ELEMENTARY COMBINATORICS & PROBABILITY

Lecture 4

Permutations and Combinations II

LAST TIME

- Permutations
  - ordering a set of n objects: $n!$ ways;
  - permutations with repetitions;
  - r-permutations.
- Combinations
  - when order doesn't matter;
  - $C(n, k)$
  - binomial theorem.
- Problem set 3.

TODAY

- Problem set 3
- Combinations *with* repetitions
- Problem set 4:
  - Learning to distinguish between different types of arrangements.

- Graded assignment 1 due TODAY 23:59.

WARM-UP

POSSIBLE ARRANGEMENTS

- Imagine you have n objects. How can you arrange them?

	WITHOUT REPETITIONS	WITH REPETITIONS
ORDERED	In how many ways can n people sit in a row?	How many different n-bit strings are there?
NOT ORDERED	In how many ways can we chose k out of n different candies in a bag?	In how many ways can we distribute n identical candies among k kids?

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NOT ORDERED	*COMBINATIONS* In how many ways can we chose k out of n different candies in a bag? $C(n,k) = \frac{n!}{k!\,(n-k)!}$	In how many ways can we distribute n identical candies among k kids?

- In how many ways can you arrange these 5 bottles?

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  5 ***different*** *bottles -> order* ***matters*** *-> counting* ***permutations*** -> 5!

- In how many ways can you chose 3 out of these bottles?

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  Order ***doesn't*** *matter -> counting* ***combinations*** -> $C(5,3) = 5!/(3!\,2!)$

- In how many ways can you arrange these 5 bottles?

- In how many ways can you arrange these 5 bottles? *Permutations* ***with repetitions*** -> 5!/3!

- In how many ways can you pick 3 of these bottles for 3 friends?

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  *For **different** people -> order **matters** -> counting 3-permutations ->* $5 \cdot 4 \cdot 3$

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PROBLEM SET 3

Selected problems

A ROW OF PEOPLE

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  Permute 6 objects: 6! ways to do so.

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  So, there are $3! \cdot 6!$ ways to seat 8 people with this constraint.

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  # ways to permute 2 groups:

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  \# ways to permute 2 groups: $2!$

  Thus, there are $3! \cdot 5! \cdot 2!$ ways to permute the group of 8 people with the constraints above.

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$$\frac{n!}{n} = (n-1)!$$

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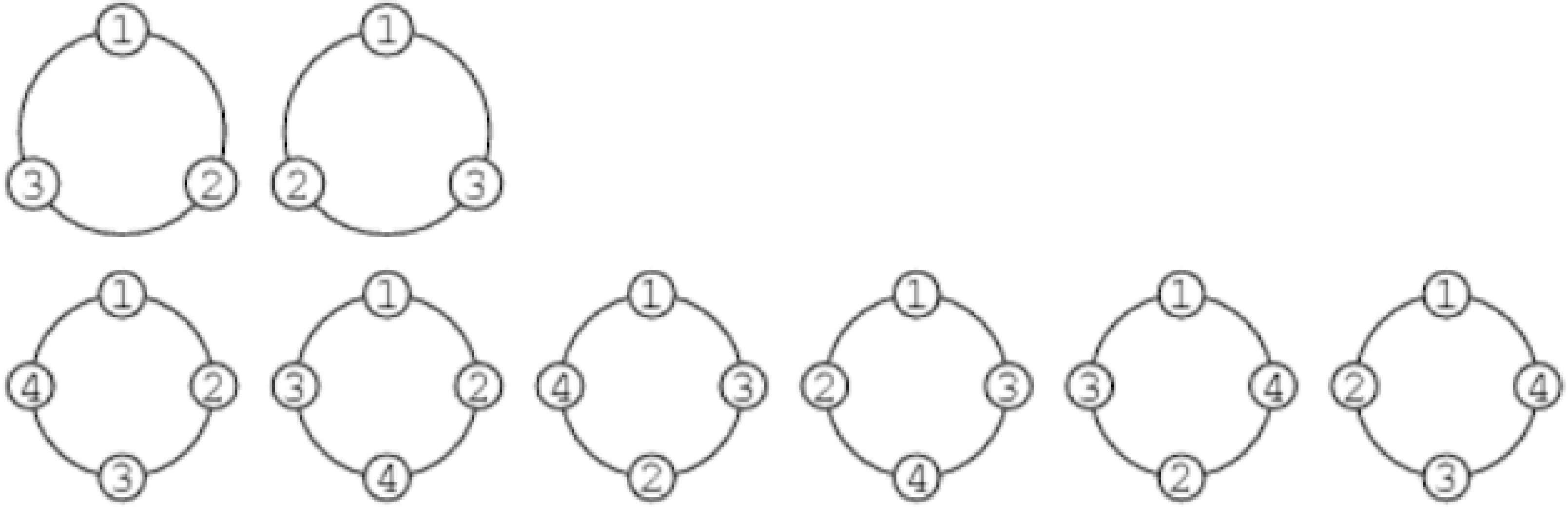
  Examples for $n = 3$ and $n = 4$:



