# PROBABILITY & STATISTICS

Lecture 4: Random variables

# LAST TIME

- Independent events
- Conditional probability
- Bayes' rule
- Law of total probability

# TODAY

- Random variables and their properties
- Probability distributions

# Random variables

https://youtu.be/S_obHZJZ5EM

# MOTIVATION

- We usually focus on some numerical aspects of the experiment
  - number of heads in 100 coin tosses;
  - number of boys among 4 kids;
  - sum on the two dice;
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- These are *random variables*
  - A real-valued variable whose value is determined by an underlying random experiment.

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- $X$ is a **random variable**.
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  - Possible values: $0, 1, 2, 3, 4, 5$

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- A set of all possible values a random variable can take is called **range**.

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# (UN)COUNTABLE SETS

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- **Probability distribution** of $X$:

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- For a discrete random variable $X$ with range $R_X = \{x_1, x_2, \dots\}$ **probability mass function (PMF)** is defined as

$$P_X(x) = \begin{cases} P(X = x), & x \in R_X \\ 0, & x \notin R_X \end{cases}$$

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# PROPERTIES OF PMF

- Let $X$ be a random variable with range $R_X$ and a PMF $P_X(x)$.
  $X$ – number of heads after 2 tosses of a coin, $R_X = \{0, 1, 2\}$

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   $P_X(0) + P_X(1) + P_X(2) = 0.25 + 0.5 + 0.25 = 1$
3. For any set $A \subseteq R_X$, $P(X \in A) = \sum_{x_k \in A} P_X(x_k)$

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   $A = \{0, 1\}$, $\quad P(X = 0 \; or \; X = 1) = 0.25 + 0.5 = 0.75$

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- You are tossing an unfair coin with the probability of heads 0.6 twice. Let random variable $X$ denote the total number of heads.

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$$P(X = 10) = P_X(10) = 0,$$
$$P(X \leq 1) =$$

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$$P(X \leq 1) = P_X(1) + P_X(0) =$$

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$$P(X = 10) = P_X(10) = 0,$$
$$P(X \leq 1) = P_X(1) + P_X(0) = 0.16 + 0.48,$$
$$P(X \leq 2) =$$

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- You are tossing an unfair coin with the probability of heads 0.6 twice. Let random variable $X$ denote the total number of heads.

PMF:

$$P_X(x) = \begin{cases} 0.4^2, & x = 0 \\ 2 \cdot 0.4 \cdot 0.6, & x = 1 \\ 0.6^2, & x = 2 \\ 0, & otherwise \end{cases}$$

$$P(X = 10) = P_X(10) = 0,$$
$$P(X \leq 1) = P_X(1) + P_X(0) = 0.16 + 0.48,$$
$$P(X \leq 2) = P_X(2) + P_X(1) + P_X(0) = 1$$

# CUMULATIVE DISTRIBUTION FUNCTION

- Another way to represent distribution.
- **Cumulative distribution function (CDF)** of a random variable $X$ is defined as follows:

$$F_X(x) = P(X \leq x) \quad \forall x \in \mathbb{R}$$

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- Example: $X$ – total number of heads after two tosses of a coin.

$$P_X(x) = P(X = x) = \begin{cases} 0.25, & x = 0 \\ 0.5, & x = 1 \\ 0.25, & x = 3 \\ 0, & otherwise \end{cases}$$

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# PMF AND CDF: EXAMPLE

- You are rolling a fair die. $Y$ – the outcome.
- Define PMF $P_Y(y)$ and CDF $F_Y(y)$ of $Y$.

$$P_Y(y) = P(Y = y) =$$

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# Expected value

# EXPECTED VALUE

- Consider the following random variable:

| $x$ | -1 | 0 | 1 |
|---|---|---|---|
| $P(x)$ | $1/3$ | $1/3$ | $1/3$ |

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- Consider the following random variable:

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| $P(x)$ | $1/3$ | $1/3$ | $1/3$ |

- Which value would you expect to get *on average*?
- And now?

| $x$ | -1 | 0 | 1 |
|---|---|---|---|
| $P(x)$ | $0.1$ | $0.1$ | $0.8$ |

# EXPECTED VALUE

- Let $X$ be a discrete random variable with a finite range $R_X = \{x_1, \dots, x_n\}$. The expected value of $X$ is defined as:

$$EX = \sum_{x_k \in R_X} x_k \cdot P(X = x_k)$$

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- Different notations: $EX, E[X], \mu_X, \ldots$

