INTRODUCTION TO STATISTICS

LECTURE 2

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  - Summarize it with
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- There is precise data! ☺

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From the data...

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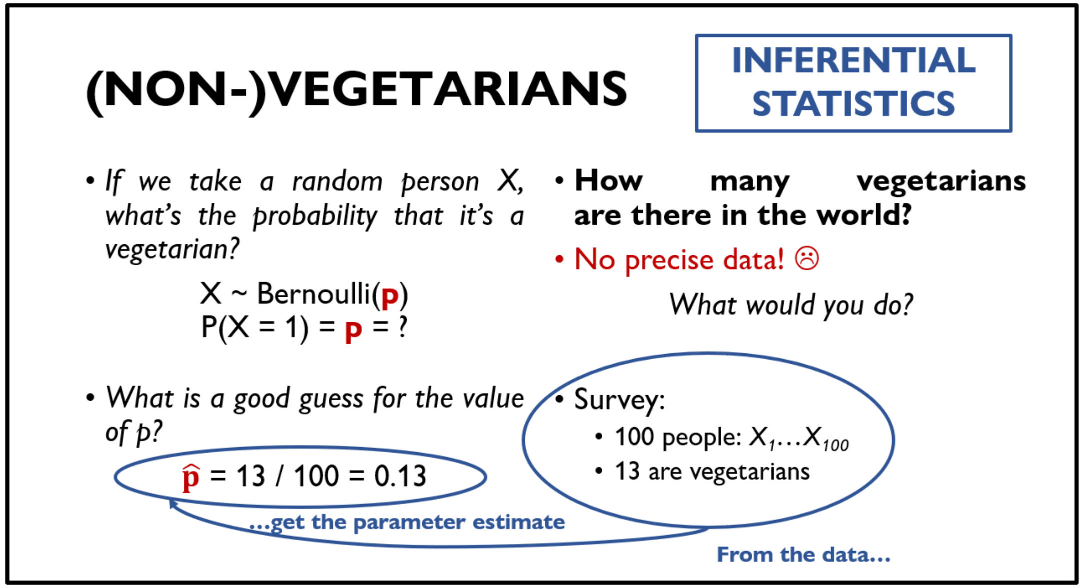

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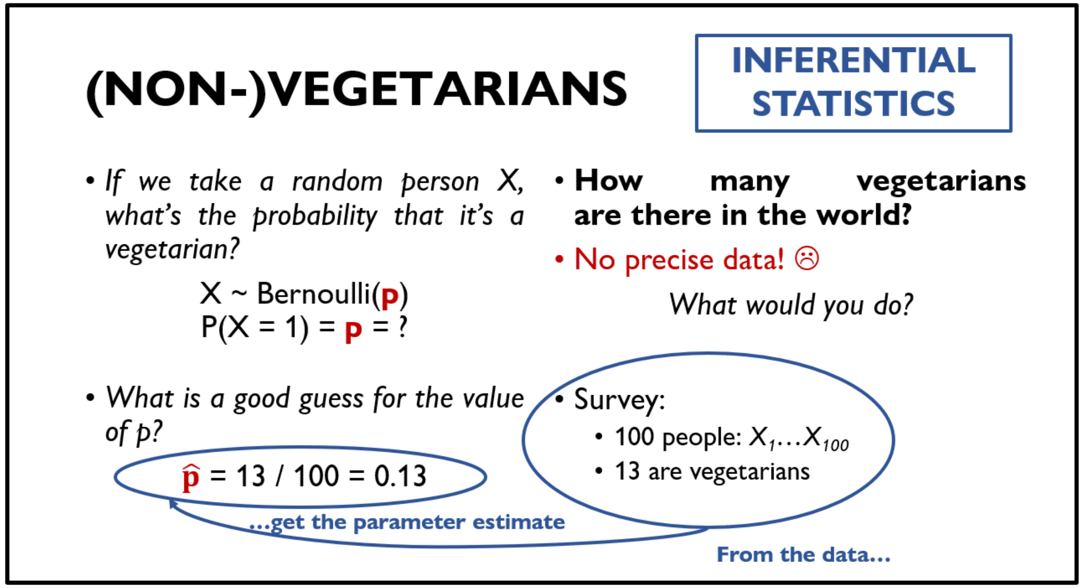

