PROBABILITY & STATITSICS

Lecture 1 – Intro & Combinatorics review

PLAN FOR TODAY

1. Quick intro
2. Course overview + Entry test
3. Combinatorics review
    - *Basic counting principles*
    - *Inclusion-Exclusion*
    - *Permutations and combinations*

Intro & Logistics

ABOUT THE COURSE

- January 9 – January 27
- 17:00 – 20:20 Barcelona time
  - *slight changes are possible,*
    I will let you know in advance
- On-campus <u>and</u> online

ABOUT ME

- EVGENIYA Korneva
  evgeniakorneva@gmail.com   Prague, Czech Republic

- Education:
  - *2015* - Bachelor of Applied Mathematics *(Moscow, Russia)*
  - *2016* - Master of Artificial Intelligence *(Leuven, Belgium)*

- *2016* - … Doctoral researcher at KU Leuven

- 2021 - … Data Scientist at ACMetric

ABOUT YOU

- 7 students from 5 different countries
- Bachelor’s *and* master’s students
- CS, Cybersecurity and Data Science tracks

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- Main topics covered:
  - Probability:
    - basic combinatorics and probability review
    - discrete and continuous random variables and their properties
  - Statistics:
    - estimators
    - confidence intervals
    - hypothesis testing

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- On-campus and online
- Final grade:
  - 40% graded assignments
  - 30% interim exam (Probability)
  - 30% final exam (Statistics)

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JOIN OUR GOOGLE CLASSROOM!

- Class materials and graded assignmnets will be posted there.
- Invites has been sent out, check your inbox.

PLEASE TAKE THE ENTRY TEST

- **NOT** graded
- Would help me adapt the materials
- Link also in Google classroom

Intro to Probability

Combinatorics review

MOTIVATION

- Some things in the world are random
  - flipping a coin;
  - roll of a die;
  - picking a random card from a shuffled deck of cards.

- Probability theory: explain the unpredictable.

IT ALL STARTED WITH GAMBLING…

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If a die is rolled four times, it's more likely that 6 would occur at least once than not at all

I should talk to mathematicians about that!

… AND WAS FORMALIZED RECENTLY

MID 17th CENTURY

Fermat

Pascal

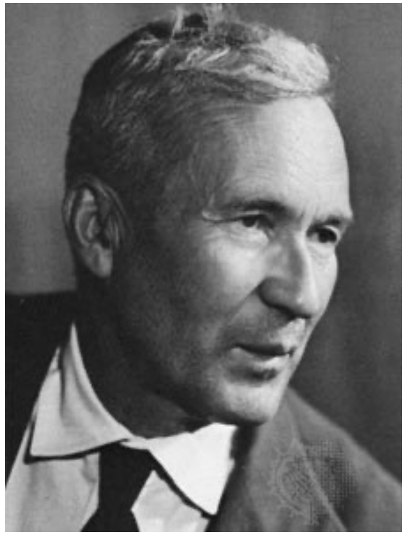
… AND WAS FORMALIZED RECENTLY

MID 17th CENTURY

Fermat

Pascal

MID 20st CENTURY



Kolmogorov

PROBABILITY IS EVERYWHERE

"Novak Djokovic will probably win Australian Open 2023"

"There is a 70% chance of rain"

"This test gives correct result with 99.9% accuracy"

"I will probably never become a US president"

INTERPRETATION DIFFERS

PROBABILITY AS LIMITING FREQUENCY

If the same experiment is repeated infinitely many times, how often doest the event happen?

- *This test gives correct result with 99.9% accuracy*
- *There is a 70% chance of rain*

PROBABILITY AS DEGREE OF BELIEF

How much are you *certain* in something?

- *Novak Djokovic will probably win Australian Open 2023*
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- The set of all possible outcomes is called the **sample space** (denoted by S).

RANDOM EXPERIMENTS: EXAMPLES

- Random experiment: flipping a coin once
  Sample space: {H, T}
  Outcome: H (we've got heads)

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- Random experiment: flipping a coin twice.
  Sample space: {HH, HT, TH, TT}
  Outcome: HT (we've got heads and tails)

- Random experiment: rolling a die
  Sample space: {1, 2, 3, 4, 5, 6}
  Outcome: 5

EVENT

- Each subset of a sampling space is called an **event** (denoted by E).

  Example: rolling a die

$$S = \{1, 2, 3, 4, 5, 6\}$$

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$$S = \{1, 2, 3, 4, 5, 6\}$$

  $E_1 = \{1\}$ – we've got number 1.

  $E_2 = \{2, 4, 6\}$ – we've got an even number.

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  $E_3 = \{3, 5\}$ – we've got 3 or 5.

What is probability

RELATIVE FREQUENCY

- Pascal and Fermat defined probability when a game is repeated a large number of times under the same conditions.

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- Example:
  If we flip a fair coin infinitely many times, it will come up heads half of the times.

COMPUTING PROBABILITY

- If a sample space S is finite, <u>and each outcome is equally likely</u>, then probability of an event E can be computed as

$$P(E) = \frac{\#\text{ ways } E \text{ can occur}}{\#\text{ possible outcomes}} = \frac{|E|}{|S|}$$

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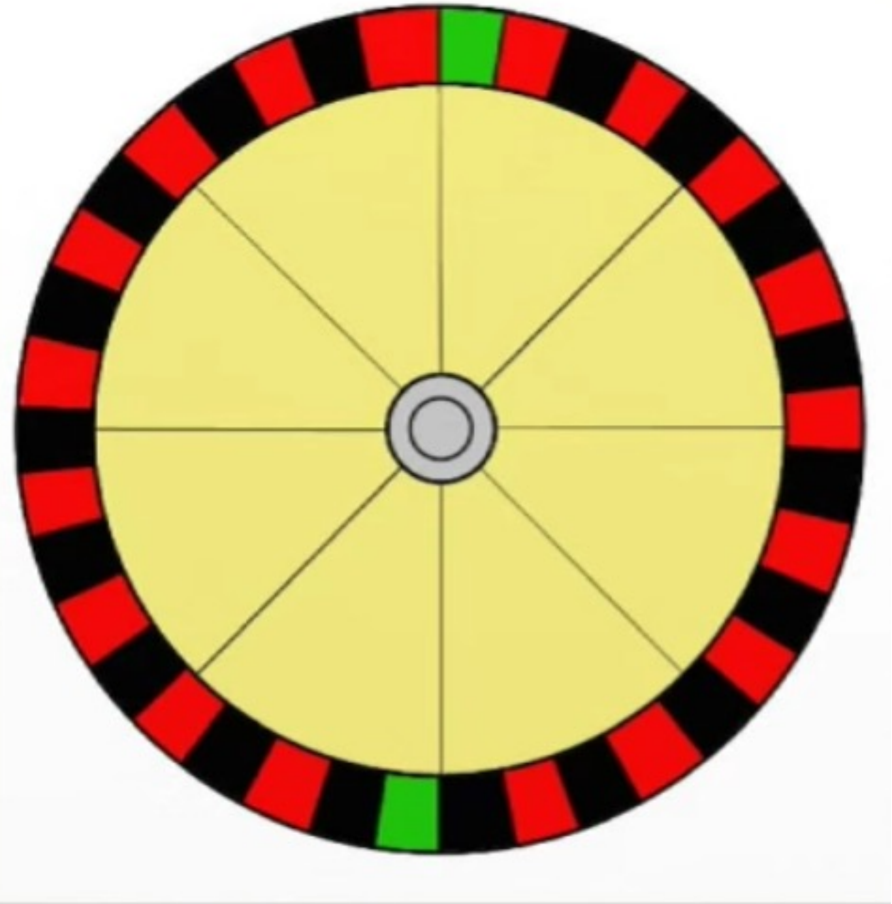
ROULETTE

- American roulette: 38 sectors
  - 18 red
  - 18 black
  - 2 green (0 and 00)
- What is the probability to win if you
  - Bet on red
  - Bet on black
  - Bet on green



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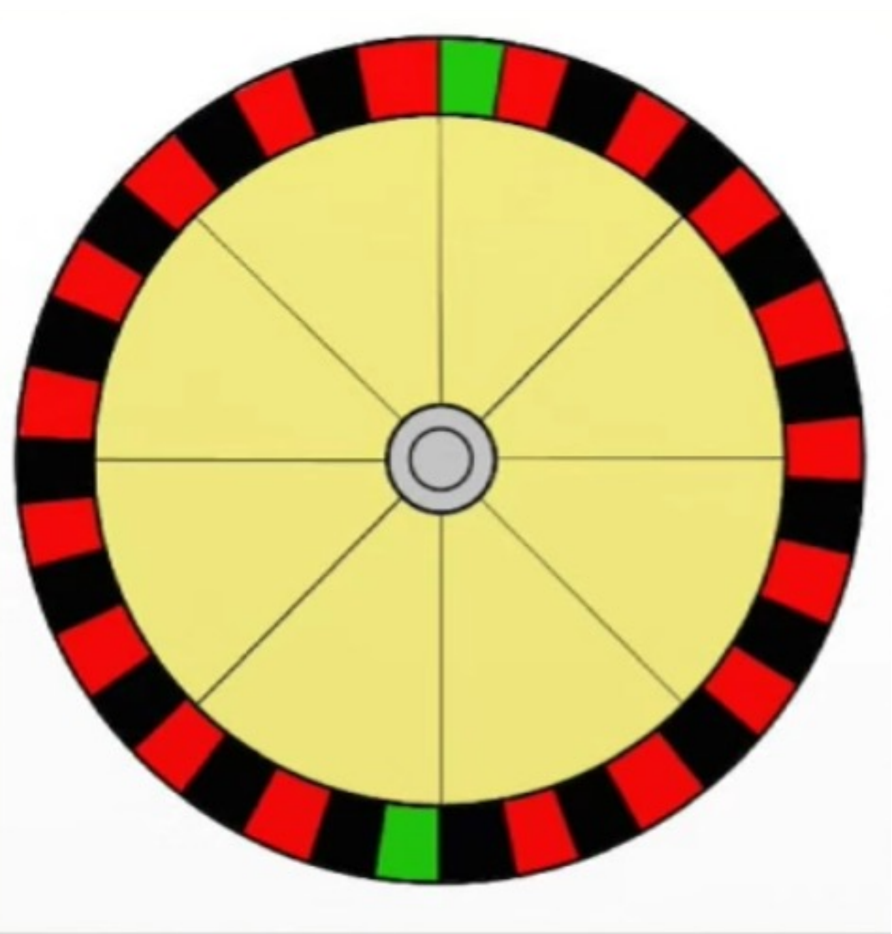

