Weakly dumped coupled oscillator model for amplitude response experiment

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Abstract

A forced under-dumped harmonic oscillator having one and two degrees of freedom is considered. In Laboratory 3 it is implemented on air-track experiment set. The goal of the experiment is to find oscillator's amplitude response and compare it with the model. Amplitude response depends on quality factor and driving frequency. Having quality factors found experimentally by measuring free oscillations decay, amplitude response in resonance points is predicted.

In this document the coupled oscillator model is described.

1 One degree of freedom oscillator

One degree of freedom oscillator having mass m_1 is connected on the right side to spring with elastic constant k_2, and on the left side by spring with elastic constant k_2 to driver moving with frequency ω and amplitude η_0 (see figure 1):

1.1 A note on effective masses

Springs are not ideal and the have their own weight. It can be taken into account by adding to the weight of cart half of weights of adjacent springs. Denoting m_{k_i} as as weight of spring k_i, we receive

$$m_{\text{effective}} = m + \frac{1}{2}(m_{k_1} + m_{k_2}) \tag{1}$$

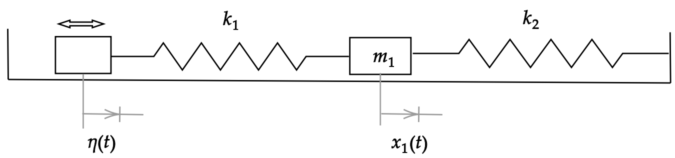


Figure 1: One degree of freedom oscillator having mass m_1 is connected on the right side to spring with elastic constant k_2, and on the left side by spring with elastic constant k_2 to driver moving with frequency ω and amplitude η_0. Parameters: $m = 209$ g, $k_1 = k_2 = 4.48$ H/m, mass of each spring $m_k = 17$ g.

Here and after for simplicity we will use m_i notation for effective masses.

1.2 Model equations

Motion equation is

$$m_1\ddot{x}_1 = -k_1(x_1 - \eta) - k_2 x_1 - b_1\dot{x}_1, \tag{2}$$

where b_1 is unknown viscous friction coefficient and η is driver displacement:

$$\eta(t) = \eta_0 \cos \omega t \tag{3}$$

Let in this section denote $k = k_1 + k_2$ and omit indexes for clarity.

$$m\ddot{x} + b\dot{x} + kx = \eta_0 e^{i\omega t} \tag{4}$$

Friction coefficient b is expressed by quality factor Q:

$$b = \frac{\sqrt{km}}{Q} \tag{5}$$

Quality factor Q is found by calculating amplitude decay in free oscillations. Rule of thumb: after πQ cycles, amplitude will be $(1/e)$ of its original value. See suggested algorithm in appendix.

1.3 Amplitude response

Look for steady-state solution in complex form:

$$x(t) = ce^{i\omega t}. \tag{6}$$

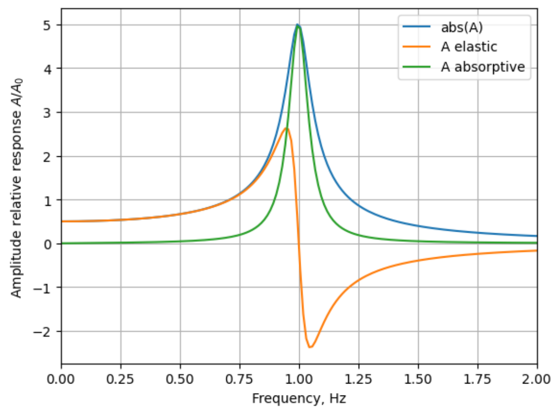


Figure 2: Elastic and absorptive amplitudes for Q=10 model.

Here c is complex amplitude with real part called elastic amplitude and imaginary part called *absorptive amplitude* (see [1, p.105]).

Substitute solution 6 into motion equation 4:

$$ce^{i\omega t}(-m\omega^2 + ib\omega + km) = \eta_0 k_1 e^{i\omega t} \tag{7}$$

Therefore relative amplitude response c/η_0 is

$$\frac{c}{\eta_0} = \frac{k_1}{-m\omega^2 + ib\omega + km}. \tag{8}$$

Note that k_1 is elasticity of left spring connected to driver, while $k = k_1 + k_2$ is effective elasticity of the oscillator.

2 Coupled oscillator

A two-degree forced oscillator as shown on Figure 4.

