# Divergence of tensors in spherical geometry up to third order (DETAILS of computation)

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January 29, 2013

**BACKGROUND READING:**

CONTINUUM MECHANICS (Lecture Notes)
Zdenek Martinec, Department of Geophysics, Faculty of Mathematics and Physics, Charles University in Prague

**Divergence of first order tensor $\nabla \cdot \mathbf{V}$**

$$\nabla (.) = \sum_n \frac{\mathbf{e_n}}{h_n} \frac{\partial (.)}{\partial x_n} \ : \text{nabla operator}$$

$$\mathbf{V} = \sum_i V_i \mathbf{e_i} \ : \text{tensor of first order (vector)} \qquad V_{i;n} = \frac{\partial V_i}{\partial x_n} + \sum_m \Gamma^i_{mn} V_m \ : \text{derivatives}$$

$$\nabla \cdot \mathbf{V} = \sum_n \frac{\mathbf{e_n}}{h_n} \frac{\partial \mathbf{V}}{\partial x_n} = \sum_n \frac{\mathbf{e_n}}{h_n} \sum_i V_{i;n} \mathbf{e_i} = \sum_{ni} \frac{V_{i;n}}{h_n} \underbrace{\mathbf{e_n} \mathbf{e_i}}_{\delta_{ni}} = \sum_i \frac{V_{i;i}}{h_i} = \sum_i \frac{1}{h_i} \left[ \frac{\partial V_i}{\partial x_i} + \sum_m \Gamma^i_{mi} V_m \right] \tag{1}$$

$$\frac{\partial \mathbf{V}}{\partial x_n} = \frac{\partial}{\partial x_n} \sum_i V_i \mathbf{e_i} = \sum_i \frac{\partial}{\partial x_n} (V_i \mathbf{e_i}) = \sum_i \left( \frac{\partial V_i}{\partial x_n} \mathbf{e_i} + V_i \frac{\partial \mathbf{e_i}}{\partial x_n} \right) = \sum_i \left( \frac{\partial V_i}{\partial x_n} \mathbf{e_i} + V_i \sum_m \Gamma^m_{in} \mathbf{e_m} \right) = \sum_i \left( \frac{\partial V_i}{\partial x_n} \mathbf{e_i} + \sum_m \Gamma^i_{mn} V_m \mathbf{e_i} \right) =$$
$$= \sum_i \left( \frac{\partial V_i}{\partial x_n} + \sum_m \Gamma^i_{mn} V_m \right) \mathbf{e_i} = \sum_i V_{i;n} \mathbf{e_i} \tag{2}$$

**Divergence of second order tensor $\nabla \cdot \mathbf{S}$**

$$\nabla(.) = \sum_n \frac{\mathbf{e_n}}{h_n}\frac{\partial(.)}{\partial x_n} \ \text{ : nabla operator}$$

$$\mathbf{S} = \sum_{ij} S_{ij}(\mathbf{e_i} \otimes \mathbf{e_j}) \ \text{ : tensor of second order} \qquad S_{ij;n} = \frac{\partial S_{ij}}{\partial x_n} + \sum_m \Gamma^i_{mn} S_{mj} + \sum_m \Gamma^j_{mn} S_{im} \ \text{ : derivatives}$$

$$\begin{aligned}\nabla \cdot \mathbf{S} &= \sum_n \frac{\mathbf{e_n}}{h_n}\frac{\partial \mathbf{S}}{\partial x_n} = \sum_n \frac{\mathbf{e_n}}{h_n}\sum_{ij} S_{ij;n}(\mathbf{e_i} \otimes \mathbf{e_j}) = \sum_{nij}\frac{S_{ij;n}}{h_n}\mathbf{e_n}(\mathbf{e_i} \otimes \mathbf{e_j}) = \sum_{nij}\frac{S_{ij;n}}{h_n}(\underbrace{\mathbf{e_n}\mathbf{e_i}}_{\delta_{ni}})\mathbf{e_j} = \sum_{ij}\frac{S_{ij;i}}{h_i} = \\ &= \sum_{ij}\frac{1}{h_i}\left(\frac{\partial S_{ij}}{\partial x_i} + \sum_m \Gamma^i_{mi} S_{mj} + \sum_m \Gamma^j_{mi} S_{im}\right)\mathbf{e_j}\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{\partial \mathbf{S}}{\partial x_n} &= \frac{\partial}{\partial x_n}\sum_{ij} S_{ij}(\mathbf{e_i} \otimes \mathbf{e_j}) = \sum_{ij}\frac{\partial}{\partial x_n} S_{ij}(\mathbf{e_i} \otimes \mathbf{e_j}) = \sum_{ij}\left[\frac{\partial S_{ij}}{\partial x_n}(\mathbf{e_i} \otimes \mathbf{e_j}) + S_{ij}\frac{\partial}{\partial x_n}(\mathbf{e_i} \otimes \mathbf{e_j})\right] = \\ &= \sum_{ij}\left[\frac{\partial S_{ij}}{\partial x_n}(\mathbf{e_i} \otimes \mathbf{e_j}) + S_{ij}\frac{\partial \mathbf{e_i}}{\partial x_n} \otimes \mathbf{e_j} + S_{ij} \otimes \mathbf{e_i}\frac{\partial \mathbf{e_j}}{\partial x_n}\right] =\end{aligned} \tag{4}$$

