

1 Alternative hydrodynamic stellar structure equations

Below is a set of alternative hydrodynamic stellar structure equations inspired by mean fields from the mean density/temperature/internal energy and pressure flux equations involving gradient term scaled by favrian Reynolds stress \tilde{R}_{rr} and the other one turbulent dilatation flux $\overline{u'_r d''}$.

$$\partial_r \bar{m} = -\bar{\rho} \bar{m} \overline{u'_r d''} / \tilde{R}_{rr} + 4\pi r^2 \bar{\rho} \quad (1)$$

$$\partial_r \bar{P} = -\Gamma_1 \bar{\rho} \bar{P} \overline{u'_r d''} / \tilde{R}_{rr} \quad (2)$$

$$\partial_r \tilde{L} = +\tilde{\epsilon}_t \partial_r 4\pi r^2 \bar{\rho} \tilde{u}_r - 4\pi r^2 \bar{\rho} \tilde{u}_r \bar{P} \overline{u'_r d''} / \tilde{R}_{rr} \quad (3)$$

$$\partial_r \bar{T} = -(\Gamma_3 - 1) \bar{\rho} \bar{T} \overline{u'_r d''} / \tilde{R}_{rr} \quad (4)$$

$$\partial_t \tilde{X}_i = \tilde{X}_i^{nuc} - (1/\bar{\rho}) \nabla_r f_i - \tilde{u}_r \partial_r \tilde{X}_i \quad (5)$$

These equations could be perfectly validated by our ransX framework and are shown on the next page in Figure 1. It appears that there is a universality relation between gradient of a mean thermodynamic variable Q and dilatation flux $\overline{u'_r d''}$, that is:

$$\partial_r \bar{Q} \sim -\bar{\rho} \bar{Q} \overline{u'_r d''} / \tilde{R}_{rr} \quad (6)$$

Moreover, we know that due to hydrostatic equilibrium, $\partial_r \bar{P} \sim -\bar{\rho} \bar{g}_r$, and based on Equation (2) we can write, that dilatation flux $\overline{u'_r d''}$ is:

$$\overline{u'_r d''} \sim \frac{\tilde{R}_{rr} \bar{g}_r}{\Gamma_1 \bar{P}} \quad (7)$$

Also, the expansion velocity \tilde{u}_r can be replaced by:

$$\tilde{u}_r = -\partial_t \bar{M} / 4\pi r^2 \bar{\rho} \quad (8)$$

So the alternative hydrodynamic stellar structure equations for hydrostatic convection become a system with only **one unknown the composition flux** f_α for which we need a proper model. See validation of these simplified alternative hydrodynamic stellar structure equation in Figure 2.

$$\partial_r \bar{m} = -\bar{\rho} \bar{m} \bar{g}_r / \Gamma_1 \bar{P} + 4\pi r^2 \bar{\rho} \quad (9)$$

$$\partial_r \bar{P} = -\bar{\rho} \bar{g}_r \quad (10)$$

$$\partial_r \tilde{L} = -4\pi r^2 \tilde{u}_r \bar{\rho} \bar{g}_r / \Gamma_1 + \tilde{\epsilon}_t \partial_r 4\pi r^2 \bar{\rho} \tilde{u}_r \quad (11)$$

$$\partial_r \bar{T} = -(\Gamma_3 - 1) \bar{\rho} \bar{T} \bar{g}_r / \Gamma_1 \bar{P} \quad (12)$$

$$\partial_t \tilde{X}_i = \tilde{X}_i^{nuc} - (1/\bar{\rho}) \nabla_r f_i - \tilde{u}_r \partial_r \tilde{X}_i \quad (13)$$

