1 Alternative hydrodynamic stellar structure equations EXPLAINED

$$\partial_r \overline{m} = -\overline{\rho}\ \overline{m}\ \overline{g}_r/\Gamma_1 \overline{P} + 4\pi r^2 \overline{\rho} \tag{1}$$
$$\partial_r \overline{P} = -\overline{\rho}\ \overline{g}_r \tag{2}$$
$$\partial_r \widetilde{L} = -4\pi r^2 \widetilde{u}_r \overline{\rho}\ \overline{g}_r/\Gamma_1 + \widetilde{\epsilon}_t \partial_r 4\pi r^2 \overline{\rho} \widetilde{u}_r \tag{3}$$
$$\partial_r \overline{T} = -(\Gamma_3 - 1)\ \overline{\rho}\ \overline{T}\ \overline{g}_r/\Gamma_1 \overline{P} \tag{4}$$
$$\partial_t \widetilde{X}_i = \widetilde{\dot{X}}_i^{nuc} - (1/\overline{\rho}) \nabla_r f_i - \widetilde{u}_r \partial_r \widetilde{X}_i \tag{5}$$

Equations (1)-(4) can be derived by four assumptions: (i) adiabatic convection $\Gamma_1 = \partial\ ln\ P/\partial\ ln\ \rho|_{ad}$, (ii) hydrostatic equilibrium $\partial_r = -\rho g_r$, (iii) $\epsilon_{nuc} << \varepsilon_i$, (iv) $\varepsilon_K << \varepsilon_i \sim \varepsilon_t$.

1.1 Continuity equation

The continuity equation can be derived from first principles for adiabatic convection in hydrostatic equilibrium. We start from the definition of $\Gamma_1 = \partial\ ln\ P/\partial\ ln\ \rho|_{ad}$. I will avoid the $_{ad}$ suffix for clarity reasons and at some point assume hydrostatic eqvilibrium $\partial_r P = -\rho g_r$.

$$\Gamma_1 = \frac{\partial\ ln\ P}{\partial\ ln\ \rho} \tag{6}$$
$$\Gamma_1 = \frac{\rho}{P} \frac{\partial_r P}{\partial_r \rho} \tag{7}$$
$$\Gamma_1\ P\ \partial_r \rho = \rho\ \partial_r P \tag{8}$$
$$\Gamma_1\ P\ \partial_r \rho = \rho\ (-\rho g_r) \tag{9}$$
$$\partial_r \rho = -\frac{\rho\ \rho\ g_r}{\Gamma_1 P} \quad \backslash .V \text{ (volume)} = 4\pi r^3/3 \tag{10}$$
$$V \partial_r \rho = -\frac{V \rho\ \rho\ g_r}{\Gamma_1 P} \tag{11}$$
$$\partial_r V \rho - \rho \partial_r V = -\frac{V \rho\ \rho\ g_r}{\Gamma_1 P} \tag{12}$$
$$\partial_r m - \rho 4\pi r^2 = -\frac{m\ \rho\ g_r}{\Gamma_1 P} \tag{13}$$
$$\partial_r m = -\frac{\rho\ m\ g_r}{\Gamma_1 P} + 4\pi r^2 \rho \tag{14}$$
$$\partial_r m = -\rho\ m\ g_r/\Gamma_1 P + 4\pi r^2 \rho \tag{15}$$

let's transform it to more familiar form (Kippenhahn & Weigert, page 4, Eq.1.6)

$$\partial_r \rho V = + V \partial_r \rho + \rho \partial_r V \tag{16}$$
$$\partial_r \rho V = + V \partial_r \rho + 4\pi r^2 \rho \tag{17}$$
$$-\rho \ m \ g_r/\Gamma_1 P + 4\pi r^2 \rho = + V \partial_r \rho + 4\pi r^2 \rho \quad (\partial_r \rho = -\rho \ \rho \ g_r/P \ \Gamma_1) \text{ can be derived from } \Gamma_1 \text{ and } \partial_r P = -\rho g_r \tag{18}$$
$$\partial_r M = + 4\pi r^2 \rho \tag{19}$$