$$= \sum_{ij}\left[\frac{\partial S_{ij}}{\partial x_n}(\mathbf{e_i} \otimes \mathbf{e_j}) + S_{ij}\sum_m \Gamma^m_{in}(\mathbf{e_m} \otimes \mathbf{e_j}) + S_{ij}\sum_m \Gamma^m_{jn}(\mathbf{e_i} \otimes \mathbf{e_m})\right] =$$

: interchaning indices m $\leftrightarrow$ i in the second term; m $\leftrightarrow$ j in the third term (see Identity 12)

$$\begin{aligned}&= \sum_{ij}\left[\frac{\partial S_{ij}}{\partial x_n}(\mathbf{e_i} \otimes \mathbf{e_j}) + \sum_m \Gamma^i_{mn} S_{mj}(\mathbf{e_i} \otimes \mathbf{e_j}) + \sum_m \Gamma^j_{nm} S_{im}(\mathbf{e_i} \otimes \mathbf{e_j})\right] = \sum_{ij}\left[\frac{\partial S_{ij}}{\partial x_n} + \sum_m \Gamma^i_{mn} S_{mj} + \sum_m \Gamma^j_{nm} S_{im}\right](\mathbf{e_i} \otimes \mathbf{e_j}) = \\ &= \sum_{ij} S_{ij;n}(\mathbf{e_i} \otimes \mathbf{e_j})\end{aligned} \tag{5}$$

**Divergence of third order tensor $\nabla \cdot \mathbf{T}$**

$$\nabla(.) = \sum_n \frac{\mathbf{e_n}}{h_n}\frac{\partial(.)}{\partial x_n} \ \text{: nabla operator}$$

$$\mathbf{T} = \sum_{ijk} T_{ijk}(\mathbf{e_i}\otimes\mathbf{e_j}\otimes\mathbf{e_k}) \text{ : tensor of third order} \qquad T_{ijk;n} = \frac{\partial T_{ijk}}{\partial x_n} + \sum_m \Gamma^i_{mn}T_{mjk} + \sum_m \Gamma^j_{mn}T_{imk} + \sum_m \Gamma^k_{mn}T_{ijm} \text{ : derivatives}$$

$$\nabla\cdot\mathbf{T} = \sum_n \frac{\mathbf{e_n}}{h_n}\frac{\partial \mathbf{T}}{\partial x_n} = \sum_n \frac{\mathbf{e_n}}{h_n}\sum_{ijk} T_{ijk;n}(\mathbf{e_i}\otimes\mathbf{e_j}\otimes\mathbf{e_k}) = \sum_{nijk}\frac{T_{ijk;n}}{h_n}\mathbf{e_n}(\mathbf{e_i}\otimes\mathbf{e_j}\otimes\mathbf{e_k}) = \sum_{nijk}\frac{T_{ijk;n}}{h_n}(\underbrace{\mathbf{e_n}\mathbf{e_i}}_{\delta_{ni}})(\mathbf{e_j}\otimes\mathbf{e_k}) = \sum_{ijk}\frac{T_{ijk;i}}{h_i} =$$
$$= \sum_{ijk}\frac{1}{h_i}\left(\frac{\partial T_{ijk}}{\partial x_i} + \sum_m \Gamma^i_{mi}T_{mjk} + \sum_m \Gamma^j_{mi}T_{imk} + \sum_m \Gamma^k_{mi}T_{ijm}\right)(\mathbf{e_j}\otimes\mathbf{e_k}) \tag{6}$$