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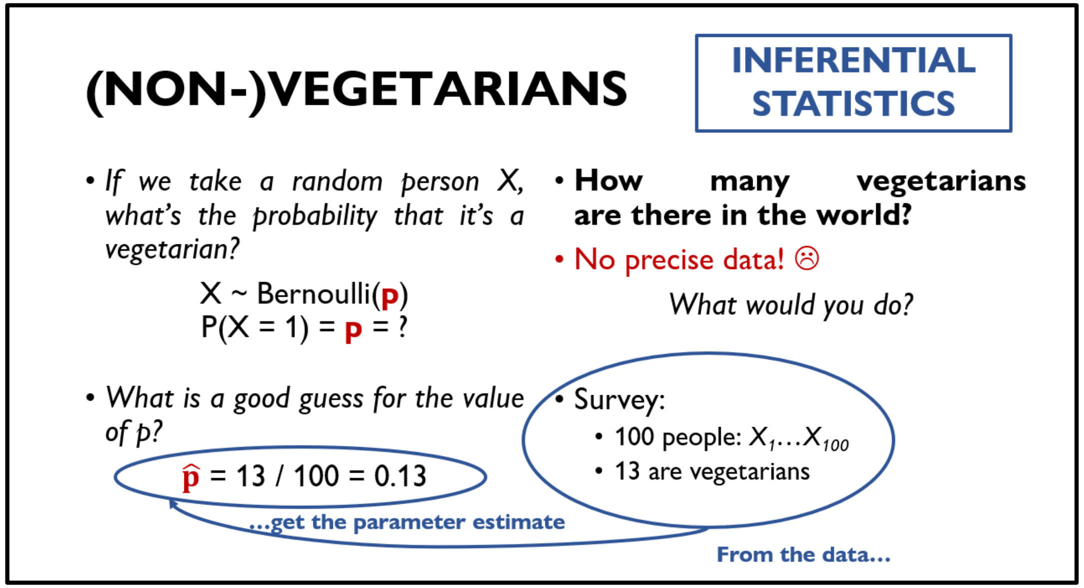

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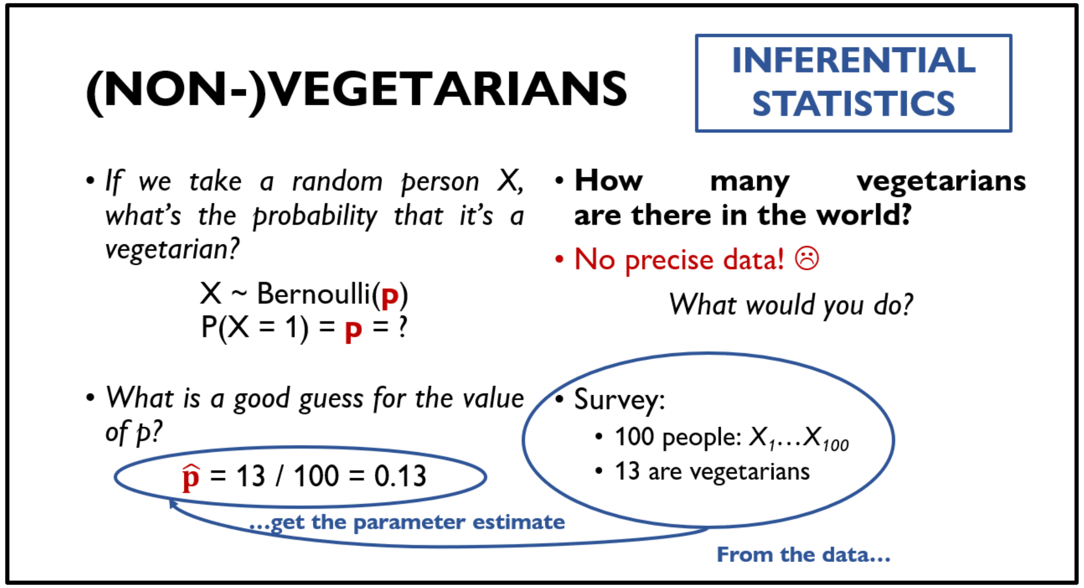

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- What is the probability to obtain such results given p? **Likelihood:**

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Likelihood function is the joint probability of realized sample given the parameters.

