Week 6 Lab. ASP3162

Q 1.1

Consider equation

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0, \quad v = \text{const}. \qquad (1)$$

Discretizing using Forward time centered space method gives

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -v\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}, \qquad (2)$$

where i is a space index, and n is a time index.

We want to find a modified PDE. Using Taylor series we get

$$u_i^{n+1} = u_i^n + \Delta t\frac{\partial u_i^n}{\partial t} + \frac{1}{2}\Delta t^2\frac{\partial^2 u_i^n}{\partial t^2} + O(\Delta t^3). \qquad (3)$$

$$u_{i\pm 1}^n = u_i^n \pm \Delta x\frac{\partial u_i^n}{\partial x} + \frac{1}{2}\Delta x^2\frac{\partial^2 u_i^n}{\partial x^2} \pm \frac{1}{6}\Delta x^3\frac{\partial^3 u_i^n}{\partial x^3} + \frac{1}{24}\Delta x^4\frac{\partial^4 u_i^n}{\partial x^4} + O(\Delta x^5). \qquad (4)$$

Substituting Eq. 3, 4 into Eq. 2 gives:

$$\frac{\partial u_i^n}{\partial t} + \frac{1}{2}\Delta t\frac{\partial^2 u_i^n}{\partial t^2} + O(\Delta t^2) = -v\left[\frac{\partial u_i^n}{\partial x} + \frac{1}{6}\Delta x^2\frac{\partial^3 u_i^n}{\partial x^3} + O(\Delta x^4)\right].$$

Rearranging and dropping indexes gives the modified PDE:

$$\boxed{\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} + \frac{1}{2}\Delta t\frac{\partial^2 u}{\partial t^2} + \frac{1}{6}v\Delta x^2\frac{\partial^3 u}{\partial x^3} + O(\Delta t^2, \Delta x^4)} \qquad (5)$$

Q 1.2

We can see from Eq. 5 that largest step sizes are Δt and Δx^2. Therefore, FTCS is first order accurate in time, and 2nd order in space

Q 1.3

Please see the code and the plots in the archive.

Q 1.4

Assume

$$u_i^n = D e^{ikx_i} e^{wt_n}, \tag{6}$$

then

$$u_i^{n+1} = D e^{ikx_i} e^{w(t_n + \Delta t)}$$

$$= D e^{ikx_i} e^{wt_n} A, \tag{7}$$

where

$$A = e^{w\Delta t}.$$

We can also write Eq. 6 as

$$u_i^n = D e^{ikx_i} A^n \tag{8}$$

Substituting u_i^n into Eq. 2 gives:

$$\frac{D e^{ikx_i} A^n A - D e^{ikx_i} A^n}{\Delta t}$$

$$= -\frac{V}{2\Delta x}\left[D e^{ik(x_i + \Delta x)} A^n - D e^{ik(x_i - \Delta x)} A^n\right]$$

Rearrange:

$$\cancel{A^n D e^{ikx_i}}\left(\frac{A-1}{\Delta t}\right) = -\frac{VD}{\Delta x} \cancel{e^{ikx_i}}\left(\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2}\right)\cancel{A^n}$$

$$\frac{A-1}{\Delta t} = -\frac{V}{\Delta x} i \sin(k\Delta x)$$

$$A = 1 - iV\frac{\Delta t}{\Delta x} \sin(k\Delta x).$$

$$|A| = \sqrt{1 + v^2 \frac{\Delta t^2}{\Delta x^2} \sin^2(k\Delta x)}$$

For stability we require $|A| \leq 1$, but $|A| > 1$.

Therefore, FTCS method is unconditionally unstable. We see in the plots of the solution that large instabilities quickly develop, which is consistent with our numerical analysis of instability.

Q. 1.5

The code and plots of the Lax method are in the archive.

Q 1.6

The reccurence relation for the Lax method is given by

$$u_i^{n+1} = \frac{1}{2}\left[u_{i-1}^n + u_{i+1}^n\right] - v\Delta t \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}. \quad (9)$$

Next we substitute u_i^n from Eq.8 and simplify:

$$\cancel{De^{ikx_i} A^n} A = \frac{1}{2}\left[\cancel{De^{ikx_i} A^n}\left(e^{-ik\Delta x} + e^{ik\Delta x}\right)\right]$$

$$- \frac{v\Delta t}{\Delta x} \cancel{De^{ikx_i} A^n}\left(\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2}\right).$$

Rearranging gives:

$$A = \cos(k\Delta x) - iv\frac{\Delta t}{\Delta x}\sin(k\Delta x).$$

$$|A|^2 = \cos^2(k\Delta x) + v^2\frac{\Delta t^2}{\Delta x^2}\sin^2(k\Delta x).$$

$$|A|^2 = 1 - \sin^2(k\Delta x) + v^2\frac{\Delta t^2}{\Delta x^2}\sin^2(k\Delta x) \quad \Big(\text{use } \cos^2 k\Delta x = 1 - \sin^2 k\Delta x$$

$$= 1 + \sin^2(k\Delta x)\left[V^2\frac{\Delta t^2}{\Delta x^2} - 1\right]$$

In order to have $|A| \leq 1$, we need

$$V^2\frac{\Delta t^2}{\Delta x^2} - 1 \leq 0$$

Rearranging gives:

$$V^2\frac{\Delta t^2}{\Delta x^2} \leq 1$$

$$\boxed{|V|\frac{\Delta t}{\Delta x} \leq 1} \qquad (10)$$

We have found that LAX method is numerically stable when Inequality 10 is satisfied. This is consistent with our calculations, where we used $V\frac{\Delta t}{\Delta x} = 0.5$, and our program produced a stable-looking solution