$$\frac{\partial\mathbf{T}}{\partial x_n} = \frac{\partial}{\partial x_n}\sum_{ijk}T_{ijk}(\mathbf{e_i}\otimes\mathbf{e_j}\otimes\mathbf{e_k}) = \sum_{ijk}\frac{\partial}{\partial x_n}T_{ijk}(\mathbf{e_i}\otimes\mathbf{e_j}\otimes\mathbf{e_k}) = \sum_{ijk}\left[\frac{\partial T_{ijk}}{\partial x_n}(\mathbf{e_i}\otimes\mathbf{e_j}\otimes\mathbf{e_k}) + T_{ijk}\frac{\partial}{\partial x_n}(\mathbf{e_i}\otimes\mathbf{e_j}\otimes\mathbf{e_k})\right] =$$
$$= \sum_{ijk}\left[\frac{\partial T_{ijk}}{\partial x_n}\big(\mathbf{e_i}\otimes\mathbf{e_j}\otimes\mathbf{e_k}\big) + T_{ijk}\big(\frac{\partial \mathbf{e_i}}{\partial x_n}\otimes\mathbf{e_j}\otimes\mathbf{e_k}\big) + T_{ijk}\big(\mathbf{e_i}\otimes\frac{\partial \mathbf{e_j}}{x_n}\big)\otimes\mathbf{e_k}\big) + T_{ijk}\big(\mathbf{e_i}\otimes\mathbf{e_j}\otimes\frac{\partial\mathbf{e_k}}{\partial x_n}\big)\right] =$$
$$= \sum_{ijk}\left[\frac{\partial T_{ijk}}{\partial x_n}\big(\mathbf{e_i}\otimes\mathbf{e_j}\otimes\mathbf{e_k}\big) + T_{ijk}\sum_m\Gamma^m_{in}(\mathbf{e_m}\otimes\mathbf{e_j}\otimes\mathbf{e_k}) + T_{ijk}\sum_m\Gamma^m_{jn}(\mathbf{e_i}\otimes\mathbf{e_m}\otimes\mathbf{e_k}) + T_{ijk}\sum_m\Gamma^m_{kn}(\mathbf{e_i}\otimes\mathbf{e_j}\otimes\mathbf{e_m})\right] =$$

: interchaning indices m $\leftrightarrow$ i in second term; m $\leftrightarrow$ j in the third term; m $\leftrightarrow$ k in fourth term (see Identity 13)

$$= \sum_{ijk}\left[\frac{\partial T_{ijk}}{\partial x_n}\big(\mathbf{e_i}\otimes\mathbf{e_j}\otimes\mathbf{e_k}\big) + \sum_m\Gamma^i_{mn}T_{mjk}(\mathbf{e_i}\otimes\mathbf{e_j}\otimes\mathbf{e_k}) + \sum_m\Gamma^j_{mn}T_{imk}(\mathbf{e_i}\otimes\mathbf{e_j}\otimes\mathbf{e_k}) + \sum_m\Gamma^k_{mn}T_{ijm}(\mathbf{e_i}\otimes\mathbf{e_j}\otimes\mathbf{e_k})\right] =$$
$$= \sum_{ijk}\left[\frac{\partial T_{ijk}}{\partial x_n} + \sum_m\Gamma^i_{mn}T_{mjk} + \sum_m\Gamma^j_{mn}T_{imk} + \sum_m\Gamma^k_{mn}T_{ijm}\right](\mathbf{e_i}\otimes\mathbf{e_j}\otimes\mathbf{e_k}) =$$
$$= \sum_{ijk}T_{ijk;n}(\mathbf{e_i}\otimes\mathbf{e_j}\otimes\mathbf{e_k}) \tag{7}$$

### Used identities

$$\frac{\partial \mathbf{e_i}}{\partial x_n} = \sum_m \Gamma^m_{in} \mathbf{e_m} \tag{8}$$

$$\mathbf{e_n}.(\mathbf{e_i} \otimes \mathbf{e_j}) = (\mathbf{e_n}.\mathbf{e_i}).\mathbf{e_j} \tag{9}$$

$$\mathbf{e_n}.(\mathbf{e_i} \otimes \mathbf{e_j} \otimes \mathbf{e_k}) = (\mathbf{e_n}.\mathbf{e_i})(\mathbf{e_j} \otimes \mathbf{e_k}) \tag{10}$$

$$V_i \sum_m \Gamma^m_{in} \mathbf{e_m} = \sum_m \Gamma^i_{mn} V_m \mathbf{e_i} \tag{11}$$

$$S_{ij} \sum_m \Gamma^m_{in} (\mathbf{e_m} \otimes \mathbf{e_j}) = \sum_m \Gamma^i_{mn} S_{mj} (\mathbf{e_i} \otimes \mathbf{e_j}) \tag{12}$$

$$T_{ijk} \sum_m \Gamma^m_{in} (\mathbf{e_m} \otimes \mathbf{e_j} \otimes \mathbf{e_k}) = \sum_m \Gamma^i_{mn} T_{mjk} (\mathbf{e_i} \otimes \mathbf{e_j} \otimes \mathbf{e_k}) \tag{13}$$