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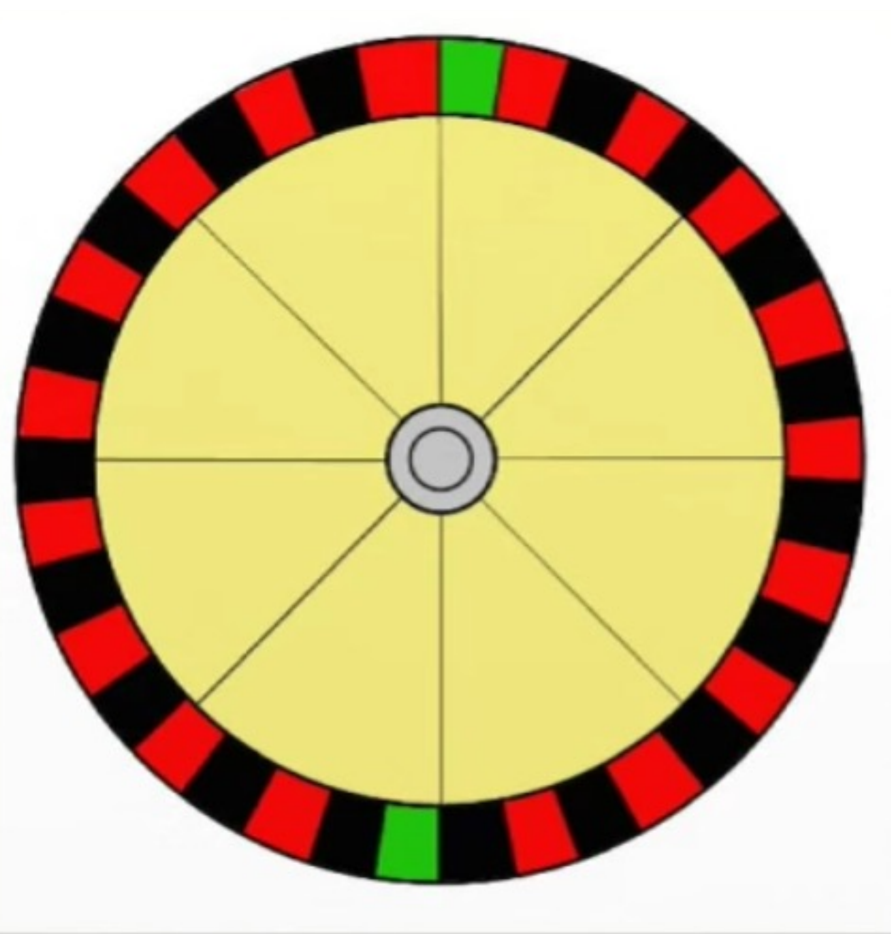

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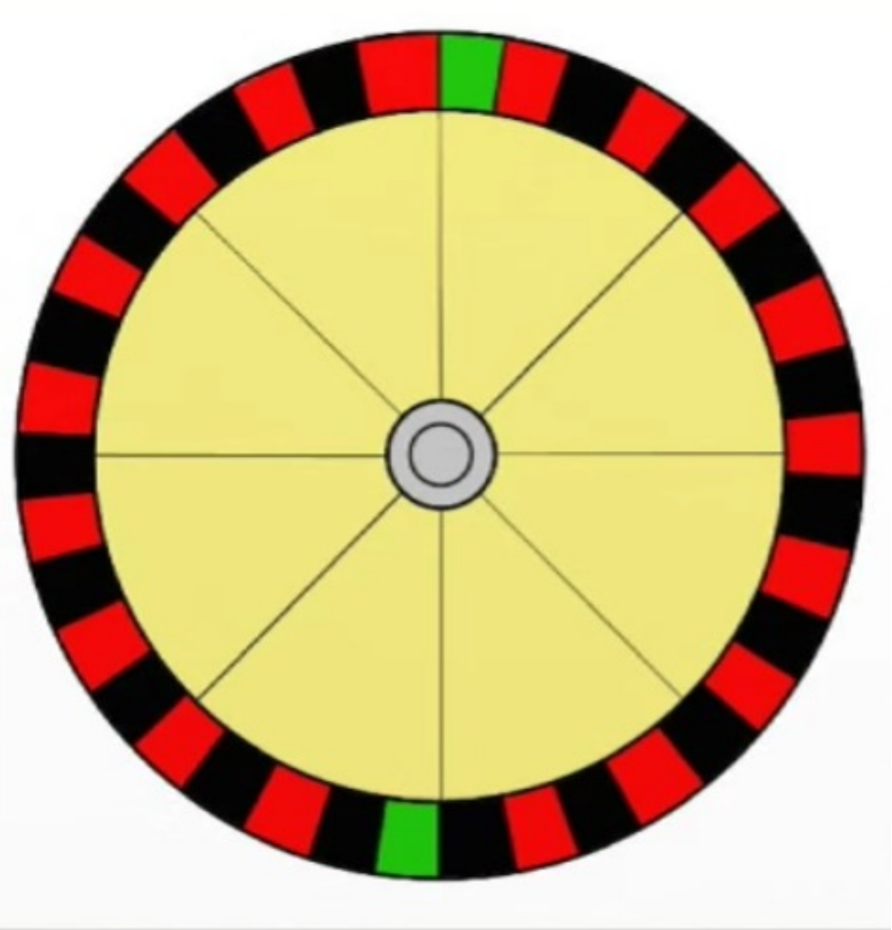
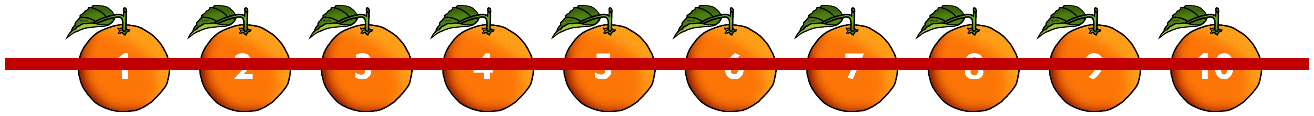
  - Bet on green
    $$P(win) = 2/38 \approx 0.05$$



Combinatorics review

COMBINATORICS: WHAT FOR?

- Combinatorics: the art of counting **without** enumerating.



- Computer science: determine time and space complexity of an algorithm.

- Counting number of elements in a set – basis for probability theory.

Fundamental counting principles

TWO FUNDAMENTAL RULES

Sum Rule

- If there are **a** ways of doing A and **b** ways of doing B and we can not do both at the same time, then there are **a + b** ways to choose <u>one of the actions</u>.

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$$|A \cup B| = |A| + |B|$$

when $A \cap B = \emptyset$

RULE OF SUM: EXAMPLE

- We want to order dinner for tonight:
  - Pizza:

    Dominos, Pizza Hut or King Slice (3 restaurants)
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  3 pizza places **+** **2** sushi places = **3** **+** **2** = 5 options in total

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$$|P| = 3, |S| = 2, \qquad |P \cup S| = |P| + |S| = 3 + 2 = 5$$

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$$|A \times B| = |A| \cdot |B|$$

RULE OF PRODUCT: EXAMPLE

- You want to order pizza:
  - first, choose the type of crust:
    thin or thick (2 choices);
  - second, choose one topping:
    cheese, pepperoni, or sausage (3 choices).
- How many different pizzas are there?

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= **2** **x** **3** = 6 different pizzas

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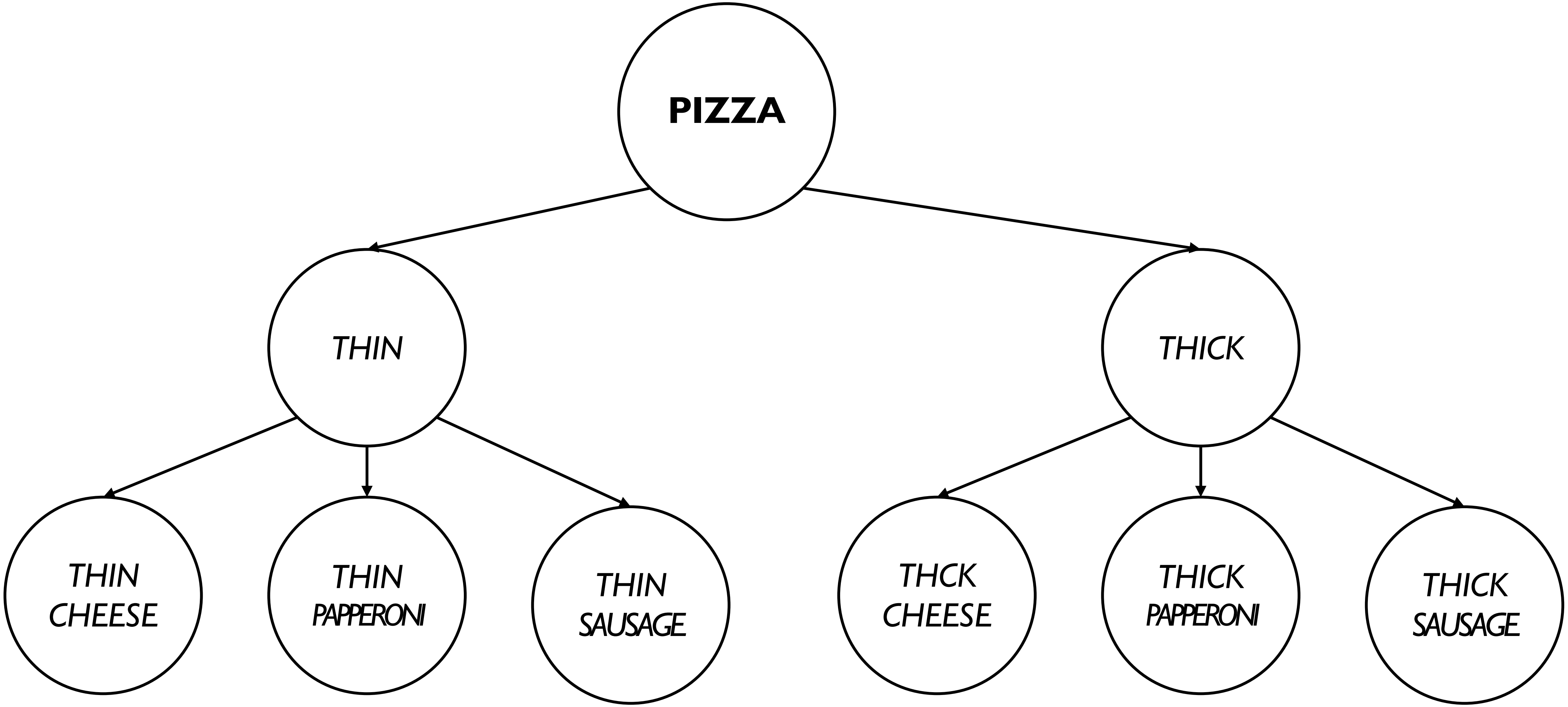

