

# Weakly damped coupled oscillator model for amplitude response experiment design

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## Abstract

A forced under-damped harmonic oscillator having one and two degrees of freedom is considered. In Laboratory 3 it is implemented on air-track experiment set. The goal of the experiment is to find oscillator's amplitude response and compare it with the model. Amplitude response depends on quality factor and driving frequency. Having quality factors found experimentally by measuring free oscillations decay, amplitude response in resonance points is predicted.

In this document the coupled oscillator model is described.

## 1 One degree of freedom oscillator

One degree of freedom oscillator having mass  $m_1$  is connected on the right side to spring with elastic constant  $k_2$ , and on the left side by spring with elastic constant  $k_2$  to driver moving with frequency  $\omega$  and amplitude  $\eta_0$  (see figure 1):

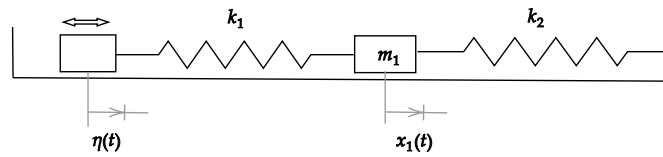


Figure 1: One degree of freedom oscillator having mass  $m_1$  is connected on the right side to spring with elastic constant  $k_2$ , and on the left side by spring with elastic constant  $k_2$  to driver moving with frequency  $\omega$  and amplitude  $\eta_0$ . Parameters:  $m = 209$  g,  $k_1 = k_2 = 4.48$  H/m, mass of each spring  $m_k = 17$  g.

### 1.1 A note on effective masses

Springs are not ideal and they have their own weight. It can be taken into account by adding to the weight of cart half of weights of adjacent springs. Denoting

$m_{k_i}$  as as weight of spring  $k_i$ , we receive

$$m_{\text{effective}} = m + \frac{1}{2}(m_{k_1} + m_{k_2}) \quad (1)$$

Here and after for simplicity we will use  $m_i$  notation for effective masses.

## 1.2 Model equations

Motion equation is

$$m_1 \ddot{x}_1 = -k_1(x_1 - \eta) - k_2 x_1 - b_1 \dot{x}_1 \quad (2)$$

, where  $b_1$  is unknown viscous friction coefficient and  $\eta$  is driver displacement:

$$\eta(t) = \eta_0 \cos \omega t \quad (3)$$

Let in this section denote  $k = k_1 + k_2$  and omit indexes for clarity.

$$m \ddot{x} + b \dot{x} + kx = \eta_0 e^{i\omega t} \quad (4)$$

Friction coefficient  $b$  is expressed by quality factor  $Q$ :

$$b = \frac{\sqrt{km}}{Q} \quad (5)$$

## 1.3 Quality factor

Quality factor is found by calculating amplitude decay in free oscillations. Rule of thumb: after  $\pi Q$  cycles, amplitude will be  $(1/e)$  of its original value.

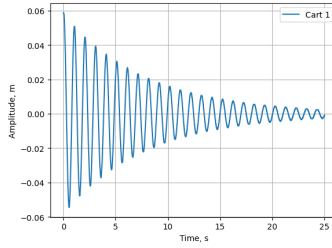


Figure 2: Free damped oscillations with quality factor  $Q = 24$

## 1.4 Amplitude response

Look for steady-state solution in complex form:

$$x(t) = ce^{i\omega t}. \quad (6)$$

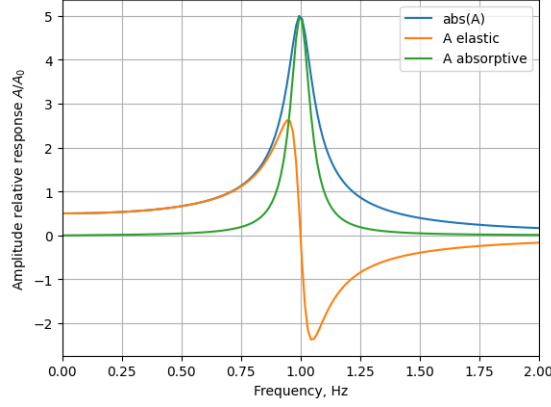


Figure 3: Elastic and absorptive amplitudes for  $Q=10$  model.

