PROBABILITY & STATITSICS

Lecture 2 – Basic probability review

LAST TIME

- Probability as limiting frequency
- Basic combinatorics review

TODAY

- Review of practice problems
- Probability of an event: more examples
- Axioms of probability

Practice problems review

COMMITTEE

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1 freshman, 2 sophomores and 1 junior

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$N_{211} + N_{121} + N_{112}$ committees with at least one student from each group

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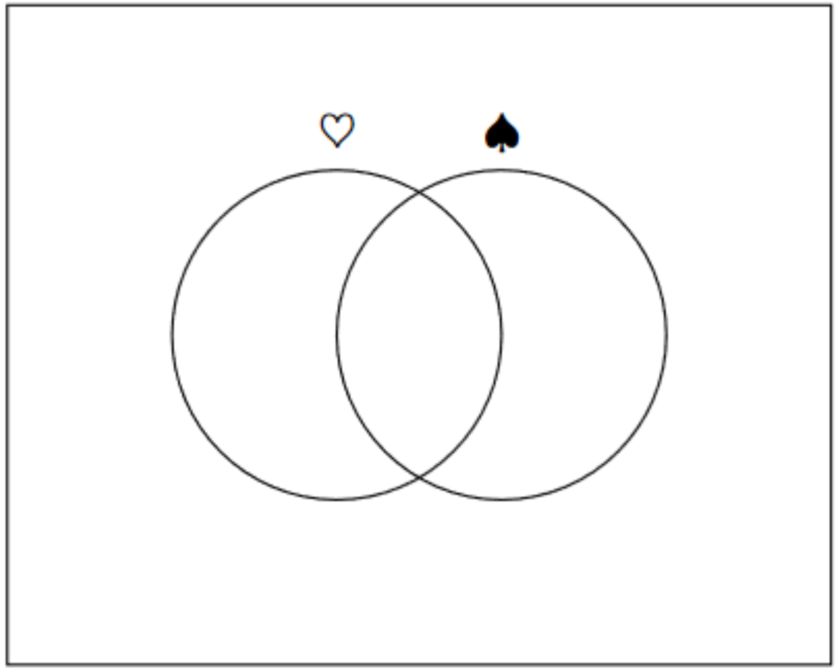

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So, the number of possible shuffles is

$$52! - (51! + 51! - 50!) = 52! - 2 \cdot 51! + 50!$$

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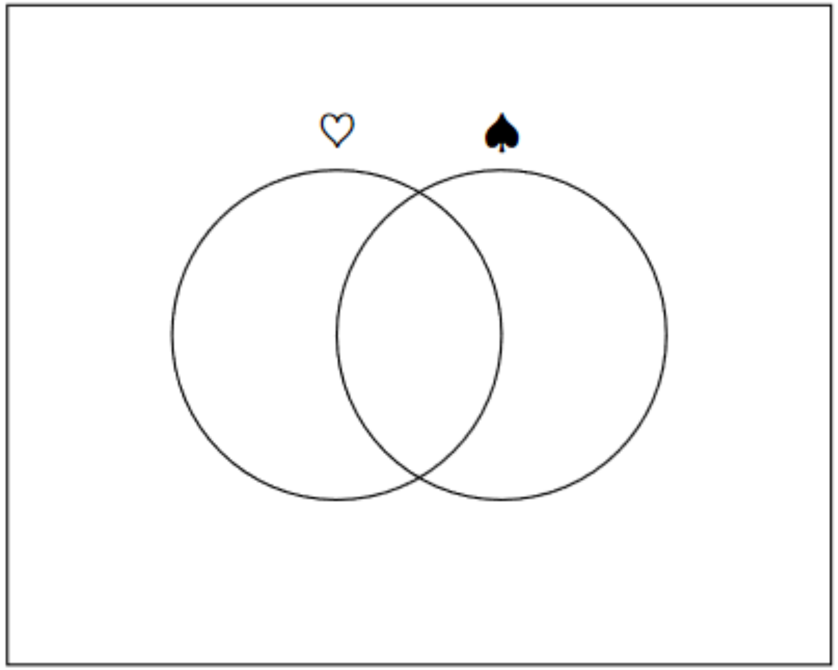