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$$|C \times T| = |C| \cdot |T| = 2 \cdot 3 = 6$$

EXAMPLE: NUMBER OF PASSWORDS

- A valid password contains 6 to 8 symbols
  - first symbol is a letter (upper- or lowercase);
  - other symbols are letters (upper- or lowercase) or digits.
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  Sum rule:

$$N = P_6 + P_7 + P_8 \qquad P_i \text{ – \# passwords of length i}$$

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- Solution:

Sum rule: $N = P_6 + P_7 + P_8$ P_i – # passwords of length i

First symbol:

upper- or lowercase letter – $2 \cdot 26 = 52$ options

Other symbols:

upper- or lowercase letter or digit

EXAMPLE: NUMBER OF PASSWORDS

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First symbol:

upper- or lowercase letter – $2 \cdot 26 = 52$ options

Other symbols:

upper- or lowercase letter or digit – $52 + 10 = 62$ options

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Product rule: $P_6 =$ $P_7 =$ $P_8 =$

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Product rule: $P_6 = 52 \cdot 62^5$ $P_7 =$ $P_8 =$

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- Solution:

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$$N = 52 \cdot 62^5 + 52 \cdot 62^6 + 52 \cdot 62^7$$

COUNT NUMBERS DIVISIBLE BY 3

- How many numbers between 1 and 100 are divisible by 3?

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- How many numbers between 1 and 100 are divisible by 3?

  Every 3rd number is divisible by 3: 3, 6, 9, …, 93, 96, 99.

  Thus, there are [100 / 3] = 33 of them.

COUNT NUMBERS DIVISIBLE BY 5

- How many numbers between 1 and 100 are divisible by 5?

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  Every 5th number is divisible by 5: 5, 10, 15, …, 95, 95, 100.

COUNT NUMBERS DIVISIBLE BY 5

- How many numbers between 1 and 100 are divisible by 5?

  Every 5th number is divisible by 5: 5, 10, 15, …, 95, 95, 100.

  Thus, there are [100 / 5] = 20 of them.

COUNT NUMBERS DIVISIBLE BY 3 & 5

- How many numbers between 1 and 100 are divisible by 3?

  33

- How many numbers between 1 and 100 are divisible by 5?

  20

- How many numbers between 1 and 100 are divisible by 3 or 5?

COUNT NUMBERS DIVISIBLE BY 3 & 5

- How many numbers between 1 and 100 are divisible by 3?

  33

- How many numbers between 1 and 100 are divisible by 5?

  20

- How many numbers between 1 and 100 are divisible by 3 or 5?

  Sum rule:

  33 numbers divisible by 3 + 20 numbers divisible by 5 =
  = 53 numbers divisible by 3 or 5.

COUNT NUMBERS DIVISIBLE BY 3 & 5

- How many numbers between 1 and 100 are divisible by 3?

  33

- How many numbers between 1 and 100 are divisible by 5?

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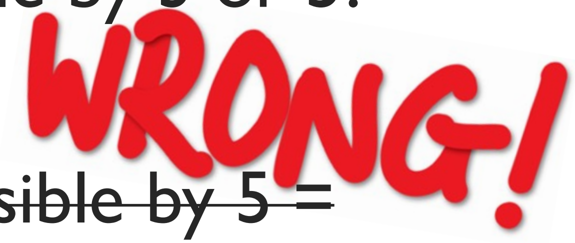
- How many numbers between 1 and 100 are divisible by 3 or 5?

  ~~*Sum rule:*~~



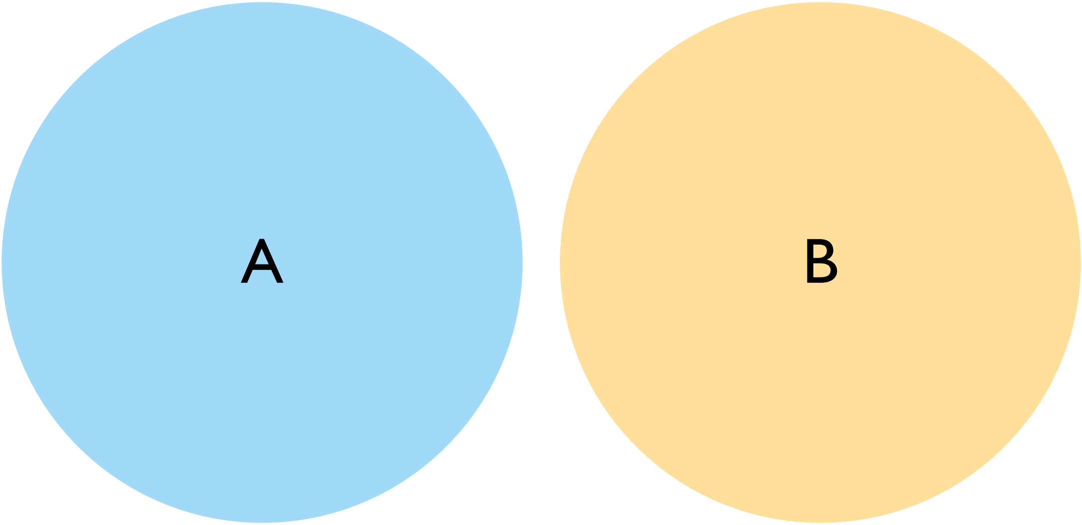
~~33 numbers divisible by 3 + 20 numbers divisible by 5 =~~
~~= 53 numbers divisible by 3 or 5.~~

Exclusion-Inclusion Principle

ADDITION PRINCIPLE

- If A and B are disjoint sets ($A \cap B = \emptyset$), then

$$|A \cup B| = |A| + |B|$$



INCLUSION-EXCLUSION (2 SETS)

- If A and B are **not** disjoint ($A \cap B = \emptyset$), then

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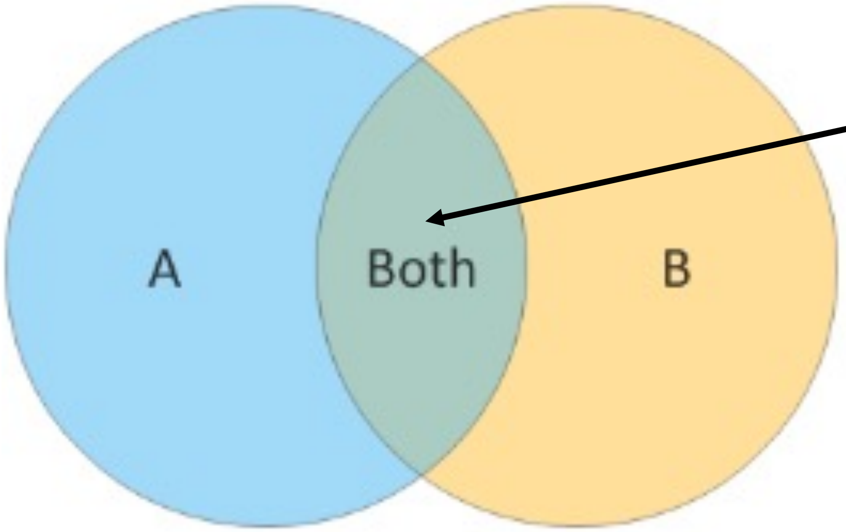

