Derivation of equation of motion for the angle θ

The derivation is used in the article https://evgenii.com/blog/earth-orbit-simulation.

We have the Lagrangian

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{GMm}{r}.$$

We need to calculate

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}. \tag{1}$$

Since L does not depend on θ,

$$\frac{\partial L}{\partial \theta} = 0.$$

Furthermore,

$$\begin{aligned}\frac{\partial L}{\partial \dot{\theta}} &= \frac{\partial}{\partial \dot{\theta}}\left(\frac{m}{2}r^2\dot{\theta}^2\right)\\ &= \frac{m}{2}r^2(2)\dot{\theta} = mr^2\dot{\theta}.\end{aligned}$$

Next, we take the time derivative:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= \frac{d}{dt}\left(mr^2\dot{\theta}\right)\\ &= m\frac{d}{dt}\left(r^2\dot{\theta}\right)\\ &= m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}). && \text{(Product rule)}\end{aligned}$$

Substituting into Equation 1 gives

$$m(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = 0.$$

Simplify:

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0.$$

Finally, we solve for $\ddot{\theta}$:

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r}.$$