Here  $c$  is complex amplitude with real part called elastic amplitude and imaginary part called *absorptive amplitude* (see [1, p.105]).

Substitute solution 6 into motion equation 4:

$$ce^{i\omega t}(-m\omega^2 + i b\omega + km) = \eta_0 k_1 e^{i\omega t} \quad (7)$$

Therefore relative amplitude response  $c/\eta_0$  is

$$\frac{c}{\eta_0} = \frac{k_1}{-m\omega^2 + i b\omega + km}. \quad (8)$$

Note that  $k_1$  is elasticity of left spring connected to driver, while  $k = k_1 + k_2$  is effective elasticity of the oscillator.

## 2 Coupled oscillator

A two-degree forced oscillator as shown on Figure 5.

Effective masses

$$\begin{aligned} m_1 &\equiv m_1 + (m_{k1} + m_{k12})/2 \\ m_2 &\equiv m_2 + (m_{k2} + m_{k12})/2 \end{aligned} \quad (9)$$

### 2.1 Model equations

$$\begin{aligned} m_1 \ddot{x}_1 + b_1 \dot{x}_1 + (k_1 + k_{12})x_1 - k_{12}x_2 &= \eta_0 e^{i\omega t}, \\ m_2 \ddot{x}_2 + b_2 \dot{x}_2 + (k_2 + k_{12})x_2 - k_{12}x_1 &= 0 \end{aligned} \quad (10)$$

The same in matrix form: Coordinates vector

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad (11)$$

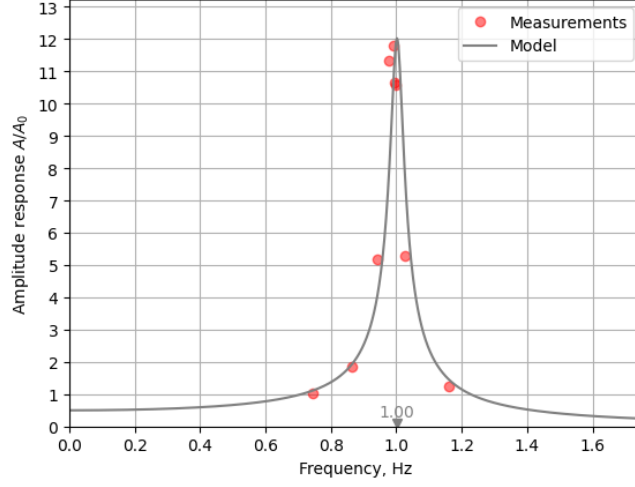


Figure 4: Amplitude response experiment with experimental points. Parameters as on fig 1, Quality factor  $Q=24$  (magnets set vertically). Natural frequency  $f=1.00$  Hz is shown by triangle marker on frequencies axis.

Masses matrix

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad (12)$$

Elastic coefficients matrix

$$K = \begin{bmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_{12} + k_2 \end{bmatrix}, \quad (13)$$

Friction matrix

$$B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}, \quad (14)$$

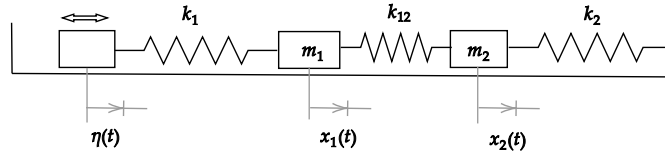


Figure 5: Coupled oscillator. Cart 2 with mass  $m_2$  is coupled with wall by string with elastic coefficient  $k_2$  and to the Cart 1 with mass  $m_1$  by spring  $k_{12}$ . On the left end Cart 1 is coupled with driver by spring with elastic coefficient  $k_1$ . Driver is moving with amplitude  $\eta_0$  and frequency  $\omega$ . Parameters:  $m_1 = 209$  g,  $m_2 = 214$  g,  $k_1 = k_2 = k_{12} = 4.48$  H/m, mass of each spring  $m_k = 17$  g. Magnets set vertically.  $Q_1 = 26.4$ ,  $Q_2 = 24.9$

