PROBABILITY & STATISTICS

Lecture 9 – Intro to Statistics, Parameter estimation

STATISTICS

LET’S STRAT!

WHAT IS STATISTICS?

- How do *you* imagine Statistics?

WHAT IS STATISTICS?

- How do *you* imagine Statistics?
  - Can you think of any example of Statistics?

WE'RE ALL FOLLOWING THESE STATISTICS…

Coronavirus Statistics

Global

59229918 Infected

1397583 2.4% Deaths

40976963 69.2% Recovered

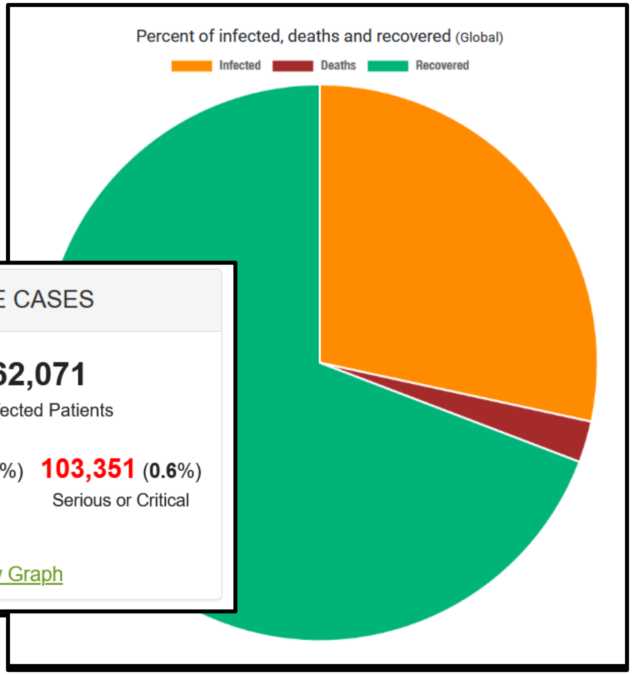
Last update: 23.11.20 17:03 GMT

Show 10 countries Search:

Country	Infected	Deaths	Recovered	Deaths Percent	Recovered Percent	Infected per million	Deaths per million
USA	12603019	262781	7453682	2.1%	59.1%	37987	792
India	9170825	134088	8592303	1.5%	93.7%	6620	97
Brazil	6073058	169213	5432505	2.8%	89.5%	28491	794
France	2140208	48732	149521	2.3%	7.0%	32760	746
Russia	2114502	36540	1611445	1.7%	76.2%	14487	250
Spain	1589219	42619	N/A	2.7%	0.0%	33985	911
UK	1527495	55230	N/A	3.6%	0.0%	22454	812
Italy	1431795	50453	584493	3.5%	40.8%	23695	835
Argentina	1370366	37002	1195492	2.7%	87.2%	30212	816
Colombia	1248417	35287	1150932	2.8%	92.2%	24433	691

Showing 1 to 10 of 209 countries

Previous 1 2 3 4 5 … 21 Next



ACTIVE CASES

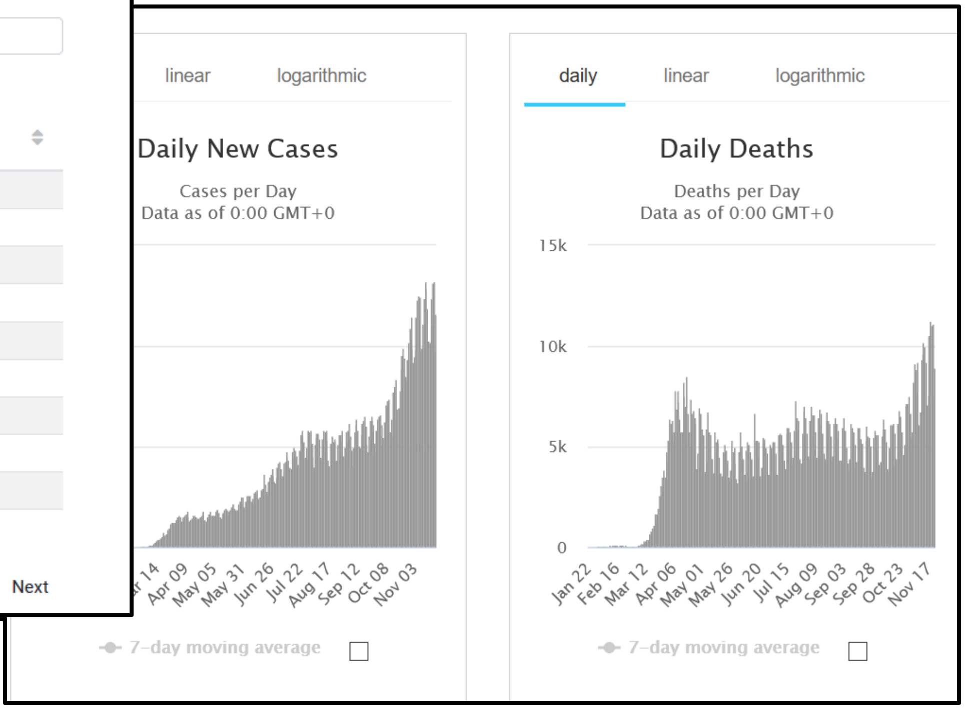
16,862,071

Currently Infected Patients

16,758,720 (99.4%) in Mild Condition

103,351 (0.6%) Serious or Critical

Show Graph



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  - Can you think of any example of Statistics?
  - What is it?
  - Why do we need it?

