INTRODUCTION TO STATISTICS

LECTURE 7

LAST TIME

- Normal distribution
    - probability of being in an interval;
    - some properties;
    - MLE for μ and σ.

LAST TIME

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  - probability of being in an interval;
  - some properties;
  - MLE for μ and σ.

- Central limit theorem – an introduction.

TODAY

- Central limit theorem
  - revision;
  - practical exercise.

TODAY

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  - revision;
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- Properties of point estimators
  - bias, variance and consistency.

TODAY

- Central limit theorem
  - revision;
  - practical exercise.
- Properties of point estimators
  - bias, variance and consistency.
- Confidence intervals

CENTRAL LIMIT THEOREM

Why normal distribution is so important

CENTRAL LIMIT THEOREM

In the video:

- Instead of measuring every single rabbit, weigh samples of size N and compute sample means.

CENTRAL LIMIT THEOREM

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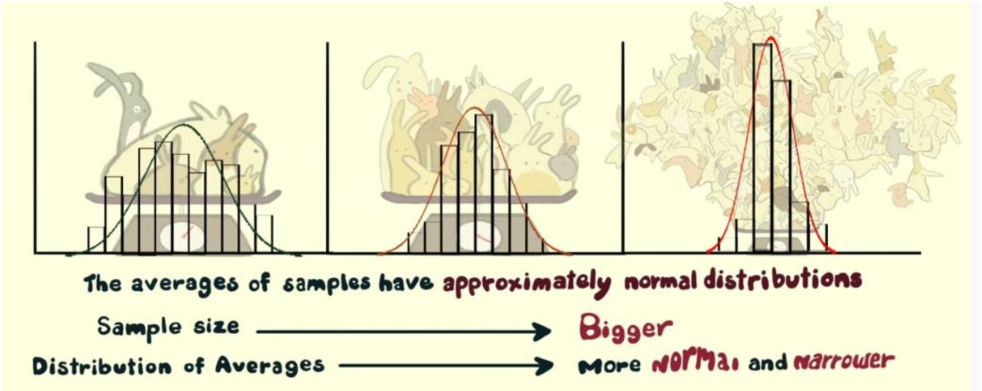
- Instead of measuring every single rabbit, weigh samples of size N and compute sample means.

- ***Central limit theorem (informally):*** the larger the N, the more "normal" the distribution of the sample averages is.



CENTRAL LIMIT THEOREM

Samples $X_1, X_2, \ldots, X_n$:

- i.i.d.
- a *finite* mean μ and *finite* variance σ^2

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Then

$$\bar{X}_n \approx N\left(\mu, \frac{\sigma^2}{n}\right)$$

CENTRAL LIMIT THEOREM

- $X \sim Po(5)$ - number of errors per computer program
  - $E(X) = Var(X) = 5$

- $X_1, X_2, \ldots, X_{125}$ - number of errors in the programs.

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$$\approx P(Z \leq 2.5) = 0.9938$$

CLT IN ACTION

Google Classroom -> Lecture 7 -> Mean of means

PROPERTIES OF ESTIMATORS

Bias, variance and consistency

PROPERTIES OF ESTIMATORS

- Parameter estimation:

  Given samples $X_1, X_2, \ldots X_n$

  Obtain estimate $\hat{\theta}$ of an unknown parameter θ.

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$$\hat{\theta}_1 = \frac{1}{n}\sum_{i=1}^{n} X_i\,, \qquad \hat{\theta}_2 = \frac{X_1 + X_n}{2}, \qquad \hat{\theta}_3 = \max(X_1, \dots, X_n)\,, \qquad \dots$$

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- We need to compare different estimators.

PROPERTIES OF ESTIMATORS

- Bias
- Variance
- Consistency

BIAS

An estimator $T(X)$ is unbiased of θ if

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Bias is defined as

$$bias\big(T(X)\big) = \theta - E\big(T(X)\big)$$

BIAS: EXAMPLE 1

$$X_1, \dots, X_n \sim Bernoulli(p)$$

$\hat{p}_{ML} = \frac{1}{n} \sum_{i=1}^{n} X_i$ − (un)biased?

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BIAS: EXAMPLE 2

$$X_1, \dots, X_n \sim N(\mu, \sigma^2)$$

$\hat{\mu}_{ML} = \frac{1}{n}\sum_{i=1}^{n} X_i$ – (un)biased?

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BIAS: EXAMPLE 3

$$X_1, \dots, X_n \sim N(\mu, \sigma^2)$$

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- How to make it **unbiased**?