Problem:
counting this area
twice

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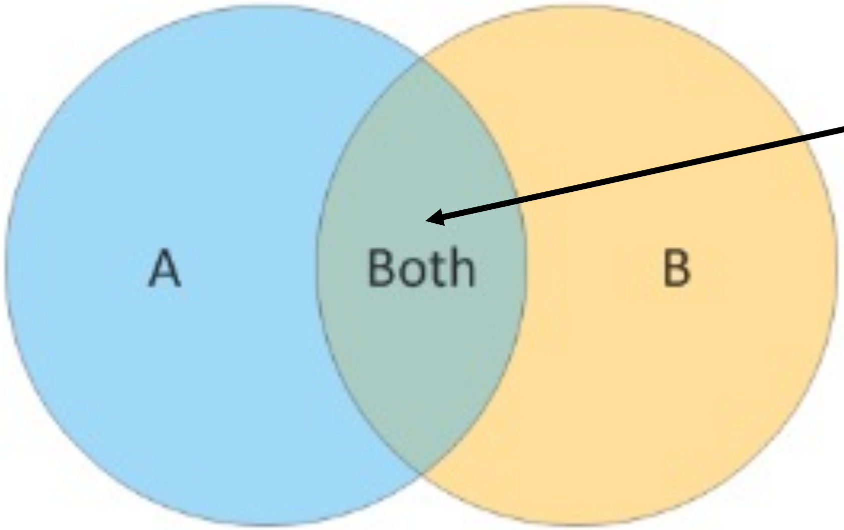
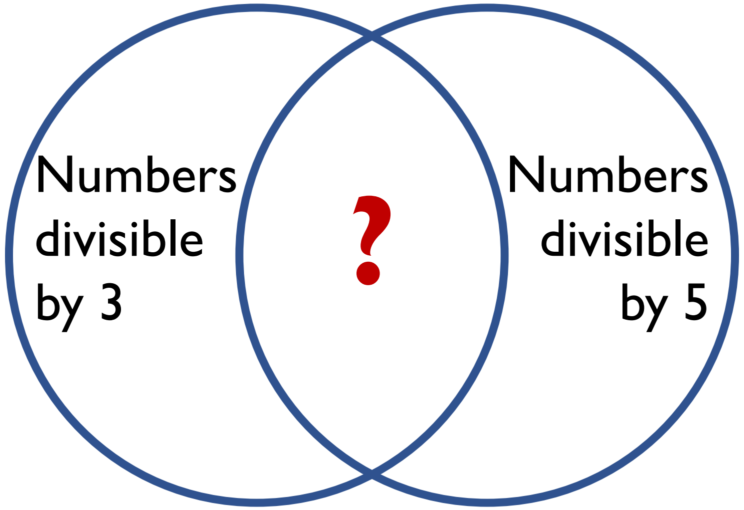


Problem: counting this area twice

BACK TO OUR EXAMPLE

- How many numbers between 1 and 100 are divisible by 3 or 5?

$$|D_3 \cup D_5| =$$



BACK TO OUR EXAMPLE

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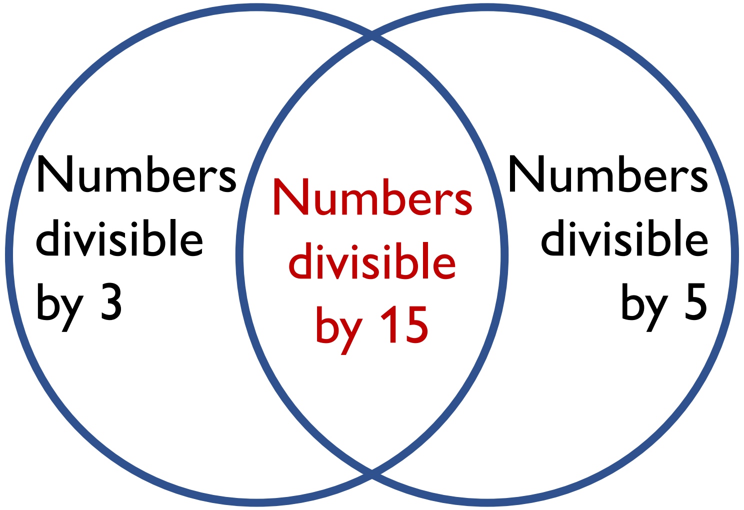

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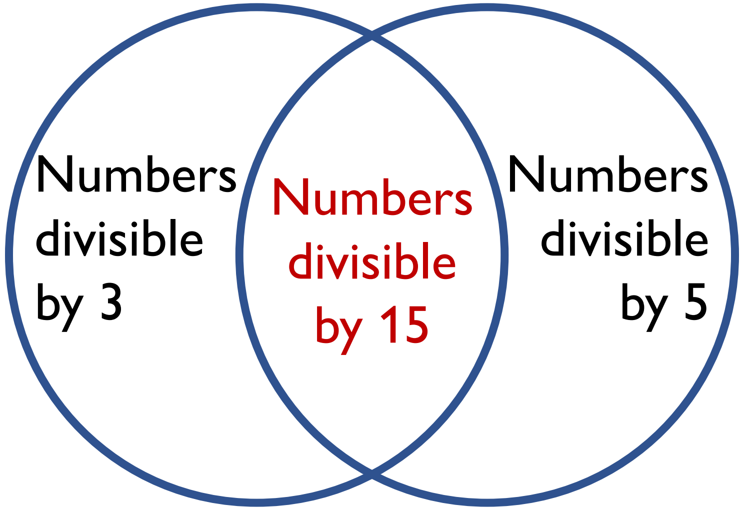

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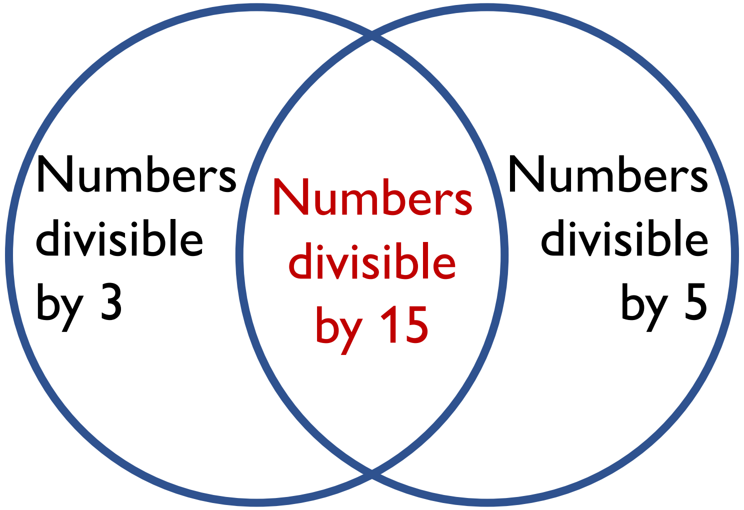
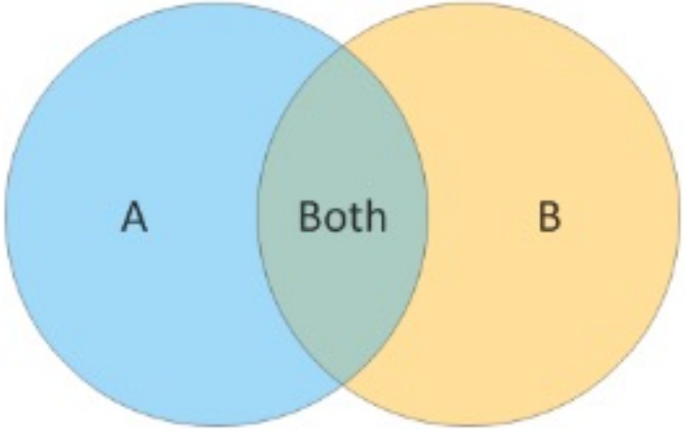
$$= 33 + 20 - 6 = 47.$$



PRINCIPLE OF INCLUSION-EXCLUSION FOR TWO SETS

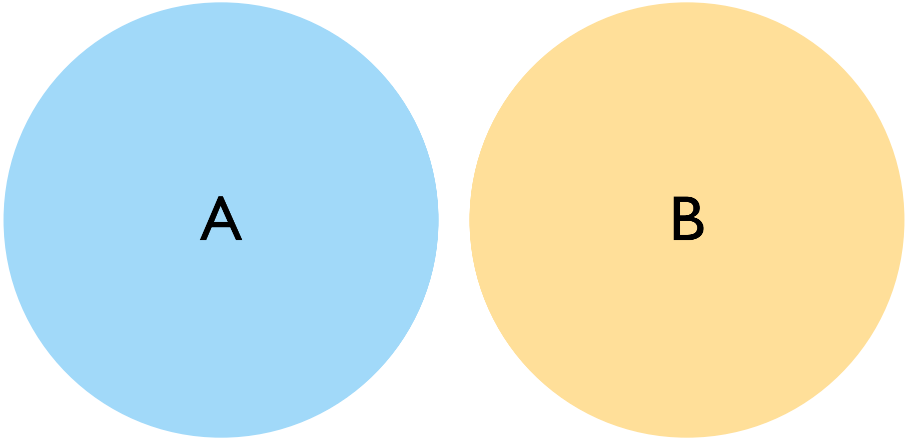
- For every two finite sets A and B

$$|A \cup B| = |A| + |B| - |A \cap B|.$$



- In particular, if A and B are disjoint, then $|A \cap B| = \emptyset$ and

$$|A \cup B| = |A| + |B|.$$



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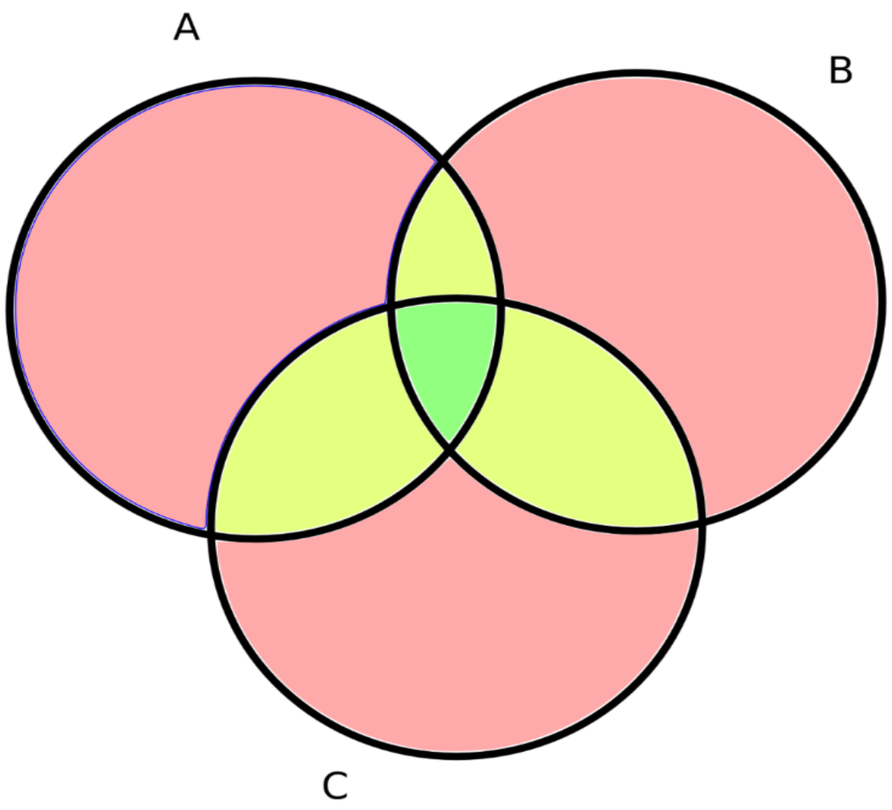