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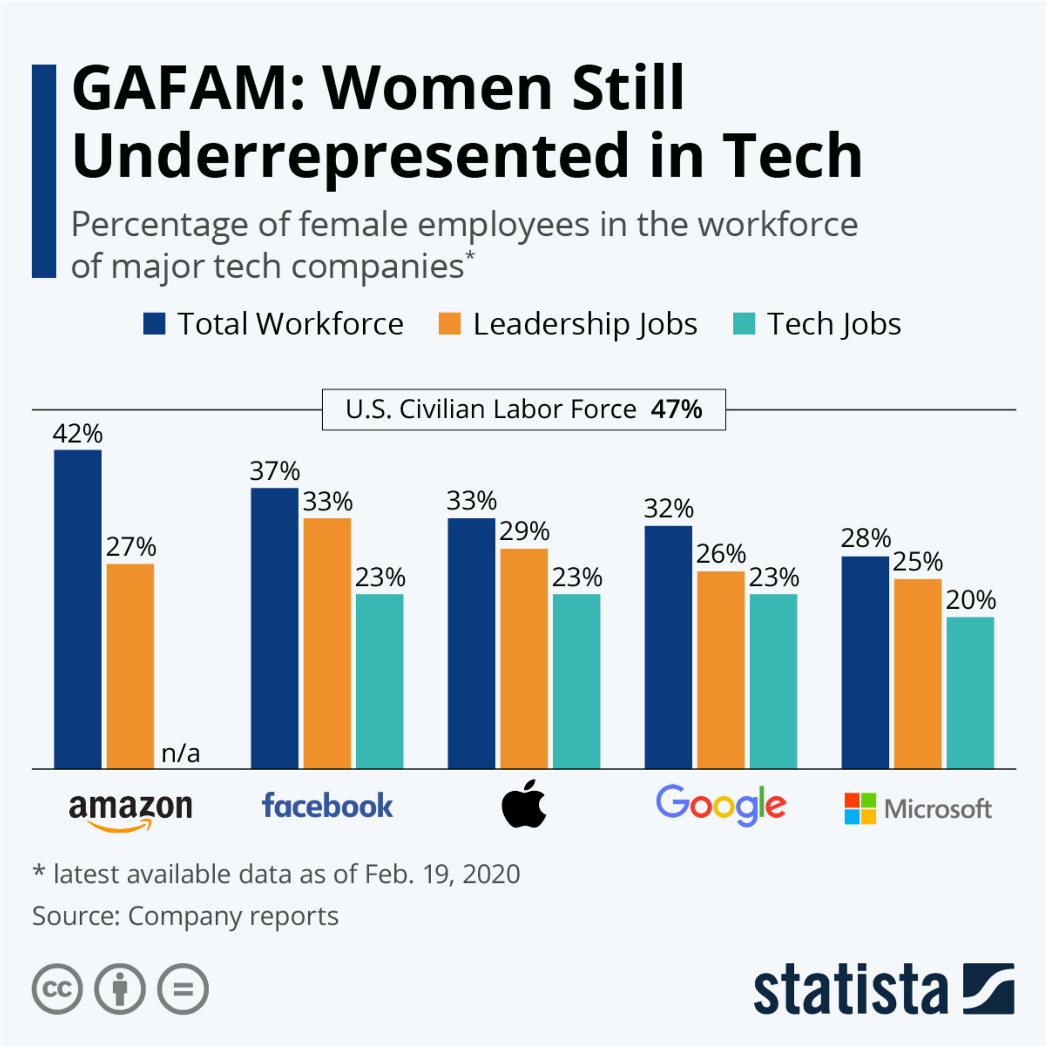
- Vital in research, politics, management, business…

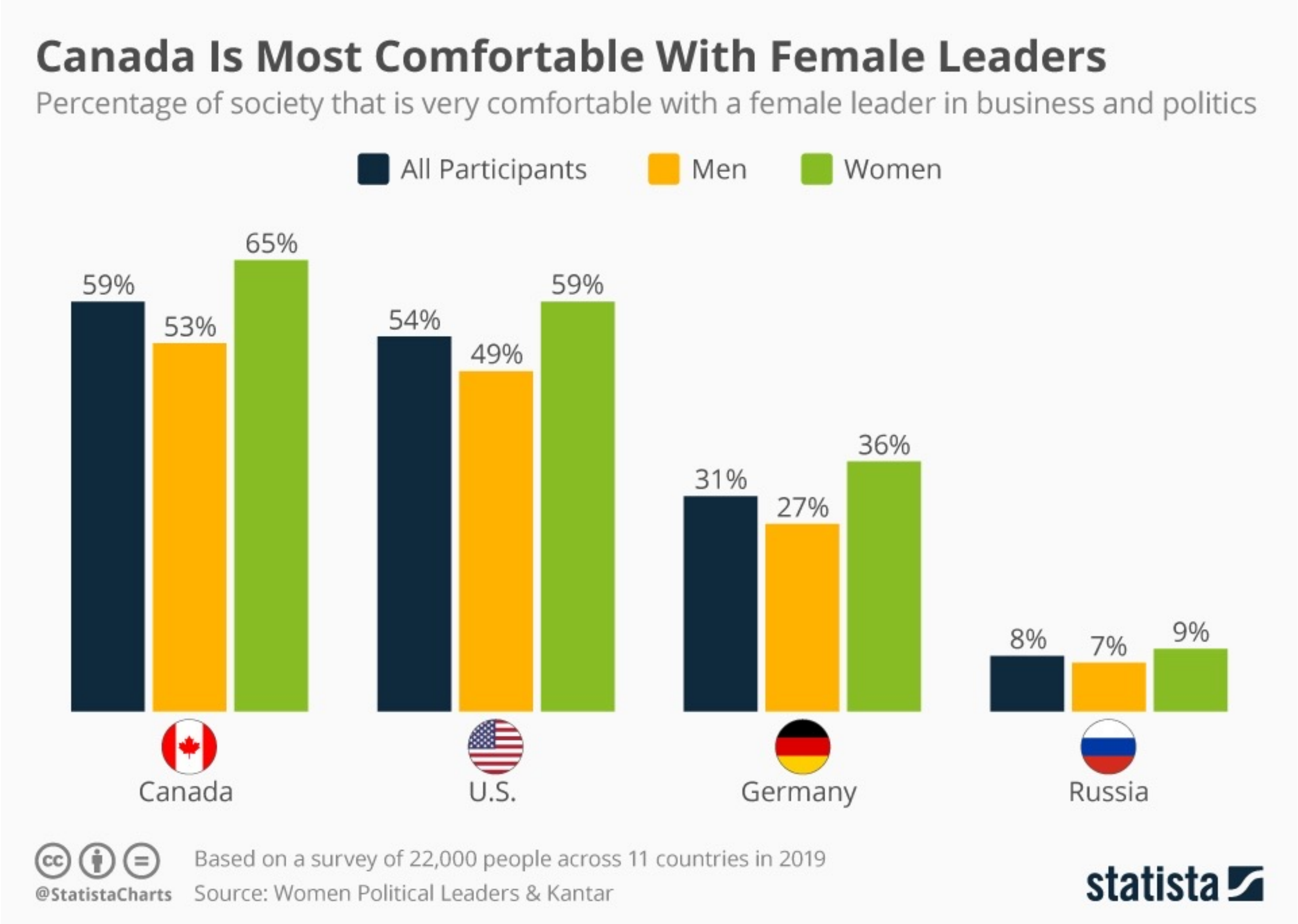
- There are different kinds of Statistics…

STATISTICS: EXAMPLE 1



Source: https://www.statista.com/chart/4467/female-employees-at-tech-companies/

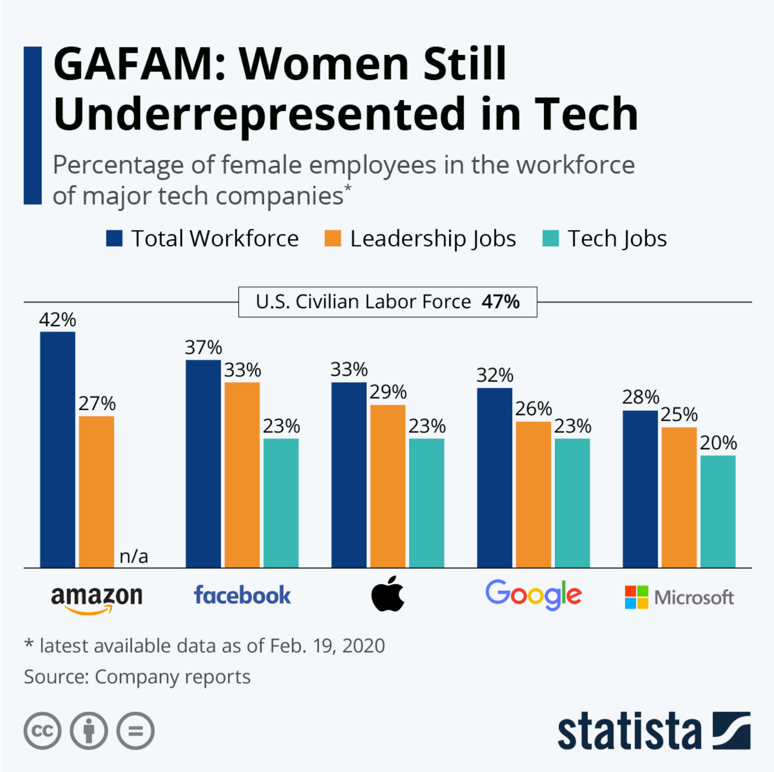
STATISTICS: EXAMPLE 2



Source: https://www.statista.com/chart/20018/canada-most-comfortable-with-female-leaders/

WHAT'S THE DIFFERENCE BETWEEN THE TWO?

