Weakly dumped coupled oscillator model for lab experiment design

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Abstract

A forced under-dumped harmonic oscillator having one and two degrees of freedom is considered. In Laboratory 3 it is implemented on air-track experiment set. The goal of the experiment is to find oscillator's amplitude response and compare it with the model. Amplitude response depends on quality factor and driving frequency. Having quality factors found experimentally by measuring free oscillations decay, amplitude response in resonance points is predicted.

In this document the coupled oscillator model is described.

1 Dumped oscillator with one degree of freedom

One degree of freedom oscillator having mass m_1 is connected on the right side to spring with elastic constant k_2, and on the left side by spring with elastic constant k_2 to driver moving with frequency ω and amplitude η_0 (see figure 1):

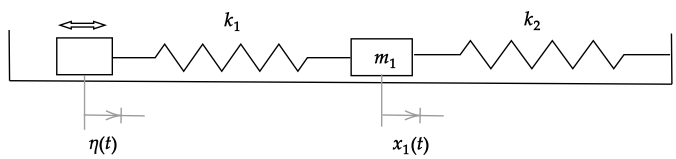

Figure 1: One degree of freedom oscillator having mass m_1 is connected on the right side to spring with elastic constant k_2, and on the left side by spring with elastic constant k_2 to driver moving with frequency ω and amplitude η_0. Parameters: $m = 209$ g, $k_1 = k_2 = 4.48$ H/m, mass of each spring $m_k = 17$ g.

1.1 A note on effective masses

Springs are not ideal and the have their own weight. It can be taken into account by adding to the weight of cart half of weights of adjacent springs. Denoting

m_{k_i} as as weight of spring k_i, we receive

$$m_{\text{effective}} = m + \frac{1}{2}(m_{k_1} + m_{k_2}) \tag{1}$$

Here and after for simplicity we will use m_i notation for effective masses.

1.2 Model equations

Motion equation is

$$m_1\ddot{x}_1 = -k_1(x_1 - \eta) - k_2 x_1 - b_1\dot{x}_1 \tag{2}$$

, where b_1 is unknown viscous friction coefficient and η is driver displacement:

$$\eta(t) = \eta_0 \cos \omega t \tag{3}$$

Let in this section denote $k = k_1 + k_2$ and omit indexes for clarity.

$$m\ddot{x} + b\dot{x} + kx = \eta_0 e^{i\omega t} \tag{4}$$

Friction coefficient b is expressed by quality factor Q:

$$b = \frac{\sqrt{km}}{Q} \tag{5}$$

1.3 Quality factor

Quality factor is found by calculating amplitude decay in free oscillations. Rule of thumb: after πQ cycles, amplitude will be $(1/e)$ of its original value.

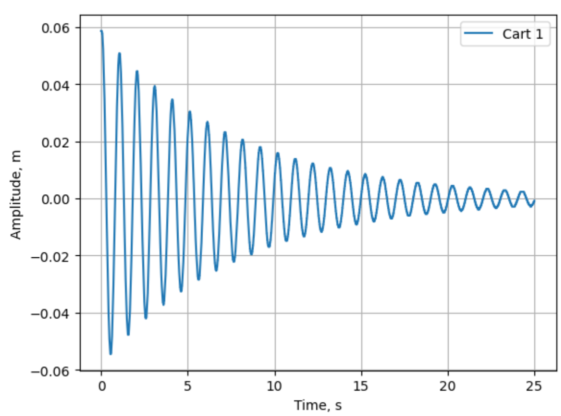

Figure 2: Free dumped oscillations with quality factor $Q = 24$

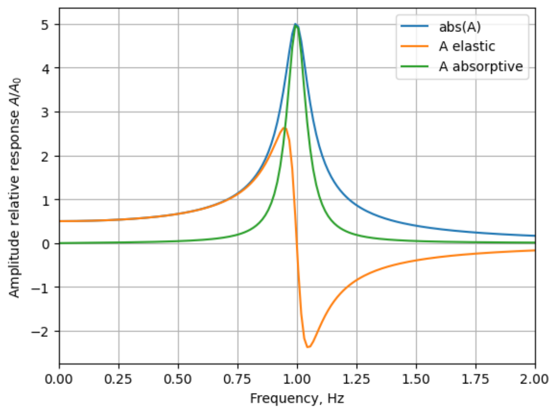

Figure 3: Elastic and absorptive amplitudes for Q=10 model.

1.4 Amplitude response in steady-state

Look for steady-state solution in complex form:

$$x(t) = ce^{i\omega t}. \tag{6}$$

Here c is complex amplitude with real part called elastic amplitude and imaginary part called *absorptive amplitude* (see [1, p.105]).