1.2 Temperature equation

The temperature equation can be derived from first principles for adiabatic convection in hydrostatic equilibrium. We start from the definition of $\Gamma_1 = \partial \ ln \ P/\partial \ ln \ \rho|_{ad}$. I will avoid the $_{ad}$ suffix for clarity reasons and at some point assume hydrostatic eqvilibrium $\partial_r P = -\rho g_r$.

$$\Gamma_1 = \frac{\partial \ ln \ P}{\partial \ ln \ \rho} \tag{20}$$
$$\Gamma_1 = \frac{\rho}{P} \frac{\partial_r P}{\partial_r \rho} \quad \backslash . \partial_r lnT \tag{21}$$
$$\partial_r lnT \ \Gamma_1 = \frac{\rho}{P} \frac{\partial_r P}{\partial_r \rho} \partial_r lnT \tag{22}$$
$$P \ \partial_r lnT \ \Gamma_1 = \rho \frac{\partial_r P}{\partial_r \rho} \partial_r lnT \tag{23}$$
$$P \ \partial_r lnT \ \Gamma_1 = \frac{\partial_r P}{\partial_r ln\rho} \partial_r lnT \tag{24}$$
$$P \ \partial_r lnT \ \Gamma_1 = \frac{-\rho g_r}{\partial_r ln\rho} \partial_r lnT \tag{25}$$
$$\frac{P \ \partial_r T \ \Gamma_1}{T} = -\frac{\rho g_r}{\partial_r ln\rho} \partial_r lnT \tag{26}$$
$$-\frac{P \ \partial_r T \ \Gamma_1}{\rho g_r T} = \frac{\partial_r lnT}{\partial_r ln\rho} \tag{27}$$
$$-\frac{P \ \partial_r T \ \Gamma_1}{\rho g_r T} = \Gamma_3 - 1 \tag{28}$$
$$\partial_r T = -\frac{(\Gamma_3 - 1)\rho g_r T}{P\Gamma_1} \tag{29}$$
$$\partial_r T = -(\Gamma_3 - 1) \ \rho \ T \ g_r/\Gamma_1 P \tag{30}$$

let's transform it to more familiar form (Kippenhahn & Weigert, page 55, Eq.7.32)

$$\frac{\partial T}{\partial r} = - \frac{T}{P}\frac{\Gamma_3 - 1}{\Gamma_1}\rho \; g_r \tag{31}$$

$$\frac{\partial T}{\partial r} = + \frac{T}{P}\frac{\Gamma_3 - 1}{\Gamma_1}\frac{Gm}{r^2}\rho \quad \backslash . \frac{1}{4\pi r^2 \rho} \tag{32}$$

$$\frac{\partial T}{\partial m} = + \frac{T}{P}\frac{Gm}{4\pi r^4}\frac{\Gamma_3 - 1}{\Gamma_1} \tag{33}$$

$$\frac{\partial T}{\partial m} = + \frac{T}{P}\frac{Gm}{4\pi r^4}\nabla_{ad} \tag{34}$$

1.3 Luminosity equation

The luminosity equation can be derived from first principles for adiabatic convection in hydrostatic equilibrium. We start from the definition of $\Gamma_1 = \partial \; ln \; P / \partial \; ln \; \rho|_{ad}$. I will avoid the $_{ad}$ suffix for clarity reasons and at some point assume hydrostatic equilibrium $\partial_r P = -\rho g_r$, nuclear energy production and kinetic energy being negligible compared to internal energy i.e. $\epsilon_{nuc} << \varepsilon_i$ and $\varepsilon_K << \varepsilon_i$.