1.1 Derivation of the alternative hydrodynamic stellar structure equations from flux evolution equations

From the pressure flux equation (Fig.5) we get, that:

$$+ \overline{u'_r u''_r} \partial_r \overline{P} = - \Gamma_1 \overline{u'_r P d} \quad \backslash \cdot \overline{\rho} \quad (14)$$

$$+ \overline{\rho} \overline{u'_r u''_r} \partial_r \overline{P} = - \overline{\rho} \Gamma_1 \overline{u'_r P d} \quad P = \overline{P} + P' \quad d = \overline{d} + d' \quad (15)$$

$$+ \tilde{R}_{rr} \partial_r \overline{P} = - \overline{\rho} \Gamma_1 (\overline{P u'_r d} + \overline{P' u'_r d'} + \overline{P' u'_r d'}) \quad \overline{P u'_r d'} \gg \overline{P u'_r d} + \overline{P' u'_r d'} \quad \text{and} \quad \overline{a' b'} = \overline{a' b''} = \overline{a'' b'} \quad (16)$$

$$+ \tilde{R}_{rr} \partial_r \overline{P} = - \overline{\rho} \Gamma_1 \overline{P u'_r d''} \quad (17)$$

$$\partial_r \overline{P} = - \overline{\rho} \Gamma_1 \overline{P u'_r d''} / \tilde{R}_{rr} \quad (18)$$

$$\partial_r \overline{P} = - \overline{\rho} \tilde{g}_r \quad \text{hydrostatic equilibrium} \quad (19)$$

$$\overline{u'_r d''} = + \tilde{R}_{rr} \tilde{g}_r / \Gamma_1 \overline{P} \quad \text{model for dilatation flux used later} \quad (20)$$

From the turbulent mass flux equation (Fig.5) we get, that:

$$+ \tilde{R}_{rr} / \overline{\rho} \partial_r \overline{\rho} = - \overline{\rho} \overline{u'_r d''} \quad (21)$$

$$\partial_r \overline{\rho} = - \overline{\rho} \overline{\rho u'_r d''} / \tilde{R}_{rr} \quad \backslash \cdot V \text{ (volume)} = 4\pi r^3 / 3 \quad (22)$$

$$V \partial_r \overline{\rho} = - \overline{\rho} V \overline{\rho u'_r d''} / \tilde{R}_{rr} \quad (23)$$

$$\partial_r \overline{\rho} V - \rho \partial_r V = - \overline{\rho} V \overline{\rho u'_r d''} / \tilde{R}_{rr} \quad (24)$$

$$\partial_r \overline{m} - \rho 4\pi r^2 = - \overline{m} \overline{\rho u'_r d''} / \tilde{R}_{rr} \quad (25)$$

$$\partial_r \overline{m} = + \rho 4\pi r^2 - \overline{m} \overline{\rho u'_r d''} / \tilde{R}_{rr} \quad (26)$$

$$\partial_r \overline{m} = + \rho 4\pi r^2 - \overline{m} \tilde{g}_r / \Gamma_1 \overline{P} \quad (27)$$

$$\partial_r \overline{m} = - \overline{m} \tilde{g}_r / \Gamma_1 \overline{P} + \rho 4\pi r^2 \quad (28)$$

From the temperature flux equation (Fig.5) we get, that:

$$- \overline{u'_r u''_r} \partial_r \overline{T} = + (\Gamma_3 - 1) \overline{T u'_r d''} \quad \backslash \cdot \overline{\rho} \quad (29)$$

$$- \overline{\rho} \overline{u'_r u''_r} \partial_r \overline{T} = + (\Gamma_3 - 1) \overline{\rho T u'_r d''} \quad (30)$$

$$- \tilde{R}_{rr} \partial_r \overline{T} = + (\Gamma_3 - 1) \overline{\rho T u'_r d''} \quad (31)$$

$$\partial_r \overline{T} = - (\Gamma_3 - 1) \overline{\rho T u'_r d''} / \tilde{R}_{rr} \quad (32)$$

$$\partial_r \overline{T} = - (\Gamma_3 - 1) \overline{\rho T (\tilde{R}_{rr} \tilde{g}_r / \Gamma_1 \overline{P})} / \tilde{R}_{rr} \quad (33)$$

$$\partial_r \overline{T} = - (\Gamma_3 - 1) \overline{\rho T \tilde{g}_r} / \Gamma_1 \overline{P} \quad (34)$$

