0.1 Momentum equation transfer from Euler to Lagrangian formulation and Lagrangian derivative D

Hydrodynamic momentum equation in spherical coordinates (r, θ, ϕ):

$$
\begin{align}
\partial_t(\rho u_r) &= -\left(\frac{1}{r^2}\partial_r\big(r^2[\rho u_r^2 - \tau_{rr}]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_r u_\theta - \tau_{r\theta}]\big) + \frac{1}{r\sin\theta}\partial_\phi\big([\rho u_r u_\phi - \tau_{r\phi}]\big) + G_r^M + \partial_r P\right) - \rho\partial_r\Phi \tag{1}\\
\partial_t(\rho u_r) &= -\left(\frac{1}{r^2}\partial_r\big(r^2[\rho u_r u_r]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\rho u_r u_\theta]\big) + \frac{1}{r\sin\theta}\partial_\phi\big([\rho u_r u_\phi]\big)\right) \tag{2}\\
&\quad + \left(\frac{1}{r^2}\partial_r\big(r^2[\tau_{rr}]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\tau_{r\theta}]\big) + \frac{1}{r\sin\theta}\partial_\phi\big([\tau_{r\phi}]\big)\right) - G_r^M - \partial_r P - \rho\partial_r\Phi \tag{3}\\
\partial_t(\rho u_r) &= -\left(\frac{1}{r^2}\big(r^2 u_r\partial_r[\rho u_r] + \rho u_r\partial_r[r^2 u_r]\big) + \frac{1}{r\sin\theta}\big(\sin\theta u_\theta\partial_\theta[\rho u_r] + \rho u_r\partial_\theta[\sin\theta u_\theta]\big) + \frac{1}{r\sin\theta}\big(u_\phi\partial_\phi[\rho u_r] + \rho u_r\partial_\phi u_\phi\big)\right) \tag{4}\\
&\quad + \left(\frac{1}{r^2}\partial_r\big(r^2[\tau_{rr}]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\tau_{r\theta}]\big) + \frac{1}{r\sin\theta}\partial_\phi\big([\tau_{r\phi}]\big)\right) - G_r^M - \partial_r P - \rho\partial_r\Phi \tag{5}\\
\partial_t(\rho u_r) &= -\big(u_r\partial_r[\rho u_r] + \rho u_r\frac{1}{r^2}\partial_r[r^2 u_r]\big) - \big(\frac{u_\theta}{r}\partial_\theta[\rho u_r] + \rho u_r\frac{1}{r\sin\theta}\partial_\theta[\sin\theta u_\theta]\big) - \big(\frac{u_\phi}{r\sin\theta}\partial_\phi[\rho u_r] + \rho u_r\frac{1}{r\sin\theta}\partial_\phi u_\phi\big) \tag{6}\\
&\quad + \left(\frac{1}{r^2}\partial_r\big(r^2[\tau_{rr}]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\tau_{r\theta}]\big) + \frac{1}{r\sin\theta}\partial_\phi\big([\tau_{r\phi}]\big)\right) - G_r^M - \partial_r P - \rho\partial_r\Phi \tag{7}\\
\partial_t(\rho u_r) &= -\big(u_r\partial_r[\rho u_r] + \frac{u_\theta}{r}\partial_\theta[\rho u_r] + \frac{u_\phi}{r\sin\theta}\partial_\phi[\rho u_r]\big) - \big(\rho u_r\frac{1}{r^2}\partial_r[r^2 u_r] + \rho u_r\frac{1}{r\sin\theta}\partial_\theta[\sin\theta u_\theta] + \rho u_r\frac{1}{r\sin\theta}\partial_\phi u_\phi\big) \tag{8}\\
&\quad + \left(\frac{1}{r^2}\partial_r\big(r^2[\tau_{rr}]\big) + \frac{1}{r\sin\theta}\partial_\theta\big(\sin\theta[\tau_{r\theta}]\big) + \frac{1}{r\sin\theta}\partial_\phi\big([\tau_{r\phi}]\big)\right) - G_r^M - \partial_r P - \rho\partial_r\Phi \tag{9}\\
\rho\partial_t u_r + u_r\partial_t\rho &= -\mathbf{u}\cdot\nabla[\rho u_r] - \rho u_r\nabla\cdot\mathbf{u} + \nabla\cdot\tau_r - G_r^M - \partial_r P - \rho\partial_r\Phi \tag{10}\\
\rho\partial_t u_r + u_r\partial_t\rho &= -\rho\mathbf{u}\cdot\nabla u_r - u_r\mathbf{u}\cdot\nabla\rho - \rho u_r\nabla\cdot\mathbf{u} + \nabla\cdot\tau_r - G_r^M - \partial_r P - \rho\partial_r\Phi \tag{11}\\
\rho\partial_t u_r + \rho\mathbf{u}\cdot\nabla u_r &= -u_r\partial_t\rho - u_r\mathbf{u}\cdot\nabla\rho - \rho u_r\nabla\cdot\mathbf{u} + \nabla\cdot\tau_r - G_r^M - \partial_r P - \rho\partial_r\Phi \tag{12}\\
\rho(\partial_t u_r + \mathbf{u}\cdot\nabla u_r) &= -u_r\overbrace{\big(\partial_t\rho + \mathbf{u}\nabla\rho + \rho\nabla\cdot\mathbf{u}\big)}^{=0\text{ continuity equation}} + \nabla\cdot\tau_r - G_r^M - \partial_r P - \rho\partial_r\Phi \tag{13}\\
\rho D_t(u_r) &= +\nabla\cdot\tau_r - G_r^M - \partial_r P + \rho g_r \tag{14}
\end{align}
$$