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  - **Maximize L(p) w.r.t. p!**

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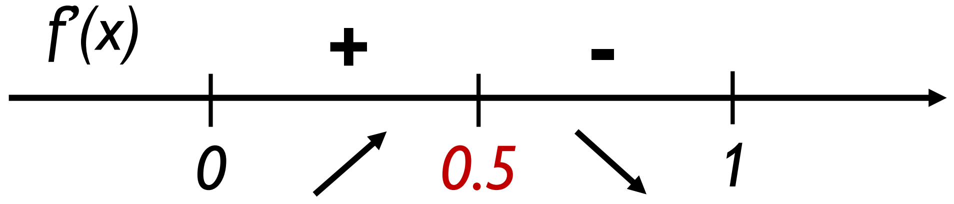

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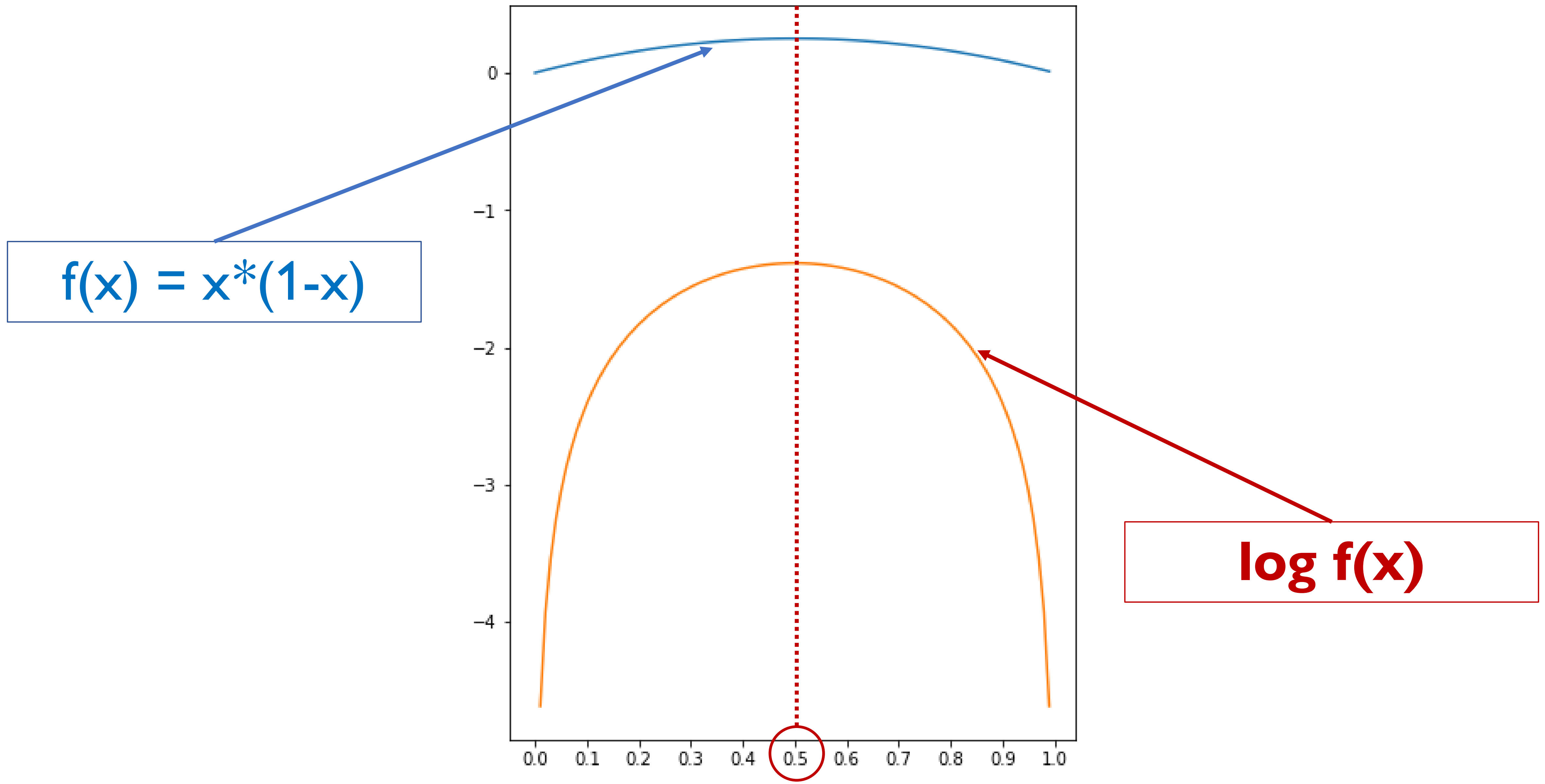

f(x) = x*(1-x)
log f(x)
0
−1
−2
−3
−4
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0