$\mathrm{S}_{unbiased}^2 =$

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VARIANCE

Variance of an estimator $T(X)$ is defined as

$$Var\big(T(X)\big) = E\big\{T(X) - E\big(T(X)\big)\big\}^2$$

VARIANCE: EXAMPLE

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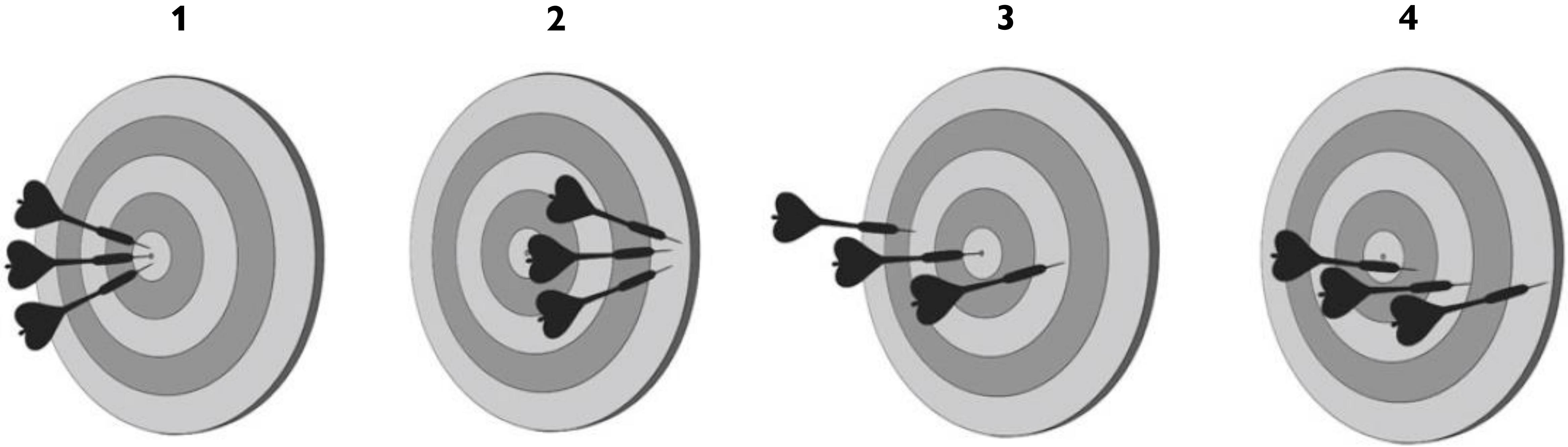
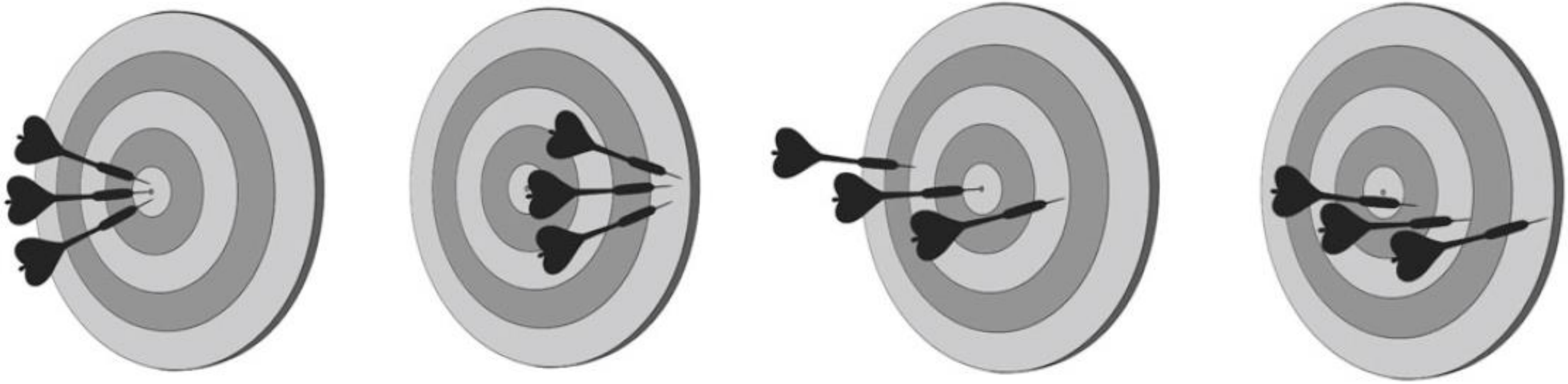
BIAS VS VARIANCE