From the internal energy flux equation (Fig.5) we get, that:

$$-\tilde{R}_{rr} \partial_r \tilde{\varepsilon}_i = + \overline{u_r'' P d} \quad P = \bar{P} + P' \quad d = \bar{d} + d' \quad (35)$$

$$-\tilde{R}_{rr} \partial_r \tilde{\varepsilon}_i = + \overline{P u_r''} \bar{d} + \overline{P' u_r'' d'} + \overline{P' u_r'' d'} \quad \overline{P' u_r'' d'} \gg \overline{P u_r''} \bar{d} + \overline{P' u_r'' d'} \quad \text{and} \quad \overline{a' b'} = \overline{a' b''} = \overline{a'' b'} \quad (36)$$

$$-\tilde{R}_{rr} \partial_r \tilde{\varepsilon}_i = + \overline{P' u_r'' d'} \quad \bar{P} = \bar{\rho} \tilde{\varepsilon}_i \quad (37)$$

$$-\tilde{R}_{rr} \partial_r \tilde{\varepsilon}_i = + \bar{\rho} \tilde{\varepsilon}_i \overline{u_r'' d'} \quad (38)$$

$$\partial_r \tilde{\varepsilon}_i = - \bar{\rho} \tilde{\varepsilon}_i \overline{u_r'' d'} / \tilde{R}_{rr} \quad (39)$$

$$\partial_r \tilde{\varepsilon}_i = - \bar{\rho} \tilde{\varepsilon}_i (\tilde{R}_{rr} \tilde{g}_r / \Gamma_1 \bar{P}) / \tilde{R}_{rr} \quad (40)$$

$$\partial_r \tilde{\varepsilon}_i = - \bar{\rho} \tilde{\varepsilon}_i \tilde{g}_r / \Gamma_1 \bar{P} \quad \bar{P} = \bar{\rho} \tilde{\varepsilon}_i \quad (41)$$

$$\partial_r \tilde{\varepsilon}_i = - \bar{\rho} \tilde{\varepsilon}_i \tilde{g}_r / \Gamma_1 \bar{\rho} \tilde{\varepsilon}_i \quad (42)$$

$$\partial_r \tilde{\varepsilon}_i = - \tilde{g}_r / \Gamma_1 \quad \setminus 4\pi r^2 \bar{\rho} \tilde{u}_r \quad (43)$$

$$4\pi r^2 \bar{\rho} \tilde{u}_r \partial_r \tilde{\varepsilon}_i = - 4\pi r^2 \bar{\rho} \tilde{u}_r \tilde{g}_r / \Gamma_1 \quad (44)$$

$$\partial_r 4\pi r^2 \bar{\rho} \tilde{u}_r \tilde{\varepsilon}_i - \tilde{\varepsilon}_i \partial_r 4\pi r^2 \bar{\rho} \tilde{u}_r = - 4\pi r^2 \bar{\rho} \tilde{u}_r \tilde{g}_r / \Gamma_1 \quad \varepsilon_K \ll \varepsilon_i \quad \text{and} \quad \varepsilon_i \sim \varepsilon_t \quad (45)$$

$$\partial_r 4\pi r^2 \bar{\rho} \tilde{u}_r \tilde{\varepsilon}_t - \tilde{\varepsilon}_t \partial_r 4\pi r^2 \bar{\rho} \tilde{u}_r = - 4\pi r^2 \bar{\rho} \tilde{u}_r \tilde{g}_r / \Gamma_1 \quad (46)$$

$$\partial_r \tilde{L} - \tilde{\varepsilon}_t \partial_r 4\pi r^2 \bar{\rho} \tilde{u}_r = - 4\pi r^2 \bar{\rho} \tilde{u}_r \tilde{g}_r / \Gamma_1 \quad (47)$$

$$\partial_r \tilde{L} = - 4\pi r^2 \bar{\rho} \tilde{u}_r \tilde{g}_r / \Gamma_1 + \tilde{\varepsilon}_t \partial_r 4\pi r^2 \bar{\rho} \tilde{u}_r \quad (48)$$