Effective masses

$$\begin{aligned} m_1 &\equiv m_1 + (m_{k1} + m_{k12})/2 \\ m_2 &\equiv m_2 + (m_{k2} + m_{k12})/2 \end{aligned} \tag{9}$$

2.1 Model equations

$$\begin{aligned} m_1\ddot{x}_1 + b_1\dot{x}_1 + (k_1 + k_{12})x_1 - k_{12}x_2 &= \eta_0 e^{i\omega t}, \\ m_2\ddot{x}_2 + b_2\dot{x}_2 + (k_2 + k_{12})x_2 - k_{12}x_1 &= 0 \end{aligned} \tag{10}$$

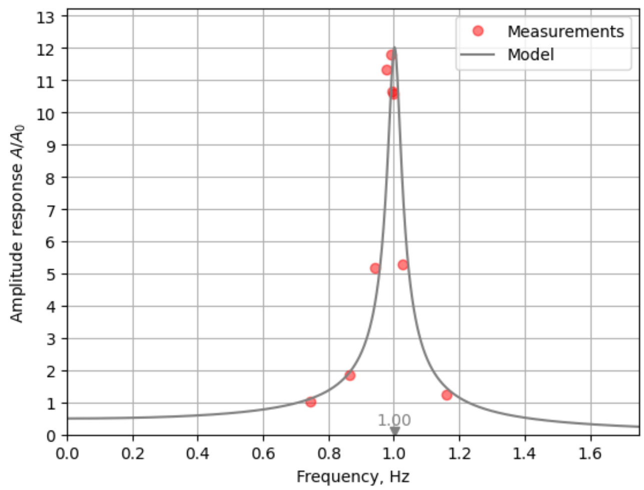


Figure 3: Amplitude response experiment with experimental points. Parameters as on fig 1, Quality factor Q=24 (magnets set vertically). Natural frequency f=1.00 Hz is shown by triangle marker on frequencies axis.

The same in matrix form: Coordinates vector

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{11}$$

Masses matrix

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \tag{12}$$

Elastic coefficients matrix

$$K = \begin{bmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_{12} + k_2 \end{bmatrix}, \tag{13}$$

Friction matrix

$$B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}, \tag{14}$$

Force amplitude vector

$$F = \eta_0 \begin{bmatrix} k_1 \\ 0 \end{bmatrix}, \tag{15}$$

Equation for forced oscillations with friction

$$M\ddot{X}(t) + B\dot{X}(t) + KX(t) = Fe^{i\omega t} \tag{16}$$

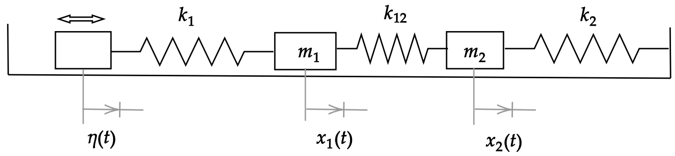


Figure 4: Coupled oscillator. Cart 2 with mass m_2 is coupled with wall by string with elastic coefficient k_2 and to the Cart 1 with mass m_1 by spring k_{12}. On the left end Cart 1 is coupled with driver by spring with elastic coefficient k_1. Driver is moving with amplitude η_0 and frequency ω. Parameters: $m_1 = 209$ g, $m_2 = 214$ g, $k_1 = k_2 = k_{12} = 4.48$ H/m, mass of each spring $m_k = 17$ g. Magnets set vertically. $Q_1 = 26.4$, $Q_2 = 24.9$

2.2 Natural frequencies and normal coordinates

Equation of free oscillations without friction

$$M\ddot{X}(t) + KX(t) = 0 \tag{17}$$

Let I - diagonal identity matrix, and $Z = M^{-1}K$ - mechanical impedance matrix. Then eigenvalues λ_i and eigenvectors matrix E are found from equation

$$\det|Z - \lambda I| = \begin{vmatrix} \frac{k_1+k_{12}}{m_1} - \lambda & -\frac{k_{12}}{m_1} \\ -\frac{k_{12}}{m_2} & \frac{k_{12}+k_2}{m_2} - \lambda \end{vmatrix} = 0 \tag{18}$$

Eigenvectors are squares of the oscillator natural frequencies:

$$\omega_i^2 = \lambda_i, \tag{19}$$

. For this experimental system natural frequencies converted from radians per second to cycles per second are 1.22 Hz and 0.71 Hz.

Conversion from original to normal coordinates and vise versa is linear transformation by eigenmatrix:

$$x = Ey,\ y = E^{-1}x \tag{20}$$

The physical sense of normal coordinates for weakly dumped case is that the first normal coordinate is system center mass $y_1 = x_1 + x_2$, while the other is distance between masses $y_2 = x_2 - x_1$.

In normal coordinates weakly dumped oscillations are decoupled for any initial conditions.