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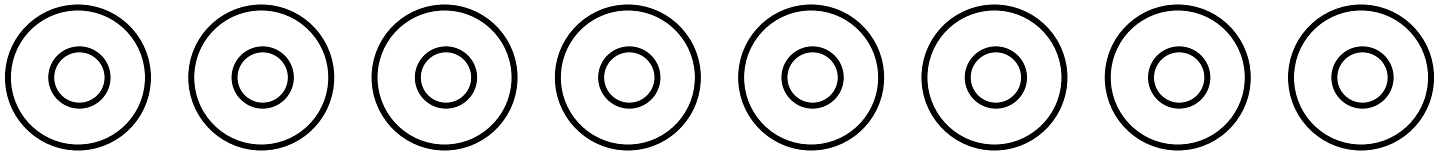

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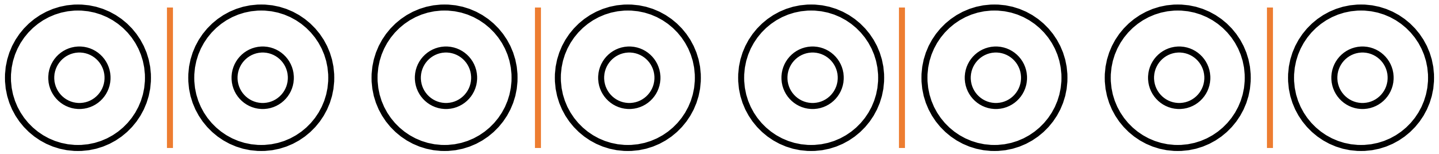

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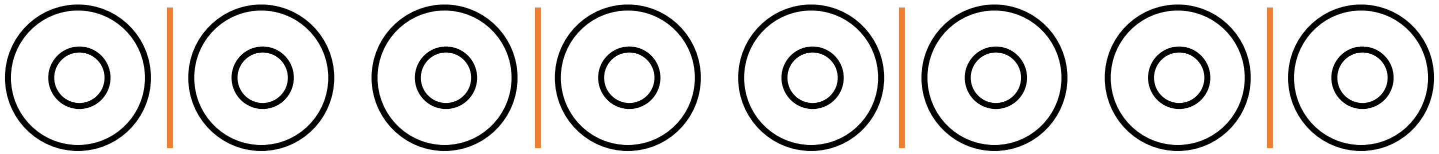

$$C(\qquad, 5-1) =$$

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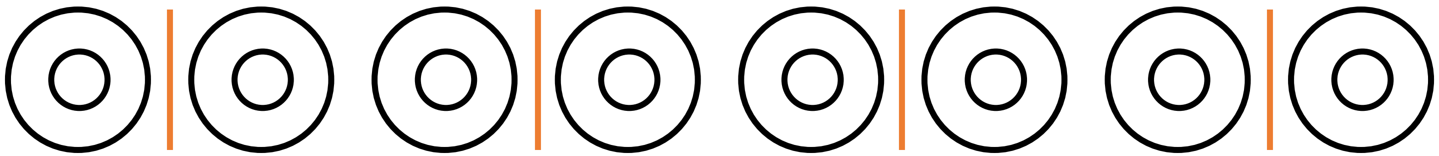

$$C(8+5-1, 5-1) =$$

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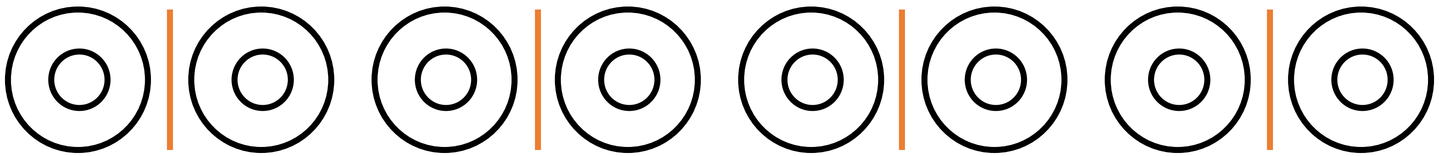

$$C(8+5-1, 5-1) = C(12,4) = \frac{12!}{4!8!} = 495$$

ways to fill a box of 8 doughnuts

Back to probability

REMINDER: BASIC DEFINITIONS

- A **random experiment** is a process by which we observe something uncertain.

- The result of a random experiment is called the **outcome**.

- The set of all possible outcomes is called the **sample space** (denoted by S).

REMINDER: RELATIVE FREQUENCY

- Probability of an event is defined by the **limiting frequency** with which this event appears in a long series of similar experiments.

- Example:
  If we flip a fair coin infinitely many times, it will come up heads half of the times.