### Tensor of second order expressed by dyadic products

$$\sum_{kl} S_{kl} (\mathbf{e_k} \otimes \mathbf{e_l}) = S_{11} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + S_{12} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + S_{13} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \tag{14}$$

$$S_{21} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + S_{22} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + S_{23} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \tag{15}$$

$$S_{31} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + S_{32} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + S_{33} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{16}$$

**Geometry, scale factors and Christoffel symbols**

$$x_1 = r \qquad x_2 = \theta \qquad x_3 = \phi \qquad \text{(coordinates)} \tag{17}$$

$$\mathbf{e_1} = \mathbf{e_r} \qquad \mathbf{e_2} = \mathbf{e}_\theta \qquad \mathbf{e_3} = \mathbf{e}_\phi \qquad \text{(unit base vectors)} \tag{18}$$

$$h_1 = h_r = 1 \qquad h_2 = h_\theta = r \qquad h_3 = h_\phi = r\sin\theta \qquad \text{(scale factors)} \tag{19}$$

$$\begin{pmatrix} \Gamma^\theta_{r\theta} = \ 1 & \Gamma^\phi_{r\phi} = \ \sin\theta & \Gamma^\phi_{\theta\phi} = \ \cos\theta \\ \Gamma^r_{\theta\theta} = -1 & \Gamma^r_{\phi\phi} = -\sin\theta & \Gamma^\theta_{\phi\phi} = -\cos\theta \end{pmatrix} \tag{20}$$

**Summary**

$$\nabla(.) = \sum_n \frac{\mathbf{e_n}}{h_n}\frac{\partial(.)}{\partial x_n} \quad : \text{ nabla operator} \qquad \mathbf{V} = \sum_i V_i \mathbf{e_i} \quad : \text{ tensor of first order (vector)} \tag{21}$$

$$\mathbf{S} = \sum_{ij} S_{ij}(\mathbf{e_i} \otimes \mathbf{e_j}) \quad : \text{ tensor of second order} \tag{22}$$

$$\mathbf{T} = \sum_{ijk} T_{ijk}(\mathbf{e_i} \otimes \mathbf{e_j} \otimes \mathbf{e_k}) \quad : \text{ tensor of third order} \tag{23}$$

$$\nabla.\mathbf{V} = \sum_i \frac{1}{h_i}\left[\frac{\partial V_i}{\partial x_i} + \sum_m \Gamma^i_{mi} V_m\right] \quad : \text{ div of first order tensor (vector)} \tag{24}$$

$$\nabla.\mathbf{S} = \sum_{ij} \frac{1}{h_i}\left[\frac{\partial S_{ij}}{\partial x_i} + \sum_m \Gamma^i_{mi} S_{mj} + \sum_m \Gamma^j_{mi} S_{im}\right]\mathbf{e_j} \quad : \text{ div of second order tensor} \tag{25}$$

$$\nabla.\mathbf{T} = \sum_{ijk} \frac{1}{h_i}\left[\frac{\partial T_{ijk}}{\partial x_i} + \sum_m \Gamma^i_{mi} T_{mjk} + \sum_m \Gamma^j_{mi} T_{imk} + \sum_m \Gamma^k_{mi} T_{ijm}\right](\mathbf{e_j} \otimes \mathbf{e_k}) \quad : \text{ div of third order tensor} \tag{26}$$

**Divergence of first order tensor $\nabla \cdot \mathbf{V}$**

$$\frac{1}{r^2}\frac{\partial(r^2 V_r)}{\partial_r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi} \tag{27}$$