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$$C(15,4) = \frac{15!}{4!\,(15-4)!} = \frac{15 \cdot 14 \cdot 13 \cdot 12}{4 \cdot 3 \cdot 2} = 1365$$

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committees without freshmen:

$$C(9,4) = \frac{9!}{5! \cdot 4!} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2} = 126$$

all possible committees:

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$$1365 - 126 = 1239$$

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$N_{211} + N_{121} + N_{112}$ committees with at least one student from each group

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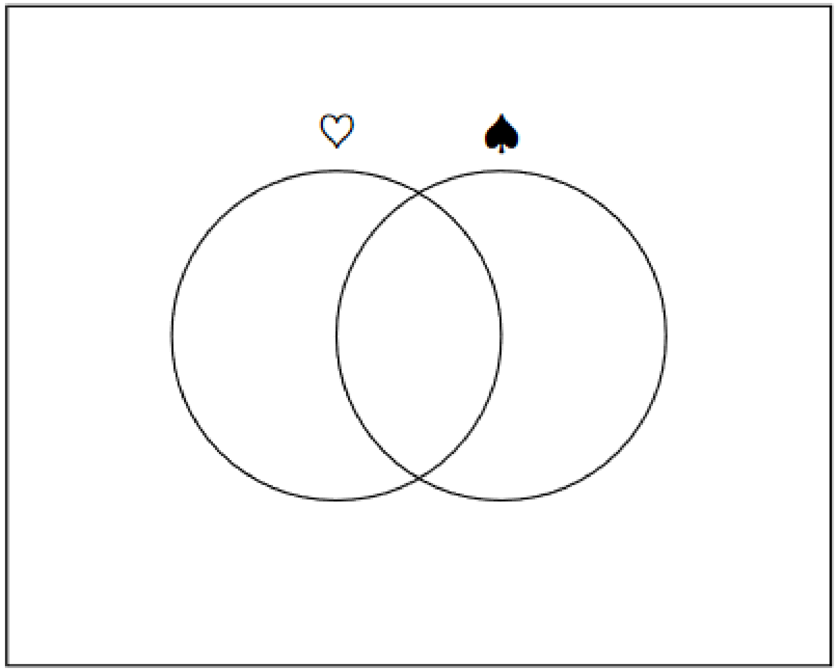

$\heartsuit$ A set of shuffles where ace of hearts **is** directly on top of king of hearts.

$\spadesuit$ A set of shuffles where ace of spades **is** directly on top of king of spades.

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shuffles where both is true

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So the number of possible shuffles is

$$52! - (51! + 51! - 50!) = 52! - 2 \cdot 51! + 50!$$

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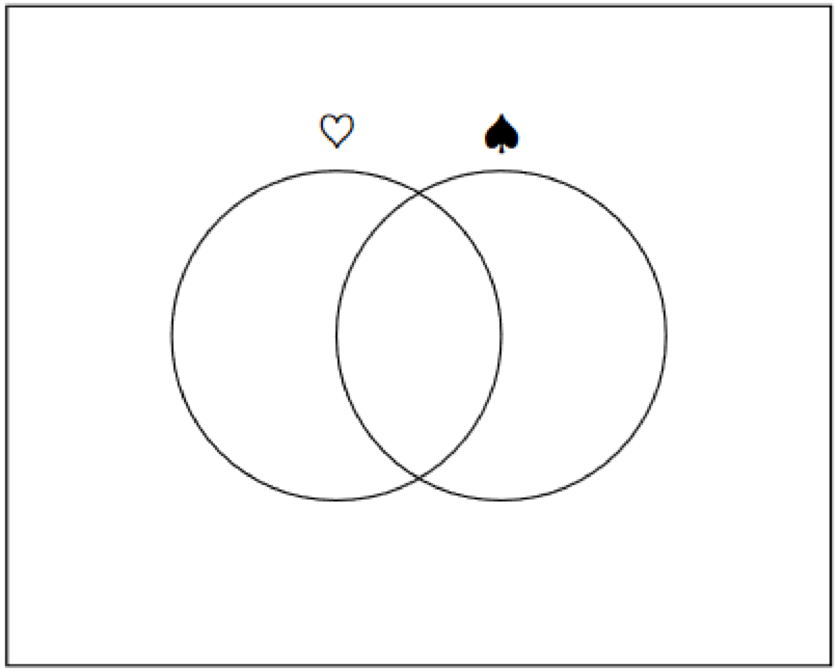

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M (2 times)	I
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unique permutations.

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  1. cannot be together
  2. cannot be in front
  3. cannot be at the end

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unique permutations with 2 spaces next to each other.

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  - # start with a space
  - # end with a space
  - # start and end with a space

TUPLES

Ordered sequences with repetitions

POSSIBLE ARRANGEMENTS

- Imagine you have n objects. How can you arrange them?