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$$\frac{\mathrm{d}}{\mathrm{dp}} \log \mathrm{L(p)} = \frac{13}{\mathrm{p}} - \frac{87}{1-\mathrm{p}} = 0$$

$$\hat{p} = \frac{13}{100}$$

MAXIMUM LIKELIHOOD ESTIMATE

1. Write down the likelihood function:

$$L(\theta) = P(X_1, \ldots, X_n \mid \theta) = \prod_{i=1}^{n} P(Xi \mid \theta)$$

2. Find its maximum w.r.t. the unknown parameter θ:

$$\widehat{\Theta} = \text{argmax } L(\theta) \quad \text{w.r.t. } \theta$$

*(!) In many cases, it's easier to maximize **log-likelihood**:*

$$\log L(\theta) = \log \prod_{i=1}^{n} P(Xi \mid \theta) = \sum_{i=1}^{n} \log P(Xi \mid \theta)$$

$$\widehat{\Theta} = \text{argmax } \log L(\theta)$$

Maximum Likelihood Estimate (MLE) is the value which …

Maximum Likelihood Estimate (MLE) is the value which maximizes the probability of observing the realized sample.

BREAK

MLE FOR SOME DISCRETE DISTRIBUTIONS

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Var(X) = p(p-1)

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- What's the MLE of p?

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$$\boxed{\hat{p} = \frac{1}{N} \sum_{i=1}^{N} X_i}$$

POISSON DISTRIBUTION

$X \sim Po(\lambda), \quad \lambda > 0$

$$P(X = k) = \begin{cases} \frac{e^{-\lambda} \cdot \lambda^k}{k!}, k \geq 0 \\ 0, \text{otherwise} \end{cases}$$

$E(X) = \lambda, \quad Var(X) = \lambda$

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- Assumptions: events occur
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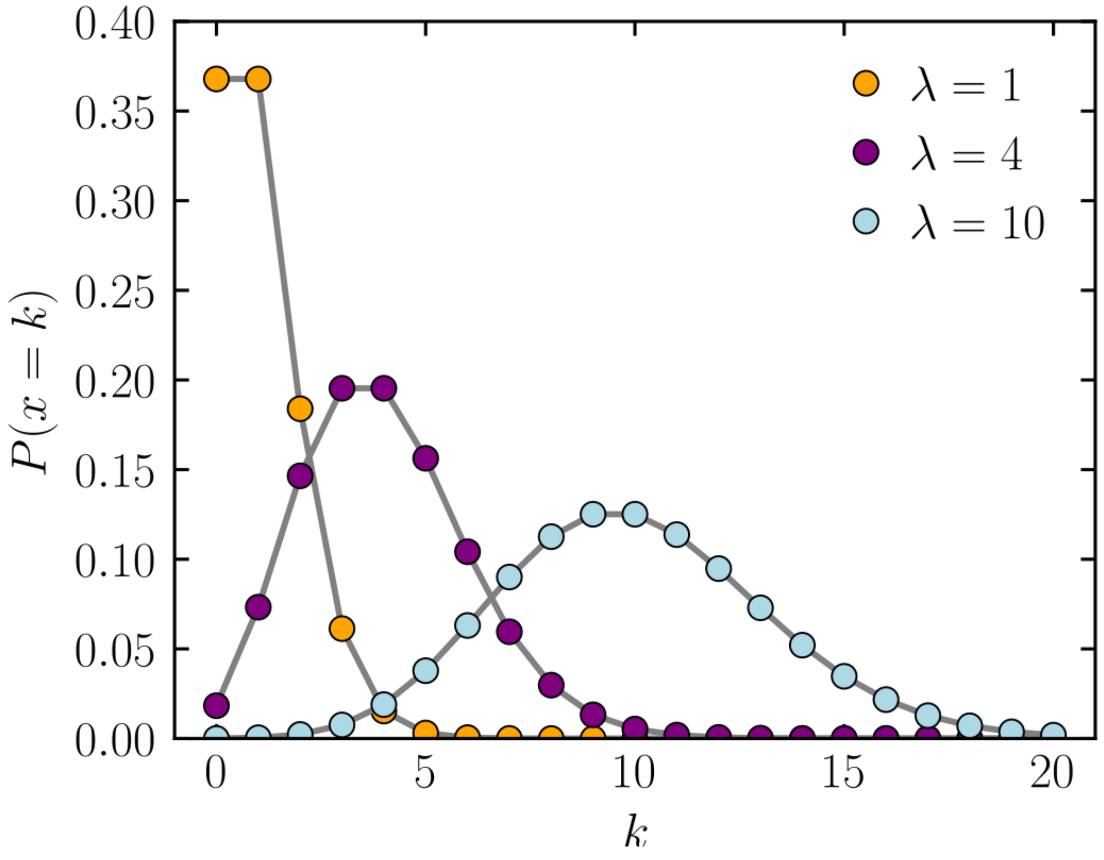