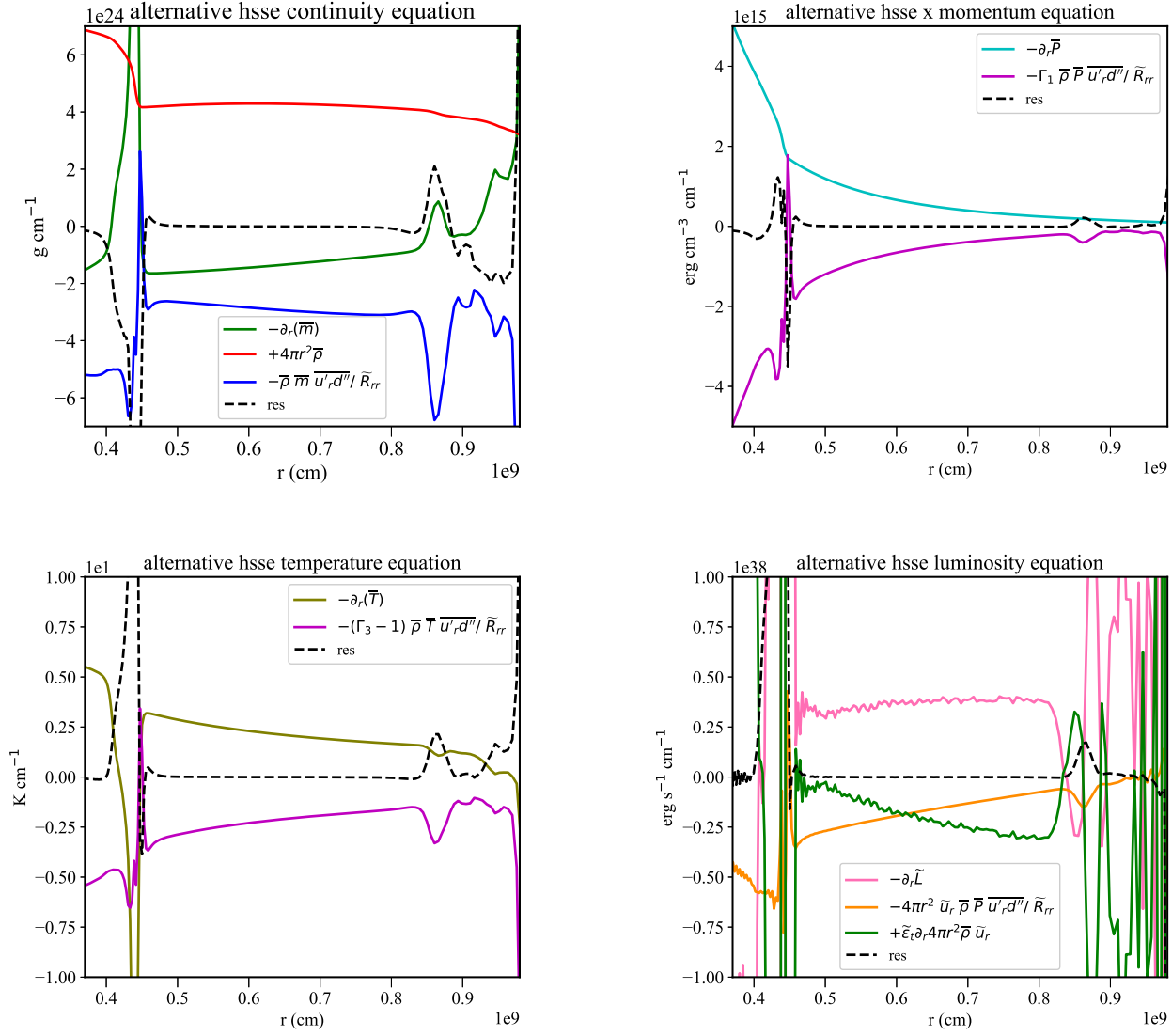


Figure 1: Alternative hydrodynamic stellar structure continuity.

