PROBABILITY & STATISTICS

Lecture 11 – Confidence Intervals

MLE FOR NORMAL DISTRIBUTIONS

estimating μ and σ

MLE FOR μ AND σ

- X1, X2, …, Xn – samples from the normal distribution.
- Parameters μ and σ are unknown.

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$$L(\mu, \sigma) =$$

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$$\log L(\mu, \sigma) =$$

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  - $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = S^2$ - **MLE for σ^2 is sample variance**

PROPERTIES OF ESTIMATORS

Bias, variance and consistency

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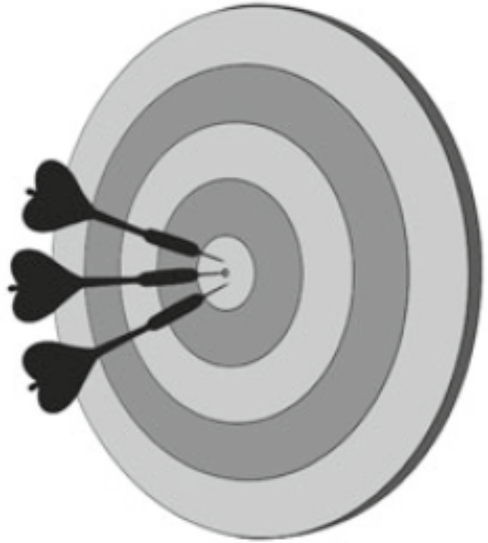
- We need to compare different estimators.

PROPERTIES OF ESTIMATORS

- Bias
- Variance
- Consistency

BIAS VS VARIANCE

1



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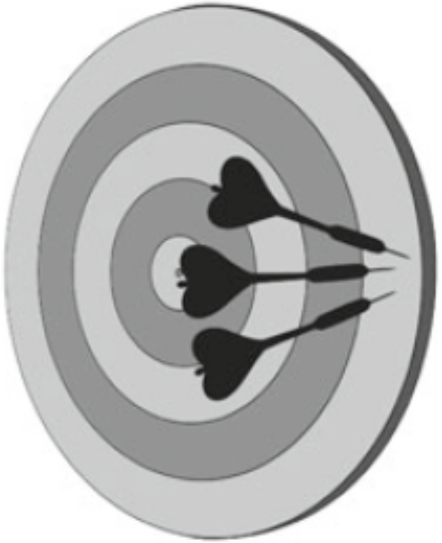