where $D_t(.) = \partial_t(.) + \mathbf{u}\cdot\nabla(.)$ and $g_r = -\partial_r\Phi$. This is exactly Equation (8) in the paper.

0.2 Second-order moments

In order to calculate evolution equations for correlations of two arbitrary fluctuations, we can derive the following general formula.

$$
\begin{aligned}
\overline{\rho}\widetilde{D}_t\widetilde{c''d''} - \overline{\rho D_t c''d''} &= \overline{\rho}\big(\partial_t\widetilde{c''d''} + \widetilde{u}_n\partial_n\widetilde{c''d''}\big) - \overline{\rho\big(\partial_t c''d'' + u_n\partial_n c''d''\big)} = \overline{\rho}\partial_t\widetilde{c''d''} + \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''} - \overline{\rho\partial_t c''d''} - \overline{\rho u_n\partial_n c''d''} = \\
&= \overline{\rho}\partial_t\widetilde{c''d''} + \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''} - \big(\overline{\partial_t\rho c''d''} - \overline{c''d''\partial_t\rho}\big) - \overline{\rho u_n\partial_n c''d''} = \qquad (15) \\
&= \overline{\rho}\partial_t\widetilde{c''d''} + \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''} - \partial_t\overline{\rho}\widetilde{c''d''} - \overline{c''d''\partial_n\rho u_n} - \overline{\rho u_n\partial_n c''d''} = \qquad (16) \\
&= \overline{\rho}\partial_t\widetilde{c''d''} + \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''} - \big(\overline{\rho}\partial_t\widetilde{c''d''} + \widetilde{c''d''}\partial_t\overline{\rho}\big) - \overline{c''d''\partial_n\rho u_n} - \overline{\rho u_n\partial_n c''d''} = \qquad (17) \\
&= \overline{\rho}\partial_t\widetilde{c''d''} + \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''} - \big(\overline{\rho}\partial_t\widetilde{c''d''} - \widetilde{c''d''}\partial_n\overline{\rho}\widetilde{u}_n\big) - \overline{c''d''\partial_n\rho u_n} - \overline{\rho u_n\partial_n c''d''} = \qquad (18) \\
&= \overline{\rho}\partial_t\widetilde{c''d''} - \overline{\rho}\partial_t\widetilde{c''d''} + \overline{\rho}\widetilde{u}_n\partial_n\widetilde{c''d''} + \widetilde{c''d''}\partial_n\overline{\rho}\widetilde{u}_n - \overline{\partial_n\rho u_n c''d''} = \qquad (19) \\
&= \partial_n\overline{\rho}\widetilde{u}_n\widetilde{c''d''} - \big(\overline{\partial_n\rho\widetilde{u}_n c''d''} + \overline{\partial_n\rho u''_n c''d''}\big) = -\overline{\partial_n\rho u''_n c''d''} \qquad (20)
\end{aligned}
$$