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- You've been recording it for the past *n* hours:

X1, X2, …, Xn

POISSON DISTRIBUTION

$X \sim Po(\lambda), \quad \lambda > 0$

$$P(X = k) = \begin{cases} \frac{e^{-\lambda} \cdot \lambda^k}{k!}, k \geq 0 \\ 0, \text{otherwise} \end{cases}$$

$E(X) = \lambda, \quad Var(X) = \lambda$

- Models number of events occurring in a fixed interval of time.
- Assumptions: events occur
  - with a known constant mean rate;
  - independently of the time since the last event.

- Suppose you want to model the number of phone calls a call center receives in an hour.
- You've been recording it for the past *n* hours:

X1, X2, …, Xn

- What's the MLE of λ?

MLE FOR POISSON DISTRIBUTION

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$$L(\lambda) = P(X_1, \dots, X_n | \lambda) =$$

MLE FOR POISSON DISTRIBUTION

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MLE FOR POISSON DISTRIBUTION

$$\mathrm{L}(\lambda) = P(X_1, \ldots, X_2|\lambda) = \prod_{\mathrm{i}=1}^{\mathrm{n}} \frac{\mathrm{e}^{-\lambda}\lambda^{\mathrm{X_i}}}{\mathrm{X_i}!}$$

$$\log \mathrm{L}(\lambda) = \sum_{\mathrm{i}=1}^{\mathrm{n}} [-\lambda \log \mathrm{e} + \mathrm{X}_1 \log \lambda - \log \mathrm{X_i}!]$$

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MLE: YET ANOTHER EXAMPLE

- Random variable X takes three values with unknown probabilities:

X	1	2	3
P(X)	?	?	?

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- What is the MLE of the parameters?

CALCULUS 101

- **How to optimize a function of several variables?**

CALCULUS 101

- **How to optimize a function of several variables?**

1. Compute partial derivatives.
2. Set them to zero.

3. Solve the equations and get the critical points.

CALCULUS 101

- **How to optimize a function of several variables?**

$$f(x,y) = x^2 + 2xy - 2x - 4y$$

1. Compute partial derivatives.
2. Set them to zero.

$$\frac{d}{dx} f(x,y) = 2x + 2y - 2 = 0$$

$$\frac{d}{dy} f(x,y) = 2x - 4 = 0$$

3. Solve the equations and get the critical points.

$$x^* = 2, \qquad y^* = 1$$

MLE: YET ANOTHER EXAMPLE

X	1	2	3
P(X)	p	q	1-p-q

Value	1	2	3
#	n_1	n_2	n_3

- What's the probability of observing such data? Likelihood:

$$L(p, q) =$$

$$\Leftrightarrow$$

MLE: YET ANOTHER EXAMPLE

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P(X)	p	q	1-p-q

Value	1	2	3
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- What's the probability of observing such data? Likelihood:

$$L(p, q) = p^{n_1} \cdot q^{n_2} \cdot (1 - p - q)^{n_3}$$

maximize L(p, q) w.r.t. p, q

$\Leftrightarrow$

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maximize L(p, q) w.r.t. p, q