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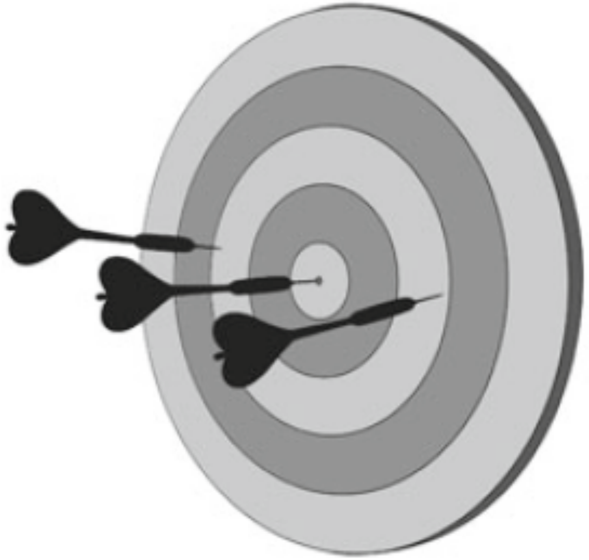
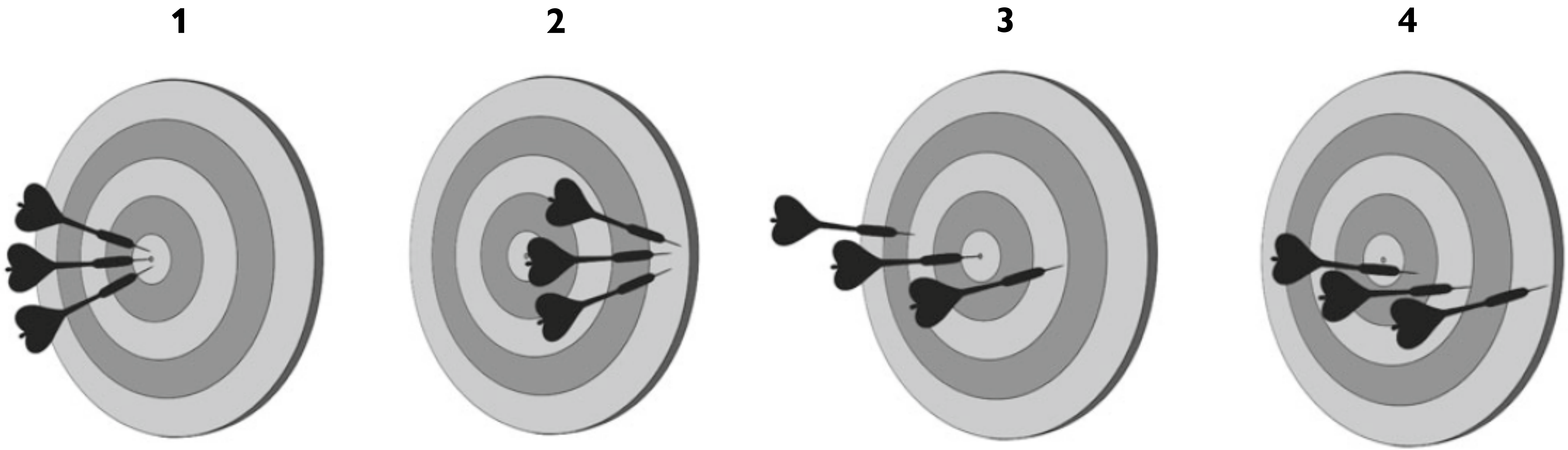