Programming Exercise

Google Classroom -> Day 2

Axioms of probability

AXIOMS OF PROBABILITY

1. **For any event E, $P(E) \geq 0$.**

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Example: rolling a die

E_1: 'get 1' $\quad P(E_1) = \frac{1}{6}$

E_2: 'get an even number' $\quad P(E_2) = \frac{1}{2}$

E_3: 'get 27' $\quad P(E_3) = 0$

AXIOMS OF PROBABILITY

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Example: tossing a coin

$$S = \{H, T\}$$

$E = S$: ‘get Heads or Tails’ $\qquad P(E) = 1$

AXIOMS OF PROBABILITY

3. **If $E_1, E_2, E_3, \dots$ are mutually exclusive events, then**

$$P(E_1 \cup E_2 \cup E_3 \dots) = P(E_1) + P(E_2) + P(E_3) + \cdots$$

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$$P(E_A \cup E_B) = P(E_A) + P(E_B) = 0.2 + 0.4 = 0.6$$

Complementary events

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$$\rightarrow P(E) = 1 - P(E^C) = 1 - 1/2^{10}$$

Union of events

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DRAWING NAMES

A total of 20 students participate in a drawing in which one name is chosen at random. There are 6 seniors (2 men and 4 women), 5 juniors (3 men and 2 women) and 9 freshmen (5 men and 4 women). What is the probability that the name that is drawn belongs to **either a woman or to a junior**?

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$$= \frac{4+2+4}{20} + \frac{5}{20} - \frac{2}{20} = 0.5 + 0.25 - 0.1 = 0.65$$

More examples

PROBABILITY OF AN EVENT

- Consider a random experiment where all outcomes are equally likely.
- Probability $P(E)$ of an event E is such an experiment is

$$P(E) = \frac{\#\, ways\ E\ can\ happen}{\#\, possible\ outcomes}$$

- Let E^C be the complementary event. Then

$$P(E^C) = 1 - P(E)$$

- Let F be some other event. Then

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

ROLLING DICE

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  Compute the probability that the sum of the faces is 3.

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$$P(E) = \frac{2}{36} = \frac{1}{18} \approx 0.056$$

Practice problems

Google Classroom -> Lecture 2

BALLS FROM A BOWL

- A bowl contains 15 balls: 6 are red, 5 are blue and 4 are green. What is the probability that two balls selected at random from the bowl **have the same color?**

  Method 1: combinations *(grabbing 2 balls at the same time)*

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  $$|S| = C(15,2) = \frac{15!}{2!13!} = 105 \text{ ways to choose 2 balls out of 15}$$

BALLS FROM A BOWL

- A bowl contains 15 balls: 6 are red, 5 are blue and 4 are green. What is the probability that two balls selected at random from the bowl **have the same color?**

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ways to choose 2 same-color balls

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$$P(E) = 31/105$$

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  Method 2: 2-permutations *(grabbing 2 balls one-by-one)*

$|S| = 15 \cdot 14 = 210$ ways to pick 2 balls out of 15 one-by-one

$$|E| = |E_{red,red}| + |E_{blue,\, blue}| + |E_{green,green}| =$$
$$= 6 \cdot 5 + 5 \cdot 4 + 4 \cdot 3 = 30 + 20 + 12 = 62$$
ways to pick two same-color balls one by one

$$P(E) = 62/210 = 31/105$$

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$$=$$

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$$= \frac{|E_2|}{|S|} + \frac{|E_5|}{|S|} - \frac{|E_2 \cap E_5|}{|S|} =$$

$$= \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = 0.6$$

POKER

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$$C(52,5) = \frac{52!}{5! \cdot 47!} = 2{,}598{,}960$$

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- What is the probability of being dealt a poker hand that consists of the 3♥, 5♦, 6♠, 10♣ and Q♦?

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$$P(E) = \frac{1}{2{,}598{,}960} \approx 0.000000385$$

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$$= \frac{33}{100} + \frac{14}{100} - \frac{4}{100} = \frac{43}{100} = 0.43$$

CLASS PICTURE

- At school, 4 girls and 6 boys are lining up for a class picture in a random order. What is the probability that …
- **… all girls are standing next to each other?**
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CANDIES

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$$P(E) = \frac{C(14,6)}{C(21,6)} = \frac{3003}{54264} = \ 0.055$$

Monty Hall

https://youtu.be/4Lb-6rxZxx0

MONTY HALL