BIAS VS VARIANCE

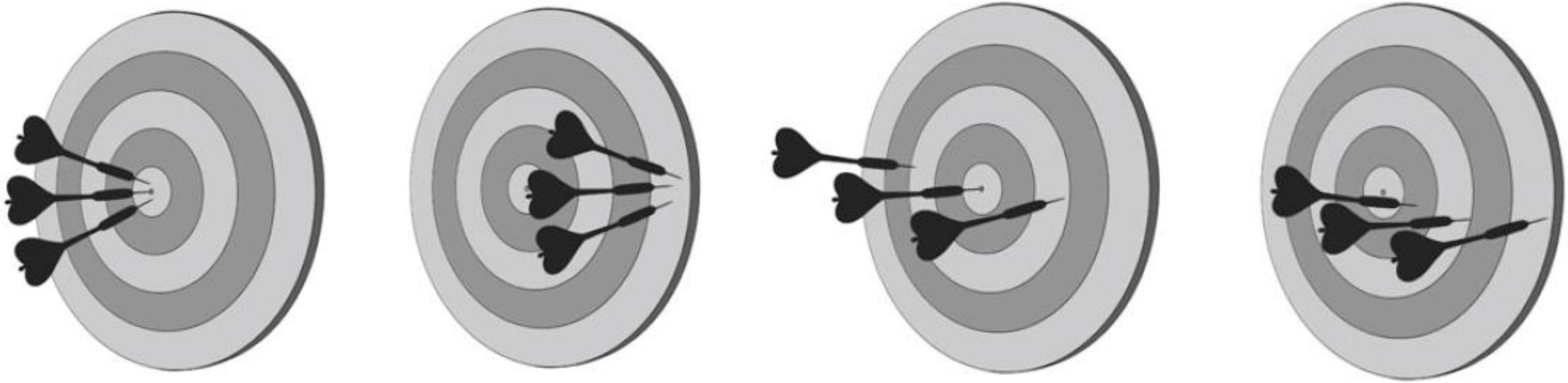
1 **2** **3** **4**



LOW BIAS
LOW VARIANCE

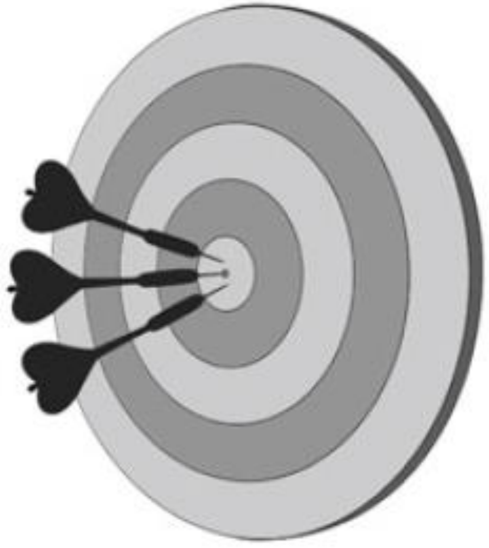
BIAS VS VARIANCE

1 2 3 4



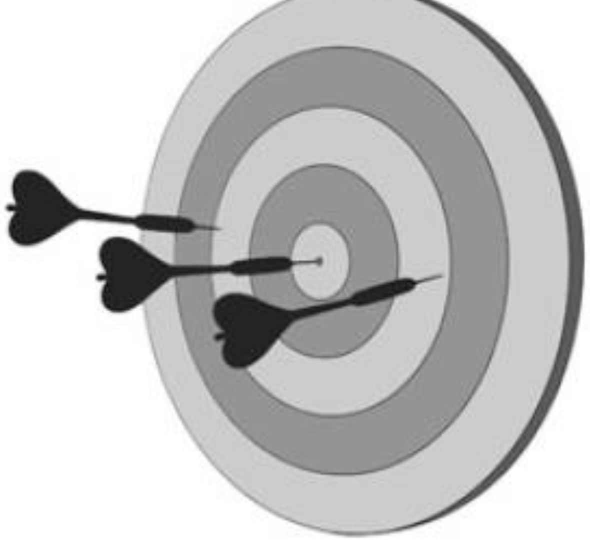
BIAS VS VARIANCE

1



2

3

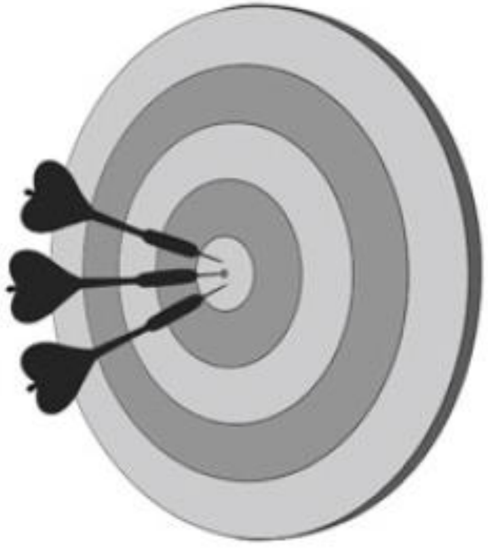


LOW BIAS
HIGH VARIANCE

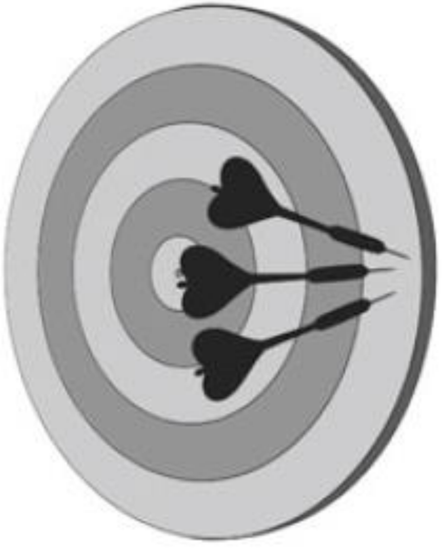
4

BIAS VS VARIANCE

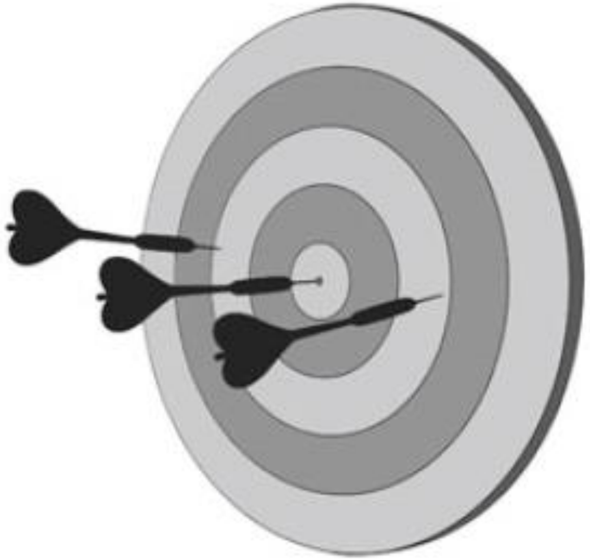
1



2



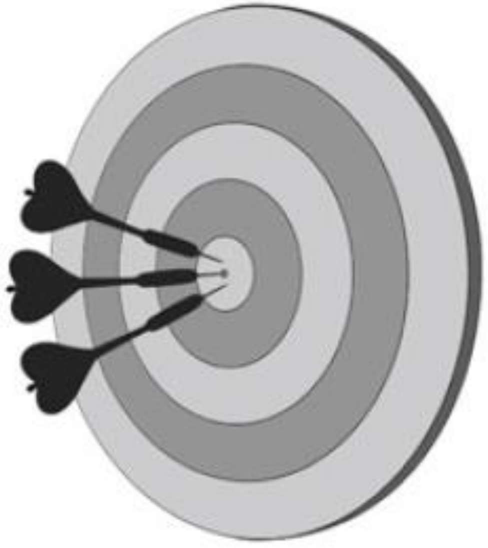
3