EXAMPLE 1



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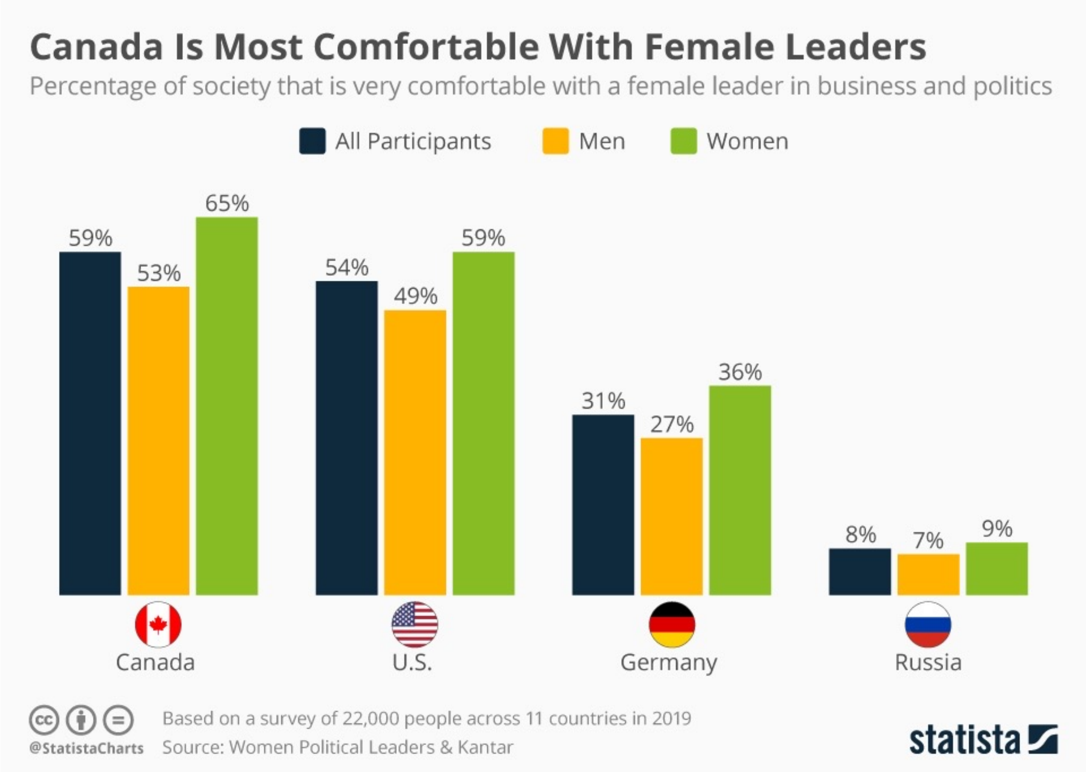

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WATCH THE VIDEO:

https://bit.ly/3fy8nzd

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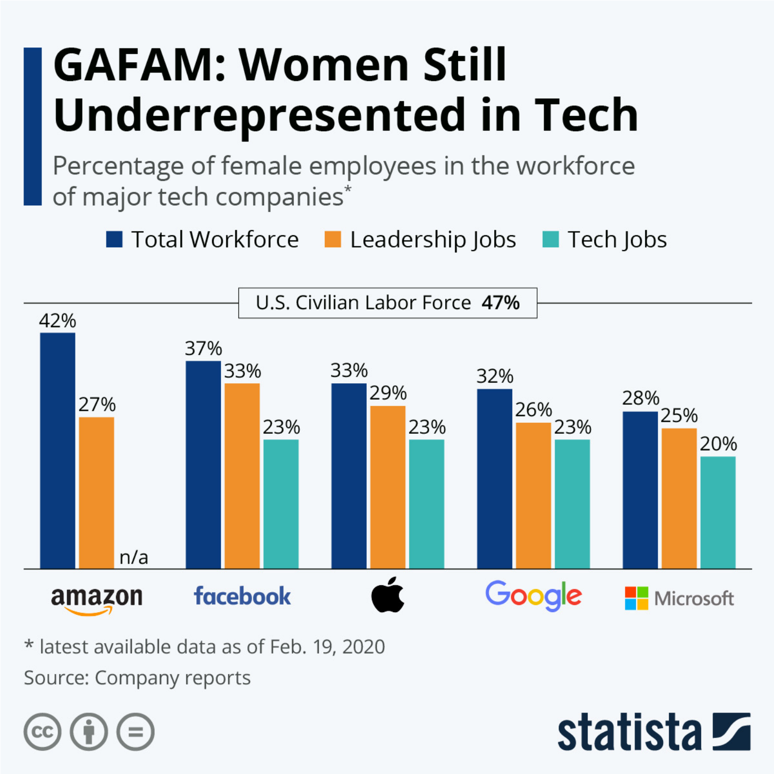

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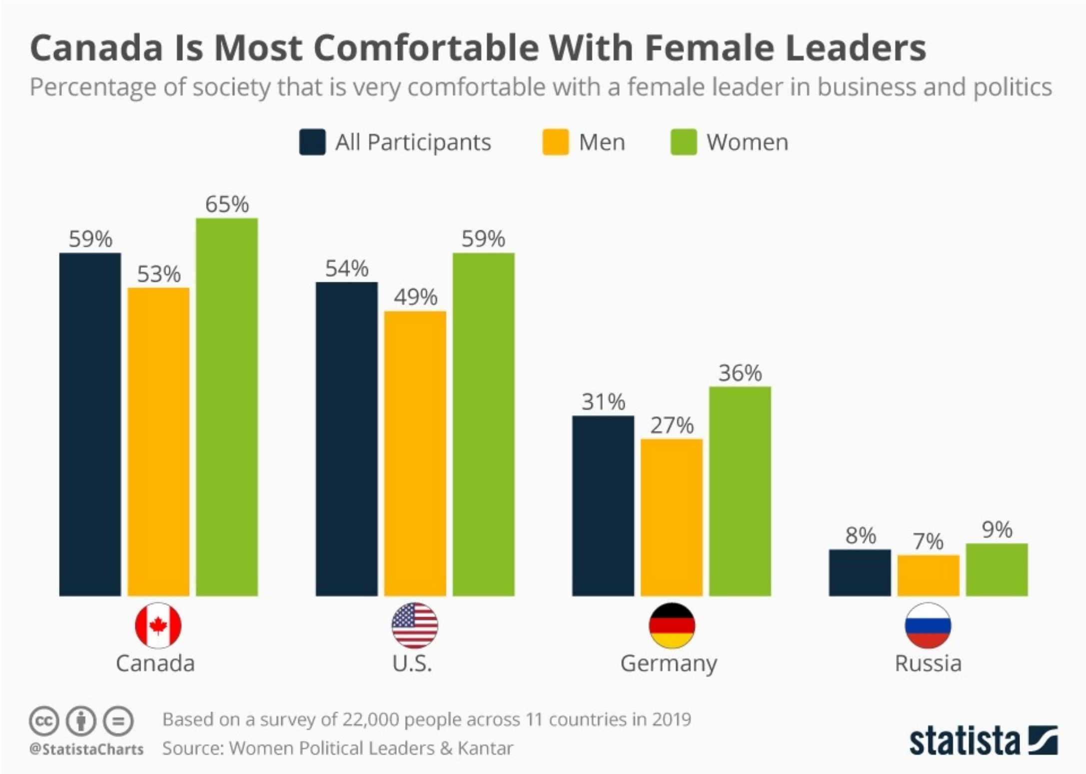

