Divergence of third order tensor in spherical geometry

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$$\mathbf{T} = \sum_{ijk} T_{ijk} (\mathbf{e_i} \otimes \mathbf{e_j} \otimes \mathbf{e_k}) \tag{1}$$

$$\nabla \cdot \mathbf{T} = \sum_{ijk} \frac{1}{h_i} \left[\frac{\partial T_{ijk}}{\partial x_i} + \sum_m \Gamma^i_{mi} T_{mjk} + \sum_m \Gamma^j_{mi} T_{imk} + \sum_m \Gamma^k_{mi} T_{ijm} \right] (\mathbf{e_j} \otimes \mathbf{e_k}) = \tag{2}$$

$$= \sum_{jk} \left\{ \frac{1}{h_r} \left[\frac{\partial T_{rjk}}{\partial r} + \sum_m \Gamma^r_{mr} T_{mjk} + \sum_m \Gamma^j_{mr} T_{rmk} + \sum_m \Gamma^k_{mr} T_{rjm} \right] \right\} (\mathbf{e_j} \otimes \mathbf{e_k}) + \tag{3}$$

$$+ \sum_{jk} \left\{ \frac{1}{h_\theta} \left[\frac{\partial T_{\theta jk}}{\partial \theta} + \sum_m \Gamma^\theta_{m\theta} T_{mjk} + \sum_m \Gamma^j_{m\theta} T_{\theta mk} + \sum_m \Gamma^k_{m\theta} T_{\theta jm} \right] \right\} (\mathbf{e_j} \otimes \mathbf{e_k}) + \tag{4}$$

$$+ \sum_{jk} \left\{ \frac{1}{h_\phi} \left[\frac{\partial T_{\phi jk}}{\partial \phi} + \sum_m \Gamma^\phi_{m\phi} T_{mjk} + \sum_m \Gamma^j_{m\phi} T_{\phi mk} + \sum_m \Gamma^k_{m\phi} T_{\phi jm} \right] \right\} (\mathbf{e_j} \otimes \mathbf{e_k}) = \tag{5}$$

$$= \sum_{jk} \left\{ \frac{1}{h_r} \left[\frac{\partial T_{rjk}}{\partial r} + \left(\Gamma^r_{rr} T_{rjk} + \Gamma^r_{\theta r} T_{\theta jk} + \Gamma^r_{\phi r} T_{\phi jk} \right) + \left(\Gamma^j_{rr} T_{rrk} + \Gamma^j_{\theta r} T_{r\theta k} + \Gamma^j_{\phi r} T_{r\phi k} \right) + \left(\Gamma^k_{rr} T_{rjr} + \Gamma^k_{\theta r} T_{rj\theta} + \Gamma^k_{\phi r} T_{rj\phi} \right) \right] \right\} (\mathbf{e_j} \otimes \mathbf{e_k}) + \tag{6}$$

$$+ \sum_{jk} \left\{ \frac{1}{h_\theta} \left[\frac{\partial T_{\theta jk}}{\partial \theta} + \left(\Gamma^\theta_{r\theta} T_{rjk} + \Gamma^\theta_{\theta\theta} T_{\theta jk} + \Gamma^\theta_{\phi\theta} T_{\phi jk} \right) + \left(\Gamma^j_{r\theta} T_{\theta rk} + \Gamma^j_{\theta\theta} T_{\theta\theta k} + \Gamma^j_{\phi\theta} T_{\theta\phi k} \right) + \left(\Gamma^k_{r\theta} T_{\theta jr} + \Gamma^k_{\theta\theta} T_{\theta j\theta} + \Gamma^k_{\phi\theta} T_{\theta j\phi} \right) \right] \right\} (\mathbf{e_j} \otimes \mathbf{e_k}) + \tag{7}$$

$$+ \sum_{jk} \left\{ \frac{1}{h_\phi} \left[\frac{\partial T_{\phi jk}}{\partial \phi} + \left(\Gamma^\phi_{r\phi} T_{rjk} + \Gamma^\phi_{\theta\phi} T_{\phi jk} + \Gamma^\phi_{\phi\phi} T_{\phi jk} \right) + \left(\Gamma^j_{r\phi} T_{\phi rk} + \Gamma^j_{\theta\phi} T_{\phi\theta k} + \Gamma^j_{\phi\phi} T_{\phi\phi k} \right) + \left(\Gamma^k_{r\phi} T_{\phi jr} + \Gamma^k_{\theta\phi} T_{\phi j\theta} + \Gamma^k_{\phi\phi} T_{\phi j\phi} \right) \right] \right\} (\mathbf{e_j} \otimes \mathbf{e_k}) = \tag{8}$$