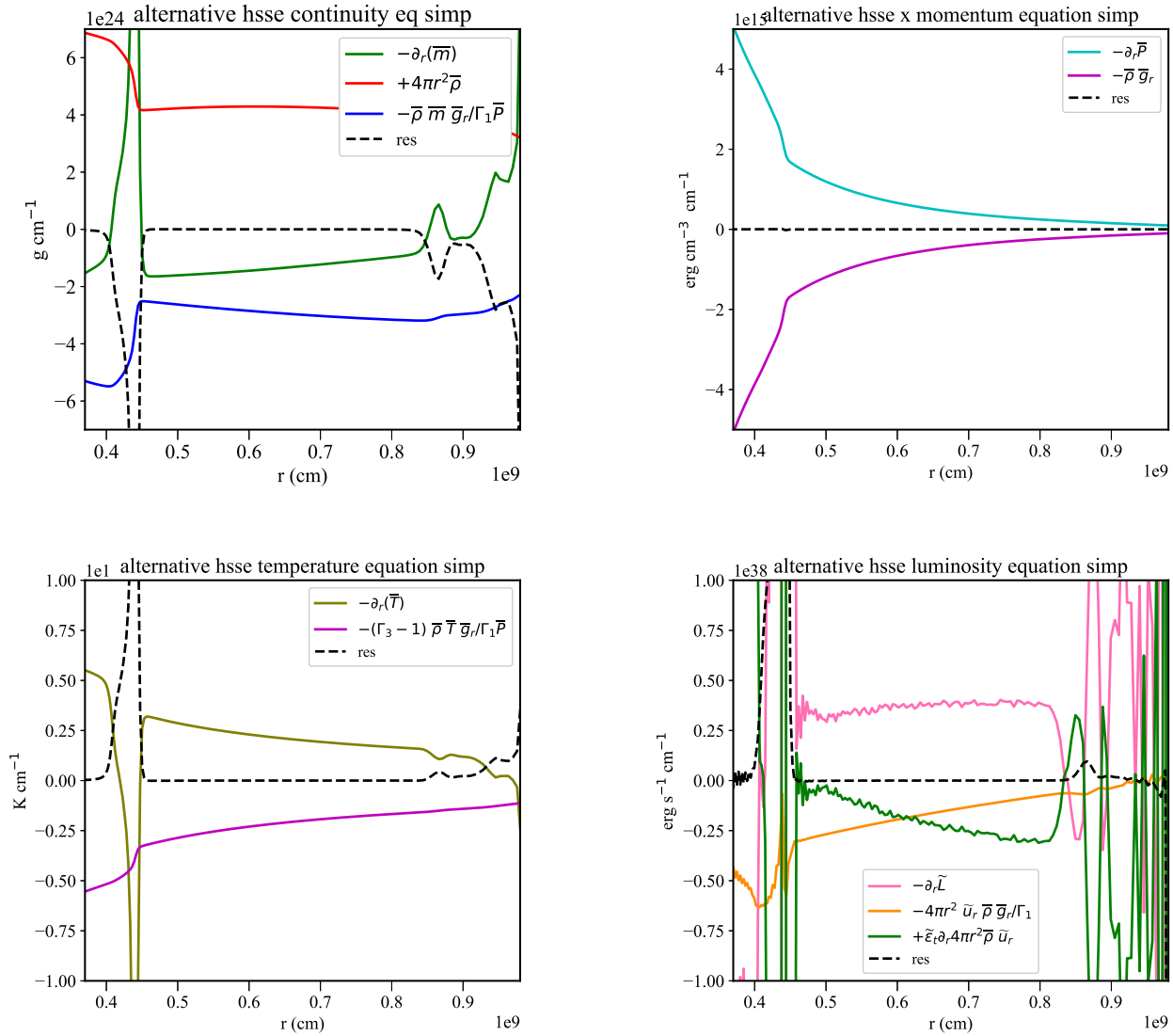


Figure 2: Simplified alternative hydrodynamic stellar structure continuity.