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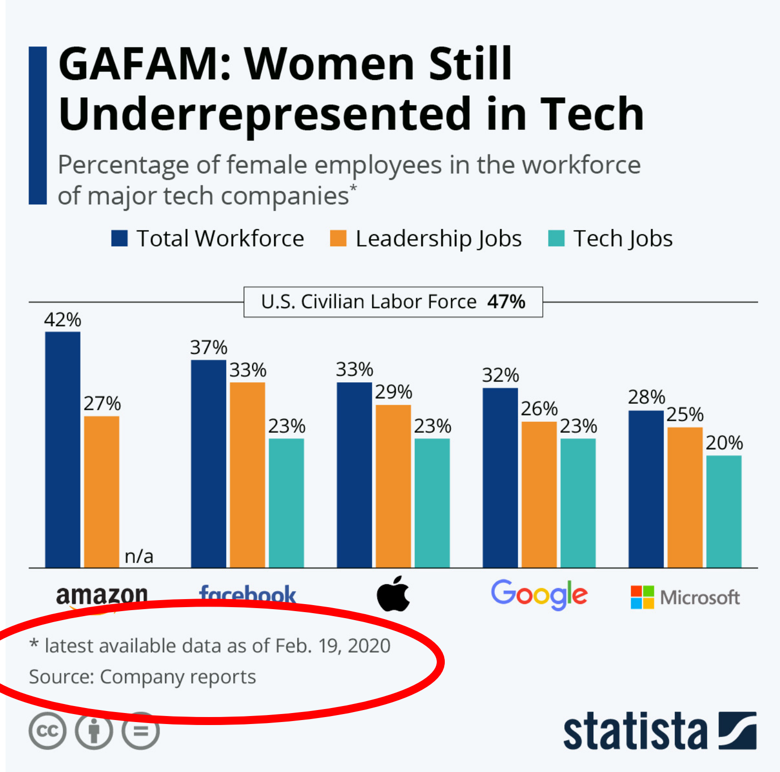

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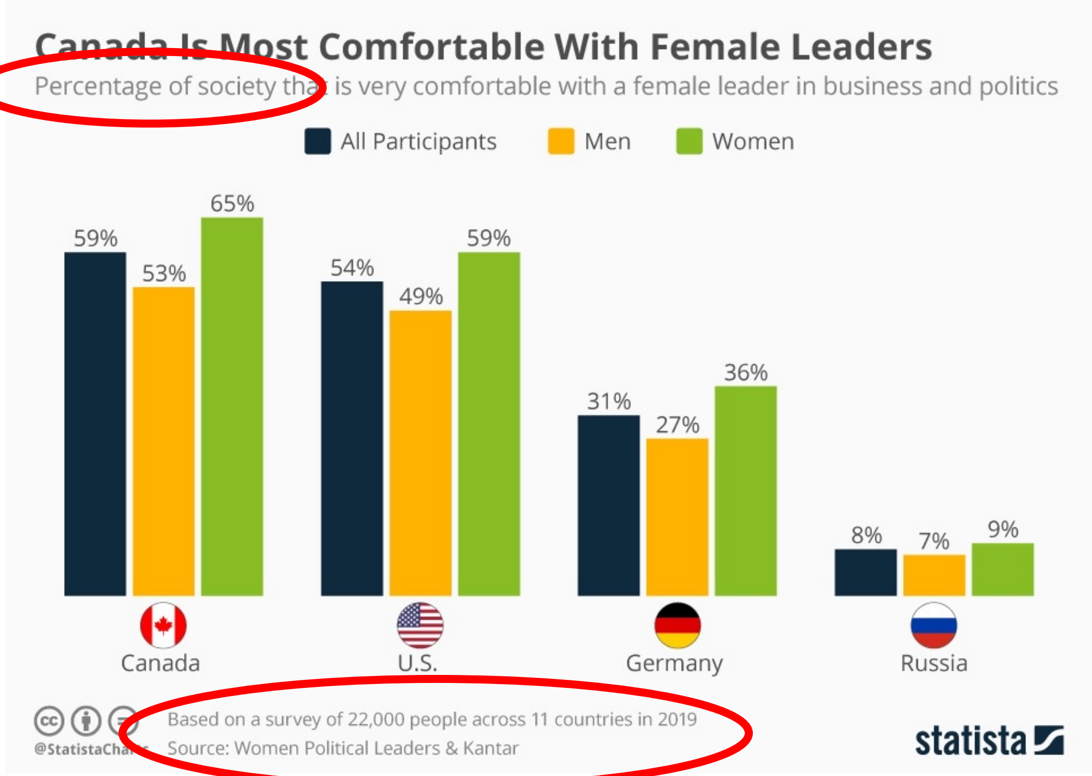
INFERENTIAL STATISTICS