Substitute solution 6 into motion equation 4:

$$ce^{i\omega t}(-m\omega^2 + ib\omega + km) = \eta_0 k_1 e^{i\omega t} \tag{7}$$

Therefore relative amplitude response c/η_0 is

$$\frac{c}{\eta_0} = \frac{k_1}{-m\omega^2 + ib\omega + km}. \tag{8}$$

Note that k_1 is elasticity of left spring connected to driver, while $k = k_1 + k_2$ is effective elasticity of the oscillator.

2 Coupled oscillator

A two-degree forced oscillator as shown on Figure 5.

Effective masses

$$\begin{aligned} m_1 &\equiv m_1 + (m_{k1} + m_{k12})/2 \\ m_2 &\equiv m_2 + (m_{k2} + m_{k12})/2 \end{aligned} \tag{9}$$

2.1 Model equations

$$\begin{aligned} m_1\ddot{x}_1 + b_1\dot{x}_1 + (k_1 + k_{12})x_1 - k_{12}x_2 &= \eta_0 e^{i\omega t}, \\ m_2\ddot{x}_2 + b_2\dot{x}_2 + (k_2 + k_{12})x_2 - k_{12}x_1 &= 0 \end{aligned} \tag{10}$$

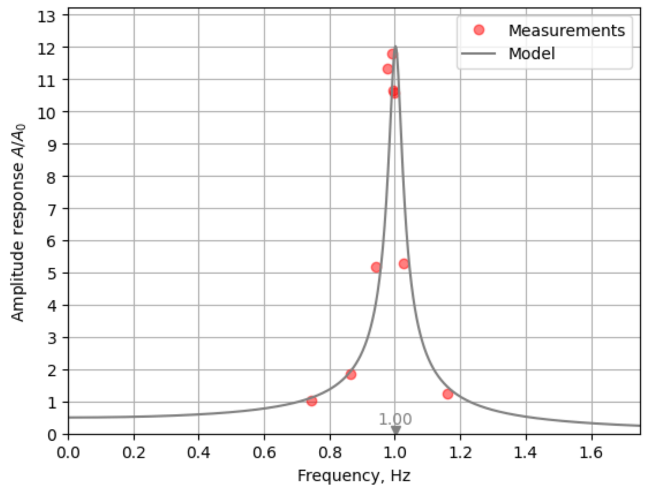

Figure 4: Amplitude response experiment with experimental points. Parameters as on fig 1, Quality factor Q=24 (magnets set vertically). Natural frequency f=1.00 Hz is shown by triangle marker on frequencies axis.

The same in matrix form: Coordinates vector

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{11}$$

Masses matrix

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \tag{12}$$

Elastic coefficients matrix

$$K = \begin{bmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_{12} + k_2 \end{bmatrix}, \tag{13}$$

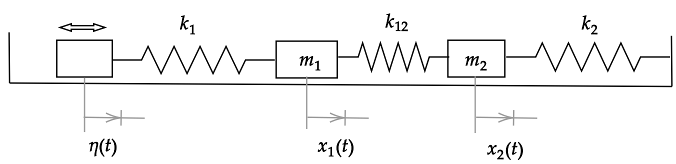

Figure 5: Coupled oscillator. Cart 2 with mass m_2 is coupled with wall by string with elastic coefficient k_2 and to the Cart 1 with mass m_1 by spring k_{12}. On the left end Cart 1 is coupled with driver by spring with elastic coefficient k_1. Driver is moving with amplitude η_0 and frequency ω. Parameters: $m_1 = 209$ g, $m_2 = 214$ g, $k_1 = k_2 = k_{12} = 4.48$ H/m, mass of each spring $m_k = 17$ g. Magnets set vertically. $Q_1 = 26.4$, $Q_2 = 24.9$

Friction matrix

$$B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}, \tag{14}$$

Force amplitude vector

$$F = \eta_0 \begin{bmatrix} k_1 \\ 0 \end{bmatrix}, \tag{15}$$

Equation for forced oscillations with friction

$$M\ddot{X}(t) + B\dot{X}(t) + KX(t) = Fe^{i\omega t} \tag{16}$$

2.2 Natural frequencies and normal coordinates

Equation of free oscillations without friction

$$M\ddot{X}(t) + KX(t) = 0 \tag{17}$$

Let I - diagonal identity matrix, and $Z = M^{-1}K$ - mechanical impedance matrix. Then eigenvalues λ_i and eigenvectors matrix E are found from equation

$$\det|\mathrm{Z} - \lambda\mathrm{I}| = \begin{vmatrix} \frac{k_1+k_{12}}{m_1} - \lambda & -\frac{k_{12}}{m_1} \\ -\frac{k_{12}}{m_2} & \frac{k_{12}+k_2}{m_2} - \lambda \end{vmatrix} = 0 \tag{18}$$

Eigenvectors are squares of the oscillator natural frequencies:

$$\omega_i^2 = \lambda_i, \tag{19}$$

. For this experimental system natural frequencies converted from radians per second to cycles per second are 1.22 Hz and 0.71 Hz.