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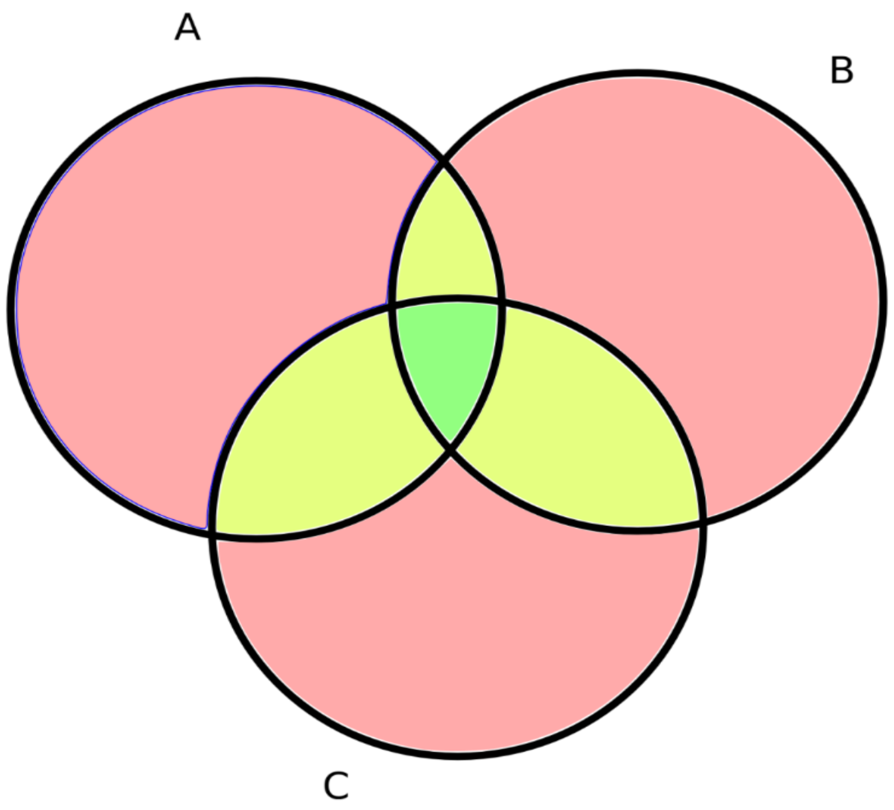

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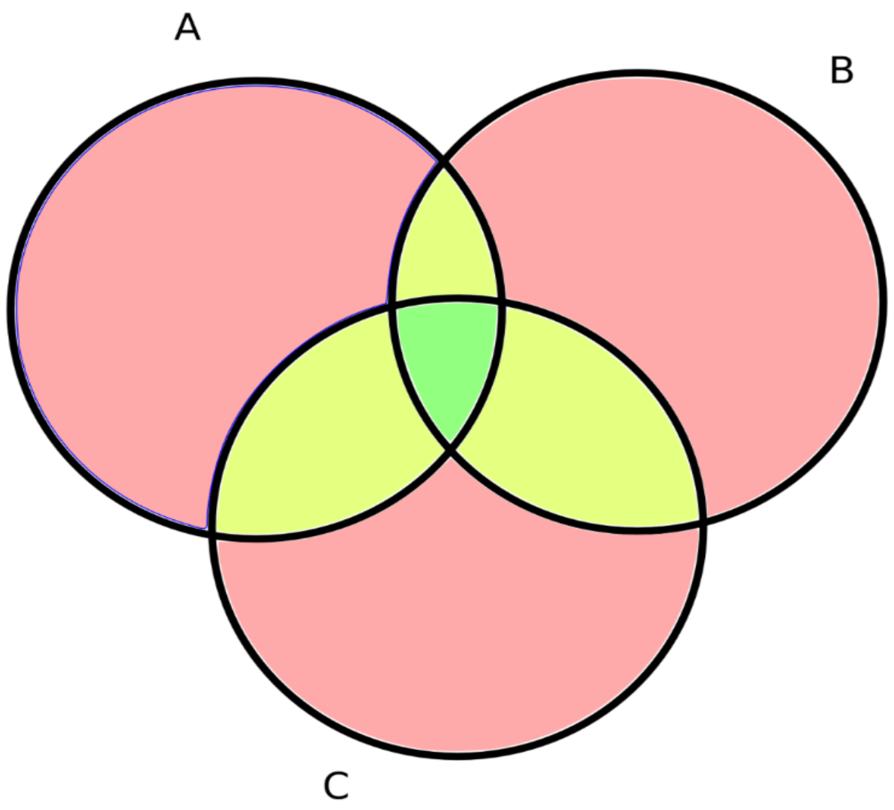

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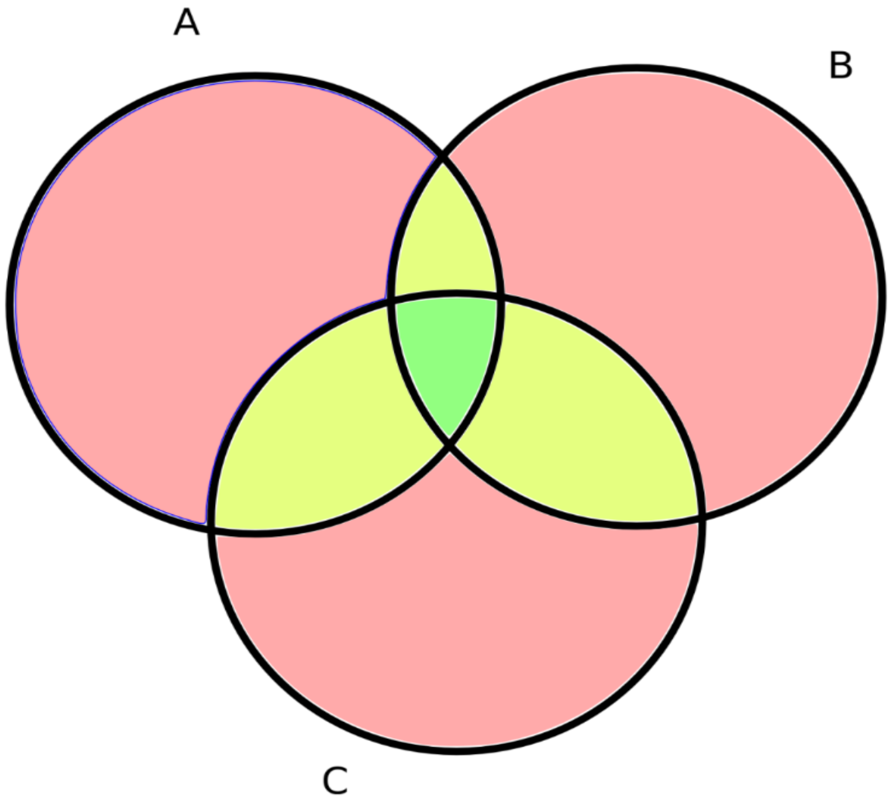

Practice problems

Google classroom -> Day 1

STUDENTS AND COURSES

- A total of 36 students plan to take at least one of Discrete Mathematics, Algebra and Calculus during the coming semester:

Discrete Mathematics	23
Algebra	19
Calculus	18
Discrete Mathematics & Algebra	7
Discrete Mathematics & Calculus	9
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How many students plan to take *all three* courses?

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$36 = 23 + 19 + 18 - 7 - 9 - 11 + N$ $\qquad$ $N = 3$ students

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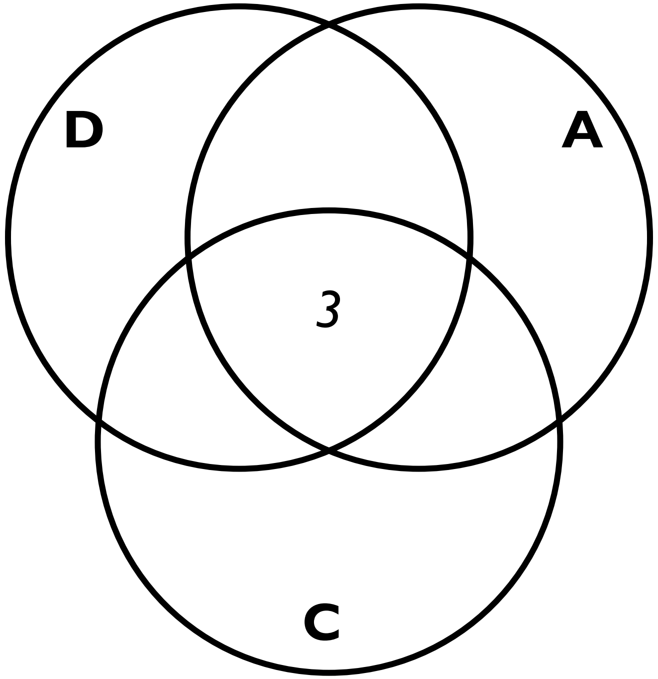