$$\Gamma_1 = \frac{\partial \; ln \; P}{\partial \; ln \; \rho} \tag{35}$$

$$\Gamma_1 = \frac{\rho}{P}\frac{\partial_r P}{\partial_r \rho} \tag{36}$$

$$\Gamma_1 = \frac{\rho}{P}\frac{-\rho g_r}{\partial_r \rho} = -\frac{\rho^2}{P}\frac{g_r}{\partial_r \rho} \tag{37}$$

$$\Gamma_1 = \; - \frac{g_r}{P \partial_r \rho / \rho^2} \tag{38}$$

$$dq = du + Pdv \quad dq = \epsilon_{nuc} << du = d\varepsilon_i \quad v = 1/\rho \quad \text{1st thermodynamic law, e.g. Kippenhahn and Weigert, 1994, p.19} \tag{39}$$

$$d\varepsilon_i = - \, Pdv = -Pd(1/\rho) = Pd\rho/\rho^2 \tag{40}$$

$$\Gamma_1 = \; - \frac{g_r}{\partial_r \varepsilon_i} \tag{41}$$

$$\partial_r \varepsilon_i = \; - \frac{g_r}{\Gamma_1} \quad \varepsilon_K << \varepsilon_i \text{ and } \varepsilon_i \sim \varepsilon_t \tag{42}$$

$$\partial_r \varepsilon_t = \; - \frac{g_r}{\Gamma_1} \quad \backslash . 4\pi r^2 \rho u_r \tag{43}$$

$$4\pi r^2 \rho u_r \partial_r \varepsilon_t = -\frac{4\pi r^2 \rho u_r g_r}{\Gamma_1} \tag{44}$$

$$\partial_r 4\pi r^2 \rho u_r \varepsilon_t - \varepsilon_t \partial_r 4\pi r^2 \rho u_r = -\frac{4\pi r^2 \rho u_r g_r}{\Gamma_1} \tag{45}$$

$$\partial_r L - \varepsilon_t \partial_r 4\pi r^2 \rho u_r = -\frac{4\pi r^2 \rho u_r g_r}{\Gamma_1} \tag{46}$$

$$\partial_r L = -\frac{4\pi r^2 \rho u_r g_r}{\Gamma_1} + \varepsilon_t \partial_r 4\pi r^2 \rho u_r \tag{47}$$

$$\partial_r L = -4\pi r^2 u_r \rho g_r / \Gamma_1 + \varepsilon_t \partial_r 4\pi r^2 \rho u_r \tag{48}$$

Table 1: Definitions:

ρ density	g_r radial gravitational acceleration
$m = \rho V = \rho \frac{4}{3}\pi r^3$ mass	$M = \int \rho(r) dV = \int \rho(r) 4\pi r^2 dr$ integrated mass
T temperature	$\mathcal{S} = \rho \epsilon_{\rm nuc}(q)$ nuclear energy production (cooling function)
P pressure	$\tau_{ij} = 2\mu S_{ij}$ viscous stress tensor (μ kinematic viscosity)
u_r, u_θ, u_ϕ velocity components	$S_{ij} = (1/2)(\partial_i u_j + \partial_j u_i)$ strain rate
$\mathbf{u} = u(u_r, u_\theta, u_\phi)$ velocity	$\widetilde{R}_{ij} = \overline{\rho} \widetilde{u''_i u''_j}$ Reynolds stress tensor
$j_z = r \sin\theta \ u_\phi$ specific angular momentum	$F_T = \chi \partial_r T$ heat flux
$d = \nabla \cdot \mathbf{u}$ dilatation	$\Gamma_1 = (d \ ln \ P / d \ ln \ \rho)\vert_s$
ϵ_I specific internal energy	$\Gamma_2/(\Gamma_2 - 1) = (d \ ln \ P / d \ ln \ T)\vert_s$
h specific enthalpy	$\Gamma_3 - 1 = (d \ ln \ T / d \ ln \ \rho)\vert_s$
$k = (1/2)\widetilde{u''_i u''_i}$ turbulent kinetic energy	$\widetilde{k}^r = (1/2)\widetilde{u''_r u''_r} = (1/2)\widetilde{R}_{rr}/\overline{\rho}$ radial turbulent kinetic energy
ϵ_k specific kinetic energy	$\widetilde{k}^\theta = (1/2)\widetilde{u''_\theta u''_\theta} = (1/2)\widetilde{R}_{\theta\theta}/\overline{\rho}$ angular turbulent kinetic energy
ϵ_t specific total energy	$\widetilde{k}^\phi = (1/2)\widetilde{u''_\phi u''_\phi} = (1/2)\widetilde{R}_{\phi\phi}/\overline{\rho}$ angular turbulent kinetic energy
s specific entropy	$\widetilde{k}^h = \widetilde{k}^\theta + \widetilde{k}^\phi$ horizontal turbulent kinetic energy
$v = 1/\rho$ specific volume	$f_k = (1/2)\overline{\rho} \widetilde{u''_i u''_i u''_r}$ turbulent kinetic energy flux
X_α mass fraction of isotope α	$f_k^r = (1/2)\overline{\rho} \widetilde{u''_r u''_r u''_r}$ radial turbulent kinetic energy flux
$\dot{X}_\alpha^{nuc}$ rate of change of X_α	$f_k^\theta = (1/2)\overline{\rho} \widetilde{u''_\theta u''_\theta u''_r}$ angular turbulent kinetic energy flux
A_α number of nucleons in isotope α	$f_k^\phi = (1/2)\overline{\rho} \widetilde{u''_\phi u''_\phi u''_r}$ angular turbulent kinetic energy flux
Z_α charge of isotope α	$f_k^h = f_k^\theta + f_k^\phi$ horizontal turbulent kinetic energy flux
A mean number of nucleons per isotope	$W_p = \overline{P' d''}$ turbulent pressure dilatation
Z mean charge per isotope	$W_b = \overline{\rho} \overline{u''_r} \widetilde{g}_r$ buoyancy
$f_P = \overline{P' u'_r}$ acoustic flux	$f_T = -\overline{\chi \partial_r T}$ heat flux (χ thermal conductivity)