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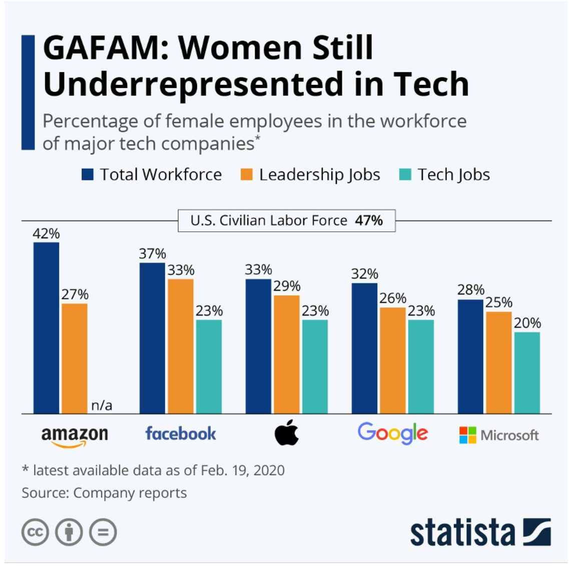
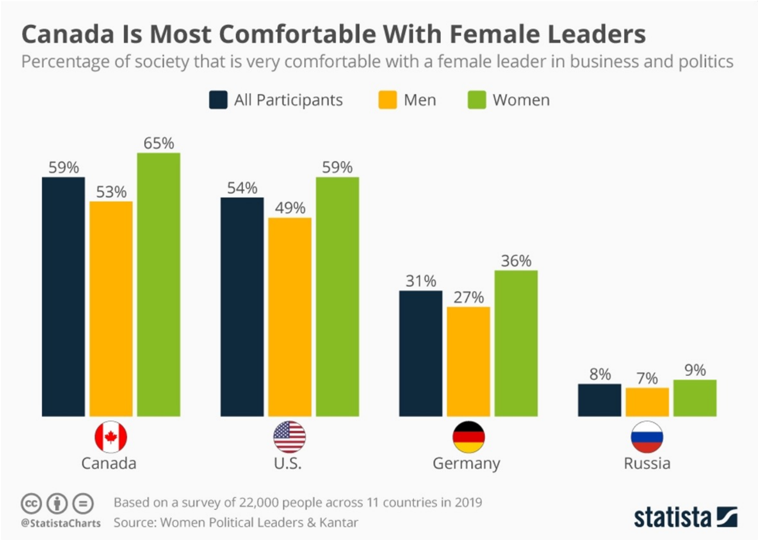
DESCRIPTIVE VS INFERENTIAL STATISTICS

DESCRIPTIVE STATISTICS



- *Describe* the data at hand.

INFERENTIAL STATISTICS



- From the data at hand, *make conclusions about a larger group.*

SO...

- DESCRIPTIVE STATISTICS
  - Data about **the whole population** is available.
  - Summarize it with
    - *summary statistics;*
    - *tables;*
    - *plots.*

- INFERENTIAL STATISTICS
  - Data about ***a sample* of the whole population** is available.
  - Reason about the whole population.

SO…

Explorative Data Analysis (EDA) in ML, Business Analytics, …

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OUR FOCUS

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Parameter estimation

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World Male Population	World Female Population
3,720,696 thousands	3,659,101 thousands

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$$E(N_{men}) = 100 * \mathbf{p} = 50$$

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From the data…

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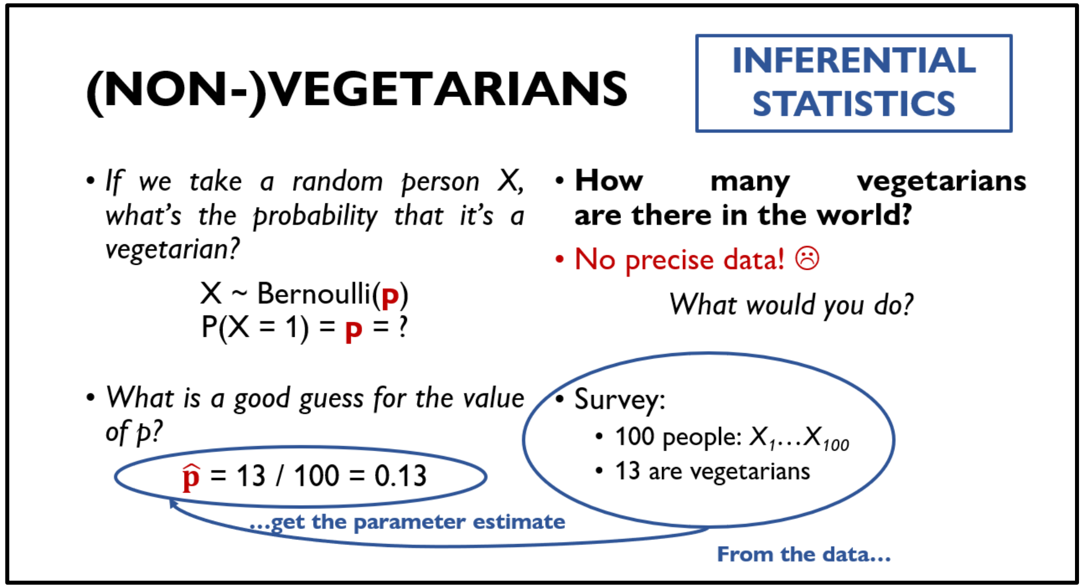

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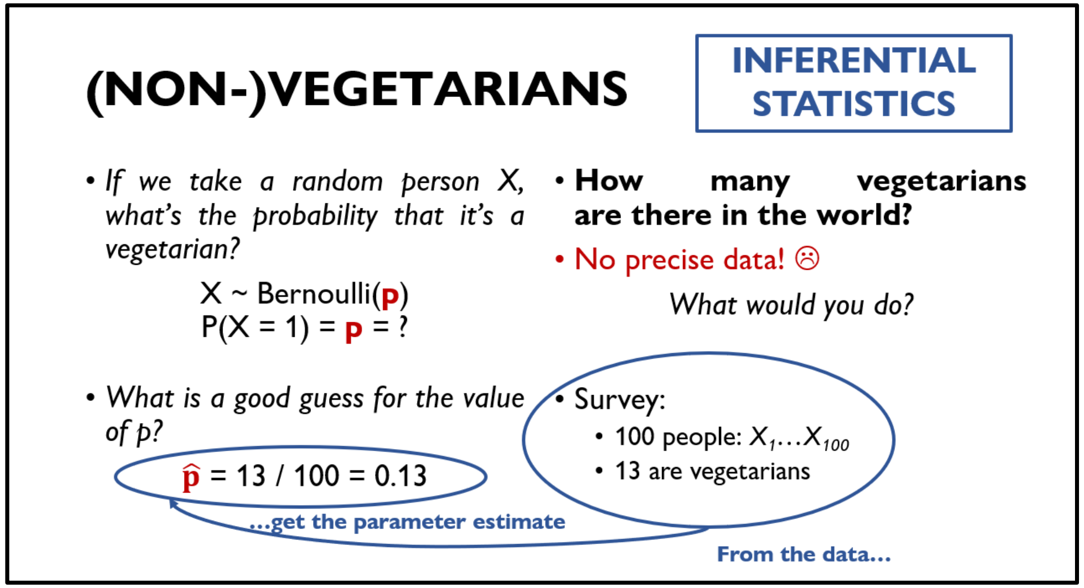