Force amplitude vector

$$F = \eta_0 \begin{bmatrix} k_1 \\ 0 \end{bmatrix}, \quad (15)$$

Equation for forced oscillations with friction

$$M\ddot{X}(t) + B\dot{X}(t) + KX(t) = Fe^{i\omega t} \quad (16)$$

## 2.2 Natural frequencies and normal coordinates

Equation of free oscillations without friction

$$M\ddot{X}(t) + KX(t) = 0 \quad (17)$$

Let  $I$  - diagonal identity matrix, and  $Z = M^{-1}K$  - mechanical impedance matrix. Then eigenvalues  $\lambda_i$  and eigenvectors matrix  $E$  are found from equation

$$\det |Z - \lambda I| = \begin{vmatrix} \frac{k_1+k_{12}}{m_1} - \lambda & -\frac{k_{12}}{m_1} \\ -\frac{k_{12}}{m_2} & \frac{k_{12}+k_2}{m_2} - \lambda \end{vmatrix} = 0 \quad (18)$$

Eigenvectors are squares of the oscillator natural frequencies:

$$\omega_i^2 = \lambda_i, \quad (19)$$

. For this experimental system natural frequencies converted from radians per second to cycles per second are 1.22 Hz and 0.71 Hz.

Conversion from original to normal coordinates and vice versa is linear transformation by eigenmatrix:

$$x = Ey, \quad y = E^{-1}x \quad (20)$$

The physical sense of normal coordinates for weakly damped case is that the first normal coordinate is system center mass  $y_1 = x_1 + x_2$ , while the other is distance between masses  $y_2 = x_2 - x_1$ .

In normal coordinates weakly damped oscillations are decoupled for any initial conditions.

## 2.3 Amplitude response

### 2.3.1 Model

Steady-state solution

$$X(t) = Ce^{i\omega t}, \quad (21)$$

where  $C$  is vector of complex amplitudes:

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}. \quad (22)$$

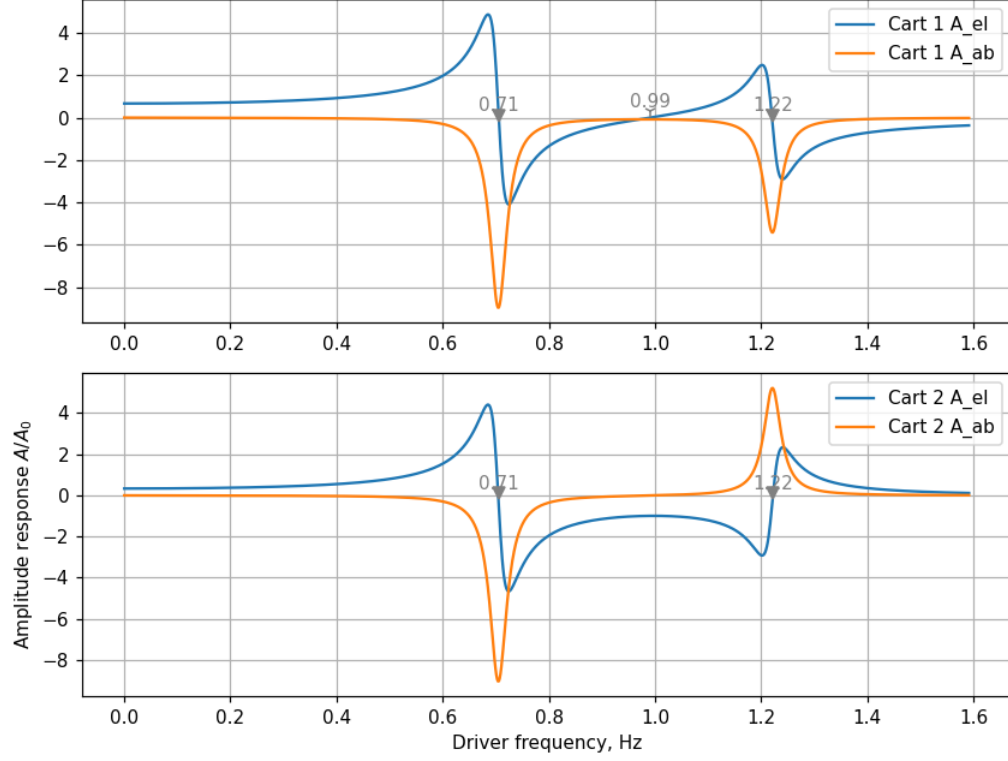


Figure 6: Amplitude response. The negative sign of amplitude means that the phase is opposite to driver displacement phase. Note the still point of Cart 1 between resonances: at this frequency (0.99Hz) amplitude of mass 1 becomes nearly zero, while mass 2 oscillates with the amplitude of driver.

Substitute steady-state solution to motion equation to find  $C$ :

$$(-\omega^2 M + i\omega B + K) \times C = F \quad (23)$$

Let

$$Z = -\omega^2 M + i\omega B + K \quad (24)$$

Relative amplitudes response

$$\frac{C}{\eta_0} = Z^{-1} \frac{F}{\eta_0} \quad (25)$$

### 2.3.2 Experiments results

Friction constants  $b_i$  and quality factors  $Q_i$  are found by measuring free oscillations decay for each mass and adjacent springs separately as described in

subsection 1.3. Quality factors found are  $Q_1 = 26.4$ ,  $Q_2 = 24.9$ .

A maximum relative displacement is expected from the model response with known quality. Note that system with two degrees of freedom has lower maximal response at resonance than the system with one oscillator of the same quality. Not also a "still point" - a point on driver frequency axis between low and high resonance frequencies, where amplitude  $C_1$  becomes nearly zero.

The results of experiment agree well with the model (see figure 7).

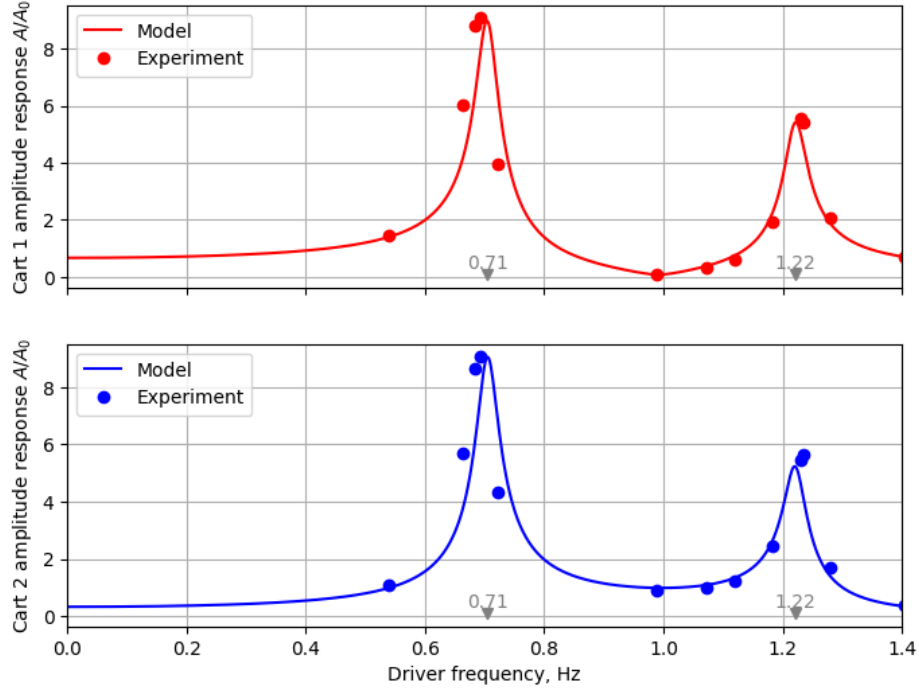


Figure 7: Amplitude response experiment. Red and blue lines are theoretical response amplitudes for mass 1 and 2 respectively. Red and blue points are experimental data for masses 1 and 2 respectively. Parameters:  $m_1 = 209$  g,  $m_2 = 214$  g,  $k_1 = k_2 = k_{12} = 4.48$  H/m, mass of each spring  $m_k = 17$  g. Magnets set vertically.  $Q_1 = 26.4$ ,  $Q_2 = 24.9$ .