Conversion from original to normal coordinates and vise versa is linear transformation by eigenmatrix:

$$x = Ey, \; y = E^{-1}x \tag{20}$$

The physical sense of normal coordinates for weakly dumped case is that the first normal coordinate is system center mass $y_1 = x_1 + x_2$, while the other is distance between masses $y_2 = x_2 - x_1$.

In normal coordinates weakly dumped oscillations are decoupled for any initial conditions.

2.3 Amplitude response to periodic displacement model

Steady-state solution

$$X(t) = Ce^{i\omega t}, \tag{21}$$

where C is vector of complex amplitudes:

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}. \tag{22}$$

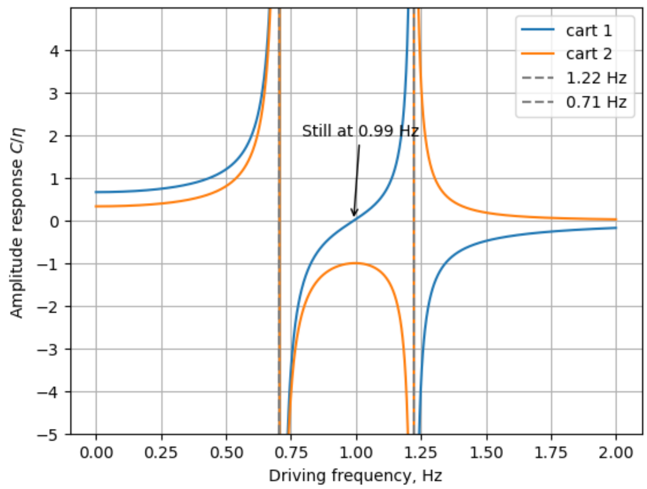

Figure 6: Amplitude response for high quality Q case. The negative sign of amplitude means that the phase is opposite to driver displacement phase. Note the still point between resonances: at this point amplitude of mass 1 becomes zero, while mass 2 oscillates with the amplitude of driver.

Substitute steady-state solution to motion equation to find C:

$$(-\omega^2 M + i\omega B + K) \times C = F \tag{23}$$

Let $Z = -\omega^2 M + i\omega B + K$.

Relative amplitudes response

$$\frac{C}{\eta_0} = Z^{-1} \frac{F}{\eta_0} \tag{24}$$

2.4 Amplitude response experiment for system with friction

Friction constants b_i and quality factors Q_i are found by measuring free oscillations decay for each mass and adjacent springs separately as described in subsection 1.3. Quality factors found are $Q_1 = 26.4$, $Q_2 = 24.9$.

A maximum relative displacement is expected form the model response with known quality (see figure 7). Note that 2-dof system has lower maximal response at resonance than the system with one oscillator of the same quality. Not also a "still point" - a point on driver frequency axis between low and high resonance, where amplitude C_1 becomes nearly zero.

The results of experiment agree well with the model (see figure 8.

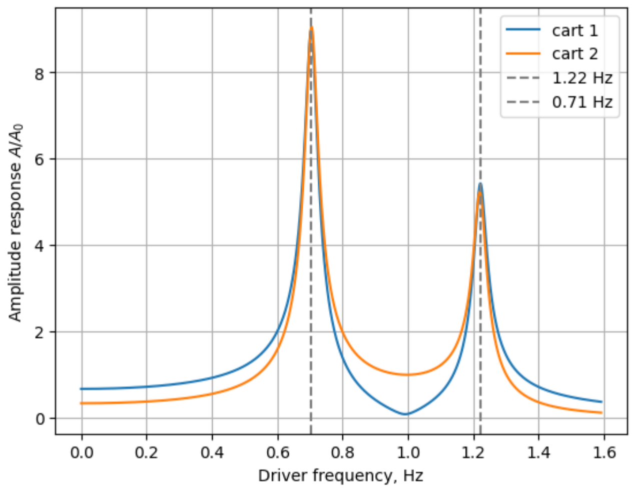

Figure 7: Amplitude response model for low quality Q=25. System parameters as described above.