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = \overline{\rho D_t c''d''} - \overline{\partial_n\rho u''_n c''d''} = \overline{c''\rho D_t d''} + \overline{d''\rho D_t c''} - \overline{\partial_n\rho u''_n c''d''} \qquad (21)$$

$$\rho D_t c'' = \rho D_t c - \rho D_t\widetilde{c} = \rho D_t c - \rho\widetilde{D}_t\widetilde{c} - \rho u''_n\partial_n\widetilde{c} = \rho D_t c - \frac{\rho}{\overline{\rho}}\left[\overline{\rho}\widetilde{D}_t\widetilde{c}\right] - \rho u''_n\partial_n\widetilde{c} \qquad (22)$$

$$\rho D_t d'' = \rho D_t d - \rho D_t\widetilde{d} = \rho D_t d - \rho\widetilde{D}_t\widetilde{d} - \rho u''_n\partial_n\widetilde{d} = \rho D_t d - \frac{\rho}{\overline{\rho}}\left[\overline{\rho}\widetilde{D}_t\widetilde{d}\right] - \rho u''_n\partial_n\widetilde{d} \qquad (23)$$

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = \overline{c''\left(\rho D_t d - \frac{\rho}{\overline{\rho}}\left[\overline{\rho}\widetilde{D}_t\widetilde{d}\right] - \rho u''_n\partial_n\widetilde{d}\right)} + \overline{d''\left(\rho D_t c - \frac{\rho}{\overline{\rho}}\left[\overline{\rho}\widetilde{D}_t\widetilde{c}\right] - \rho u''_n\partial_n\widetilde{c}\right)} - \overline{\partial_n\rho c''d''u''_n} \qquad (24)$$

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = \overline{c''\left(\rho D_t d - \rho\left[\widetilde{D}_t\widetilde{d}\right] - \rho u''_n\partial_n\widetilde{d}\right)} + \overline{d''\left(\rho D_t c - \rho\left[\widetilde{D}_t\widetilde{c}\right] - \rho u''_n\partial_n\widetilde{c}\right)} - \overline{\partial_n\rho c''d''u''_n} \qquad (25)$$

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = +\overline{c''\rho D_t d} - \cancel{\overline{\rho c''}}^{\text{0 by definition; see Sect.0.4}}\left[\widetilde{D}_t\widetilde{d}\right] - \overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{d} + \overline{d''\rho D_t c} - \cancel{\overline{\rho d''}}^{\text{0 by definition; see Sect.0.4}}\left[\widetilde{D}_t\widetilde{c}\right] - \overline{\rho}\widetilde{d''u''_n}\partial_n\widetilde{c} - \overline{\partial_n \rho c''d''u''_n} \tag{26}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = +\overline{c''\rho D_t d} - \overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{d} + \overline{d''\rho D_t c} - \overline{\rho}\widetilde{d''u''_n}\partial_n\widetilde{c} - \overline{\partial_n \rho c''d''u''_n} \tag{27}$$

This is exactly the Equation (7) in the paper.

0.3 Mean composition flux equation of chemical element X_i

We can derive the composition flux equation using the general formula for second order moments, where we substitute c with X_i and d with u_i.

$$\overline{\rho}\widetilde{D}_t\widetilde{c''d''} = \overline{c''\rho D_t d} - \overline{\rho}\widetilde{c''u''_n}\partial_n\widetilde{d} + \overline{d''\rho D_t c} - \overline{\rho}\widetilde{d''u''_n}\partial_n\widetilde{c} - \overline{\partial_n \rho c''d''u''_n} \tag{28}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{X''_i u''_i} = \overline{X''_i \rho D_t u_i} - \overline{\rho}\widetilde{X''_i u''_n}\partial_n\widetilde{u}_i + \overline{u''_i \rho D_t X_i} - \overline{\rho}\widetilde{u''_i u''_n}\partial_n\widetilde{X}_i - \overline{\partial_n \rho X''_i u''_i u''_n} \tag{29}$$

Please not that the last term $\overline{\partial_n \rho X''_i u''_i u''_n} \equiv \overline{\nabla\cdot(\rho X''_i \mathbf{u}''\mathbf{u}'')}$ is div of 2nd order tensor; see Sect.0.5 and $\overline{\nabla\cdot(\rho X''_i \mathbf{u}''\mathbf{u}'')}(\mathbf{e_r}) = \nabla_r \overline{\rho X_i u''_r u''_r} - \overline{\rho X''_i u''_\theta u''_\theta / r} - \overline{\rho X''_i u''_\phi u''_\phi / r}$