4

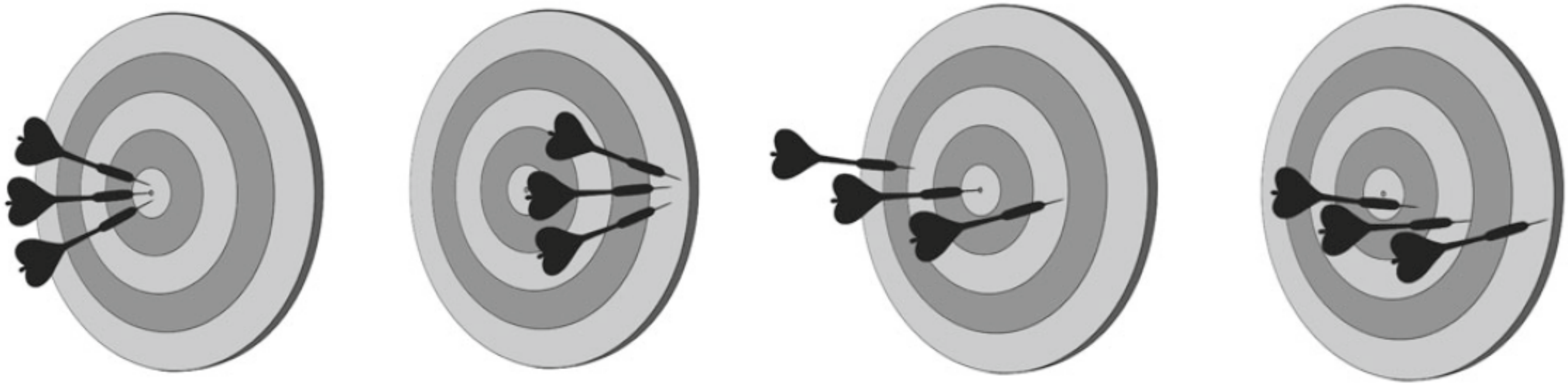
BIAS VS VARIANCE



LOW BIAS
LOW VARIANCE

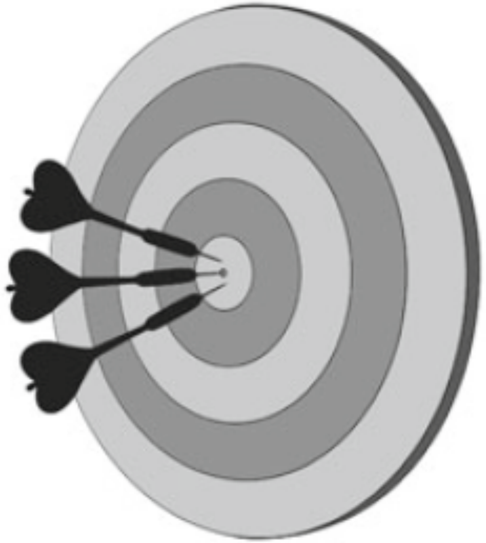
BIAS VS VARIANCE

1 2 3 4



BIAS VS VARIANCE

1



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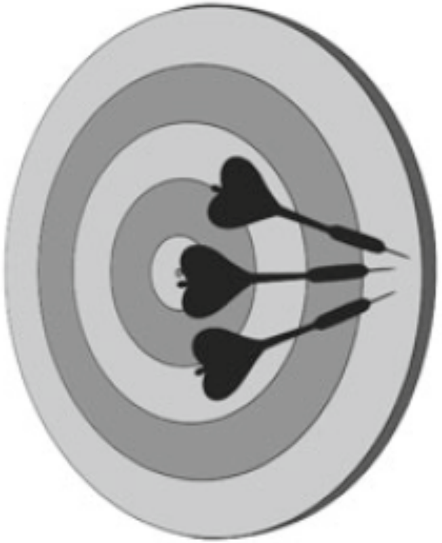

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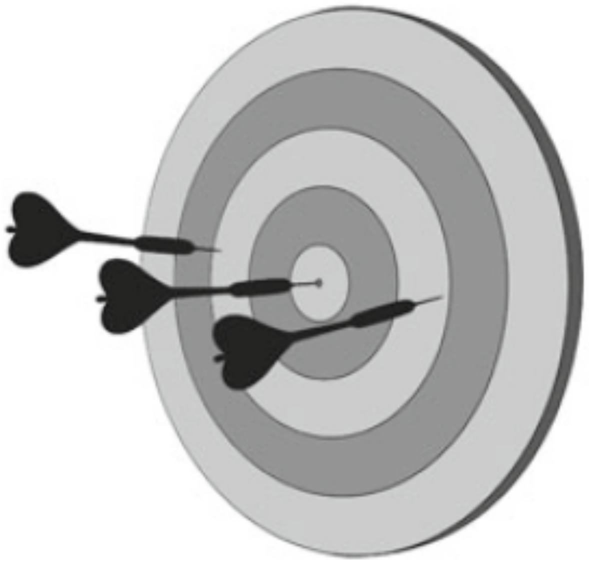
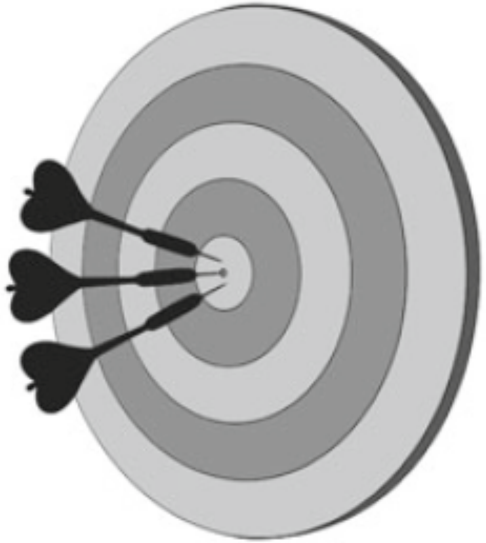