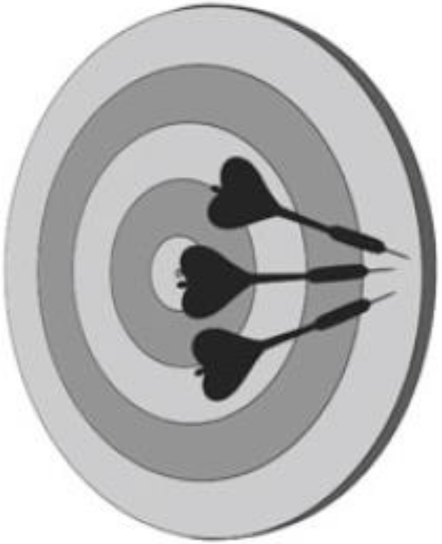
4

BIAS VS VARIANCE

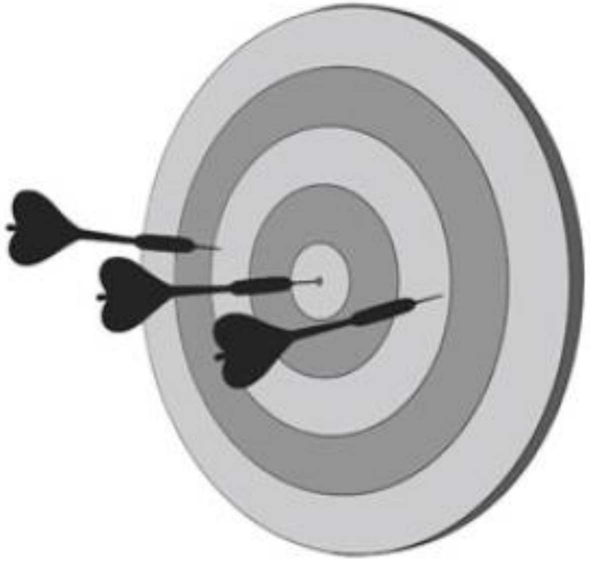
1



2



3

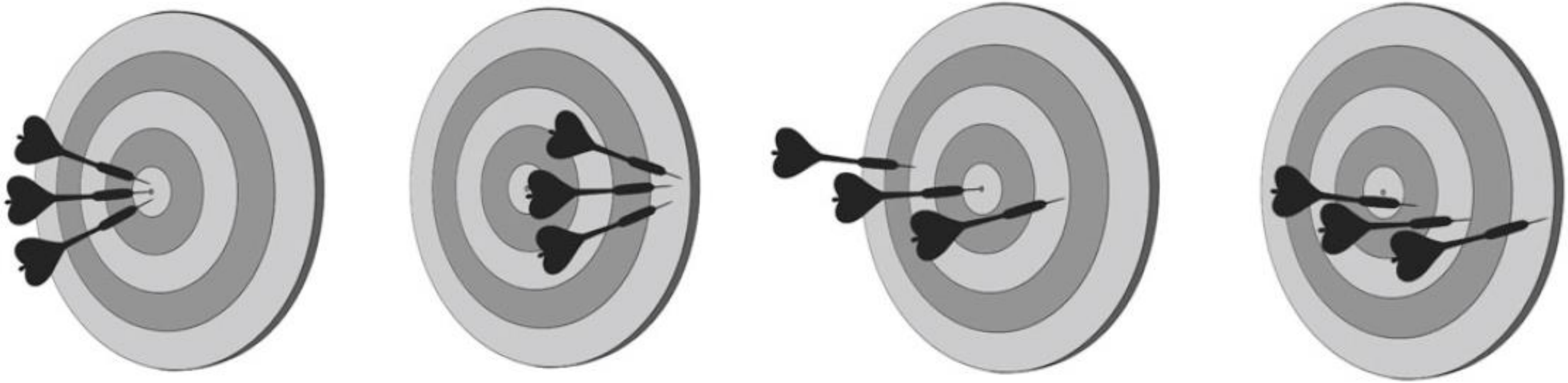


4

HIGH BIAS
LOW VARIANCE

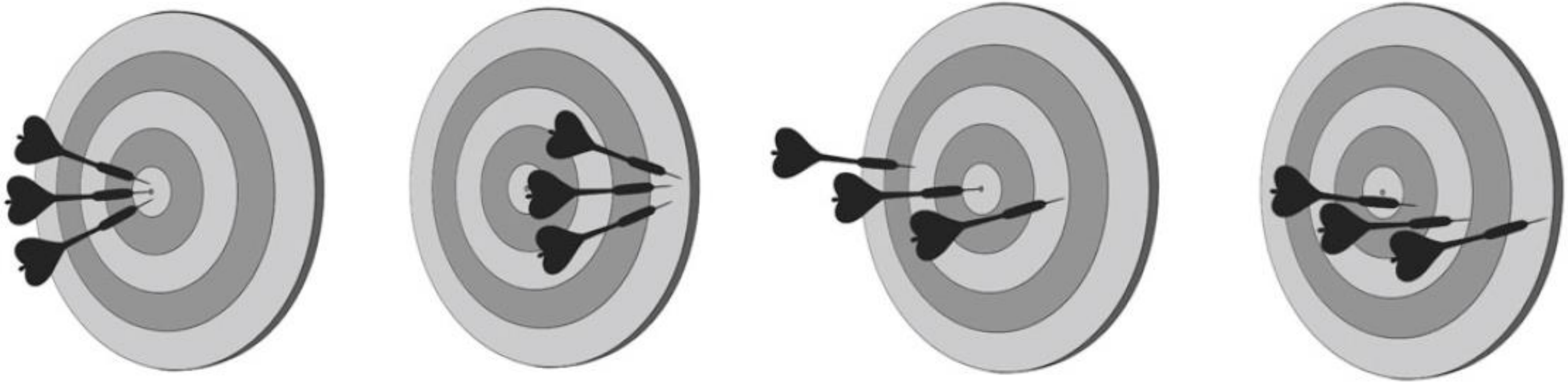
BIAS VS VARIANCE

1 2 3 4



BIAS VS VARIANCE

1 **2** **3** **4**



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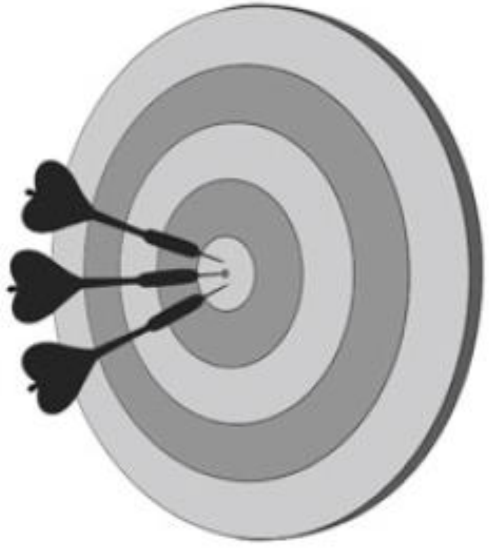
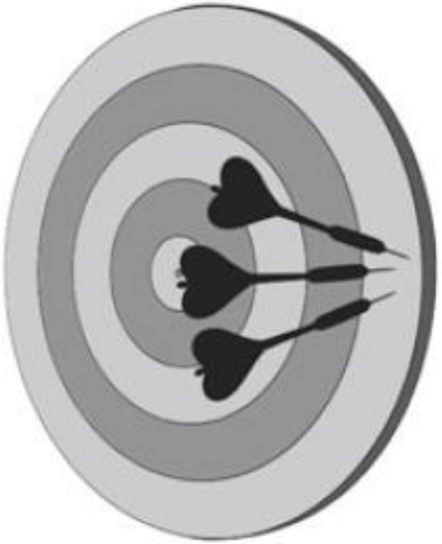


LOW BIAS
LOW VARIANCE

2