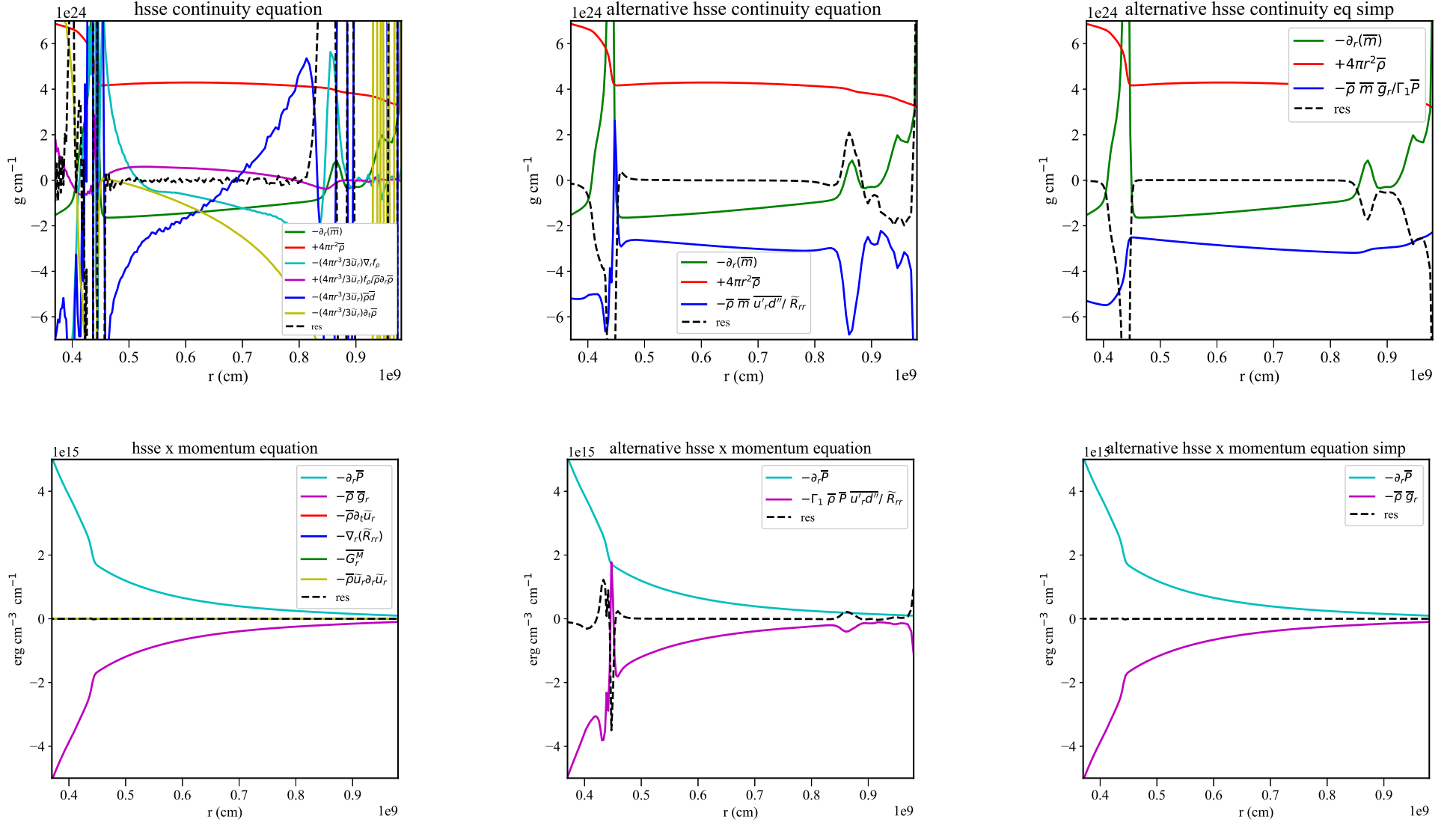


Figure 3: Comparison of all versions of continuity and momentum hydrodynamic stellar structure equations.