LOW BIAS
HIGH VARIANCE

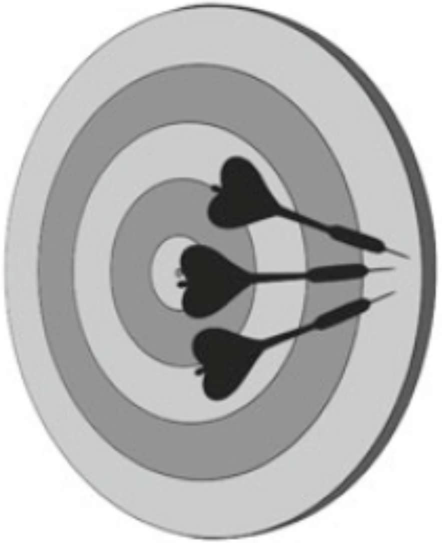
4

BIAS VS VARIANCE

1



2



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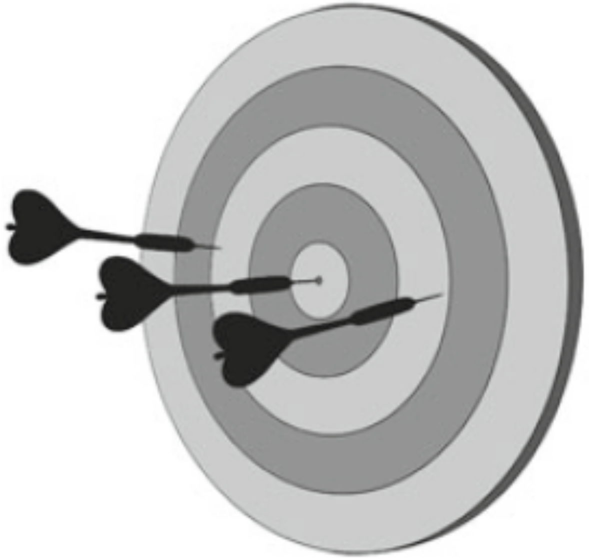
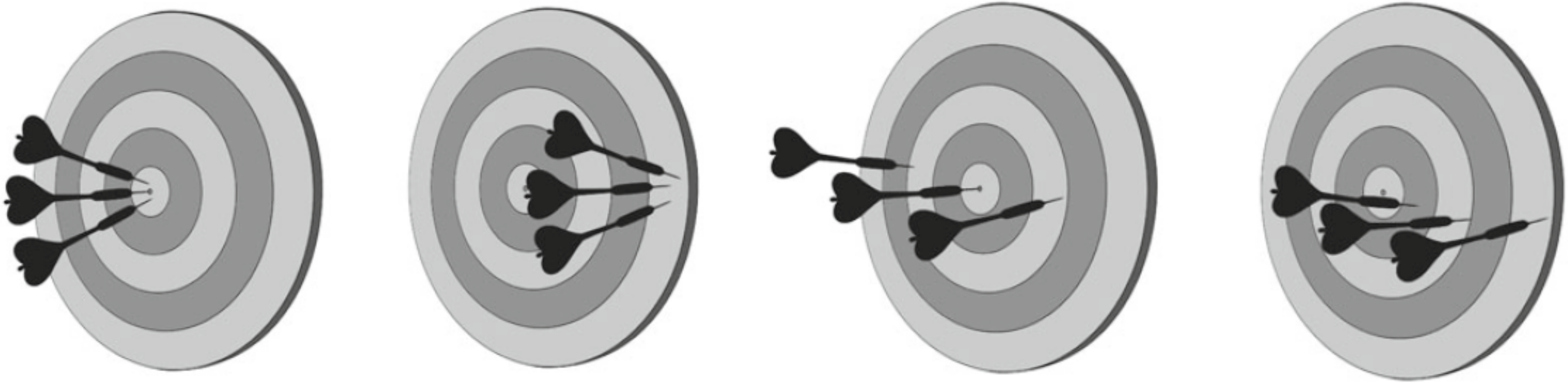