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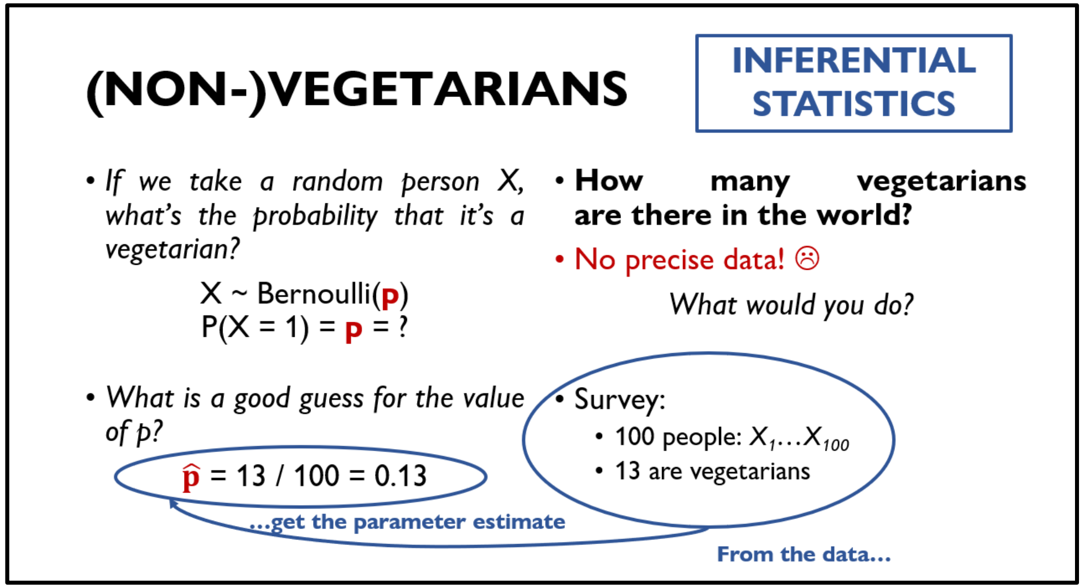

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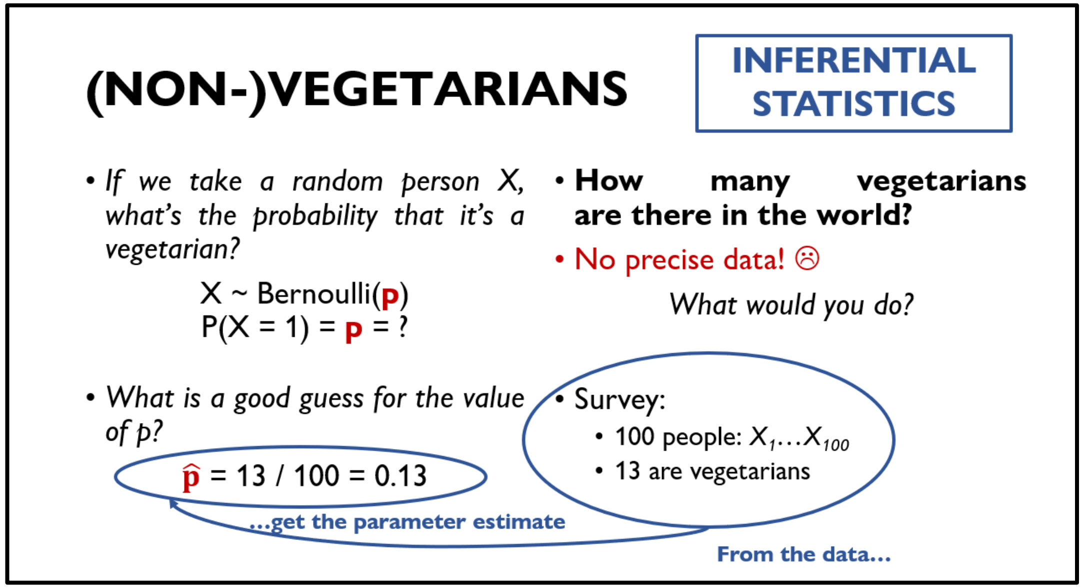

THE METHOD OF MAXIMUM LIKELIHOOD

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- What is the probability to obtain such results given p? **Likelihood:**

$$L(p) = P(X_1 = 1, \ldots, X_{13} = 1, X_{14} = 0, \ldots, X_{100} = 0 \mid p) =$$

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Likelihood function is the joint probability of realized sample given the parameters.

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  - We want the data that we obtained to be the most likely one.
  - **Maximize L(p) w.r.t. p!**

CALCULUS 101

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2. Set it to zero.
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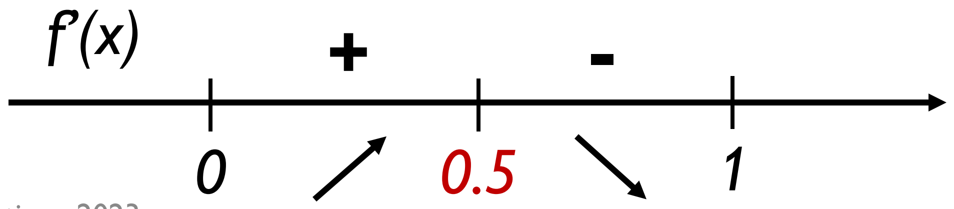

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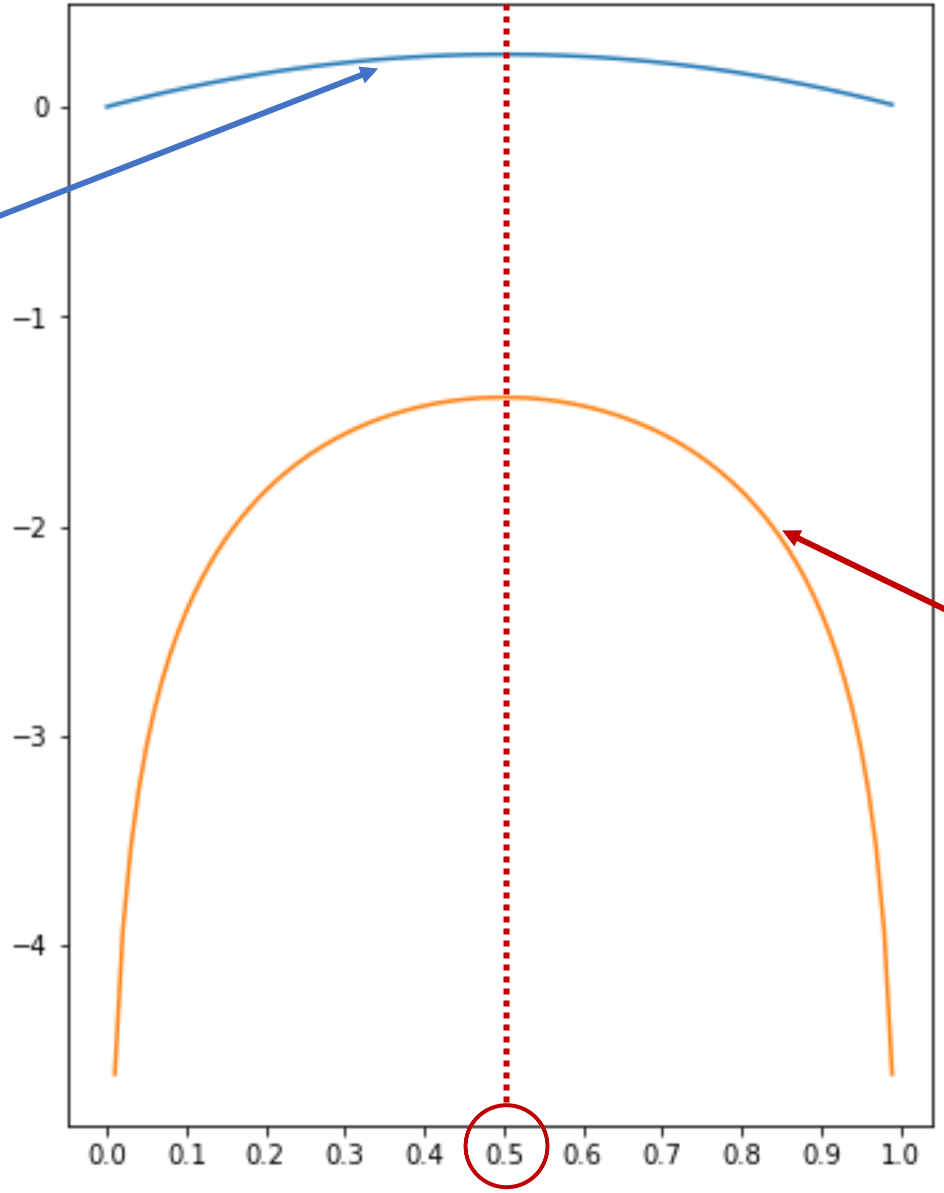
USEFUL TRICK: MAXIMIZE LOG-LIKELIHOOD INSTEAD