**Divergence of second order tensor $\nabla \cdot \mathbf{S}$**

$$S_r(\mathbf{e_r}): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 S_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ S_{\theta r}) + \frac{1}{r\sin\theta}\frac{\partial S_{\phi r}}{\partial\phi} - \frac{S_{\theta\theta}}{r} - \frac{S_{\phi\phi}}{r} \tag{28}$$

$$S_\theta(\mathbf{e}_\theta): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 S_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ S_{\theta\theta}) + \frac{1}{r\sin\theta}\frac{\partial S_{\phi\theta}}{\partial\phi} + \frac{S_{\theta r}}{r} - \frac{S_{\phi\phi}\cos\theta}{r\sin\theta} \tag{29}$$

$$S_\phi(\mathbf{e}_\phi): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 S_{r\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ S_{\theta\phi}) + \frac{1}{r\sin\theta}\frac{\partial S_{\phi\phi}}{\partial\phi} + \frac{S_{\phi r}}{r} + \frac{S_{\phi\theta}\cos\theta}{r\sin\theta} \tag{30}$$

**Divergence of third order tensor $\nabla \cdot \mathbf{T}$**

$$T_{rr}\ (\mathbf{e_r}\otimes\mathbf{e_r}): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{rrr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta rr}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi rr}}{\partial\phi} - \frac{T_{\theta\theta r}}{r} - \frac{T_{\theta r\theta}}{r} - \frac{T_{\phi\phi r}}{r} - \frac{T_{\phi r\phi}}{r} \tag{31}$$

$$T_{r\theta}\ (\mathbf{e_r}\otimes\mathbf{e}_\theta): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{rr\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta r\theta}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi r\theta}}{\partial\phi} - \frac{T_{\theta\theta\theta}}{r} + \frac{T_{\theta rr}}{r} - \frac{T_{\phi\phi\theta}}{r} - \frac{T_{\phi r\phi}\cos\theta}{r\sin\theta} \tag{32}$$

$$T_{r\phi}\ (\mathbf{e_r}\otimes\mathbf{e}_\phi): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{rr\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta r\phi}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi r\phi}}{\partial\phi} + \frac{T_{\theta\theta\phi}}{r} - \frac{T_{\phi\phi\phi}}{r} + \frac{T_{\phi r\phi}\cos\theta}{r\sin\theta} \tag{33}$$

$$T_{\theta r}\ (\mathbf{e}_\theta\otimes\mathbf{e_r}): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\theta r}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\theta r}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta r}}{\partial\phi} + \frac{T_{\theta rr}}{r} - \frac{T_{\theta\theta\theta}}{r} - \frac{T_{\phi\phi r}\cos\theta}{r\sin\theta} - \frac{T_{\phi\theta\phi}}{r} \tag{34}$$

$$T_{\theta\theta}\ (\mathbf{e}_\theta\otimes\mathbf{e}_\theta): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\theta\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\theta\theta}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta\theta}}{\partial\phi} + \frac{T_{\theta r\theta}}{r} + \frac{T_{\theta\theta r}}{r} - \frac{T_{\phi\phi\theta}\cos\theta}{r\sin\theta} - \frac{T_{\phi\theta\phi}\cos\theta}{r\sin\theta} \tag{35}$$

$$T_{\theta\phi}\ (\mathbf{e}_\theta\otimes\mathbf{e}_\phi): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\theta\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\theta\phi}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta\phi}}{\partial\phi} + \frac{T_{\theta r\phi}}{r} + \frac{T_{\phi\theta r}}{r} + \frac{T_{\phi\theta\theta}\cos\theta}{r\sin\theta} \tag{36}$$

$$T_{\phi r}\ (\mathbf{e}_\phi\otimes\mathbf{e_r}): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\phi r}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\phi r}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi r}}{\partial\phi} - \frac{T_{\theta\phi\theta}}{r} + \frac{T_{\phi rr}}{r} + \frac{T_{\phi\theta r}\cos\theta}{r\sin\theta} - \frac{T_{\phi\phi\phi}}{r} \tag{37}$$

$$T_{\phi\theta}\ (\mathbf{e}_\phi\otimes\mathbf{e}_\theta): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\phi\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\phi\theta}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi\theta}}{\partial\phi} + \frac{T_{\theta\phi r}}{r} + \frac{T_{\phi r\theta}}{r} + \frac{T_{\phi\theta\theta}\cos\theta}{r\sin\theta} - \frac{T_{\phi\theta\phi}\cos\theta}{r\sin\theta} \tag{38}$$

$$T_{\phi\phi}\ (\mathbf{e}_\phi\otimes\mathbf{e}_\phi): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\phi\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\phi\phi}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi\phi}}{\partial\phi} + \frac{T_{\phi r\phi}}{r} + \frac{T_{\phi\theta\phi}\cos\theta}{r\sin\theta} + \frac{T_{\phi\phi r}}{r} + \frac{T_{\phi\phi\theta}\cos\theta}{r\sin\theta} \tag{39}$$