4

BIAS VS VARIANCE

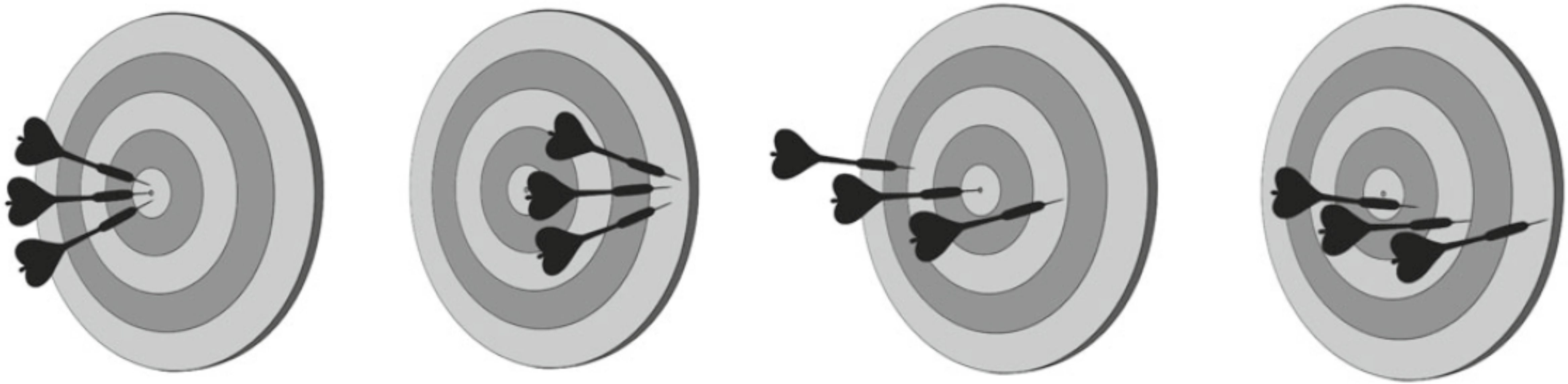
1 2 3 4



HIGH BIAS
LOW VARIANCE

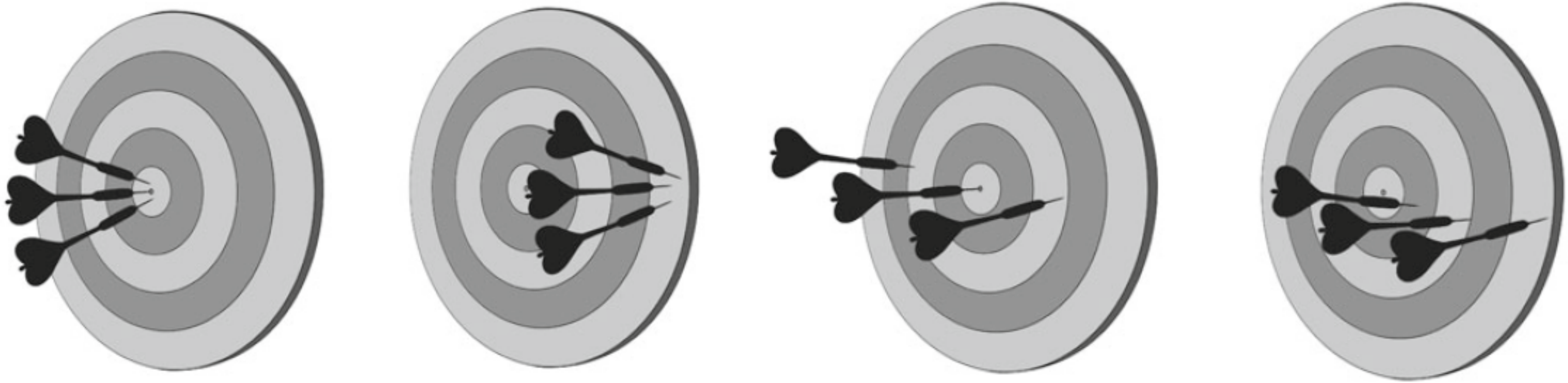
BIAS VS VARIANCE

1 2 3 4



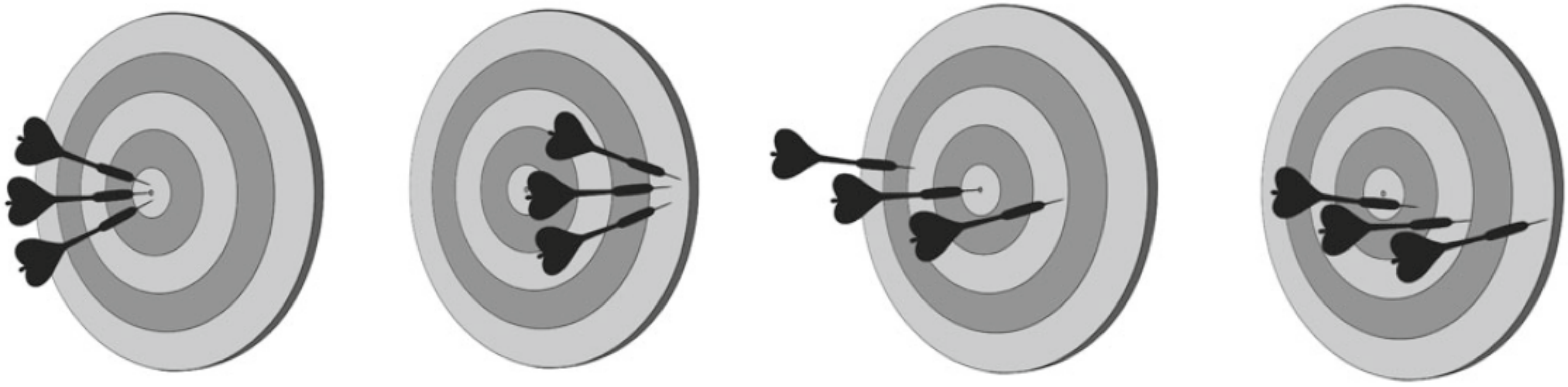
BIAS VS VARIANCE

1 2 3 4