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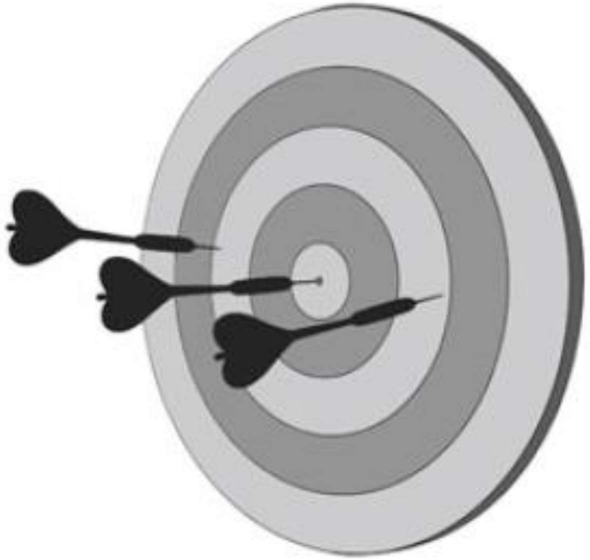

LOW BIAS
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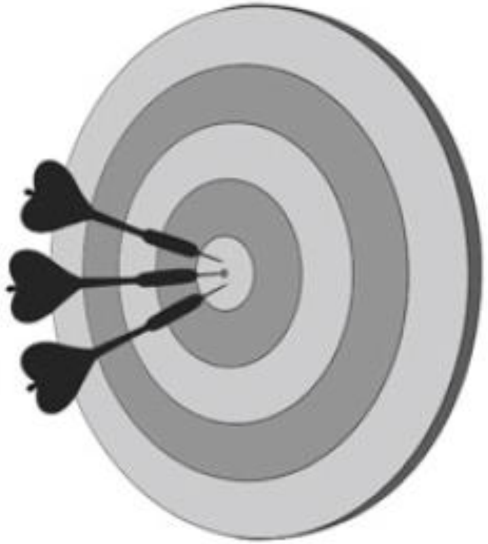

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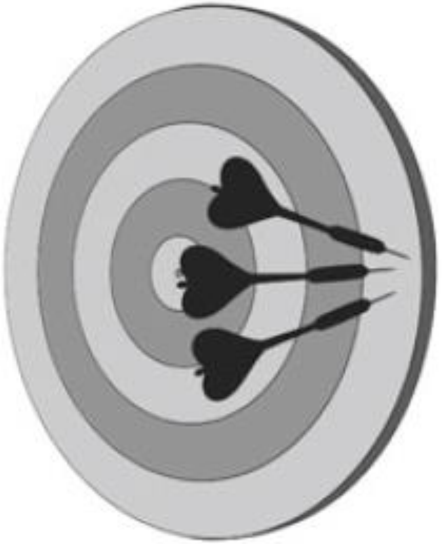

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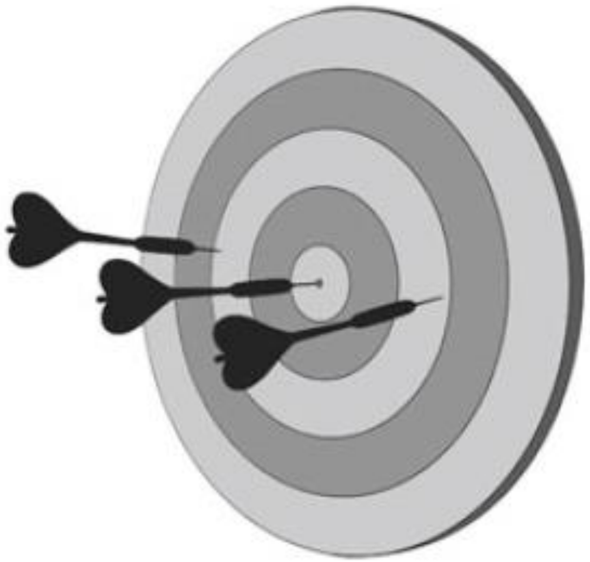

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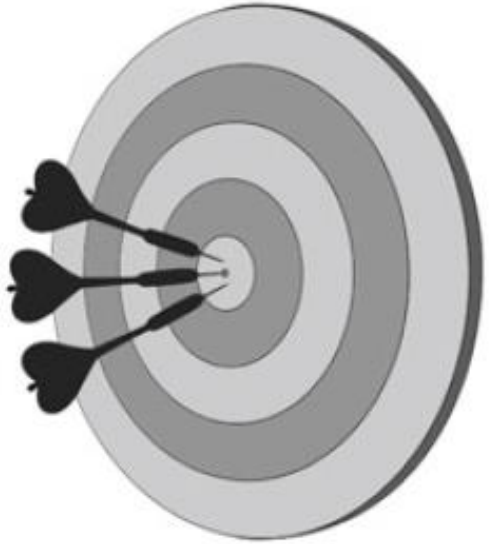