	WITHOUT REPETITIONS	**WITH REPETITIONS**
ORDERED	*PERMUTATIONS* Seating n people in a row $n!$	Counting different n-bit strings are there?
NOT ORDERED	*COMBINATIONS* Choosing k out of n different candies in a bag $C(n,k) = \frac{n!}{k!\,(n-k)!}$	Distributing k identical candies among n kids

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$$3 \cdot 3 \cdot \ldots \cdot 3 = k^n = 3^6 \text{ different sequences}$$

TUPLES

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  # string ending with 01:

  # strings beginning with 002 and ending with 01:

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  # strings beginning with 022: $3^3 = 27$

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  # strings beginning with 022: $3^3 = 27$

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  # strings beginning with 002 and ending with 01:

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  # strings beginning with 022: $3^3 = 27$

  # string ending with 01: $3^4 = 81$

  # strings beginning with 002 and ending with 01: 3

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  Inclusion-exclusion

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  # string ending with 01: $3^4 = 81$

  # strings beginning with 002 and ending with 01: 3

$$27 + 81 - 3 = 105$$

different sequences that start with 002 or end with 01

COMBINATIONS WITH REPETITIONS

POSSIBLE ARRANGEMENTS

- Imagine you have n objects. How can you arrange them?

	WITHOUT REPETITIONS	**WITH REPETITIONS**
ORDERED	*PERMUTATIONS* Seating n people in a row $n!$	*TUPLES* Counting different n-bit strings are there? k^n
NOT ORDERED	*COMBINATIONS* Choosing k out of n different candies in a bag $C(n,k) = \frac{n!}{k!\,(n-k)!}$	Distributing k identical candies among n kids

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- There are $C(n, k)$ ways to choose k distinct elements without regard to order from a set of n elements.

- How many ways are there to choose k elements without regard to order from a set of n elements **if repetition is allowed**?

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  - several balls can be given the same colour (assume there are at least 4 balls of each colour available);
  - we don't have to use all colours.

COMBINATIONS WITH REPETITIONS

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TRY TO LIST ALL OPTIONS
(AND COUNT THEM)

HOW MANY DID YOU OBTAIN?

COMBINATIONS WITH REPETITIONS

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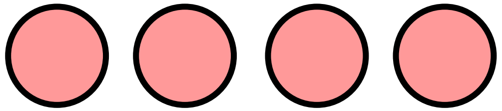

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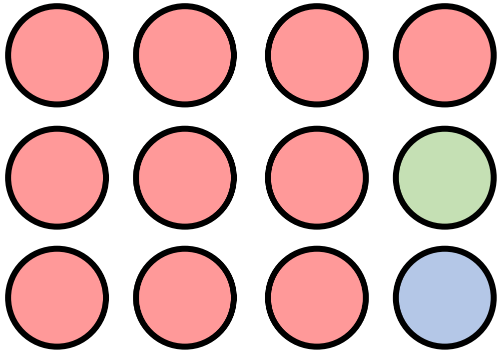

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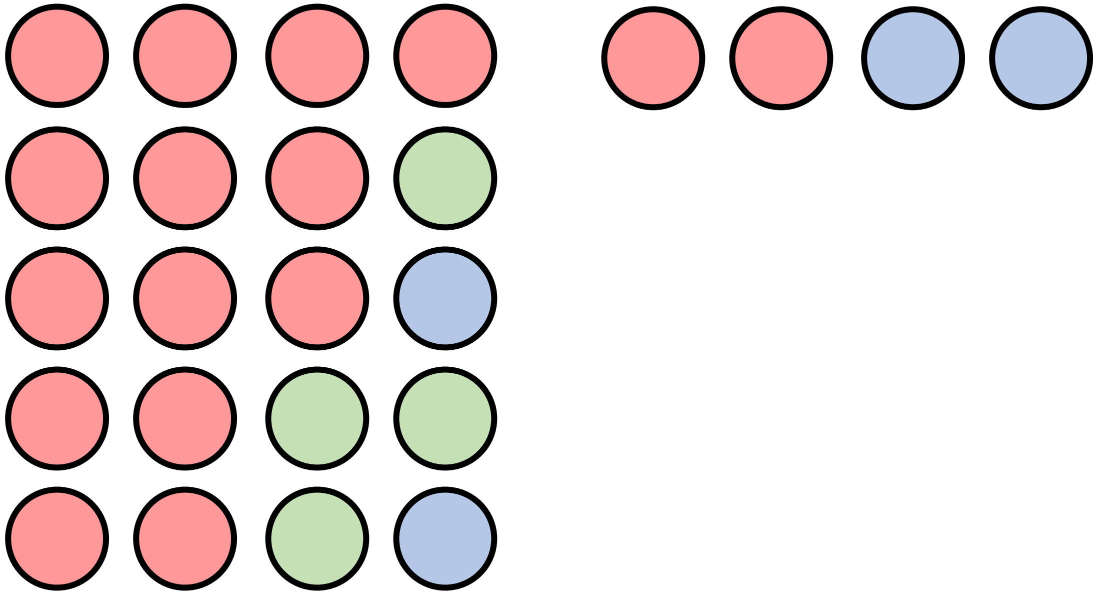