Table 2: Definitions (continued):

$f_I = \overline{\rho}\widetilde{\epsilon''_I u''_r}$ internal energy flux	$f_\alpha = \overline{\rho}\widetilde{X''_\alpha u''_r}$ X_α flux
$f_s = \overline{\rho}\widetilde{s'' u''_r}$ entropy flux	$f_{jz} = \overline{\rho}\widetilde{j''_z u''_r}$ angular momentum flux
$f_T = \overline{u'_r T'}$ turbulent heat flux	$f_A = \overline{\rho}\widetilde{A'' u''_r}$ A (mean number of nucleons per isotope) flux
$f_h = \overline{\rho}\widetilde{h'' u''_r}$ enthalpy flux	$f_Z = \overline{\rho}\widetilde{Z'' u''_r}$ Z (mean charge per isotope) flux
$b = \overline{v'\rho'}$ density-specific volume covariance	$\mathcal{N}_\rho, \mathcal{N}_{ur}, \mathcal{N}_{u\theta}, \mathcal{N}_{u\phi}, \mathcal{N}_{jz}, \mathcal{N}_\alpha, \mathcal{N}_A, \mathcal{N}_Z$ numerical effect
$f_\tau = f^r_\tau + f^\theta_\tau + f^\phi_\tau$ viscous flux	$\mathcal{N}_{\epsilon I} = -\nabla_r f_\tau + \varepsilon_k$ numerical effect
$f^r_\tau = -\overline{\tau'_{rr} u'_r}$ viscous flux	$\mathcal{N}_{\epsilon k} = -\varepsilon_k$ numerical effect
$f^\theta_\tau = -\overline{\tau'_{\theta r} u'_\theta}$ viscous flux	$\mathcal{N}_{\epsilon t} = -\nabla_r f_\tau$ numerical effect
$f^\phi_\tau = -\overline{\tau'_{\phi r} u'_\phi}$ viscous flux	$\mathcal{N}_s = \overline{-\varepsilon_k/T}$ numerical effect
$f^h_\tau = f^\theta_\tau + f^\phi_\tau$ viscous flux	$\mathcal{N}_h = -\nabla_r f_\tau + (\Gamma_3 - 1)\varepsilon_k$ numerical effect
$f^r_I = \overline{\rho}\widetilde{\epsilon''_I u''_r u''_r}$ radial flux of f_I	$\mathcal{N}_P = +(\Gamma_3 - 1)\varepsilon_k$ numerical effect
$f^r_s = \overline{\rho}\widetilde{s'' u''_r u''_r}$ radial flux of f_s	$\mathcal{N}_T = +\overline{(\tau_{ij}\partial_j u_i)/(c_v\rho)}$ numerical effect
$f^r_h = \overline{\rho}\widetilde{h'' u''_r u''_r}$ radial flux of f_h	$\mathcal{N}_{Rrr} = -2\nabla_r f^r_\tau - 2\varepsilon^r_k$ numerical effect
$f^r_T = \overline{T' u'_r u'_r}$ radial flux of f_T	$\mathcal{N}_{R\theta\theta} = -2\nabla_r f^\theta_\tau - 2\varepsilon^\theta_k$ numerical effect
$f^r_{jz} = \overline{\rho}\widetilde{j''_z u''_r u''_r}$ radial flux of f_{jz}	$\mathcal{N}_{R\phi\phi} = -2\nabla_r f^\phi_\tau - 2\varepsilon^\phi_k$ numerical effect
$f^r_\alpha = \overline{\rho}\widetilde{X''_\alpha u''_r u''_r}$ radial flux of f_α	$\mathcal{N}_k = -\nabla_r f_\tau - \varepsilon_k$ numerical effect
$f^r_A = \overline{\rho}\widetilde{A'' u''_r u''_r}$ radial flux of f_A	$\mathcal{N}_{kr} = -\nabla_r f^r_\tau - \varepsilon^r_k$ numerical effect
$f^r_Z = \overline{\rho}\widetilde{Z'' u''_r u''_r}$ radial flux of f_Z	$\mathcal{N}_{kh} = -\nabla_r f^h_\tau - \varepsilon^h_k$ numerical effect
$\mathcal{G}^r_k = -(1/2)\overline{G^R_{rr}} - \overline{u''_r G^M_r}$	$\mathcal{N}_a = -\varepsilon_a$ numerical effect

