 6 eligible candidates;
  - students can be chosen from 5 eligible candidates.
- How many different committees are possible?

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Product rule: 20 x 10 different committees are possible

COMMITTEE 2

- A total of 6 seniors and 5 juniors have been nominated for a 6-person committee, which should consist of 3 seniors and 3 juniors. One senior will be chosen as president, one as vice-president, and one as secretary. The three juniors will not have any special title.
- How many such committees are possible?

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$$P(6,4) = \ 6 \cdot 5 \cdot 4 = 120$$

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Product rule: 120 x 10 = 1200 different committees possible

TO SUM UP

- Imagine you have n objects. How can you arrange them?

	WITHOUT REPETITIONS	**WITH REPETITIONS**
ORDERED	*PERMUTATIONS* In how many ways can n people sit in a row?	How many different n-bit strings are there?
NOT ORDERED	*COMBINATIONS* In how many ways can we chose k out of n different candies in a bag?	In how many ways can we distribute n identical candies among k kids?

PRACTICE

https://docs.google.com/document/d/19EoiAc5JypihiBKOUOFcarzgiCXHAUbv9sQBg47Kelg/edit?usp=sharing