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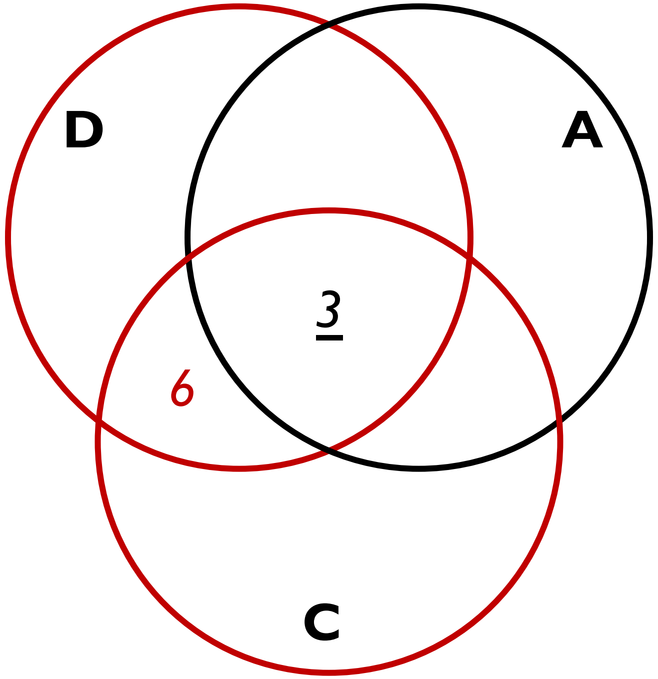

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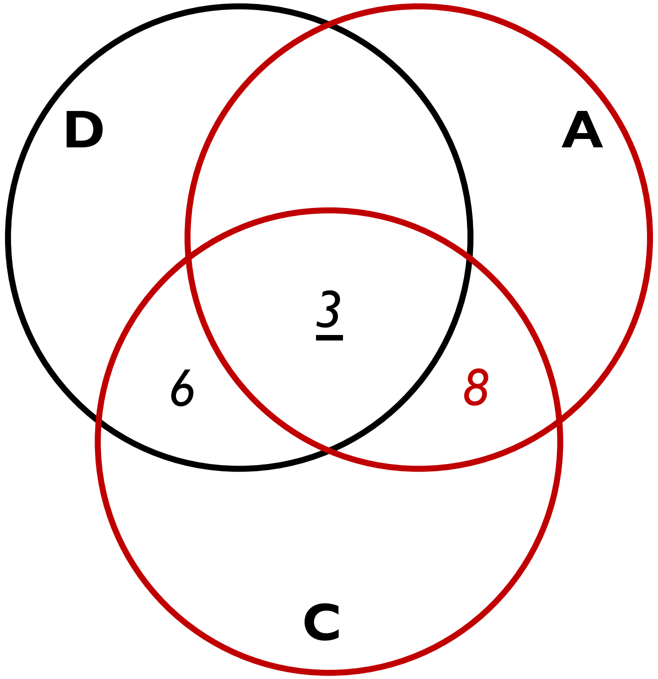

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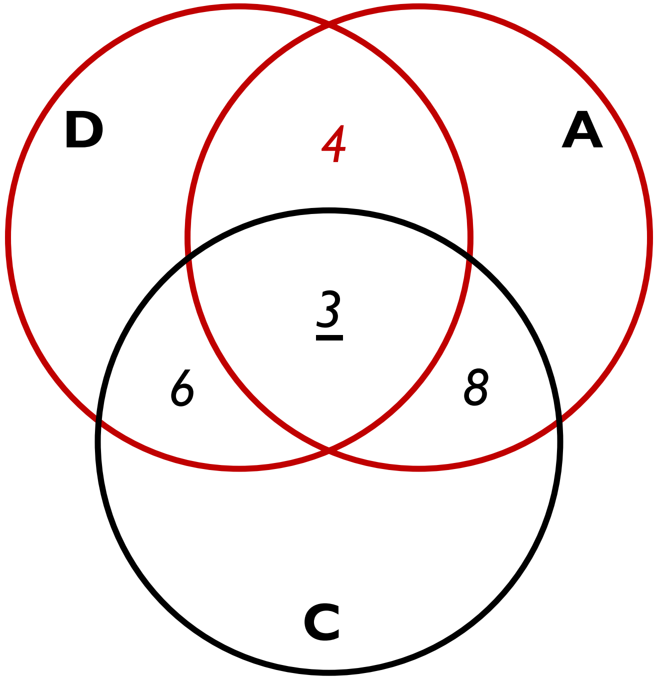

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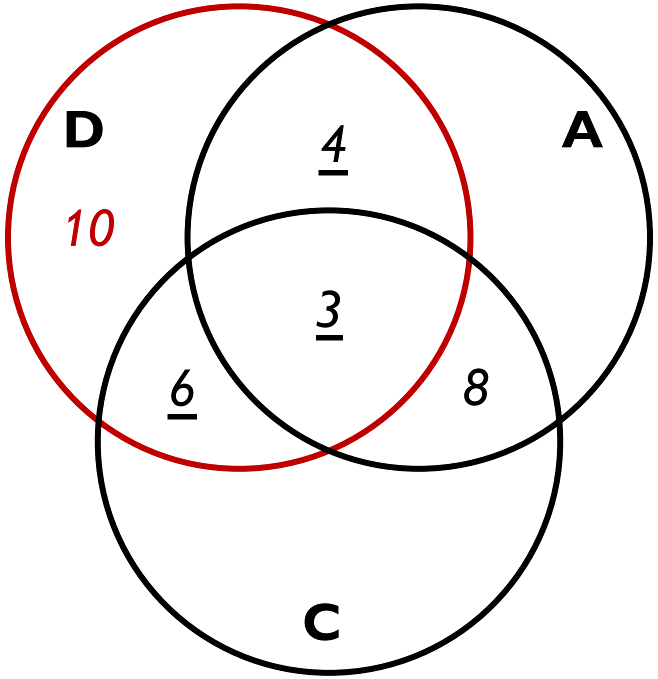

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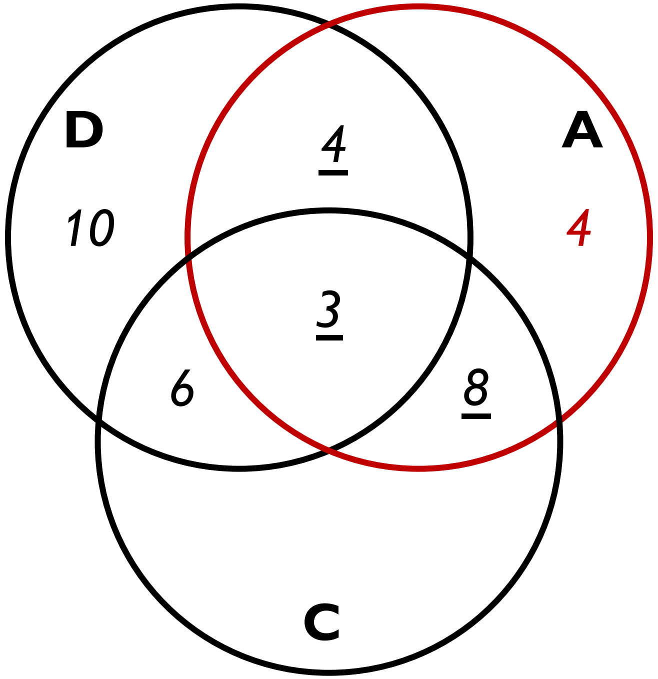

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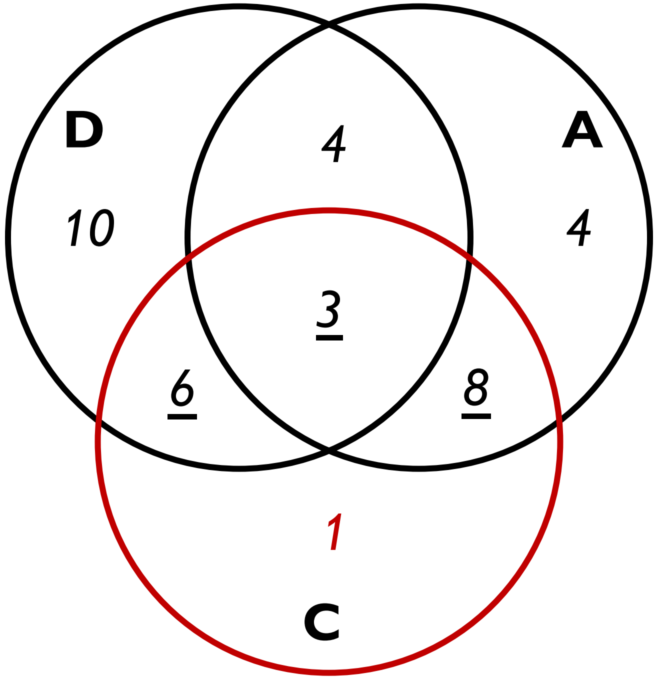

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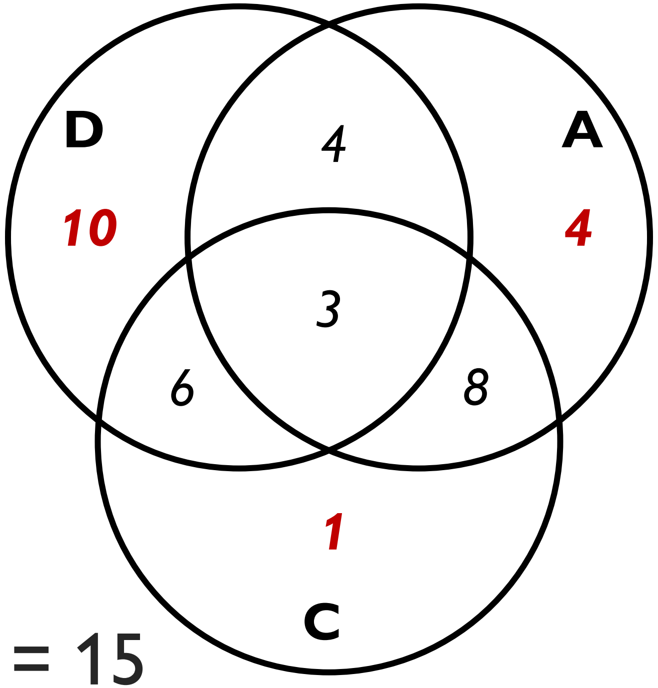