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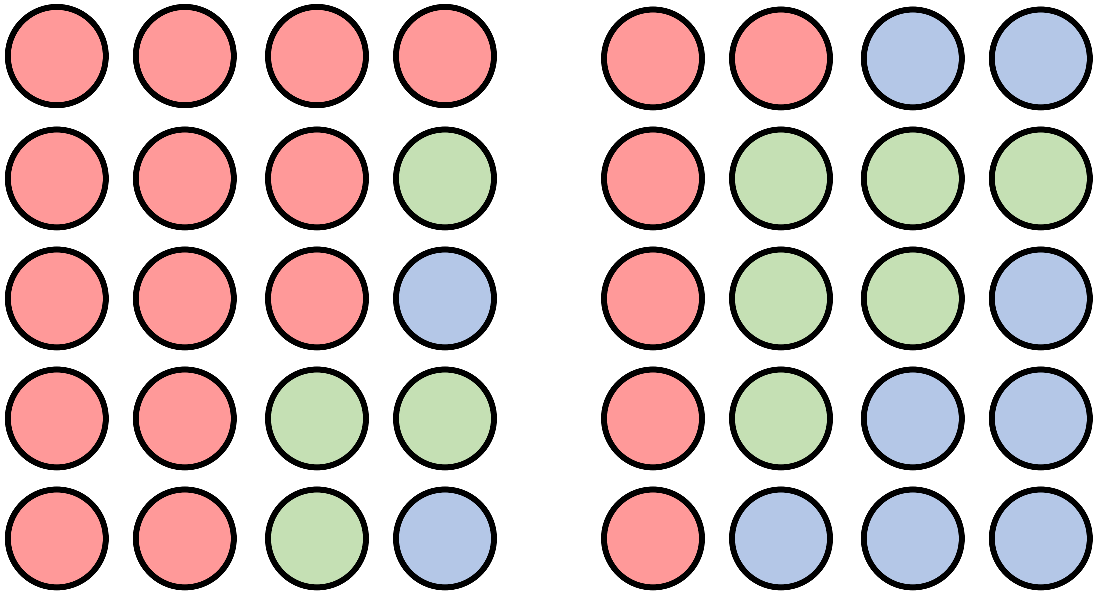

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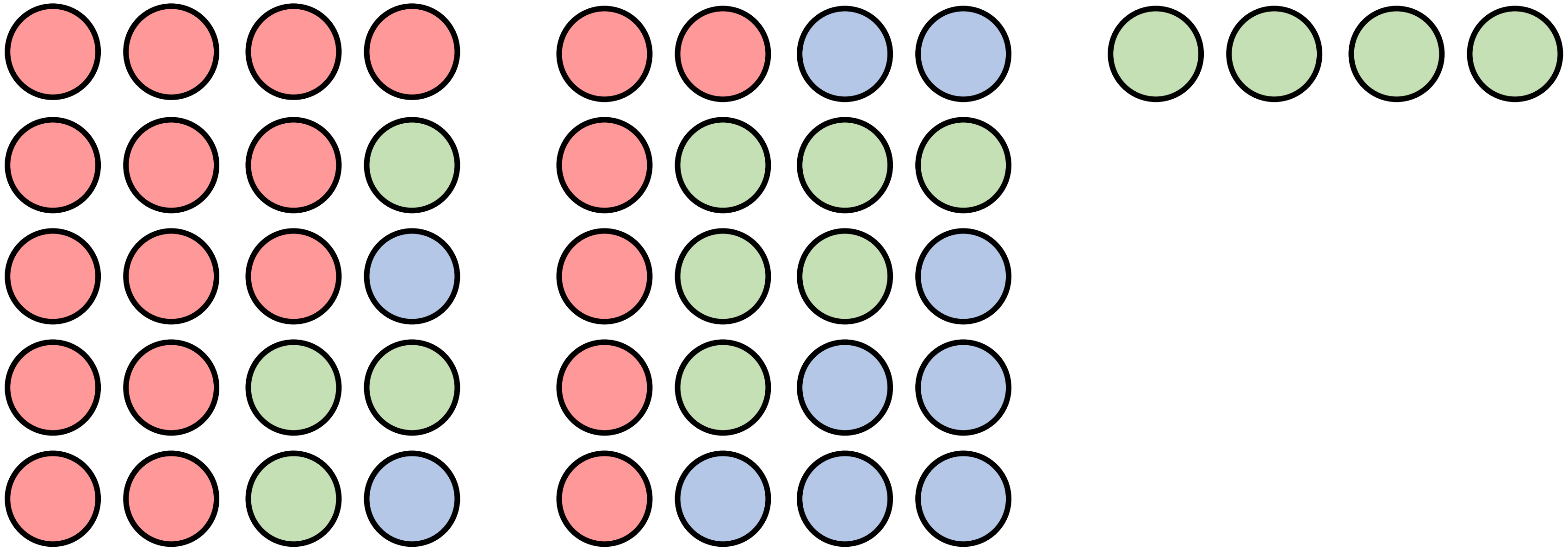

