Weakly dumped coupled oscillator model for lab experiment design

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Abstract

A forced weakly dumped harmonic oscillator having two degrees of freedom is considered. In Laboratory 3 it is implemented on air-track experiment set. The goal of the experiment is to find oscillator's natural frequencies and compare it with the model. Having natural frequencies found theoretically, the system response to forced oscillations is tested. It is expected that the system will response on forced oscillations frequencies close to natural frequencies.

In this document the coupled oscillator model is described.

1 No friction model

A two-degree forced oscillator as shown on Figure 1.

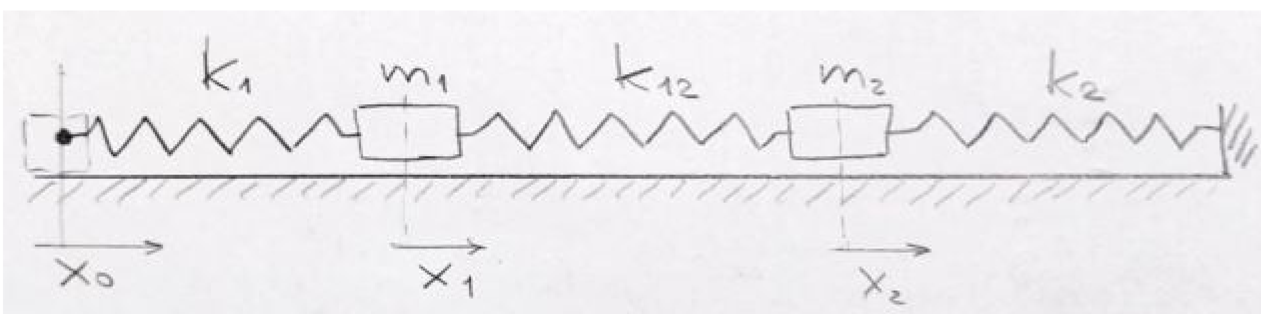

Figure 1: Forced oscillations with a coupled oscillator. Starting from right to the left: Cart 2 with mass m_2 is coupled with wall by string with elastic coefficient k_2 and to the Cart 1 with mass m_1 by spring k_{12}. On the left end Cart 1 is coupled with driver by spring with elastic coefficient k_1

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The system is described by coordinates x_0 of driver, by coordinates x_1 and x_2

of Carts 1 and 2 respectively. Driver applies periodical harmonic displacement $x_0(t) = x_0 \cos(\omega t)$. Origin of coordinates $x_i(t)$ is set on the equilibrium point.

Springs are not ideal and the have their own weight. It can be taken into account by adding to the weight of cart half of weights of adjacent springs. Denoting m_{k_i} as as weight of spring k_i, we receive

$$\begin{aligned} m_{1\text{effective}} &= m_1 + \frac{1}{2}(m_{k_1} + m_{k_{12}}), \\ m_{2\text{effective}} &= m_2 + \frac{1}{2}(m_{k_{12}} + m_{k_2}), \end{aligned} \tag{1}$$

Here and after for simplicity we will use m_i notation for effective masses.

Forces equation

$$\begin{aligned} m_1\ddot{x}_1 + (k_1 + k_{12})x_1 - k_{12}x_2 = x_0 \cos \omega t, \\ m_2\ddot{x}_2 + (k_2 + k_{12})x_2 - k_{12}x_1 = 0 \end{aligned} \tag{2}$$

1.1 Free oscillations solution

For free oscillations problem the motion equation in matrix is

$$[m]\ddot{x} + [k]x = 0 \tag{3}$$

where

$$m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \tag{4}$$

and

$$[k] = \begin{bmatrix} k_l + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{bmatrix}. \tag{5}$$

Squared natural frequencies ω_i^2 are found as eigenvalues of the matrix Z *mechanical impedance matrix*

$$\det \mathrm{Z} = \begin{vmatrix} \frac{k_1+k_{12}}{m_1} - \lambda & -\frac{k_{12}}{m_1} \\ -\frac{k_2}{m_2} & \frac{k_2+k_{12}}{m_2} - \lambda \end{vmatrix} = 0 \tag{6}$$

Elements z_{ij} of mechanical impedance matrix Z have physical dimensions of squared frequency.

Eigenvalues of matrix Z are squares of natural frequencies:

$$\lambda_i = \omega_i^2 \tag{7}$$

Canonical coordinates y_i are found by rotation of original coordinates x_i by eigenmatrix E

$$y_i = Ex_i \tag{8}$$

In canonical coordinates oscillations are decoupled:

$$y_i = A_i \cos(\omega_i t + \phi_i) \tag{9}$$

1.2 Forced oscillations solution

Forced oscillations solution is a sum of free oscillations with natural frequencies ω_i and forced oscillation with the frequency of driving force ω.

$$y_i(t) = A_i \cos(\omega_i t + \phi_i) + B_i \cos(\omega t + \varphi_i) \tag{10}$$

2 Energy

In main coordinates y_i energy E_i stored in oscillations $y_i(t)$ (see at [1, p.106 eq.22]):

$$E_i \approx \frac{1}{2} m_i A_i^2 \omega_i^2, \tag{11}$$

where m_i is mass (the same as in original coordinates?), A_i is magnitude of oscillations, ω_i is natural frequency.

3 Conclusion

The experiment with coupled oscillator can be organized in two steps.

1. On the first step student directly measures masses of carts and elastic coefficients of springs. Having this, expected natural frequencies of the system ω_1 and ω_2 is found.

2. On the next step response observed of the coupled oscillator to the force applied with range of frequencies ω. It is expected that the system will response to force by increased amplitudes when ω is close to the resonance. The response is considered in terms of steady-state energy.

References

[1] Frank S Crawford. *Waves.* eng. Berkeley physics course vol. 3. New York: McGraw-Hill, 1968. ISBN: 0070048606.