HIGH BIAS
HIGH VARIANCE

BIAS VS VARIANCE



1 — LOW BIAS, LOW VARIANCE

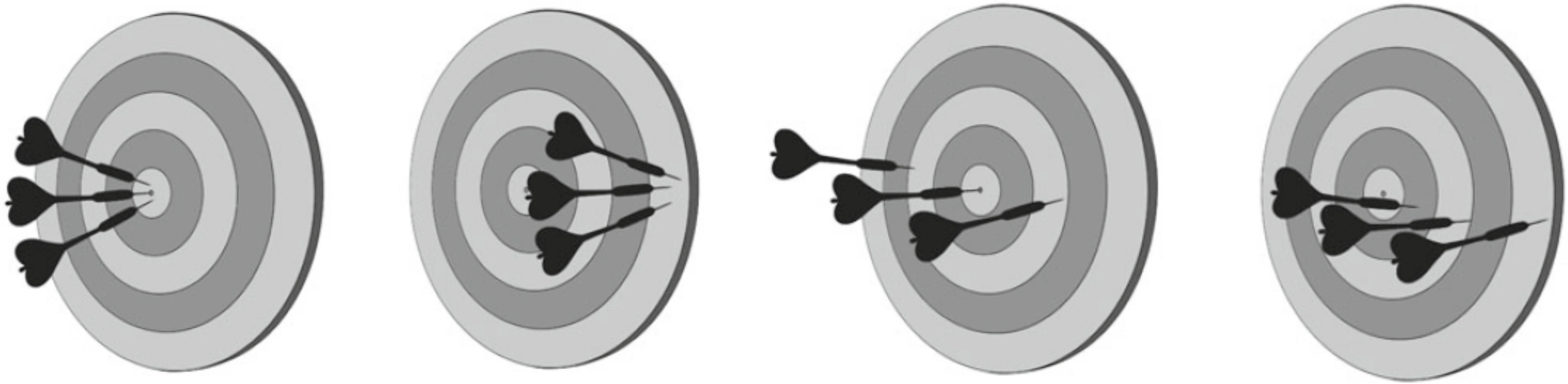
2 — HIGH BIAS, LOW VARIANCE

3 — LOW BIAS, HIGH VARIANCE

4 — HIGH BIAS, HIGH VARIANCE

BIAS VS VARIANCE

1	2	3	4