$f(x) = x*(1-x)$



log f(x)

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MAXIMUM LIKELIHOOD

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$$\frac{d}{dp} \log L(p) =$$

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$$\frac{d}{dp}\log L(p) = \frac{13}{p} - \frac{87}{1-p} = 0$$

MAXIMUM LIKELIHOOD

maximize $L(p) = p^{13}(1-p)^{87}$ w.r.t. p

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$$\hat{p} = \frac{13}{100}$$

MAXIMUM LIKELIHOOD ESTIMATE

1. Write down the likelihood function:

$$L(\theta) = P(X_1, \ldots, X_n \mid \theta) = \prod_{i=1}^{n} P(Xi \mid \theta)$$

2. Find its maximum w.r.t. the unknown parameter θ:

$$\widehat{\Theta} = \text{argmax } L(\theta) \quad \text{w.r.t. } \theta$$

(!) In many cases, it's easier to maximize ***log-likelihood****:*

$$\log L(\theta) = \log \prod_{i=1}^{n} P(Xi \mid \theta) = \sum_{i=1}^{n} \log P(Xi \mid \theta)$$

$$\widehat{\Theta} = \text{argmax } \log L(\theta)$$

Maximum Likelihood Estimate (MLE) is the value which …

Maximum Likelihood Estimate (MLE) is the value which maximizes the probability of observing the realized sample.

MLE FOR SOME DISCRETE DISTRIBUTIONS

BERNOULLI DISTRIBUTION

$X \sim \text{Bernoulli}(p)$

$E(X) = p$

$Var(X) = p(p-1)$

- Models the probability of success in an experiment with two outcomes.

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- You want to estimate the proportion of the vegetarians.

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- Survey: you ask N people whether they are vegetarians and get responses

$$X_1, X_2, \ldots X_N$$

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MLE FOR BERNOULLI DISTRIBUTION

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POISSON DISTRIBUTION

$X \sim Po(\lambda), \quad \lambda > 0$

$$P(X = k) = \begin{cases} \frac{e^{-\lambda} \cdot \lambda^k}{k!}, k \geq 0 \\ 0, \text{otherwise} \end{cases}$$

$E(X) = \lambda, \quad Var(X) = \lambda$

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- Assumptions: events occur
  - with a known constant mean rate;
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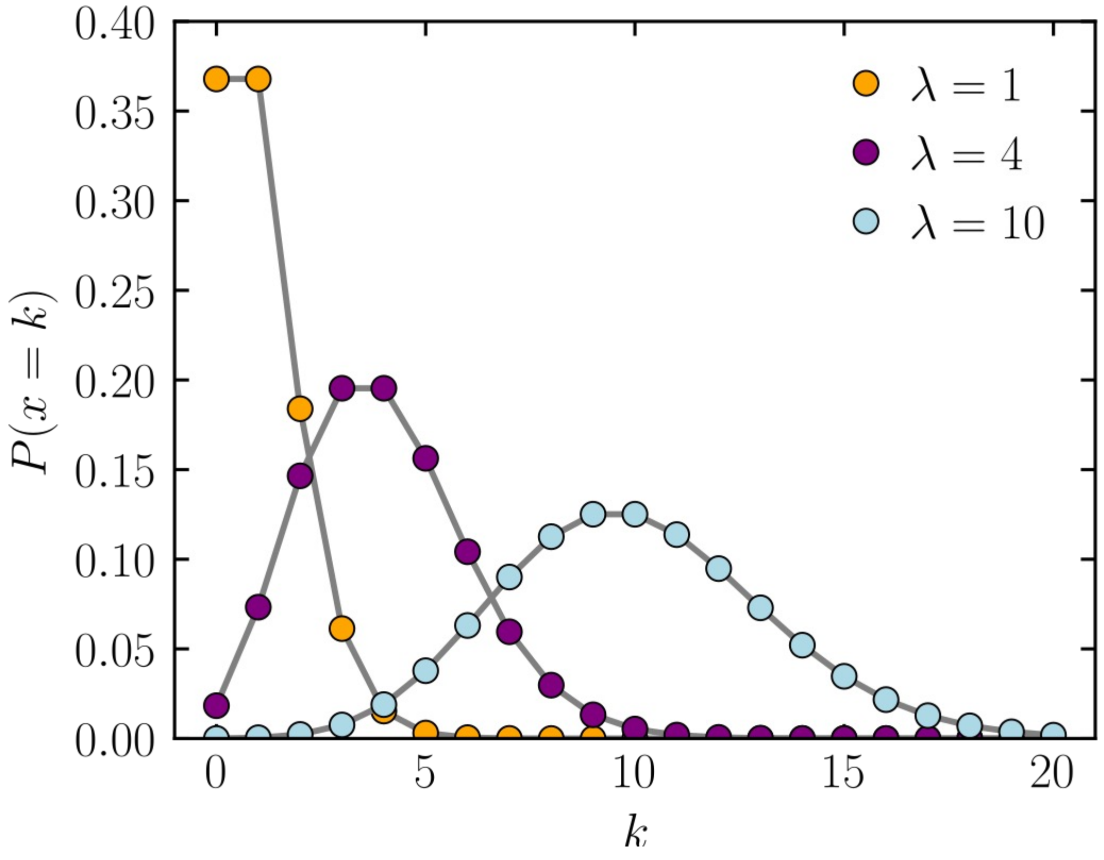

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MLE FOR POISSON DISTRIBUTION

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MLE: YET ANOTHER EXAMPLE

- Random variable X takes three values with unknown probabilities:

X	1	2	3
P(X)	?	?	?

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- What is the MLE of the parameters?