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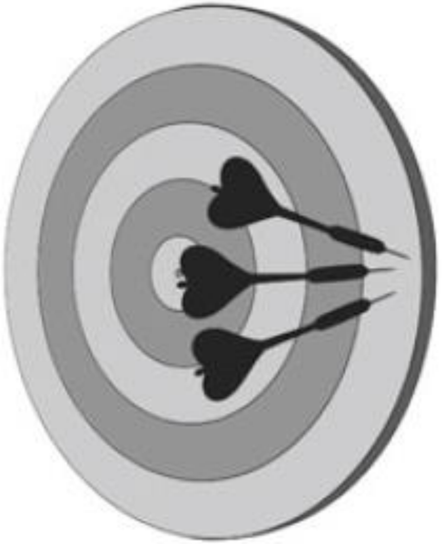

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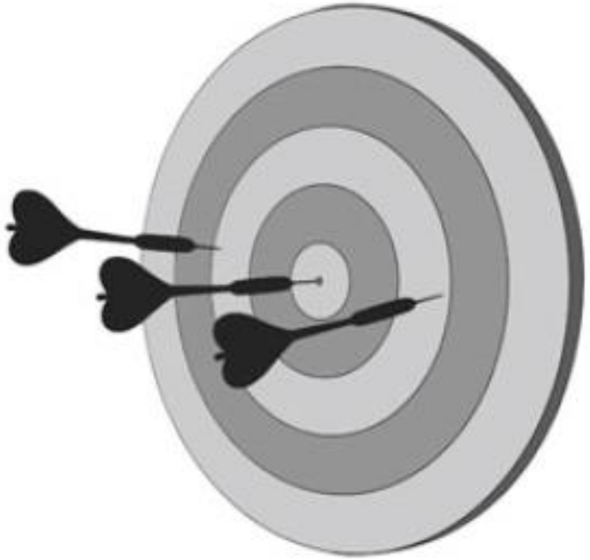


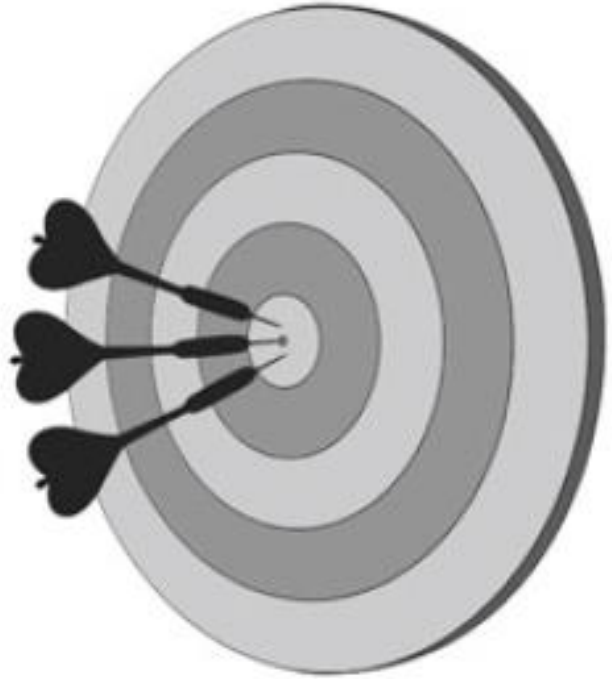
LOW BIAS
HIGH VARIANCE

4

HIGH BIAS
HIGH VARIANCE

BIAS-VARIANCE TRADE-OFF

- Impossible to simultaneously optimize bias and variance.
- Related to *under*- and *overfitting* in Machine Learning.



LOW BIAS
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CONSISTENCY

For a good estimator, as the sample size increases, the values of the estimator should get closer to the parameter being estimated.

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Let $T_1, T_2, \ldots T_n$ be a sequence of estimators for $\theta, T_k = T(X_1, \ldots, X_k)$.

Then $\{T_n\}$ is consistent if $\forall \epsilon > 0$

$$\lim_{n \to \infty} P(|T_n - \theta| < \epsilon) = 1$$

CONSISTENCY: EXAMPLE

$$\hat{\mu} = \bar{X}$$

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$$\hat{\mu} = \bar{X}, \qquad E\hat{\mu} = \mu, \qquad Var(\hat{\mu}) = \sigma^2/n$$

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Chebyshev's inequality:
$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

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3. MLE is **asymptotically efficient**:
   roughly speaking, among well-behaved estimators, it has the smallest variance, at least for large samples.

CONFIDENCE INTERVALS

MOTIVATION

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- Point estimation provides a single “best guess”.
- How confident can we be in our estimation?
- CI proposes a range of plausible values.

DEFINITION

A $1-\alpha$ confidence interval for a parameter θ is an interval $C_n = (a, b)$ such that $\mathrm{T}_1 = t_1(X_1, \dots, X_n)$, $T_2 = t_2(X_1, \dots, X_n)$ and

$$P(T_1 < \theta < T_2) \geq 1 - \alpha$$

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- **Random** intervals: T_1 and T_2 are functions of random samples.
- θ is unknown, but fixed
  T_1 and T_2 are random

INTERPRETATION

$$C_n = (T_1, T_2):\ \ P(T_1 < \theta < T_2) \geq 1 - \alpha$$

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- **Not** a probability statement about θ since it's fixed.

- Common interpretation:

If I repeat the experiment many times, the interval will contain the true value of θ 95% of the time (α=0.05).

CI FOR MEAN (VARIANCE IS KNOWN)

CI FOR μ (σ IS KNOWN)

- Suppose that $X_1, X_2, \dots, X_n$ - i.i.d. samples from a distribution with unknown mean μ and known variance σ^2.

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- How to compute a confidence interval?