Table 3: Definitions (continued):

$\mathcal{G}_k^\theta = -(1/2)\overline{G_{\theta\theta}^R} - \overline{u''_\theta G_\theta^M}$	$\mathcal{N}_b$ numerical effect
$\mathcal{G}_k^\phi = -(1/2)\overline{G_{\phi\phi}^R} - \overline{u''_\phi G_\phi^M}$	$\mathcal{N}_{fI} = -\nabla_r(\overline{\epsilon''_I \tau'_{rr}}) + \overline{u''_r \tau_{ij}\partial_i u_j} - \varepsilon_I$ numerical effect
$\mathcal{G}_k^h = +\mathcal{G}_k^\theta + \mathcal{G}_k^\phi$	$\mathcal{N}_{fh} = -\nabla_r(\overline{h''\tau'_{rr}}) + \overline{u''_r(\Gamma_3 - 1)\tau_{ij}\partial_i u_j} - \overline{u''_r \nabla_i u_i \tau_{ji}} - \varepsilon_h$ numerical effect
$\mathcal{G}_a = +\overline{\rho' v G_r^M}$	$\mathcal{N}_{fs} = -\nabla_r(\overline{s''\tau'_{rr}}) + \overline{u''_r \tau_{ij}\partial_i u_j / T} - \varepsilon_s$ numerical effect
$\mathcal{G}_I = -\overline{G_r^I} - \overline{\epsilon''_I G_r^M}$	$\mathcal{N}_{fA} = -\nabla_r(\overline{A''\tau'_{rr}}) - \varepsilon_A$ numerical effect
$\mathcal{G}_\alpha = -\overline{G_r^\alpha} - \overline{X''_\alpha G_r^M}$	$\mathcal{N}_{fZ} = -\nabla_r(\overline{Z''\tau'_{rr}}) - \varepsilon_Z$ numerical effect
$\mathcal{G}_A = -\overline{G_r^A} - \overline{A'' G_r^M}$	$\mathcal{N}_{f\alpha} = -\nabla_r(\overline{\alpha''\tau'_{rr}}) - \varepsilon_\alpha$ numerical effect
$\mathcal{G}_Z = -\overline{G_r^Z} - \overline{Z'' G_r^M}$	$\mathcal{N}_{fjz} = -\nabla_r(\overline{j''_z \tau'_{rr}}) - \varepsilon_{jz}$ numerical effect
$\mathcal{G}_h = -\overline{G_r^h} - \overline{h'' G_r^M}$	$\mathcal{N}_{fT} = +\overline{T'\partial_i \tau_{ri}/\rho} + \overline{u'_r \tau_{ij}\partial_i u_j / \rho c_v}$ numerical effect
$\mathcal{G}_T = -\overline{G_r^T} - \overline{T' G_r^M}$	
$\mathcal{G}_s = -\overline{G_r^s} - \overline{s'' G_r^M}$	
$\mathcal{G}_{jz} = -\overline{G_r^{jz}} - \overline{j''_z G_r^M}$	
$\sigma_\rho = \overline{\rho'\rho'}$	$\mathcal{N}_{\sigma_\rho}$ numerical effect
$\sigma_P = \overline{P'P'}$	$\mathcal{N}_{\sigma_P} = +2(\Gamma_3 - 1)\overline{P'\tau_{ij}\partial_i u_j}$ numerical effect
$\sigma_T = \overline{T'T'}$	$\mathcal{N}_{\sigma_T} = +\overline{2T'\tau_{ij}\partial_i u_j / \rho c_v}$ numerical effect
$\sigma_{ur} = \widetilde{u''_r u''_r}$	$\mathcal{N}_{\sigma_{ur}} = +2\nabla_r f_\tau^r - 2\varepsilon_k^r$ numerical effect
$\sigma_s = \widetilde{s'' s''}$	$\mathcal{N}_{\sigma_s} = +2\overline{s''\tau_{ij}\partial_j u_i / T}$ numerical effect