⇔

$$\log L(p, q) = n_1 \log p + n_2 \log q + n_3 \log(1 - p - q)$$

maximize log L(p, q) w.r.t. p, q

MLE: EXAMPLE 2

X	1	2	3
P(X)	p	q	1-p-q

Value	1	2	3
#	n_1	n_2	n_3

maximize $\mathrm{n}_1 \log \mathrm{p} + \mathrm{n}_2 \log \mathrm{q} + \mathrm{n}_3 \log(1 - \mathrm{p} - \mathrm{q})$ w.r.t. p, q

$$\frac{d}{dp} \log L(p,q) = \frac{n_1}{p} - \frac{n_3}{1-p-q} = 0$$

$$\frac{d}{dq} \log L(p,q) = \frac{n_2}{q} - \frac{n_3}{1-p-q} = 0$$

MLE: EXAMPLE 2

X	1	2	3
P(X)	p	q	1-p-q

Value	1	2	3
#	n_1	n_2	n_3

maximize $\mathrm{n}_1 \log \mathrm{p} + \mathrm{n}_2 \log \mathrm{q} + \mathrm{n}_3 \log(1 - \mathrm{p} - \mathrm{q})$ w.r.t. p, q

$$\frac{d}{dp} \log L(p,q) = \frac{n_1}{p} - \frac{n_3}{1-p-q} = 0$$

$$\frac{d}{dq} \log L(p,q) = \frac{n_2}{q} - \frac{n_3}{1-p-q} = 0$$

$$\Rightarrow$$

$$\hat{p} = \frac{n_1}{N}$$

$$\hat{q} = \frac{n_2}{N}$$

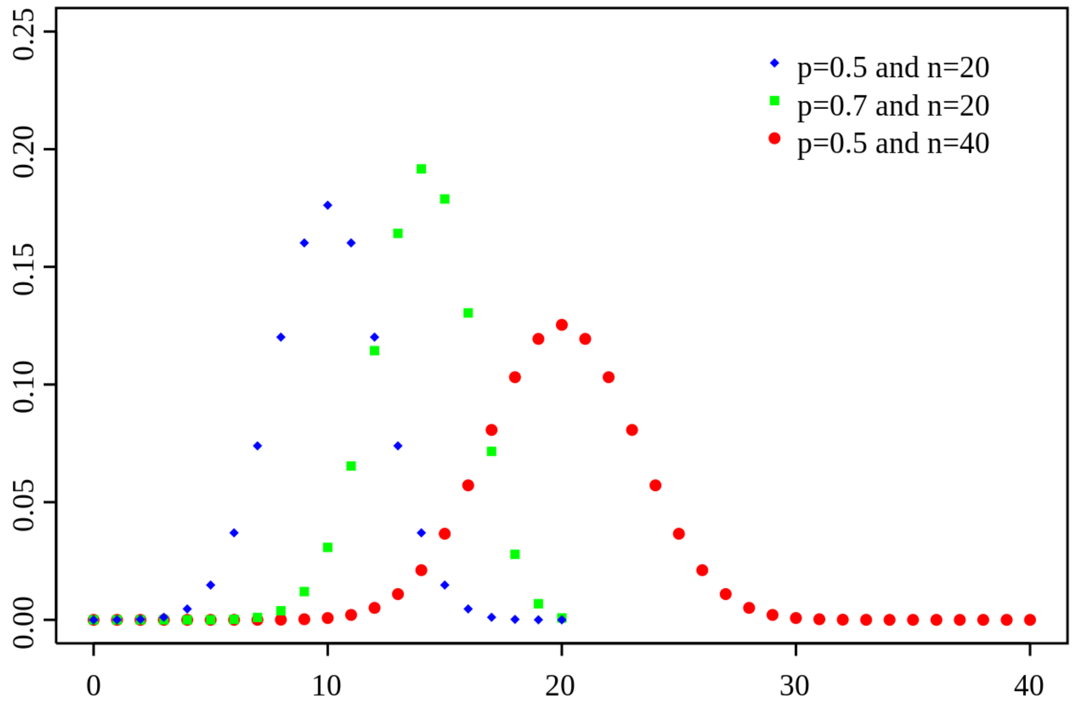
BINOMIAL DISTRIBUTION

$X \sim Bi(n, p), \;\; n = 1, 2, \ldots, \;\; 0 < p < 1$

$P(X = k) = C_n^k \, p^k (1 - p)^{n-k}$

$E(X) = np, \quad Var(X) = np(1-p)$

- Models the number of successes in a series of n independent Bernoulli trials, each of which has a success probability p.



BINOMIAL DISTRIBUTION

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$P(X = k) = C_n^k \, p^k (1 - p)^{n-k}$

$E(X) = np, \quad Var(X) = np(1-p)$

- Models the number of successes in a series of *n* independent Bernoulli trials, each of which has a success probability *p*.

- You sell two types of sandwiches: chicken and vegetarian. Which one people like more?