CI FOR μ (σ IS KNOWN)

Central Limit Theorem:

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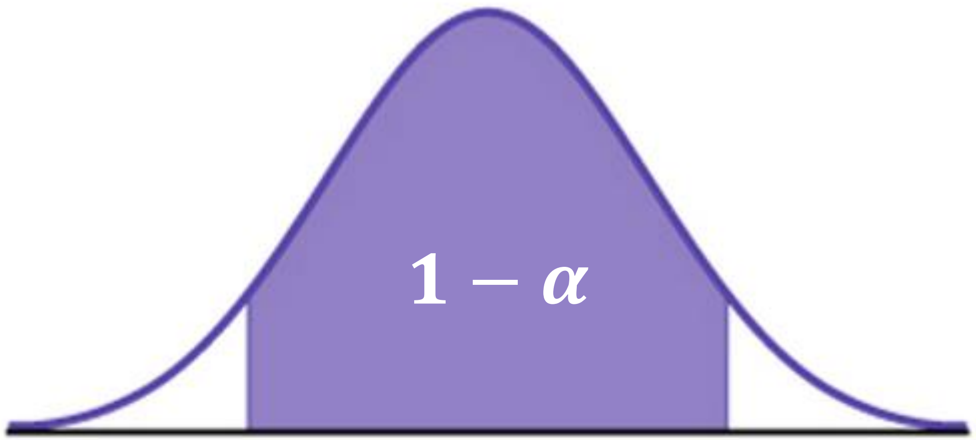

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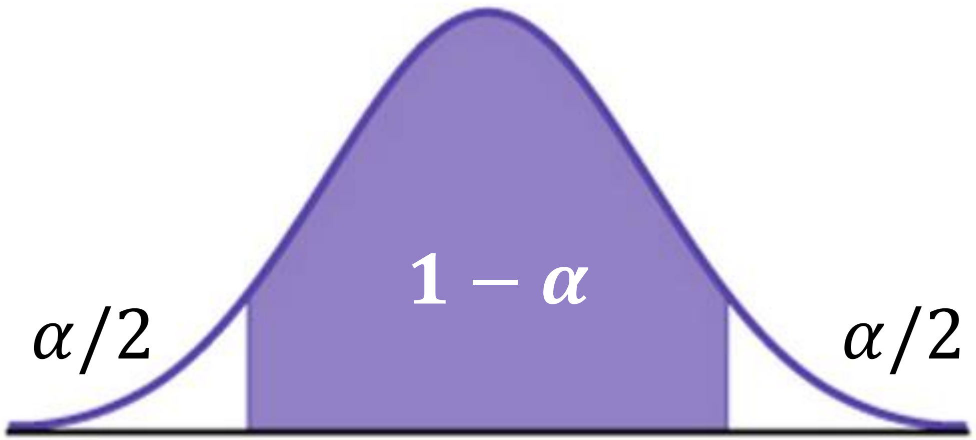

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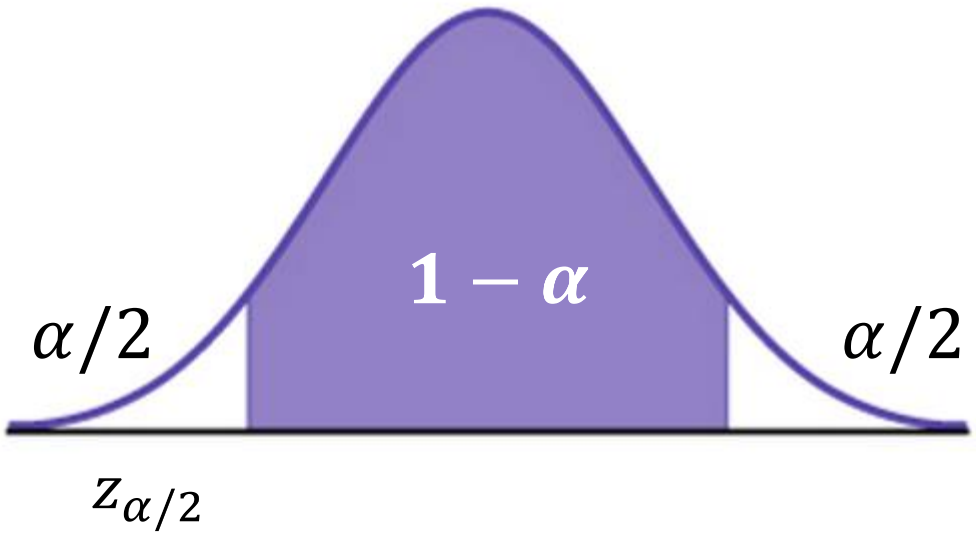

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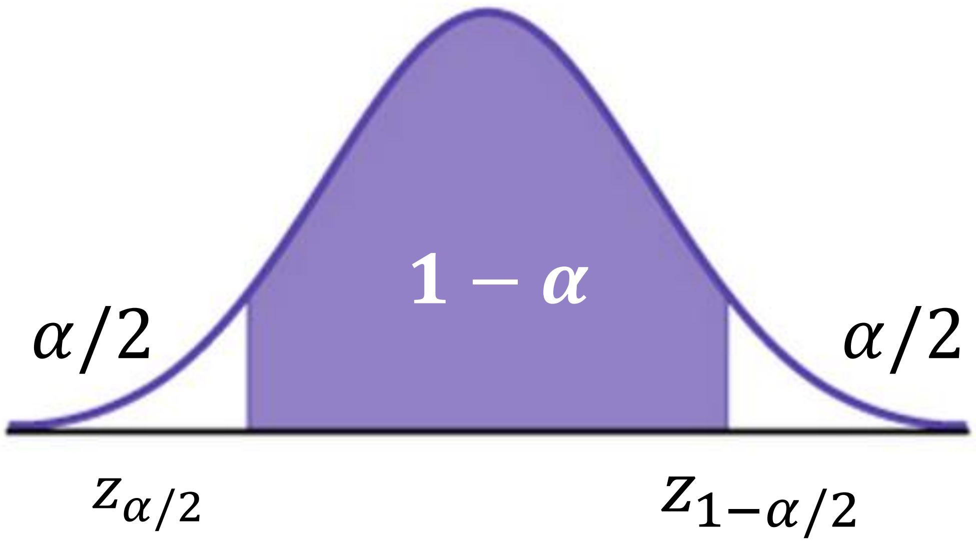