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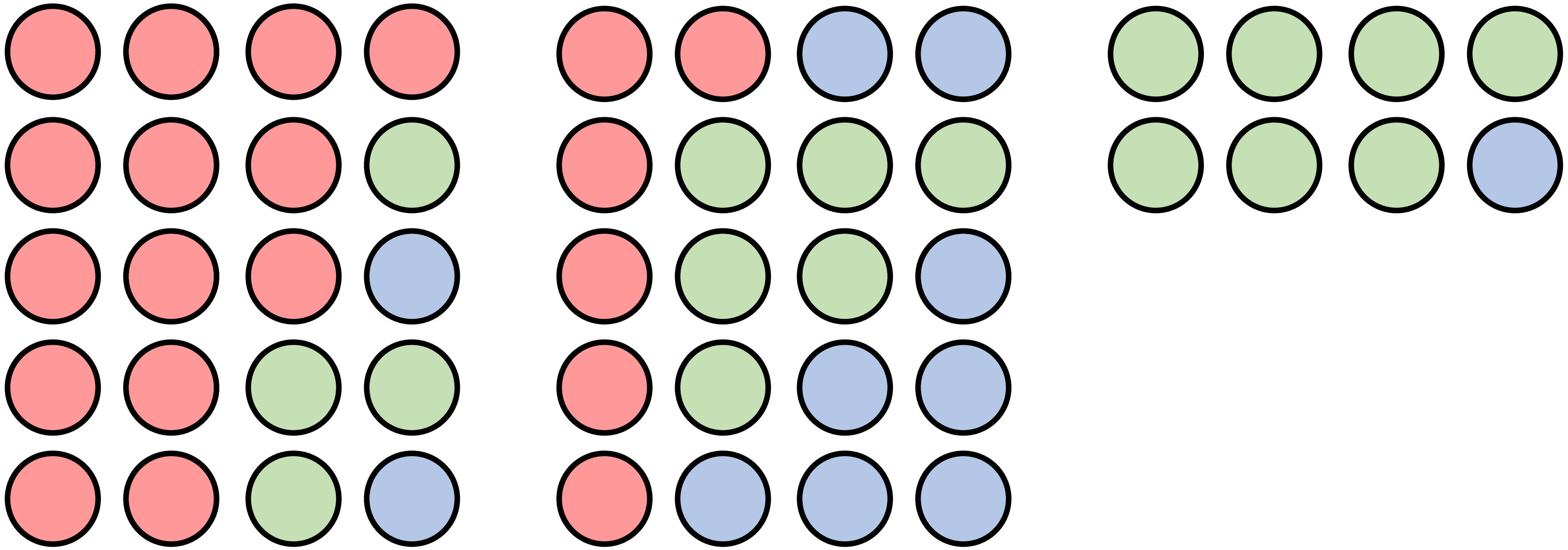

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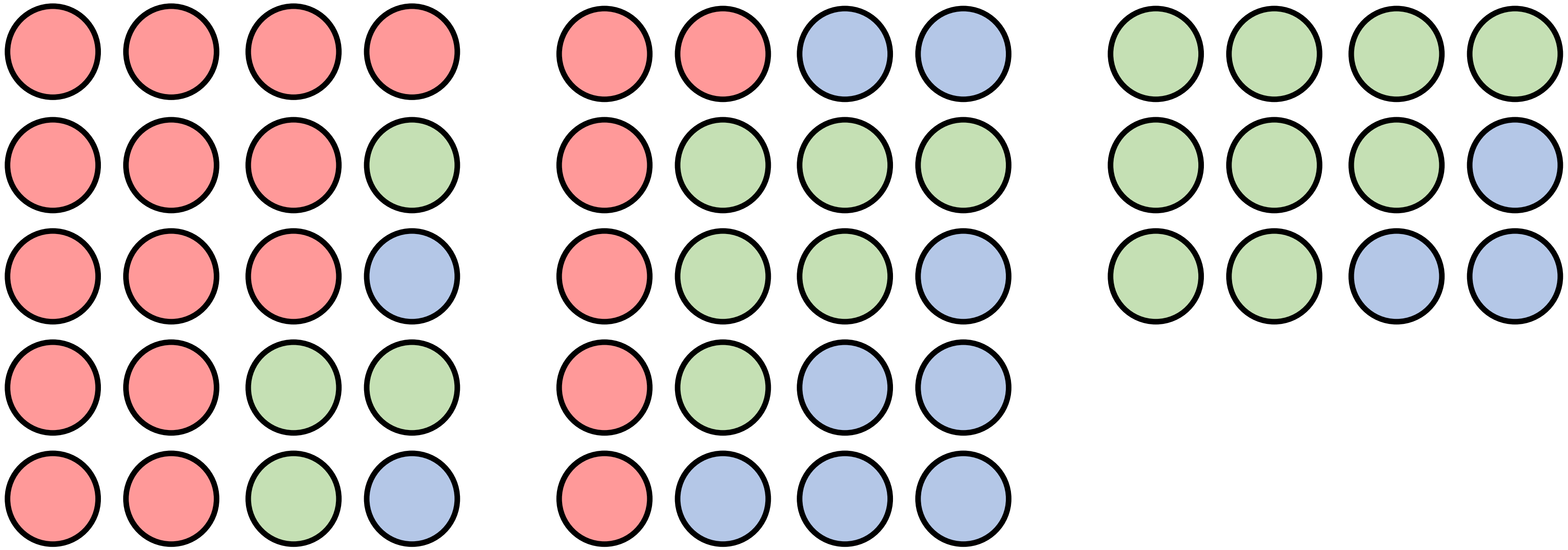