- For the past N days, you were selling n=100 sandwiches every day and recorded the number of the chicken ones:

$$X_1, X_2, \ldots, X_N$$

- What is the MLE of the p parameter?

MLE FOR BINOMIAL DISTRIBUTION

$$\mathrm{L}(\mathrm{p}) =$$

MLE FOR BINOMIAL DISTRIBUTION

$$L(p) = \prod_{i=1}^{N} C_n^{X_i} p^{X_i} (1 - p)^{100 - X_i}$$

MLE FOR BINOMIAL DISTRIBUTION

$$L(p) = \prod_{i=1}^{N} C_n^{X_i} p^{X_i} (1 - p)^{100 - X_i}$$

$$\log L(p) =$$

MLE FOR BINOMIAL DISTRIBUTION

$$\mathrm{L(p)} = \prod_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{C}_{\mathrm{n}}^{\mathrm{X_i}} \mathrm{p}^{\mathrm{X_i}} (1 - \mathrm{p}\ )^{100 - \mathrm{X_i}}$$

$$\log \mathrm{L(p)} = \sum_{\mathrm{i=1}}^{\mathrm{N}} \left[\log \mathrm{C}_{\mathrm{n}}^{\mathrm{X_i}} + \mathrm{X_i} \log \mathrm{p} + (100 - \mathrm{X_i}) \log(1 - \mathrm{p})\right]$$

MLE FOR BINOMIAL DISTRIBUTION

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$$\frac{d}{dp} L(p) =$$

MLE FOR BINOMIAL DISTRIBUTION

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$$\frac{d}{dp} L(p) = \sum_{i=1}^{N} \frac{X_i}{p} - \frac{100 - X_i}{1 - p} = \qquad\qquad = 0$$

MLE FOR BINOMIAL DISTRIBUTION

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$$\frac{d}{dp} L(p) = \sum_{i=1}^{N} \frac{X_i}{p} - \frac{100 - X_i}{1 - p} = \frac{\sum_{i=1}^{N} X_i}{p} - \frac{100N - \sum_{i=1}^{N} X_i}{1 - p} = 0$$

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$$\hat{p} =$$

MLE FOR BINOMIAL DISTRIBUTION

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$$\hat{p} = \frac{\sum_{i=1}^{N} X_i}{100N}$$

MLE FOR BINOMIAL DISTRIBUTION

$$L(p) = \prod_{i=1}^{N} C_n^{X_i} p^{X_i} (1-p)^{100-X_i}$$

$$\log L(p) = \sum_{i=1}^{N} \left[\log C_n^{X_i} + X_i \log p + (100 - X_i) \log(1-p)\right]$$

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$$\boxed{\hat{p} = \frac{\sum_{i=1}^{N} X_i}{100N}}$$

BREAK

Task: find a coin ☺

RANDOMIZED RESPONSE

Asking embarrassing questions

MOTIVATION

- BEFORE:
    - How many vegetarians are in the world?
    - Survey: ask "*are you a vegetarian?*", estimate the true proportion.

MOTIVATION

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  - How many vegetarians are in the world?
  - Survey: ask "*are you a vegetarian?*", estimate the true proportion.
- BUT WHAT IF the question is very sensitive?
  - People won't tell the truth.

MOTIVATION

- BEFORE:
  - How many vegetarians are in the world?
  - Survey: ask *"are you a vegetarian?"*, estimate the true proportion.
- BUT WHAT IF the question is very sensitive?
  - People won't tell the truth.
- EXAMPLE: *Do you find this course boring?*
  - How do I find out what my students *actually* think?

STRATEGY: RANDOMIZED RESPONSE

- Watch the video about the randomized response strategy.
- In groups, discuss the strategy and complete the assignment.
- See Google Classroom.

LET’S TRY THIS OUT!

- Toss a coin…

LET'S TRY THIS OUT!

- Toss a coin…
- Now, answer one of the following questions:

LET’S TRY THIS OUT!

- Toss a coin…
- Now, answer one of the following questions:

If you got HEADS:	DO YOU FIND THIS CLASS BORING?
If you got TAILS:	ARE YOU AT THE STATISTICS CLASS RIGHT NOW?

TO SUM UP

- **Likelihood function** is the joint probability of realized sample given the parameters.

- **Maximum Likelihood Estimate (MLE)** is the value which maximizes the probability of observing the realized sample.