Further calculation requires the following hydrodynamic equations:

$$\rho D_t(u_r) = +\left(\frac{1}{r^2}\partial_r\left(r^2[\tau_{rr}]\right) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta[\tau_{r\theta}]) + \frac{1}{r\sin\theta}\partial_\phi\left([\tau_{r\phi}]\right) - G_r^M - \partial_r P\right) + \rho g_r \tag{30}$$

$$\rho D_t(X_i) = +\rho\dot{X}_i^n \qquad i = 1 \ldots N_{nuc} \tag{31}$$

For radial component of the flux, where $u_i = u_r$ and changing order of terms in Eq.29 for clarity reasons we get:

$$\overline{\rho}\widetilde{D}_t\widetilde{X_i''u_r''} = \left(-\overline{\partial_n \rho X_i''u_r''u_n''}\right) - \overline{\rho}\widetilde{X_i''u_n''}\partial_n\widetilde{u}_r - \overline{\rho}\widetilde{u_r''u_n''}\partial_n\widetilde{X}_i + \overline{X_i''\rho D_t u_r} + \overline{u_r''\rho D_t X_i} \tag{32}$$

$$\overline{\rho}\widetilde{D}_t(f_i/\overline{\rho}) = \left(-\nabla_r f_i^r - \overline{\rho X_i''u_\theta''u_\theta''/r} - \overline{\rho X_i''u_\phi''u_\phi''/r}\right) - f_i\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_i + \overline{X_i''(\nabla\cdot\tau_r - G_r^M - \partial_r P + \rho g_r)} + \overline{u_r''\rho\dot{X}_i^{\mathrm{nuc}}} \tag{33}$$

Let's reorganize the terms again and split $P = \overline{P} + P'$ to get the equation to shape it has in the paper:

$$\overline{\rho}\widetilde{D}_t(f_i/\overline{\rho}) = -\nabla_r f_i^r - f_i\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_i - \overline{X_i''}\partial_r\overline{P} - \overline{X_i''\partial_r P'} + \cancelto{0}{\overline{X_i''\rho g_r}} + \overline{u_r''\rho\dot{X}_i^{\mathrm{nuc}}} \tag{34}$$

$$- \overline{\rho X_i''u_\theta''u_\theta''/r} - \overline{\rho X_i''u_\phi''u_\phi''/r} - \overline{X_i''G_r^M} - \overline{X''\nabla\cdot\tau_r} \tag{35}$$

$$\overline{\rho}\widetilde{D}_t(f_i/\overline{\rho}) = -\nabla_r f_i^r - f_i\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_i - \overline{X_i''}\partial_r\overline{P} - \overline{X_i''\partial_r P'} + \overline{u_r''\rho\dot{X}_i^{\mathrm{nuc}}} \tag{36}$$

$$- \overline{\rho X_i''u_\theta''u_\theta''/r} - \overline{\rho X_i''u_\phi''u_\phi''/r} - \overline{X_i''G_r^M} - \overline{\nabla\cdot X_i''\tau_r} + \overline{\tau_r\nabla X_i''} \tag{37}$$

$$\tag{38}$$

The viscoity τ is 2nd order tensor and $-\overline{\nabla\cdot X_i''\tau_r} = -\nabla_r\overline{X_i''\tau_{rr}} + \overline{X''\tau_{\theta\theta}\tau_{\theta\theta}/r} + \overline{X''\tau_{\phi\phi}\tau_{\phi\phi}/r}$

$$\overline{\rho}\widetilde{D}_t(f_i/\overline{\rho}) = -\nabla_r f_i^r - f_i\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_i - \overline{X_i''}\partial_r\overline{P} - \overline{X_i''\partial_r P'} + \overline{u_r''\rho\dot{X}_i^{\mathrm{nuc}}} \tag{39}$$

$$- \overline{\rho X_i''u_\theta''u_\theta''/r} - \overline{\rho X_i''u_\phi''u_\phi''/r} - \overline{X_i''G_r^M} - \nabla_r(\overline{X_i''\tau_{rr}}) + \overline{X''\tau_{\theta\theta}\tau_{\theta\theta}/r} + \overline{X''\tau_{\phi\phi}\tau_{\phi\phi}/r} \tag{40}$$

$$- \overline{\tau_{rr}\partial_r X_i''} - \overline{\tau_{r\theta}(1/r)\partial_\theta X_i''} - \overline{\tau_{r\phi}(1/r\sin\theta)\partial_\phi X_i''} \tag{41}$$

$$\overline{\rho}\widetilde{D}_t(f_i/\overline{\rho}) = -\nabla_r f_i^r - f_i\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_i - \overline{X_i''}\partial_r\overline{P} - \overline{X_i''\partial_r P'} + \overline{u_r''\rho\dot{X}_i^{\mathrm{nuc}}} \tag{42}$$