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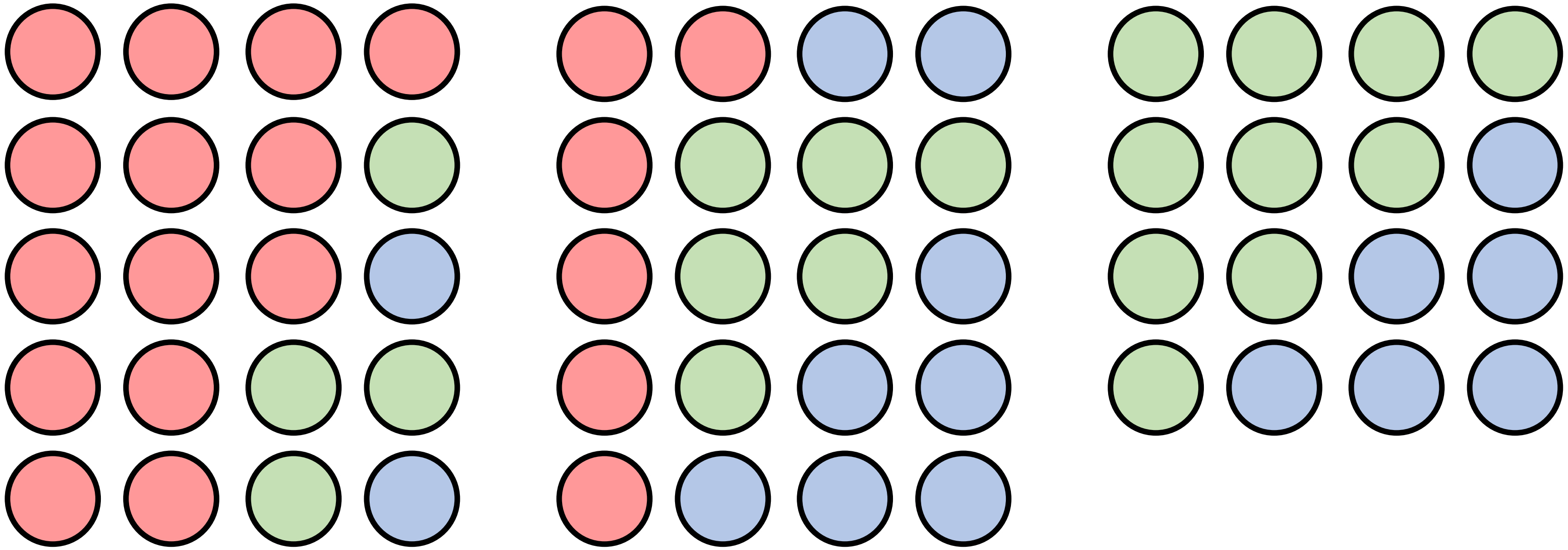

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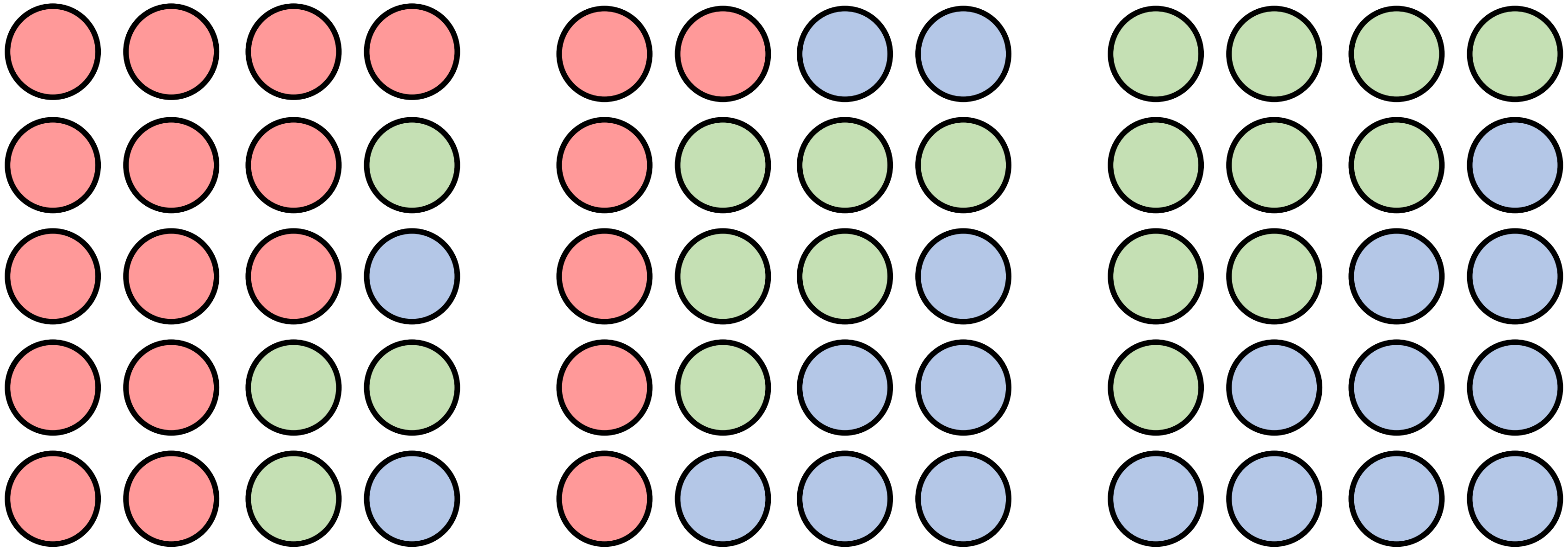