$\sigma_\alpha = \widetilde{X''_\alpha X''_\alpha}$	$\mathcal{N}_{\sigma_\alpha}$ numerical effect numerical effect
$\sigma_{\epsilon I} = \widetilde{\epsilon''_I \epsilon''_I}$	$N_{\sigma\epsilon_I} = +2\overline{\epsilon''_I \tau_{ij}\partial_j u_i}$ numerical effect

Table 4: Definitions (continued):

$$
\begin{aligned}
\varepsilon_k^r &= \overline{\tau'_{rr}\partial_r u''_r} + \overline{\tau'_{r\theta}(1/r)\partial_\theta u''_r} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi u''_r} \\
\varepsilon_k^\theta &= \overline{\tau'_{\theta r}\partial_r u''_\theta} + \overline{\tau'_{\theta\theta}(1/r)\partial_\theta u''_\theta} + \overline{\tau'_{\theta\phi}(1/r\sin\theta)\partial_\phi u''_\theta} \\
\varepsilon_k^\phi &= \overline{\tau'_{\phi r}\partial_r u''_\phi} + \overline{\tau'_{\phi\theta}(1/r)\partial_\theta u''_\phi} + \overline{\tau'_{\phi\phi}(1/r\sin\theta)\partial_\phi u''_\phi} \\
\varepsilon_k &= \varepsilon_k^r + \varepsilon_k^\theta + \varepsilon_k^\phi \\
\varepsilon_k^h &= \varepsilon_k^\theta + \varepsilon_k^\phi \\
\varepsilon_a &= \overline{\rho' v \nabla_r \tau'_{rr}} \\
\varepsilon_I &= \overline{\tau'_{rr}\partial_r \epsilon''_I} + \overline{\tau'_{r\theta}(1/r)\partial_\theta \epsilon''_I} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi \epsilon''_I} \\
\varepsilon_s &= \overline{\tau'_{rr}\partial_r s''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta s''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi s''} \\
\varepsilon_\alpha &= \overline{\tau'_{rr}\partial_r X''_\alpha} + \overline{\tau'_{r\theta}(1/r)\partial_\theta X''_\alpha} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi X''_\alpha} \\
\varepsilon_A &= \overline{\tau'_{rr}\partial_r A''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta A''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi A''} \\
\varepsilon_Z &= \overline{\tau'_{rr}\partial_r Z''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta Z''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi Z''} \\
\varepsilon_h &= \overline{\tau'_{rr}\partial_r h''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta h''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi h''} \\
\varepsilon_{jz} &= \overline{\tau'_{rr}\partial_r j''_z} + \overline{\tau'_{r\theta}(1/r)\partial_\theta j''_z} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi j''_z}
\end{aligned}
$$