$$- \overline{\rho X_i''u_\theta''u_\theta''/r} - \overline{\rho X_i''u_\phi''u_\phi''/r} - \overline{X_i''G_r^M} - \nabla_r(\overline{X_i''\tau_{rr}}) - \varepsilon_i \tag{43}$$

$$\overline{\rho}\widetilde{D}_t(f_i/\overline{\rho}) = -\nabla_r f_i^r - f_i\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_i - \overline{X_i''}\partial_r\overline{P} - \overline{X_i''\partial_r P'} + \overline{u_r''\rho\dot{X}_i^{\mathrm{nuc}}} \tag{44}$$

$$- \overline{\rho X_i''u_\theta''u_\theta''/r} - \overline{\rho X_i''u_\phi''u_\phi''/r} - \overline{X_i''G_r^M} + \mathcal{N}_{fi} \tag{45}$$

$$\overline{\rho}\widetilde{D}_t(f_i/\overline{\rho}) = -\nabla_r f_i^r - f_i\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_i - \overline{X_i''}\partial_r\overline{P} - \overline{X_i''\partial_r P'} + \overline{u_r''\rho\dot{X}_i^{\mathrm{nuc}}} \tag{46}$$

$$+ \overline{G_r^i} - \overline{X_i'' G_r^M} + \mathcal{N}_{fi} \tag{47}$$

$$\overline{\rho}\widetilde{D}_t(f_i/\overline{\rho}) = -\nabla_r f_i^r - f_i\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_i - \overline{X_i''}\partial_r\overline{P} - \overline{X_i''\partial_r P'} + \overline{u_r''\rho\dot{X}_i^{\mathrm{nuc}}} \tag{48}$$

$$+ \mathcal{G}_i + \mathcal{N}_{fi} \tag{49}$$

where:

$$\mathcal{G}_i = \overline{G_r^i} - \overline{X_i'' G_r^M} \tag{50}$$

$$\mathcal{N}_{fi} = -\nabla_r(\overline{X_i''\tau_{rr}}) - \overline{\tau_{rr}\partial_r X_i''} - \overline{\tau_{r\theta}(1/r)\partial_\theta X_i''} - \overline{\tau_{r\phi}(1/r\sin\theta)\partial_\phi X_i''} + \overline{X''\tau_{\theta\theta}\tau_{\theta\theta}/r} + \overline{X''\tau_{\phi\phi}\tau_{\phi\phi}/r} \tag{51}$$

And this is the turbulent flux equation which has the same form as in the paper, with few different details with regards to viscosity terms, which I think we should mention. Although we actually decided originally not to mention them in order to keep the equation as simple as possible. The details don't affect our conclusions at all, but still, I think we should try to stay as exact as possible. Also referee spotted one of the terms too, in his "Detailed comments" , line item starting with p5 I.60L, where he says "Finally, I think you miss a term $\overline{X_i''}\nabla_r\overline{\tau_{rr}}$ in the whole business." On the other hand, it was sort of lucky guess, because he didn't realized that we rewrote the $\overline{X''\nabla\cdot\tau_r}$ to $\overline{\nabla\cdot X''\tau_r} - \overline{\tau_r\nabla X''}$. This is because he complained about the ∂_r in the ϵ_i and thinks it should be ∇_r. But we have it correct.

Note that the blue viscosity terms are in the paper viscosity fluctuations. The τ can certainly be further decomposed to $\tau = \overline{\tau} + \tau'$ in which case we get the fluctuaing terms too, but we didn't want to mention this in the paper in order to keep the equation as simple as possible. The sentence talking about this term on page 5 says: *Physically, it captures terms like* $-\nabla_r(\overline{X_i''\tau_{rr}'})$ .. *etc.* The *like* is supposed to cover us here. The other terms mentioned in the paper with τ' i.e. $-\overline{\tau_{rr}'\partial_r X_i''} - \overline{\tau_{r\theta}'(1/r)\partial_\theta X_i''} - \overline{\tau_{r\phi}'(1/r\sin\theta)\partial_\phi X_i''}$ resemble dissipation terms in TKE equation and have equivalent interpretation. The terms with full τ i.e. $-\nabla_r(\overline{X_i''\tau_{rr}}) - \overline{\tau_{rr}\partial_r X_i''} - \overline{\tau_{r\theta}(1/r)\partial_\theta X_i''} - \overline{\tau_{r\phi}(1/r\sin\theta)\partial_\phi X_i''}$, we were not exactly sure. But viscosity can be split into background and turbulent part and I think this is how we should interpret the term, as dissipation of the flux due to turbulence and background viscosity.