### 3 Conclusion

The coupled oscillator experimental system is good enough explained by the linear model described above.

## References

- [1] Frank S Crawford. *Waves*. eng. Berkeley physics course vol. 3. New York: McGraw-Hill, 1968. ISBN: 0070048606.



## Appendix

Python code used for signal acquisition and data analysis.

```
import numpy as np
import matplotlib.pyplot as plt

def get_signal(url, show=True):
    data = np.loadtxt(url).T
    t = data[1, :]
    # convert from counts to meters
    x = data[2:, :] / counts * length
    # how many signals: Kruze may return from 1 up to 4
    signals = x.shape[0]
    # shift mean to zero
    for i in range(signals):
        x[i] -= x[i].mean()
    if show:
        for i in range(signals):
            plt.plot(t, x[i], label=f'Cart {i+1}')
        plt.grid()
        plt.xlabel('Time, s')
        plt.ylabel('Amplitude, m')
        plt.legend()
    # returns t and signals x1, x2, ..
    answer = [t] + [x[i] for i in range(signals)]
    return tuple(answer)
```

Listing 1: Get signal from Kruze interface

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.signal import find_peaks, peak_prominences

def get_peaks(t, signal, show=False):
    rate = np.median(np.diff(t))
    min_period = 0.2 #s
    min_prominence = 2e-3 ## 2 mm
    peaks, _ = find_peaks(signal, prominence=min_prominence,
                          distance=min_period/rate )
    if show:
        plt.plot(signal)
        plt.plot(peaks, signal[peaks], "x")
    return t[peaks], signal[peaks]
```

Listing 2: Get signal peaks

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import linregress

def get_quality(t, harmonic_signal, show=False):
    tpeaks, peaks = get_peaks(t, harmonic_signal)
    decay = np.log(peaks / peaks[0])
    cycles = range(len(peaks))
    fit = linregress(cycles, decay)
    quality = -np.pi/fit.slope
    N = quality / np.pi
```

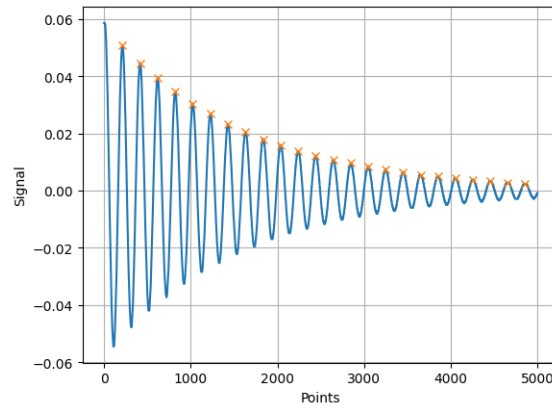


Figure 8: Getting peaks from a signal is a key to calculate harmonic signal amplitudes, find frequency and estimate uncertainty.

```
if show:
    plt.plot(cycles, decay, 'o', label='Measurements')
    plt.plot(cycles, fit.slope*cycles + fit.intercept, label=f'Fit
    $Q=\pi*N$: {quality:.1f}')
    plt.xlabel(rf'$N$ = {N:.1f} cycles for $1/e$ decay')
    plt.ylabel('Log amplitude decay')
    plt.grid()
    plt.legend()
return quality
```

Listing 3: Get harmonic oscillator quality

```
import numpy as np
def get_frequency(t, harmonic_signal):
    tpeaks, _ = get_peaks(t, harmonic_signal)
    periods = np.diff(tpeaks)
    f0 = 1 / periods
    return np.mean(f0), np.std(f0)
```

Listing 4: Get signal frequency