Weakly dumped coupled oscillator model for lab experiment design

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Abstract

A forced weakly dumped harmonic oscillator having two degrees of freedom is considered. In Laboratory 3 it is implemented on air-track experiment set. The goal of the experiment is to find oscillator's natural frequencies and compare it with the model. Having natural frequencies found theoretically, the system response to forced oscillations is tested. It is expected that the system will response on forced oscillations frequencies close to natural frequencies. The system response is considered in terms of potential energy stored in springs.

In this document the coupled oscillator model is described.

1 Natural frequencies

A two-degree forced oscillator as shown on Figure 1. (Illustration credit [1, p.93]). Three carts a and b installed on an air-track. Third cart s is moved harmonically with radian frequency ω. Carts are coupled with strings k_i.

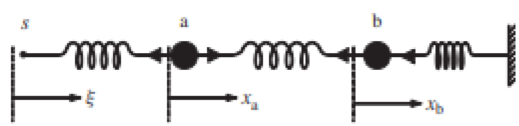

Figure 1: Forced oscillations with a coupled oscillator. The end s of the spring is moving with frequence ω.

Let's suggest that energy loss due to friction is negligible. For free oscillations problem the motion equation in matrix is

$$[m]\ddot{x} + [k]x = 0 \tag{1}$$

where

$$m = \begin{bmatrix} m_a & 0 \\ 0 & m_b \end{bmatrix}, \tag{2}$$

and

$$[k] = \begin{bmatrix} k_l + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}. \tag{3}$$

Squared natural frequencies ω_i^2 are found as eigenvalues of the matrix

$$\det A = \begin{vmatrix} \frac{k_1+k_2}{m_a} - \omega^2 & -\frac{k_2}{m_a} \\ -\frac{k_2}{m_b} & \frac{k_2+k_3}{m_b} - \omega^2 \end{vmatrix} = 0 \tag{4}$$

Springs are not ideal and the have their own weight. It can be taken into account by adding to the weight of cart half of weights of adjacent springs. Denoting m_{k_i} as as weight of spring k_i, we receive

$$\begin{aligned} m_{a\text{effective}} &= m_a + \frac{1}{2}(m_{k_1} + m_{k_2}), \\ m_{b\text{effective}} &= m_b + \frac{1}{2}(m_{k_2} + m_{k_3}), \end{aligned} \tag{5}$$

2 Steady-state solution stored potential energy

The oscillator's full stored energy E is a sum of potential energy P stored or stressed or strained springs k_i, and kinetic energy T of oscillating masses.

$$E = P + T. \tag{6}$$

In a steady state solution energy E is constant, and averaged by several periods potential energy $\hat{P}$ equals to averaged $\hat{T}$:

$$\hat{P} = \hat{T} = E/2 \tag{7}$$

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Potential energy is

$$P = k_1 \frac{(x_c - x_a)^2}{2} + k_2 \frac{(x_a - x_b)^2}{2} + k_3 \frac{x_b^2}{2}. \tag{8}$$

3 Energy in main variables

Main variables $(y_1(t), y_2(t)$ are linear transformation of original variables $(x_1(t), x_2(t))$ for those solution is decoupled. Accurate solution has to be found from equation (4). Good approximation is transformation to center mass coordinate y_1 and weighted distance from center mass coordinate y_2:

$$\begin{aligned} y_1 &= \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2}, \\ y_2 &= \frac{x_1 m_1 - x_2 m_2}{m_1 + m_2}. \end{aligned} \tag{9}$$

In main coordinates y_i energy E_i stored in oscillations of y_i is

$$E_i \approx \frac{1}{2} m_i A_i^2 \omega_i^2, \tag{10}$$

where m_i is mass (the same as originally defined), A_i is magnitude of oscillations, ω_i is natural frequency. Energy is proportional to squared magnitude and squared frequency. More accurate expression for steady-state oscillations with frequency ω see at [2, p.106 eq.22].

4 Conclusion

The experiment with coupled oscillator can be organized in two steps.

1. On the first step student directly measures masses of carts m_a, m_b , and elastic coefficients of springs k_1, k_2, k_3. Having this the student calculates expected natural frequencies of the system ω_1 and ω_2.

2. On the next step student observes response of the coupled oscillator to the force applied with range of frequencies ω. It is expected that the system will response to force by increased amplitudes when ω is close to the resonance. The response is considered in terms of averaged potential energy $\hat{P}$ in steady-state solution. The resonance pick is narrow, so one or two points near each of resonances and several points far from resonances would be acceptable for the analysis. The oscillations coordinates $x_1(t)$, $x_2(t)$ and the driver $x_3(t)$ make the raw source data for finding ω and energy.

References

[1] H. J. (Herbert John) Pain. *The physics of vibrations and waves.* John Wiley, 2005, p. 556. ISBN: 0470012951.

[2] Frank S Crawford. *Waves.* eng. Berkeley physics course vol. 3. New York: McGraw-Hill, 1968. ISBN: 0070048606.