The red terms are missing, but we did it intentionally again in order not to overcomplicate the equation.

Besides $+\overline{X''\tau_{\theta\theta}\tau_{\theta\theta}/r} + \overline{X''\tau_{\phi\phi}\tau_{\phi\phi}/r}$ has geometric origin and has nothing to do with dissipation. Therefore, the question what it actually is, remains still unanswered.

Regardless of the complexity of the flux equation, I suggest to add all of these missing terms terms into the paper too. I mean, the *like* formulations we have there for these terms now are a little bit weak. And it is better to be exact than incomplete. The paper is quite heavy on all the algebra, interested reader will have to understand.

I suggest, we present the $\mathcal{N}_{fi}$ in the following way (with not decomposed full τ) and change text a bit:

$$
\begin{aligned}
\mathcal{N}_{fi} &= -\nabla_r(\overline{X_i''\tau_{rr}}) - \overline{\tau_{rr}\partial_r X_i''} - \overline{\tau_{r\theta}(1/r)\partial_\theta X_i''} - \overline{\tau_{r\phi}(1/r\sin\theta)\partial_\phi X_i''} + \overline{X''\tau_{\theta\theta}\tau_{\theta\theta}/r} + \overline{X''\tau_{\phi\phi}\tau_{\phi\phi}/r} && (52)\\
\mathcal{N}_{fi} &= -\nabla_r(\overline{X_i''\tau_{rr}}) + \epsilon_i + \mathcal{G}_i^\tau && (53)
\end{aligned}
$$

where

$$
\begin{aligned}
\epsilon_i &= -\overline{\tau_{rr}\partial_r X_i''} - \overline{\tau_{r\theta}(1/r)\partial_\theta X_i''} - \overline{\tau_{r\phi}(1/r\sin\theta)\partial_\phi X_i''} && (54)\\
\mathcal{G}_i^\tau &= +\overline{X''\tau_{\theta\theta}\tau_{\theta\theta}/r} + \overline{X''\tau_{\phi\phi}\tau_{\phi\phi}/r} && (55)
\end{aligned}
$$

0.4 Favre decomposition

Favre decomposition:

$$F = \widetilde{F}(r) + F''(r,\theta,\phi) \tag{56}$$

Definition of the averaging operator:

$$\widetilde{F} = \frac{\overline{\rho F}}{\overline{\rho}} \tag{57}$$

Some properties of the operator:

$$\overline{\rho F''} = \widetilde{F''} = 0 \tag{58}$$

$$\widetilde{F} = \overline{F} + \frac{\overline{\rho F'}}{\overline{\rho}} \tag{59}$$

$$F'' = F' - \frac{\overline{\rho F'}}{\overline{\rho}} \rightarrow \overline{F''} = -\frac{\overline{\rho F'}}{\overline{\rho}} \tag{60}$$

$$\overline{\rho F'' G''} = \overline{\rho} \widetilde{F'' G''} \tag{61}$$

0.5 Divergence of second order tensor

More details can be found in the arXiv write-up Mocak et al., 2014

$$\mathbf{S} = \sum_{ij} S_{ij} (\mathbf{e_i} \otimes \mathbf{e_j}) \qquad : \text{ tensor of second order} \tag{62}$$

Divergence of second order tensor $\nabla \cdot \mathbf{S}$

$$S_r(\mathbf{e_r}) : \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 S_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \; S_{\theta r}) + \frac{1}{r\sin\theta}\frac{\partial S_{\phi r}}{\partial\phi} - \frac{S_{\theta\theta}}{r} - \frac{S_{\phi\phi}}{r} \tag{63}$$

$$S_\theta(\mathbf{e_\theta}) : \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 S_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \; S_{\theta\theta}) + \frac{1}{r\sin\theta}\frac{\partial S_{\phi\theta}}{\partial\phi} + \frac{S_{\theta r}}{r} - \frac{S_{\phi\phi}\cos\theta}{r\sin\theta} \tag{64}$$

$$S_\phi(\mathbf{e_\phi}) : \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 S_{r\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \; S_{\theta\phi}) + \frac{1}{r\sin\theta}\frac{\partial S_{\phi\phi}}{\partial\phi} + \frac{S_{\phi r}}{r} + \frac{S_{\phi\theta}\cos\theta}{r\sin\theta} \tag{65}$$