2.3 Amplitude response

2.3.1 Model

Steady-state solution

$$X(t) = Ce^{i\omega t}, \tag{21}$$

where C is vector of complex amplitudes:

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}. \tag{22}$$

Substitute steady-state solution to motion equation to find C:

$$(-\omega^2 M + i\omega B + K) \times C = F \tag{23}$$

Let

$$Z = -\omega^2 M + i\omega B + K \tag{24}$$

.

Relative amplitudes response

$$\frac{C}{\eta_0} = Z^{-1} \frac{F}{\eta_0} \tag{25}$$

2.3.2 Experiments results

Friction constants b_i and quality factors Q_i are found by measuring free oscillations decay for each mass and adjacent springs separately as described in subsection **??**. Quality factors found are $Q_1 = 26.4$, $Q_2 = 24.9$.

A maximum relative displacement is expected from the model response with known quality. Note that system with two degrees of freedom has lower maximal response at resonance than the system with one oscillator of the same quality. Not also a "still point" - a point on driver frequency axis between low and high resonance frequencies, where amplitude C_1 becomes nearly zero.

The results of experiment agree well with the model (see figure 6).

3 Conclusion

The coupled oscillator experimental system is good enough explained by the linear model described above.

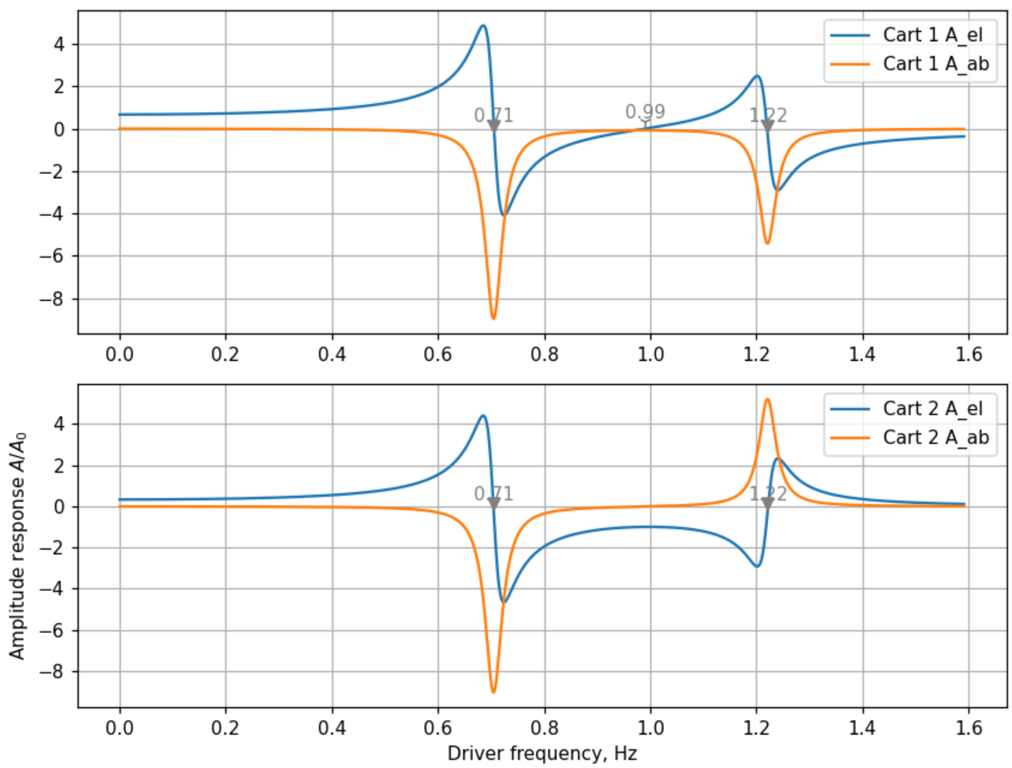
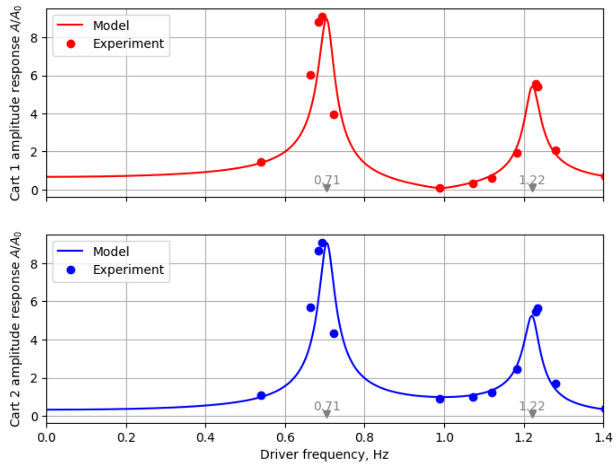
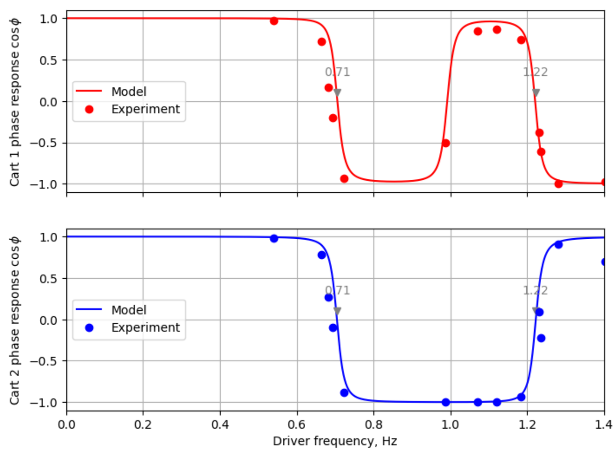