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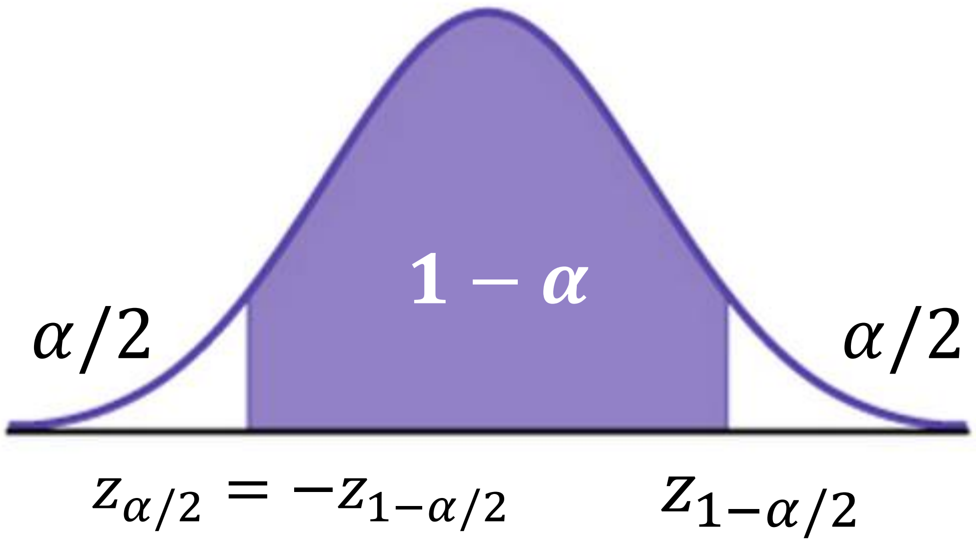

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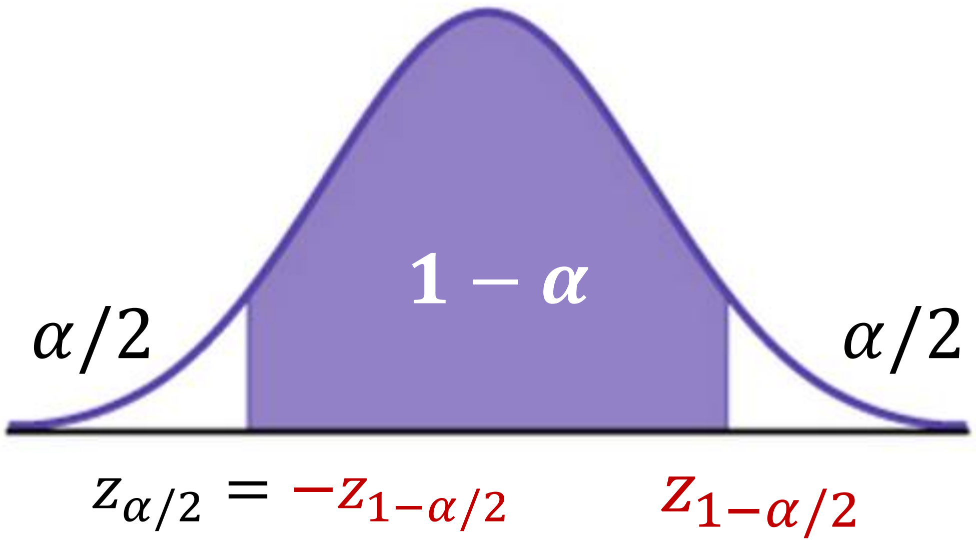

$$P\left(-z_{1-\frac{\alpha}{2}} < \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} < z_{1-\frac{\alpha}{2}}\right) = 1-\alpha$$

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CI FOR μ (σ IS KNOWN)

$$P\left(\bar{X} - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}} < \mu < \bar{X} + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right) = 1 - \alpha$$

CI FOR μ (σ IS KNOWN)

- Example:

$$n = 100, \qquad \bar{X} = 5, \qquad \sigma = 1$$

- Construct CI for μ at the level $1 - \alpha = 0.95$

$$P\left(\bar{X} - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}} < \mu < \bar{X} + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right) = 1 - \alpha$$

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$$P\left(5 - \frac{1}{10} z_{1-\frac{0.05}{2}} < \mu < 5 + \frac{1}{10} z_{1-\frac{0.05}{2}}\right) = 0.95$$

CI FOR μ (σ IS KNOWN)

Quantile (p)	$\Phi^{-1}(p, 0, 1)$
0.995	2.58
0.99	2.33
0.975	1.96
0.95	1.64
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$$P(5 - 0.196 < \mu < 5 + 0.196) = 0.95$$

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$$P(5 - 0.196 < \mu < 5 + 0.196) = 0.95$$

$$(\mathbf{5 - 0.196\,; 5 + 0.196})$$

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- Construct CI for μ at the level $1 - \alpha = 0.90$

$$P\left(\bar{X} - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}} < \mu < \bar{X} + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right) = 1 - \alpha$$

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- Construct CI for μ at the level $1 - \alpha = 0.90$

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- Example:

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- Construct CI for μ at the level $1 - \alpha = 0.90$

$$P\left(5 - \frac{1}{10} z_{0.95} < \mu < 5 + \frac{1}{10} z_{0.95}\right) = 0.90$$

$$P(5 - 0.164 < \mu < 5 + 0.164) = 0.95$$

$$(\mathbf{5 - 0.164\, ;\, 5 + 0.164})$$

TO SUM SUP

- Central Limit Theorem
- Properties of estimators
  - bias, variance, consistency;
  - properties of ML estimates.
- Confidence intervals

MID-TERM

- Tomorrow, Wednesday, December 9
- 09:00 – 12:00 (no class)
- Assignment will become available on Google Classroom
- You should submit by 12:00

Topics:

- Descriptive statistics
- Discrete distributions
- Continuous random variables (CDFs, PDFs, probabilities)
- Maximum likelihood (discrete and continuous)

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