ELEMENTARY COMBINATORICS & PROBABILITY

Review week 1

TODAY

- Problem set 5
- Review permutations & combinations
- Interim exam

- Graded assignment 2 is out
- Deadline: **TODAY, March 22, 23:59** Barcelona time
- See Google classroom

PROBLEM SET 5

https://docs.google.com/document/d/1Kf4NYeMPyABbbiWnm5sYKGgZOr3oK3XDznbb1Sf0oyY/edit?usp=sharing

THE PIGEONHOLE PRINCIPLE

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Example:

If $n = 12$ people sit on $k = 10$ chairs, then two or more people are sitting on at least one of the chairs.

GENERALIZED PIGEONHOLE PRINCIPLE

Given n items that fall into k different categories, then at least $\left\lceil \frac{n}{k} \right\rceil$ of the items must fall into the same category.

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Example:

If $n = 22$ people sit on $k = 10$ chairs, then at least $\left\lceil \frac{22}{10} \right\rceil = 3$ people are sitting on at least one of the chairs.

REMAINDERS

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  Generalized pigeonhole principle:

  At least $\lceil 7/5 \rceil = 2$ numbers have the same remainder when divided by 5.

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  At least $\lceil 32/5 \rceil = 7$ students must have received the same grade.

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  B. At least two of the numbers are **even** → the sum is even

  Thus, it is always possible numbers such that their sum is even.

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  If n numbers are distributed between $k = 2$ holes, at least $\left\lceil \frac{n}{2} \right\rceil$ of them belong to the same hole $\Leftrightarrow$

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  At least $\left\lceil \frac{n}{2} \right\rceil$ are odd or $\left\lceil \frac{n}{2} \right\rceil$ are even.

REVIEW

DIFFERENT ARRANGEMENTS

- Imagine you have n objects. How can you arrange them?

	WITHOUT REPETITIONS	**WITH REPETITIONS**
ORDERED	*PERMUTATIONS* Seating n people in a row $n!$	*TUPLES* Counting different n-bit strings are there? k^n
NOT ORDERED	*COMBINATIONS* Choosing k out of n different candies in a bag $C(n,k) = \frac{n!}{k!\,(n-k)!}$	*COMBINATIONS with repetitions* Distributing k identical candies among n kids $C(k+n-1, n-1)$

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$$10 \cdot 9 \cdot 8$$

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- How many 6-character license plates are there, if each character can be a letter or a digit?

$$36 \cdot 36 \cdot 36 \cdot 36 \cdot 36 \cdot 36 = 36^6$$

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$$C(4+10-1, 10-1) = \frac{13!}{9!\,4!}$$

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$$\rightarrow 26 \cdot 25 + 26 \cdot 25 \cdot 10$$

passwords that consist of either two different low-case letters or two different lower-case letters and a digit.

INCLUSION-EXCLUSION

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  Begin with 022 **and** end with 01:

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  Begin with 022: 3^3

  End with 01: 3^4

  Begin with 022 **and** end with 01: 3

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  Begin with 022: 3^3

  End with 01: 3^4

  Begin with 022 **and** end with 01: 3

  Begin with 022 **or** end with 01: $3^3 + 3^4 - 3$

PRODUCT RULE

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$$\rightarrow C(15,5) \cdot C(10,5) = \frac{15!}{5!10!} \cdot \frac{10!}{5!5!} = \frac{15!}{(5!)^3}$$

different sharing menus

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$C(15+5-1, 15-1) = \frac{19!}{14!5!}$ ways of choosing cold starters

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$C(15+5-1, 15-1) = \frac{19!}{14!5!}$ ways of choosing cold starters

$C(10+5-1\,, 10-1) = \frac{14!}{9!5!}$ ways of choosing cold starters

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$C(10+5-1\ , 10-1) = \frac{14!}{9!5!}$ ways of choosing cold starters

$$\rightarrow C(19,14) \cdot C(14,9) = \frac{19!}{14!5!} \cdot \frac{14!}{9!5!} = \frac{19!}{9! \cdot (5!)^2}$$

different sharing menus

INTERIM EXAM

- Exam available on Google classroom as of 10:20
- Open book
- 15 questions
  - 5 one-point questions;
  - 10 two-point questions.

- Solutions must be written in the file
  - typed or pictures;
  - should contain explanations.

- **Deadline: 12:30 Barcelona time**

GOOD LUCK!