Figure 5: Amplitude response. The negative sign of amplitude means that the phase is opposite to driver displacement phase. Note the still point of Cart 1 between resonances: at this frequency (0.99Hz) amplitude of mass 1 becomes nearly zero, while mass 2 oscillates with the amplitude of driver.



(a) Amplitude response.



(b) Phase response

Figure 6: Amplitude and phase response of coupled oscillator. Lines are theoretical response amplitudes, points are experimental data. Parameters: $m_1 = 209$ g, $m_2 = 214$ g, $k_1 = k_2 = k_{12} = 4.48$ H/m, mass of each spring $m_k = 17$ g. Magnets set vertically. $Q_1 = 26.4$, $Q_2 = 24.9$.

References

[1] Frank S Crawford. *Waves.* eng. Berkeley physics course vol. 3. New York: McGraw-Hill, 1968. ISBN: 0070048606.

Appendix

Python code used for signal acquisition and data analysis.

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import linregress
from scipy.signal import find_peaks, peak_prominences
```

Listing 1: Modules import

```python
def get_signal(url):
  data = np.loadtxt(url).T
  t = data[1, :]
  # convert from counts to meters
  x = data[2:, :] /counts * length
  # how many signals: Kruze may return  from 1 up to 4
  signals = x.shape[0]
  # shift mean to zero
  for i in range(signals):
    x[i] -= x[i].mean()
  # returns t and signals x1, x2, ..
  answer = [t] + [x[i] for i in range(signals)]
  return tuple(answer)
```

Listing 2: Get signal from Kruze interface

```python
def get_peaks(t, signal):
  rate = np.median(np.diff(t))
  min_period = 0.2 #s
  min_prominence = 2e-3 ## 2 mm
  peaks, _ = find_peaks(signal, prominence=min_prominence,
                        distance=min_period/rate )
  return t[peaks], signal[peaks]
```

Listing 3: Get signal peaks

```python
def get_quality(t, harmonic_signal):
  tpeaks, peaks = get_peaks(t, harmonic_signal)
  decay = np.log(peaks / peaks[0])
  cycles = range(len(peaks))
  fit = linregress(cycles, decay)
  quality = -np.pi/fit.slope
  return quality
```

Listing 4: Get harmonic oscillator quality

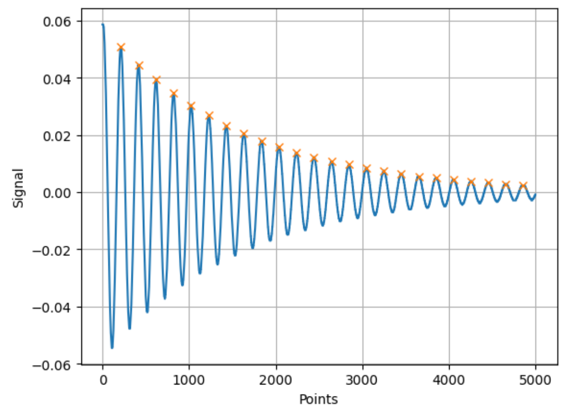


Figure 7: Getting peaks from a signal is a key to calculate harmonic signal amplitudes, find frequency and estimate uncertainty.

```
def get_frequency(t, harmonic_signal):
  tpeaks, _ = get_peaks(t, harmonic_signal)
  periods = np.diff(tpeaks)
  f0 = 1 / periods
  return np.mean(f0), np.std(f0)
```

Listing 5: Get signal frequency

```
def get_cosine(x1, x2):
# x1, x2 - vectors of harmonic signal x(t)
# Operator @ stands for scalar multiplication ov vectors
  x11 = x1 @ x1
  x12 = x1 @ x2
  x22 = x2 @ x2
  cosine = x12 / np.sqrt(x11 * x22)
  return cosine
```

Listing 6: Get cosine of phase difference between two harmonic signals, having the same frequency