Method of Maximum Likelihood: Practice Problems

January 2023

1. Consider a discrete random variable X that takes values $0, 1, 2, 3, ...$ Its probability mass function is defined as follows:

$$P(X = k) = \theta(1 - \theta)^k, \ k >= 0$$

where θ is an unknown parameter of the distribution, $0 < \theta < 1$.

   (a) Derive the maximum likelihood estimator $\hat{\theta}$ for the unknown parameter θ. First, define the likelihood function. Then, find the value of that maximises it.

   (b) A random sample of size $n = 1000$ from the distribution above gave $\sum_{i=1}^{n} X_i = 980$. What is the estimate of θ that the estimator from the previous step would provide?

2. Suppose that you want to model the maximum height of the waves (in meters) at a certain beach. You represent it by a random variable with the following density function:

$$p(x) = \frac{x}{\theta} e^{-\frac{x^2}{2\theta}}, \ x \geq 0, \ \theta > 0$$

   (a) Obtain the maximum likelihood estimator for the parameter θ from the random sample $\{X_1, X_2, \ldots, X_n\}$ by completing the following steps:

      i. Write down the likelihood function.
      ii. Now, write down log-likelihood.
      iii. Finally, maximize log-likelihood to obtain the maximum likelihood estimator for θ.

   (b) Suppose that the past six years, the following maximum heights were observed:

$$3, 2, 2.5, 2, 3, 1.5$$

   What would be the maximum likelihood estimate of θ in that case?

   (c) Show that the estimator you obtained is unbiased.