LOW BIAS LOW VARIANCE	HIGH BIAS LOW VARIANCE	LOW BIAS HIGH VARIANCE	HIGH BIAS HIGH VARIANCE

BIAS VS VARIANCE

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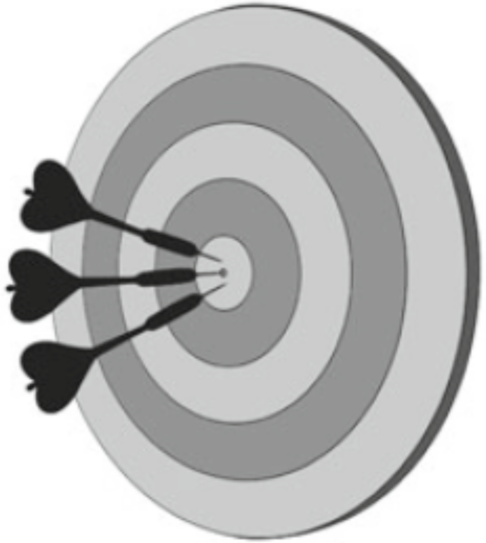
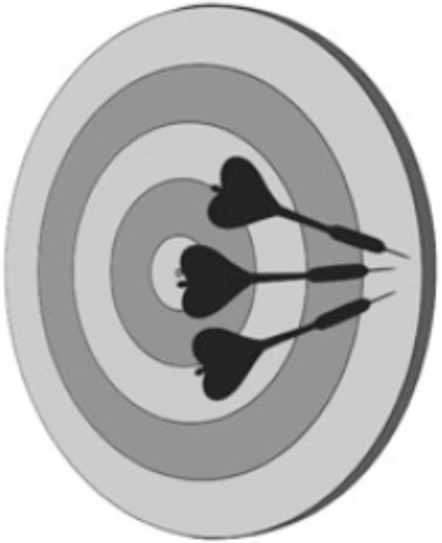