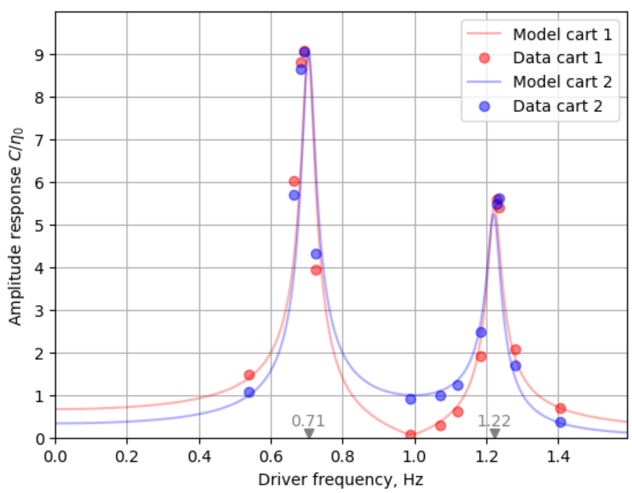

Figure 8: Amplitude response experiment. Red and blue lines are theoretical response amplitudes for mass 1 and 2 respectively. Read and blue points are experimental data for masses 1 and 2 respectively. Parameters: $m_1 = 209$ g, $m_2 = 214$ g, $k_1 = k_2 = k_{12} = 4.48$ H/m, mass of each spring $m_k = 17$ g. Magnets set vertically. $Q_1 = 26.4$, $Q_2 = 24.9$. Algorithms are listed in Appx

3 Conclusion

The experiment with coupled oscillator can be organized in following steps.

1. On the first step student directly measures masses of carts and elastic coefficients of springs. Having this, expected natural frequencies of the system ω_1 and ω_2 is found.

2. On the second step quality factor(s) estimated.

3. On the next step response observed of the coupled oscillator to the force applied with range of frequencies ω. It is expected that the system will response to force by increased amplitudes when ω is close to the resonance.

References

[1] Frank S Crawford. *Waves.* eng. Berkeley physics course vol. 3. New York: McGraw-Hill, 1968. ISBN: 0070048606.

Appendix

Python code used for signal acquisition and data analysis.

```python
import numpy as np
import matplotlib.pyplot as plt

def get_signal(url, show=True):
  data = np.loadtxt(url).T
  t = data[1, :]
  # convert from counts to meters
  x = data[2:, :] /counts * length
  # how many signals: Kruze may return  from 1 up to 4
  signals = x.shape[0]
  # shift mean to zero
  for i in range(signals):
    x[i] -= x[i].mean()
  if show:
    for i in range(signals):
      plt.plot(t, x[i], label=f'Cart {i+1}')
    plt.grid()
    plt.xlabel('Time, s')
    plt.ylabel('Amplitude, m')
    plt.legend()
  # returns t and signals x1, x2, ..
  answer = [t] + [x[i] for i in range(signals)]
  return tuple(answer)
```

Listing 1: Get signal from Kruze interface

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.signal import find_peaks, peak_prominences

def get_peaks(t, signal, show=False):
  rate = np.median(np.diff(t))
  min_period = 0.2 #s
  min_prominence = 2e-3 ## 2 mm
  peaks, _ = find_peaks(signal, prominence=min_prominence,
                        distance=min_period/rate )
  if show:
    plt.plot(signal)
    plt.plot(peaks, signal[peaks], "x")
  return t[peaks], signal[peaks]
```

Listing 2: Get signal peaks

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import linregress

def get_quality(t, harmonic_signal, show=False):
  tpeaks, peaks = get_peaks(t, harmonic_signal)
  decay = np.log(peaks / peaks[0])
  cycles = range(len(peaks))
  fit = linregress(cycles, decay)
  quality = -np.pi/fit.slope
  N = quality / np.pi
```

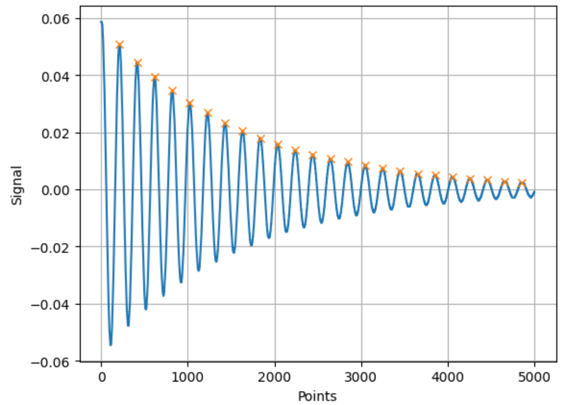

Figure 9: Getting peaks from a signal is a key to calculate harmonic signal amplitudes, find frequency and estimate uncertainty.

```
  if show:
    plt.plot(cycles, decay, 'o', label='Measurements')
    plt.plot(cycles, fit.slope*cycles + fit.intercept, label=fr'Fit $Q=\pi*N$: {quality:.1f}')
    plt.xlabel(rf'$N$ = {N:.1f} cycles for $1/e$ decay')
    plt.ylabel('Log amplitude decay')
    plt.grid()
    plt.legend()
  return quality
```

Listing 3: Get harmonit oscillator quality

```
import numpy as np
def get_frequency(t, harmonic_signal):
  tpeaks, _ = get_peaks(t, harmonic_signal)
  periods = np.diff(tpeaks)
  f0 = 1 / periods
  return np.mean(f0), np.std(f0)
```

Listing 4: Get signal frequency