$$T_{rr}\ (\mathbf{e_r}\otimes\mathbf{e_r}):\quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{rrr})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta rr})+\frac{1}{r\sin\theta}\frac{\partial T_{\phi rr}}{\partial\phi}-\frac{T_{\theta\theta r}}{r}-\frac{T_{\theta r\theta}}{r}-\frac{T_{\phi\phi r}}{r}-\frac{T_{\phi r\phi}}{r} \tag{9}$$

$$T_{r\theta}\ (\mathbf{e_r}\otimes\mathbf{e}_\theta):\quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{rr\theta})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta r\theta})+\frac{1}{r\sin\theta}\frac{\partial T_{\phi r\theta}}{\partial\phi}-\frac{T_{\theta\theta\theta}}{r}+\frac{T_{\theta rr}}{r}-\frac{T_{\phi\phi\theta}}{r}-\frac{T_{\phi r\phi}\cos\theta}{r\sin\theta} \tag{10}$$

$$T_{r\phi}\ (\mathbf{e_r}\otimes\mathbf{e}_\phi):\quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{rr\phi})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta r\phi})+\frac{1}{r\sin\theta}\frac{\partial T_{\phi r\phi}}{\partial\phi}+\frac{T_{\theta\theta\phi}}{r}-\frac{T_{\phi\phi\phi}}{r}+\frac{T_{\phi r\phi}\cos\theta}{r\sin\theta} \tag{11}$$

$$T_{\theta r}\ (\mathbf{e}_\theta\otimes\mathbf{e_r}):\quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{r\theta r})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\theta r})+\frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta r}}{\partial\phi}+\frac{T_{\theta rr}}{r}+\frac{T_{\theta\theta\theta}}{r}-\frac{T_{\phi\theta\phi}}{r} \tag{12}$$

$$T_{\theta\theta}\ (\mathbf{e}_\theta\otimes\mathbf{e}_\theta):\quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{r\theta\theta})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\theta\theta})+\frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta\theta}}{\partial\phi}+\frac{T_{\theta r\theta}}{r}+\frac{T_{\theta\theta r}}{r}-\frac{T_{\phi\phi\theta}\cos\theta}{r\sin\theta}-\frac{T_{\phi\theta\phi}\cos\theta}{r\sin\theta} \tag{13}$$

$$T_{\theta\phi}\ (\mathbf{e}_\theta\otimes\mathbf{e}_\phi):\quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{r\theta\phi})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\theta\phi})+\frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta\phi}}{\partial\phi}+\frac{T_{\theta r\phi}}{r}+\frac{T_{\phi\theta r}}{r}+\frac{T_{\phi\theta\theta}\cos\theta}{r\sin\theta} \tag{14}$$

$$T_{\theta r}\ (\mathbf{e}_\phi\otimes\mathbf{e_r}):\quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{r\phi r})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\phi r})+\frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi r}}{\partial\phi}-\frac{T_{\theta\phi\theta}}{r}+\frac{T_{\phi rr}}{r}+\frac{T_{\phi\theta r}\cos\theta}{r\sin\theta}-\frac{T_{\phi\phi\phi}}{r} \tag{15}$$

$$T_{\theta\theta}\ (\mathbf{e}_\phi\otimes\mathbf{e}_\theta):\quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{r\phi\theta})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\phi\theta})+\frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi\theta}}{\partial\phi}+\frac{T_{\theta\phi r}}{r}+\frac{T_{\phi r\theta}}{r}+\frac{T_{\phi\theta\theta}\cos\theta}{r\sin\theta}-\frac{T_{\phi\theta\phi}\cos\theta}{r\sin\theta} \tag{16}$$

$$T_{\theta\phi}\ (\mathbf{e}_\phi\otimes\mathbf{e}_\phi):\quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_{r\phi\phi})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\phi\phi})+\frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi\phi}}{\partial\phi}+\frac{T_{\phi r\phi}}{r}+\frac{T_{\phi\theta\phi}\cos\theta}{r\sin\theta}+\frac{T_{\phi\phi r}}{r}+\frac{T_{\phi\phi\theta}\cos\theta}{r\sin\theta} \tag{17}$$