# EXPECTED VALUE

- $EX = -1 \cdot$

| $x$ | -1 | 0 | 1 |
|---|---|---|---|
| $P(x)$ | $1/3$ | $1/3$ | $1/3$ |

- $EY =$

| $x$ | -1 | 0 | 1 |
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# EXPECTED VALUE

- $EX = -1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = 0$

| $x$ | -1 | 0 | 1 |
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- $EY = -1 \cdot 0.1 + 0 \cdot 0.1 + 1 \cdot 0.8 = 0.7$

| $x$ | -1 | 0 | 1 |
|---|---|---|---|
| $P(x)$ | $0.1$ | $0.1$ | $0.8$ |

# Variance

# VARIANCE

- Consider the following two random variables $X$ and $Y$:

| $x$ | -100 | 0 | 100 |
|---|---|---|---|
| $P(X = x)$ | $1/3$ | $1/3$ | $1/3$ |

| $y$ | -1 | 1 |
|---|---|---|
| $P(Y = y)$ | $1/2$ | $1/2$ |

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$EX =$ , $EY =$

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$$EX = \frac{1}{3} \cdot (-100) + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 100 = 0, \qquad EY =$$

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But $X$ and $Y$ are very different....

# VARIANCE

- The **variance** of a random variable $X$ with $EX = \mu_X$ is defined as

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- "How often does X take values far from its mean?"

# VARIANCE

$$\boldsymbol{Var(X) \geq 0}$$

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because by definition it's the expected value of

$$(X - \mu_X)^2 \geq 0$$

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$$Var(X) = E(X - EX)^2 = EX^2 = \frac{2}{3} \cdot 10^4$$

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$$Var(X) = E(X - EX)^2 = EX^2 = \frac{2}{3} \cdot 10^4$$

$$Var(Y) = E(X - EY)^2 = E(Y^2) = 1$$

# STANDARD DEVIATION

- What are the measurement units of $EX$?
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- Standard deviation:

$$std\ (X) = \sqrt{Var(X)}$$

# Practice problems

# BALLS IN A BOWL

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$$P_X(x) = P(X = x) = \begin{cases} \\ \\ \end{cases}$$

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| $P(X = x)$ | $\frac{1}{36}$ | $\frac{3}{36}$ | $\frac{5}{36}$ | $\frac{7}{36}$ | $\frac{9}{36}$ | $\frac{11}{36}$ |

- You are rolling a pair of dice. Let random variable $X$ denote the maximum of the two numbers you got.

$$R_X = \{1, 2, 3, 4, 5, 6\}$$

- CDF of $X$:

$$F_X(x) = \begin{cases} 0, & x < 1 \\ 1/36, & 1 \le x < 2 \\ & 2 \le x < 3 \\ & 3 \le x < 4 \\ & 4 \le x < 5 \\ & 5 \le x < 6 \\ & x \ge 6 \end{cases}$$

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$$E(X) = \sum_{k=1}^{+\infty} k \cdot P(X = k) = \sum_{k=1}^{+\infty} k \cdot 0.5^k = \cdots ?$$

# EXPECTATION OF $X^2$

- Consider the following random variable $X$:

| $x$ | -1 | 1 |
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- Let $Y = X^2$. Distribution of $Y$:

| $y$ | 1 |
|---|---|
| $P(y)$ | $1$ |

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|---|---|
| $P(y)$ | 1 |

# PROPERTIES OF EXPECTED VALUE 2

- Let $X$ be a random variable with range $R_X = \{x_1, \dots, x_n\}$ and $Y = g(X)$. Then

$$\mathrm{E}Y = E\big(g(X)\big) = \sum_{x_k \in R_X} g(x_k) \cdot P(X = x_k)$$

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- You roll a fair dice once, random variable $X$ – the outcome.

$$EX =$$

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$$EX^2 = \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6}$$

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$$Var(X) = (EX)^2 - E(X^2) = \frac{91}{6} - \frac{49}{9} \sim 2.92$$

$$EX^2 = \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}$$