$$
\begin{aligned}
\overline{G_r^M} &= -\overline{\rho u_\theta u_\theta/r} - \overline{\rho u_\phi u_\phi/r} \\
\overline{G_\theta^M} &= +\overline{\rho u_\theta u_r/r} - \overline{\rho u_\phi u_\phi/(r\tan\theta)} \\
\overline{G_\phi^M} &= +\overline{\rho u_\phi u_r/r} + \overline{\rho u_\phi u_\theta/(r\tan\theta)} \\
\overline{G_{rr}^R} &= -\overline{\rho u''_\theta u''_\theta u''_r/r} - \overline{\rho u''_\theta u''_r u''_\theta/r} - \overline{\rho u''_\phi u''_\phi u''_r/r} - \overline{\rho u''_\phi u''_r u''_\phi/r} \\
\overline{G_{\theta\theta}^R} &= +\overline{\rho u''_\theta u''_r u''_\theta/r} + \overline{\rho u''_\theta u''_\theta u''_r/r} - \overline{\rho u''_\phi u''_\phi u''_\theta/(r\tan\theta)} - \overline{u''_\phi u''_\theta u''_\phi/(r\tan\theta)} \\
\overline{G_{\phi\phi}^R} &= +\overline{\rho u''_\phi u''_r r_\phi/r} + \overline{\rho u''_\phi u''_\theta u''_\phi/(r\tan\theta)} + \overline{\rho u''_\phi u''_\phi u''_r/r} + \overline{\rho u''_\phi u''_\phi u''_\theta/(r\tan\theta)} \\
\overline{G_r^I} &= -\overline{\rho\epsilon''_I u''_\theta u''_\theta/r} - \overline{\rho\epsilon''_I u''_\phi u''_\phi/r} \\
\overline{G_r^s} &= -\overline{\rho s'' u''_\theta u''_\theta/r} - \overline{\rho s'' u''_\phi u''_\phi/r} \\
\overline{G_r^\alpha} &= -\overline{\rho X''_\alpha u''_\theta u''_\theta/r} - \overline{\rho X''_\alpha u''_\phi u''_\phi/r} \\
\overline{G_r^A} &= -\overline{\rho A'' u''_\theta u''_\theta/r} - \overline{\rho A'' u''_\phi u''_\phi/r} \\
\overline{G_r^Z} &= -\overline{\rho Z'' u''_\theta u''_\theta/r} - \overline{\rho Z'' u''_\phi u''_\phi/r} \\
\overline{G_r^h} &= -\overline{\rho h'' u''_\theta u''_\theta/r} - \overline{\rho h'' u''_\phi u''_\phi/r} \\
\overline{G_r^T} &= -\overline{\rho T' u'_\theta u'_\theta/r} - \overline{\rho T' u'_\phi u'_\phi/r} \\
\overline{G_r^{jz}} &= -\overline{\rho j''_z u''_\theta u''_\theta/r} - \overline{\rho j''_z u''_\phi u''_\phi/r}
\end{aligned}
$$

$$
\nabla(.) = \nabla_r(.) + \nabla_\theta(.) + \nabla_\phi(.) = \frac{1}{r^2}\partial_r(r^2 .) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta .) + \frac{1}{r\sin\theta}\partial_\phi(.)
$$