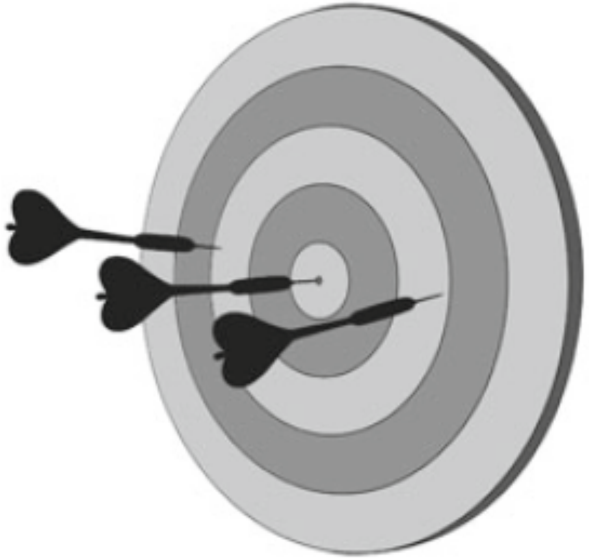
LOW BIAS
LOW VARIANCE

2



HIGH BIAS
LOW VARIANCE

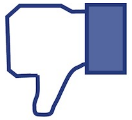
3



LOW BIAS
HIGH VARIANCE

4

HIGH BIAS
HIGH VARIANCE



BIAS

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$$E\big(T(X)\big) = \theta$$

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Bias is defined as

$$bias\big(T(X)\big) = \theta - E\big(T(X)\big)$$

BIAS: EXAMPLE 1

$$X_1, \dots, X_n \sim Bernoulli(p)$$

$\hat{p}_{ML} = \frac{1}{n}\sum_{i=1}^{n} X_i$ – (un)biased?

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$$E(\hat{p}_{ML}) =$$

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BIAS: EXAMPLE 2

$$X_1, \dots, X_n \sim N(\mu, \sigma^2)$$

$\hat{\mu}_{ML} = \frac{1}{n}\sum_{i=1}^{n} X_i$ – (un)biased?

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BIAS: EXAMPLE 3

$$X_1, \dots, X_n \sim N(\mu, \sigma^2)$$

$$\hat{\sigma}^2_{ML} = s^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 - \text{(un)biased?}$$

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$$\hat{\sigma}^2_{ML} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right)^2 =$$

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$$\mathrm{E}(\hat{\sigma}^2_{ML}) = \frac{1}{n}\sum_{i=1}^{n} EX_i^2 - \mathrm{E}\bar{X}^2$$

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Variance of an estimator $T(X)$ is defined as

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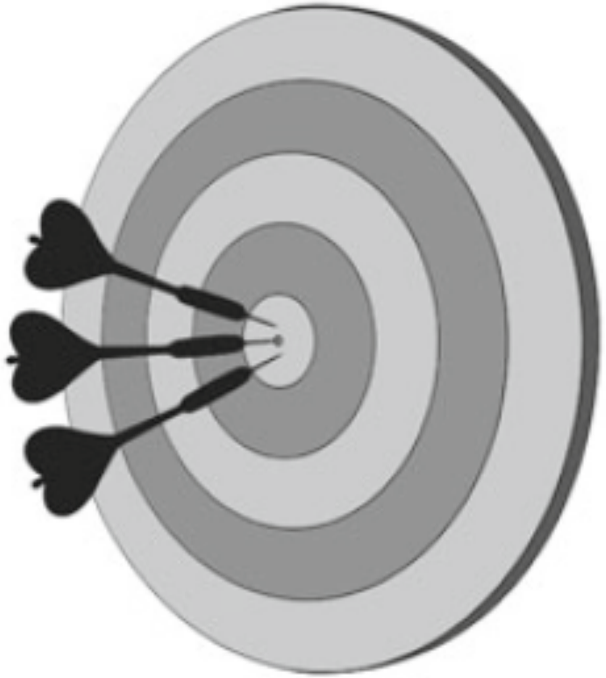
BIAS-VARIANCE TRADE-OFF

- Impossible to simultaneously optimize bias and variance.

- Related to *under*- and *overfitting* in Machine Learning.



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Let $T_1, T_2, \dots T_n$ be a sequence of estimators for $\theta, T_k = T(X_1, \dots, X_k)$.

Then $\{T_n\}$ is consistent if $\forall \epsilon > 0$

$$\lim_{n \to \infty} P(|T_n - \theta| < \epsilon) = 1$$

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roughly speaking, among well-behaved estimators, it has the smallest variance, at least for large samples.