How many students plan to take *exactly one* of the courses? 10 + 4 + 1 = 15

PRINCIPLE OF INCLUSION-EXCLUSION FOR FOUR SETS

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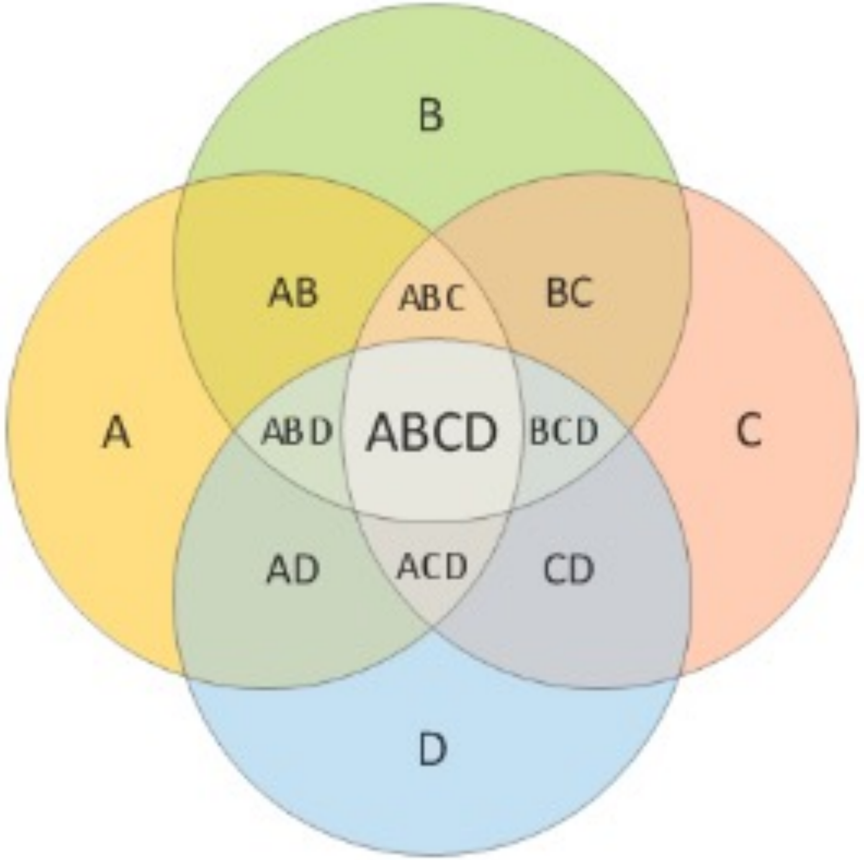

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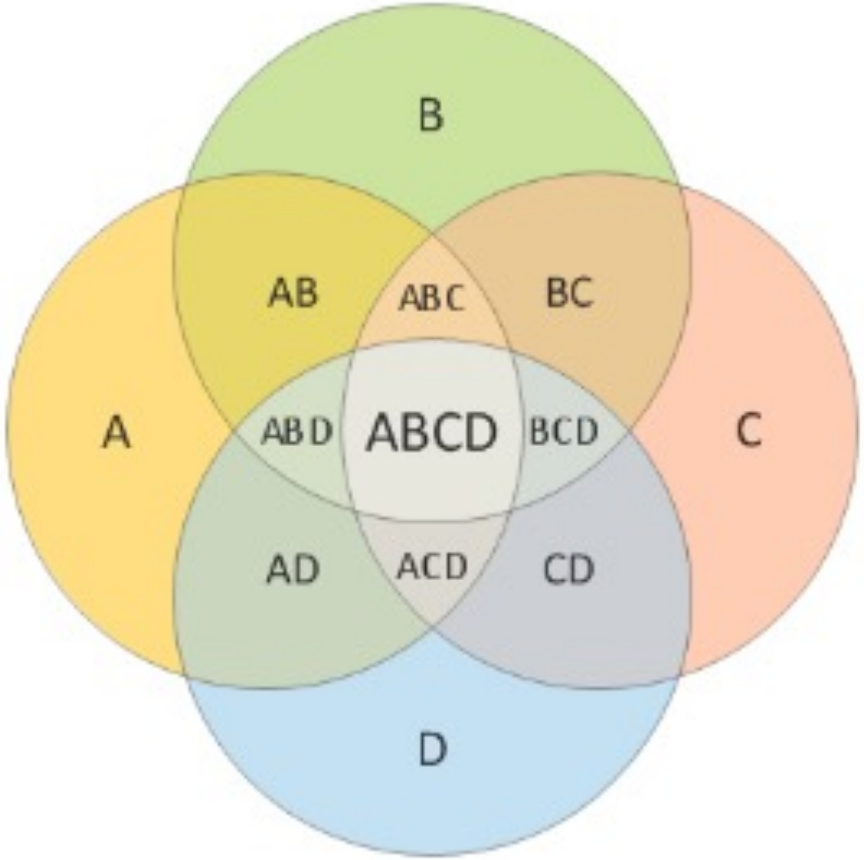

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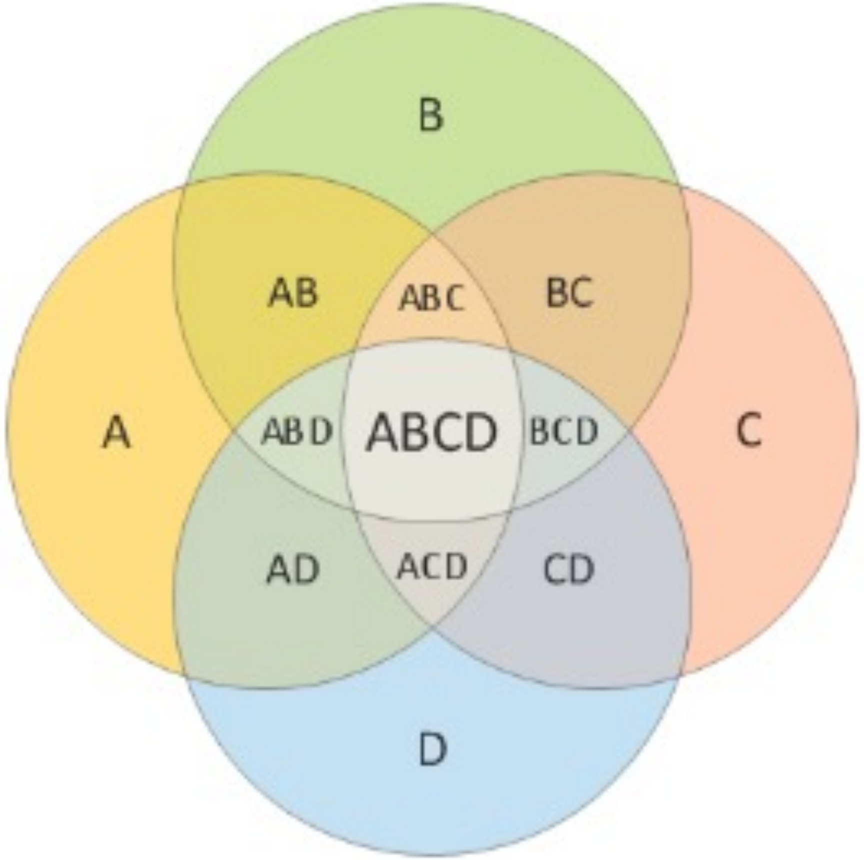

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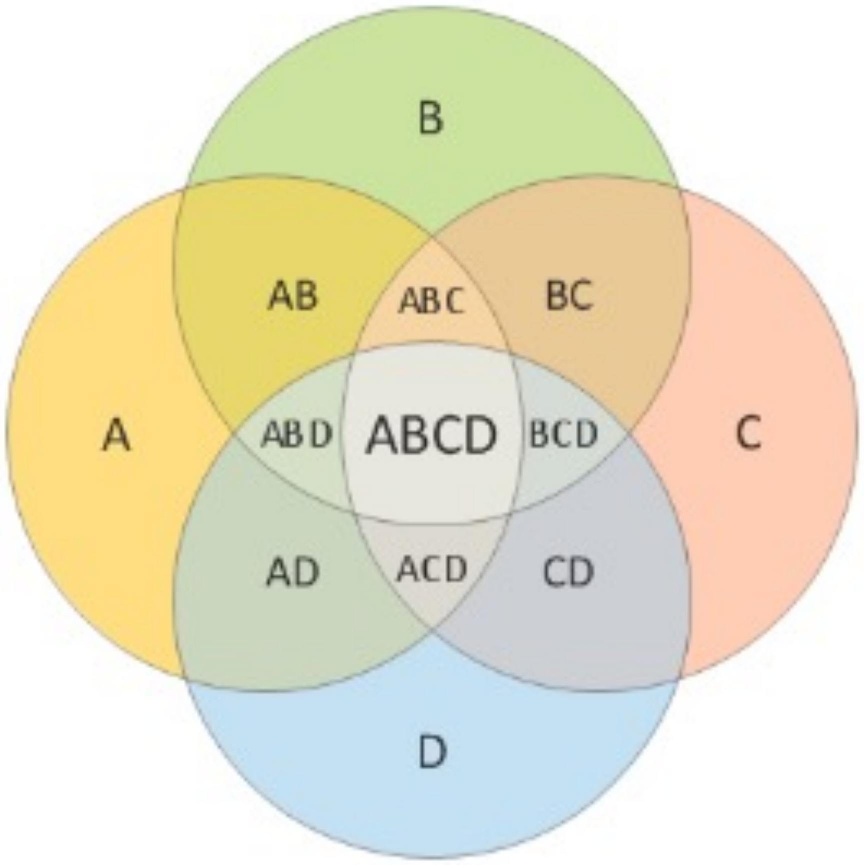