CALCULUS 101

- **How to optimize a function of several variables?**

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- **How to optimize a function of several variables?**

1. Compute partial derivatives.
2. Set them to zero.

3. Solve the equations and get the critical points.

CALCULUS 101

- **How to optimize a function of several variables?**

1. Compute partial derivatives.
2. Set them to zero.

$$f(x, y) = x^2 + 2xy - 2x - 4y$$

$$\frac{d}{dx} f(x, y) = 2x + 2y - 2 = 0$$

$$\frac{d}{dy} f(x, y) = 2x - 4 = 0$$

3. Solve the equations and get the critical points.

$$x^* = 2, \qquad y^* = 1$$

MLE: YET ANOTHER EXAMPLE

X	1	2	3
P(X)	p	q	1-p-q

Value	1	2	3
#	n_1	n_2	n_3

- What's the probability of observing such data? Likelihood:

$$L(p, q) =$$

$$\Leftrightarrow$$

MLE: YET ANOTHER EXAMPLE

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$$L(p, q) = p^{n_1} \cdot q^{n_2} \cdot (1 - p - q)^{n_3}$$

maximize L(p, q) w.r.t. p, q

$\Leftrightarrow$

MLE: YET ANOTHER EXAMPLE

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maximize L(p, q) w.r.t. p, q

$$\Leftrightarrow$$

$$\log L(p, q) = n_1 \log p + n_2 \log q + n_3 \log(1 - p - q)$$

maximize log L(p, q) w.r.t. p, q

MLE: EXAMPLE 2

X	1	2	3
P(X)	p	q	1-p-q

Value	1	2	3
#	n_1	n_2	n_3

maximize $\mathrm{n}_1 \log \mathrm{p} + \mathrm{n}_2 \log \mathrm{q} + \mathrm{n}_3 \log(1 - \mathrm{p} - \mathrm{q})$ w.r.t. p, q

$$\frac{d}{dp} \log L(p,q) = \frac{n_1}{p} - \frac{n_3}{1-p-q} = 0$$

$$\frac{d}{dq} \log L(p,q) = \frac{n_2}{q} - \frac{n_3}{1-p-q} = 0$$

MLE: EXAMPLE 2

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$$\Rightarrow$$

$$\hat{p} = \frac{n_1}{N}$$

$$\hat{q} = \frac{n_2}{N}$$

TO SUM UP

- **Likelihood function** is the joint probability of realized sample given the parameters.

- **Maximum Likelihood Estimate (MLE)** is the value which maximizes the probability of observing the realized sample.

RANDOMIZED RESPONSE

Asking embarrassing questions

MOTIVATION

- BEFORE:
  - How many vegetarians are in the world?
  - Survey: ask "*are you a vegetarian?*", estimate the true proportion.

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  - People won't tell the truth.

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- BEFORE:
  - How many vegetarians are in the world?
  - Survey: ask “*are you a vegetarian?*”, estimate the true proportion.
- BUT WHAT IF the question is very sensitive?
  - People won’t tell the truth.
- EXAMPLE: *Do you find this course boring?*
  - How do I find out what my students *actually* think?

LET'S TRY THIS OUT!

- Toss a coin…

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- Now, answer one of the following questions:

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- Now, answer one of the following questions:

If you got HEADS:	DO YOU FIND THIS CLASS BORING?
If you got TAILS:	ARE YOU AT THE STATISTICS CLASS RIGHT NOW?

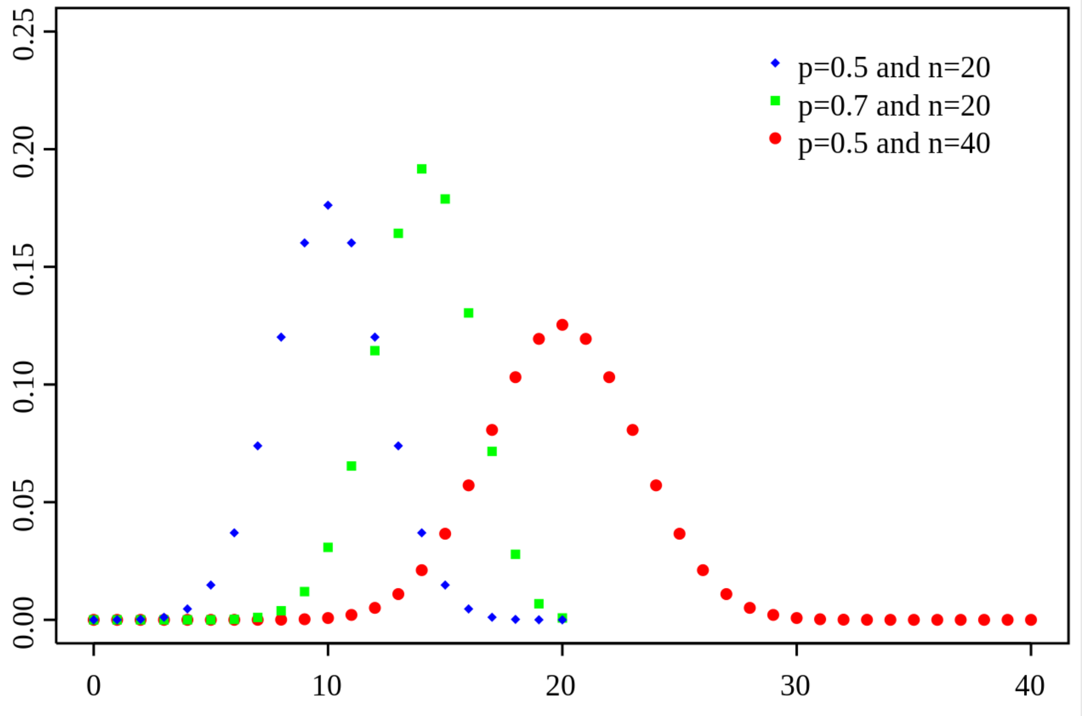
BINOMIAL DISTRIBUTION

$X \sim Bi(n, p), \quad n = 1, 2, \ldots, \quad 0 < p < 1$

$P(X = k) = C_n^k \, p^k (1 - p)^{n-k}$

$E(X) = np, \quad Var(X) = np(1\text{-}p)$

- Models the number of successes in a series of *n* independent Bernoulli trials, each of which has a success probability *p*.



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- Models the number of successes in a series of *n* independent Bernoulli trials, each of which has a success probability *p*.

- You sell two types of sandwiches: chicken and vegetarian. Which one people like more?

- For the past N days, you were selling n=100 sandwiches every day and recorded the number of the chicken ones:

$$X_1, X_2, \ldots, X_N$$

- What is the MLE of the p parameter?

MLE FOR BINOMIAL DISTRIBUTION

$$\mathrm{L(p)} =$$

MLE FOR BINOMIAL DISTRIBUTION

$$L(p) = \prod_{i=1}^{N} C_n^{X_i} p^{X_i} (1 - p)^{100 - X_i}$$

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$$\hat{p} = \frac{\sum_{i=1}^{N} X_i}{100N}$$

MLE FOR BINOMIAL DISTRIBUTION

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$$\boxed{\hat{p} = \frac{\sum_{i=1}^{N} X_i}{100N}}$$

TO SUM UP

- **Likelihood function** is the joint probability of realized sample given the parameters.

- **Maximum Likelihood Estimate (MLE)** is the value which maximizes the probability of observing the realized sample.

MAXIMUM LIKELIHOOD ESTIMATE

FOR PARAMETERS OF CONTINUOUS DISTRIBUTIONS

LIKELIHOOD FUNCTION

- Given samples from a distribution: $X_1, X_2, \dots, X_N$
  Obtain estimate of the parameter(s) θ

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