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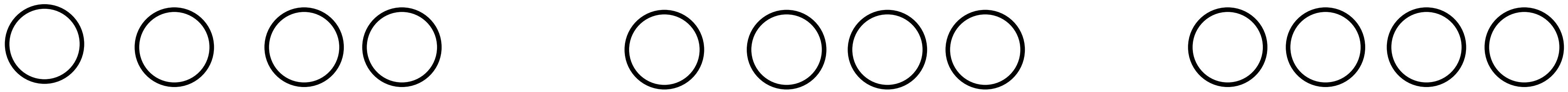

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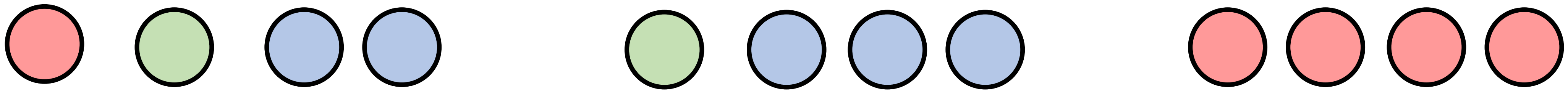

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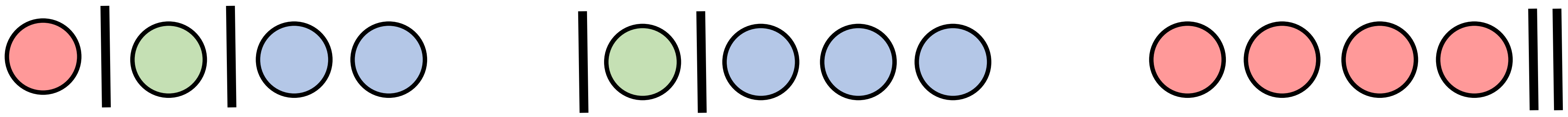

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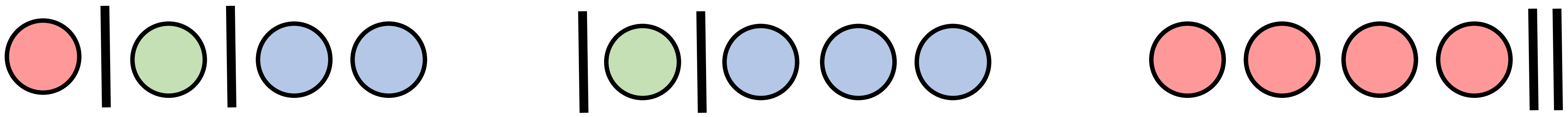
  Placing $3 - 1 = 2$ delimiters to indicate where the 3 possible colours change from one to another:



COMBINATIONS WITH REPETITIONS

- We don't even need to note the colours anymore:
  - RED before the 1st delimiter
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○|○|○○ |○|○○○   ○○○○||

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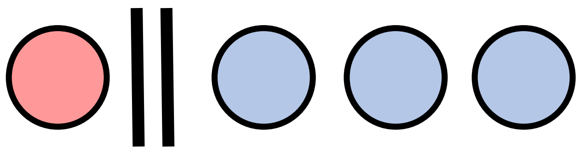

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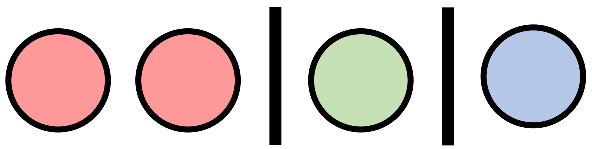

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$* \; * \; * \; * \; * \; *$