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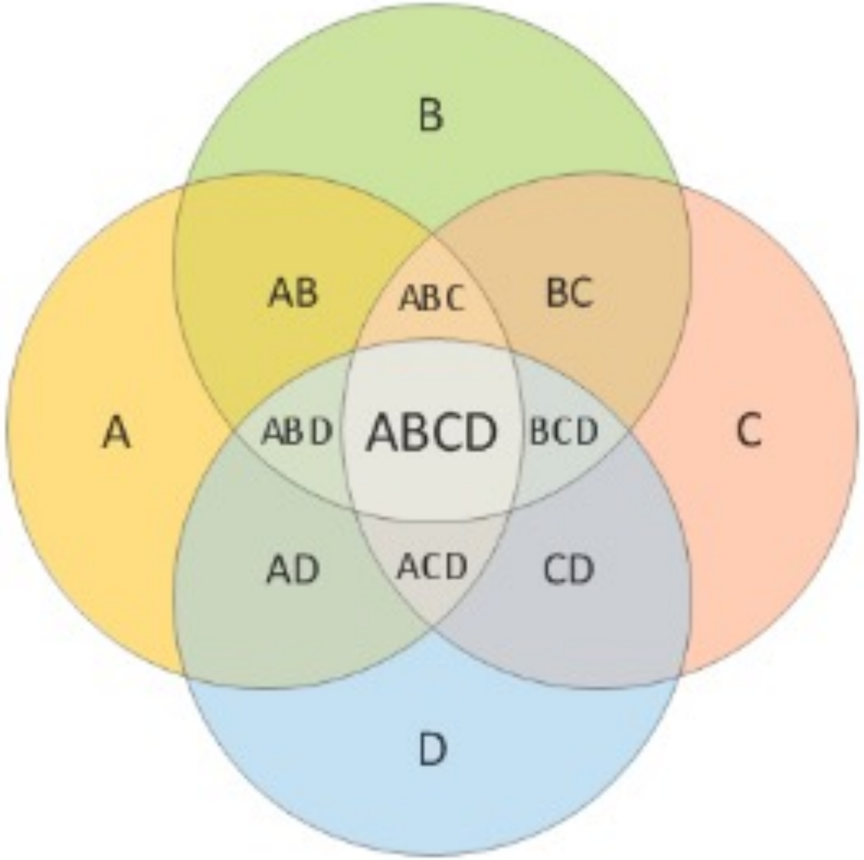

Permutations and combinations

POSSIBLE ARRANGEMENTS

- In how many ways can n people sit in a row?
- How many words of length n strings are there?

- We have n different candies.
  In how many ways can we chose $k \leq n$ out them?
- In how many ways can we distribute k identical candies among n kids?

POSSIBLE ARRANGEMENTS

	WITHOUT REPETITIONS	WITH REPETITIONS
ORDERED	In how many ways can n people sit in a row?	How many different n-bit strings are there?
NOT ORDERED	In how many ways can we chose k out of n different candies in a bag?	In how many ways can we distribute k identical candies among n kids?

POSSIBLE ARRANGEMENTS

	WITHOUT REPETITIONS	WITH REPETITIONS
ORDERED	*PERMUTATIONS* Seating n people in a row	*TUPLES* How many different n-bit strings are there?
NOT ORDERED	*COMBINATIONS* Choosing k out of n different candies in a bag	*COMBINATIONS with repetitions* Distributing k identical candies among n kids

Permutations

Ordered, each element can be used only once (no repetitions)

PERMUTATIONS OF n ELEMENTS

- In how many ways can n distinct objects be ordered?

  Example:

  In how many orders can we put 10 books on the shelf?

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$$10 \cdot$$

PERMUTATIONS OF n ELEMENTS

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  Example:

  In how many orders can we put n books on the shelf?

$$10 \cdot 9 \cdot$$

PERMUTATIONS OF n ELEMENTS

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  Example:

  In how many orders can we put n books on the shelf?

$$10 \cdot 9 \cdot 8 \cdot \ldots$$

PERMUTATIONS OF n ELEMENTS

- In how many ways can n distinct objects be ordered?

  Example:

  In how many orders can we put n books on the shelf?

$$10 \cdot 9 \cdot 8 \cdot \ldots \cdot 3 \cdot 2 \cdot 1 = 10!$$

PERMUTATIONS OF n ELEMENTS

- In how many ways can n distinct objects be ordered? Answer: $n!$

  Example:

  In how many orders can we put n books on the shelf?

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REPEATED ELEMENTS

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REPEATED ELEMENTS

If among n elements $k \leq n$ elements are not unique, with $n_1, n_2, \ldots, n_k$ repetitions respectively, then the number of possible permutations is

$$\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}$$

Tuples

Not ordered, each element can be used multiple times (repetitions are allowed)

NUMBER OF 6-BIT STRINGS

- Example of a 6-bit string:

0	**1**	**1**	**1**	**0**	**1**

- How many such strings are there?

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Each position:

NUMBER OF 6-BIT STRINGS

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Each position: 0 or 1 (2 choices)

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- Solution:

*	*	*	*	*	*

Each position: 0 or 1 (2 choices)

Product rule:

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6 = 64 \text{ different 6-bit strings.}$$

Combinations (without repetitions)

Not ordered, each element can be used only once (no repetitions)

COMBINATIONS

- How many committees of two people can be chosen from this group of four people:

Jones, Chan, Vello, Harris

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Chan, Jones	Jones, Harris	Harris, Vello	Vello, Chan
Chan, Harris	Jones, Vello	Harris, Chan	Vello, Jones
Chan, Vello	Jones, Chan	Harris, Jones	Vello, Harris.

COMBINATIONS

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- Number of 2-permutations: $4!/(4-2)! = 12$

Chan, Jones	Jones, Harris	Harris, Vello	Vello, Chan
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Chan, Vello	Jones, Chan	Harris, Jones	Vello, Harris.

- Since the roles are the same, many of these are the same…

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- Since the roles are the same, many of these are the same. How do we fix this?
- People in the committee are indistinguishable and can be re-ordered in 2! ways:

$$\frac{4!}{\mathbf{2}!(4-2)!} \text{ different committees}$$

COMBINATIONS: GENERAL CASE

- Suppose we want to choose k elements from a set with n elements, in no specific order.

  In other words: select a subset of size k from a set of size n.

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$$C(n,k) = \frac{n!}{k!\,(n-k)!}$$

NOTATION

$$C(n,k) = \binom{n}{k} = C_n^k = \frac{n!}{k!\,(n-k)!}$$

Combinations with repetitions

BOOKS

- In how many ways can we place 20 books on 5 bookshelves? Each shelf can accommodate from 0 to 20 books.

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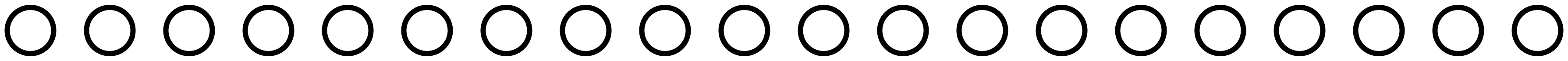

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○ ○|○ ○ ○ ○|○ ○ ○ ○ ○ ○ ○ ○ ○|○ ○ ○ ○|○

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$$C(\qquad\qquad n-1) =$$

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○ ○|○ ○ ○ ○|○ ○ ○ ○ ○ ○ ○ ○ ○|○ ○ ○ ○|○

$$C(k + n - 1, n - 1) =$$

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$$C(k+n-1, n-1) = C(24,4) =$$

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- In how many ways can we place 20 books on 5 bookshelves? Each shelf can accommodate from 0 to 20 books.

- $n = 5$ bookshelves
- $k = 20$ books

We need to assign shelves to the books

○ ○|○ ○ ○ ○|○ ○ ○ ○ ○ ○ ○ ○ ○|○ ○ ○ ○|○

$$C(k + n - 1, n - 1) = C(24,4) = \frac{24!}{4!20!} = 10626$$

To sum up

POSSIBLE ARRANGEMENTS

	WITHOUT REPETITIONS	WITH REPETITIONS
ORDERED	*PERMUTATIONS* Seating n people in a row $n!$	*TUPLES* Counting different n-bit strings are there? k^n
NOT ORDERED	*COMBINATIONS* Choosing k out of n different candies in a bag $C(n,k) = \frac{n!}{k!\,(n-k)!}$	*COMBINATIONS with repetitions* Distributing k identical candies among n kids $C(k+n-1, n-1)$

Practice problems

Google classroom -> Day 1

STRINGS

- Each entry of a string is an element of the set $S = \{0, 1, 2\}$. How many such strings of length 6 are there that begin with 022 or end with 01?

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  # strings beginning with 022:

  # string ending with 01:

  # strings beginning with 002 and ending with 01:

STRINGS

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Inclusion-exclusion

\# strings beginning with 022: $3^3 = 27$

\# string ending with 01:

\# strings beginning with 002 and ending with 01:

STRINGS

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  Inclusion-exclusion

  \# strings beginning with 022: $3^3 = 27$

  \# string ending with 01: $3^4 = 81$

  \# strings beginning with 002 and ending with 01:

STRINGS

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STRINGS

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Inclusion-exclusion

\# strings beginning with 022: $3^3 = 27$

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$$27 + 81 - 3 = 105$$

sequences that start with 002 or end with 01

NUMBER OF SUBSETS OF A SET

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- **Solution:**

  Each element can either be included in the subset (1) or not (0):

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NUMBER OF SUBSETS OF A SET

- How many subsets does a set of *n* elements have?
- **Solution:**

  Each element can either be included in the subset (1) or not (0):

0	**1**	**0**	**…**	**0**	**0**	**1**

- 2 options per each of the *n* element ***(just like bit-strings!)***.

  Product rule: $2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$ subsets