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$$* \; * \; * \; * \; * \; *$$

$$C(4 + 3 - 1, 3 - 1) =$$

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○|○|○○ |○|○○○   ○○○○||

$* * * * * *$

$$C(4+3-1, 3-1) = C(6,2) = \frac{6!}{2!\,4!} = 15$$

COMBINATIONS WITH REPETITIONS

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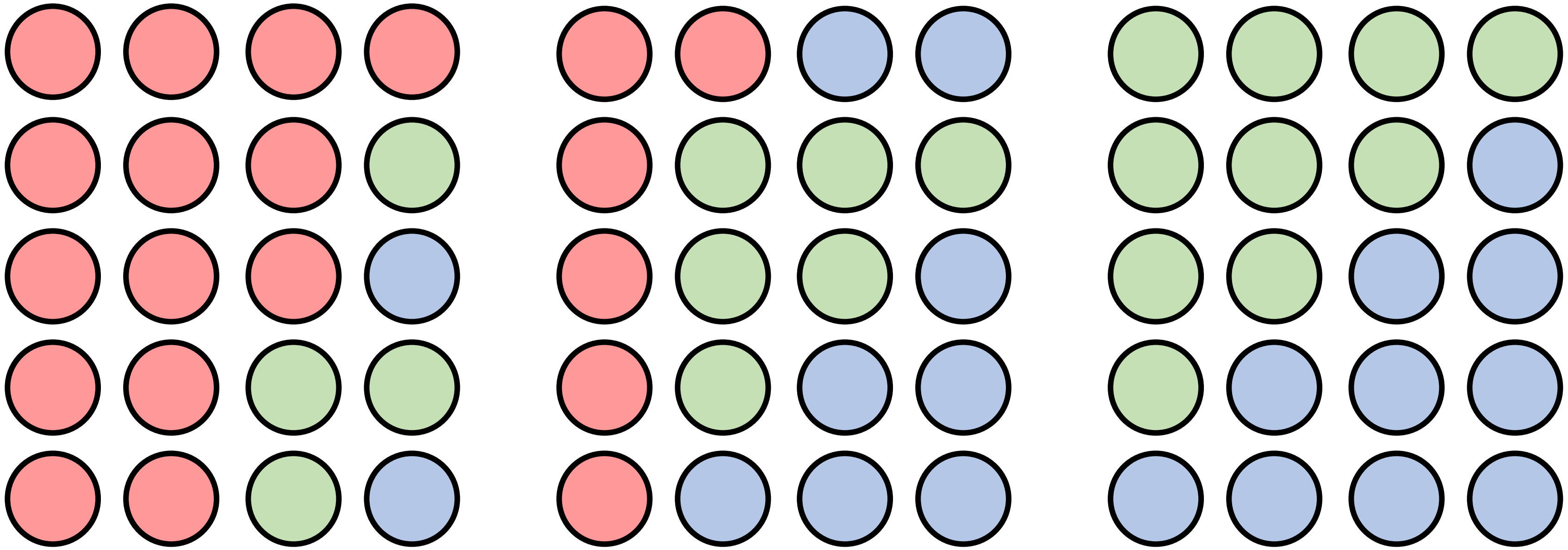

COMBINATIONS WITH REPETITIONS

- How many ways are there to place k colored balls in a bag, when each ball can be of one of the n colours?

$$C(k+n-1, n-1)$$

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- In how many ways can we place 20 books on 5 bookshelves? Each shelf can accommodate from 0 to 20 books.

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$$\circ\ \circ \mid \circ\ \circ\ \circ\ \circ \mid \circ\ \circ\ \circ\ \circ\ \circ\ \circ\ \circ\ \circ\ \circ \mid \circ\ \circ\ \circ\ \circ \mid \circ$$

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$$C(k + n - 1, n - 1) =$$

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$$C(k+n-1, n-1) = C(24,4) = \frac{24!}{4!20!} = 10626$$

EQUATIONS

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$$x_1 + x_2 + x_3 = 10, \qquad x_i \geq 0$$

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  - $x_1 = 5,\ x_2 = 2,\ x_3 = 3$
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- How is this related to combinations with repetitions?

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EQUATIONS

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$$x_1 + x_2 + x_3 = 10, \qquad x_i \geq 0$$

- 10 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1

1 1 1 1 1 1 1 1 1 1

$$C(10 + 3 - 1, 3 - 1) = C(12,2) = \frac{12!}{10!2!} = 66 \text{ solutions}$$

EQUATIONS

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- What is there are constraints:

$$x_1 \geq 1, \qquad x_2 \geq 2, \qquad x_3 \geq 3$$

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EQUATIONS

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EQUATIONS

$$y_1 + 1 + y_2 + 2 + y_3 + 3 = 10$$
$$\Longleftrightarrow$$
$$y_1 + y_2 + y_3 = 4, \qquad y_i \geq 0$$

- How many positive integer solutions are there?

EQUATIONS

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$$\Leftrightarrow$$
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EQUATIONS

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$$\Leftrightarrow$$
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- How many positive integer solutions are there?

$$C(k+n-1, n-1) = C(3+4-1, 4-1) = C(6,3) =$$

$$= \frac{6!}{3!\,3!} = 20$$

EQUATIONS

- Equation:

$$x_1 + x_2 + x_3 = 10$$

- With constraints:

$$x_1 \geq 1, \qquad x_2 \geq 2, \qquad x_3 \geq 3$$

- Has 20 positive integer solutions.

PROBLEM SET 4

https://docs.google.com/document/d/1m0Q5laEdLtz5kezjI4ZNf1o46FxUVCJCT0pjzp84C4A/edit?usp=sharing

TO SUM UP

- Imagine you have n objects. How can you arrange them?

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