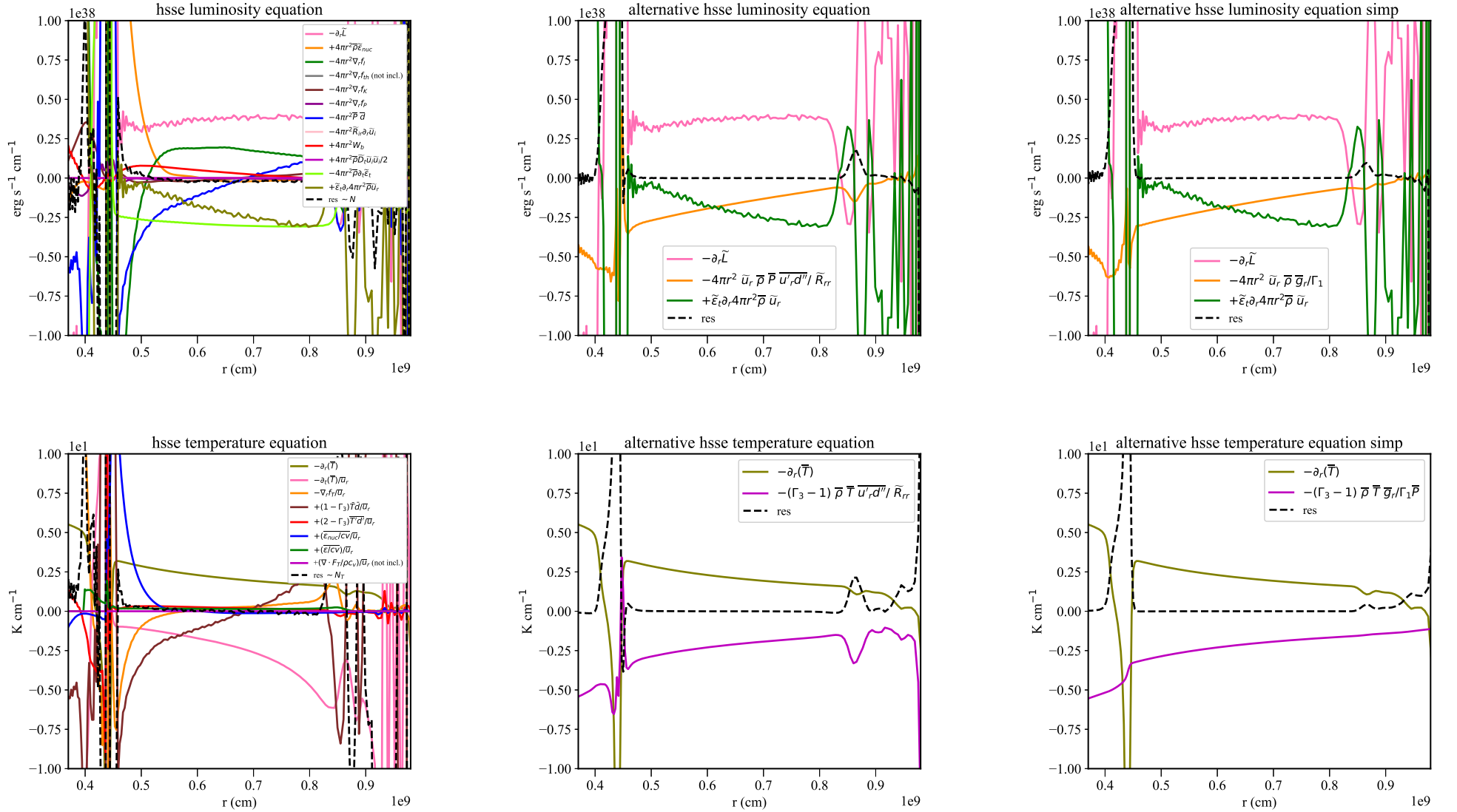


Figure 4: Comparison of all version of luminosity and temperature hydrodynamic stellar structure equations.

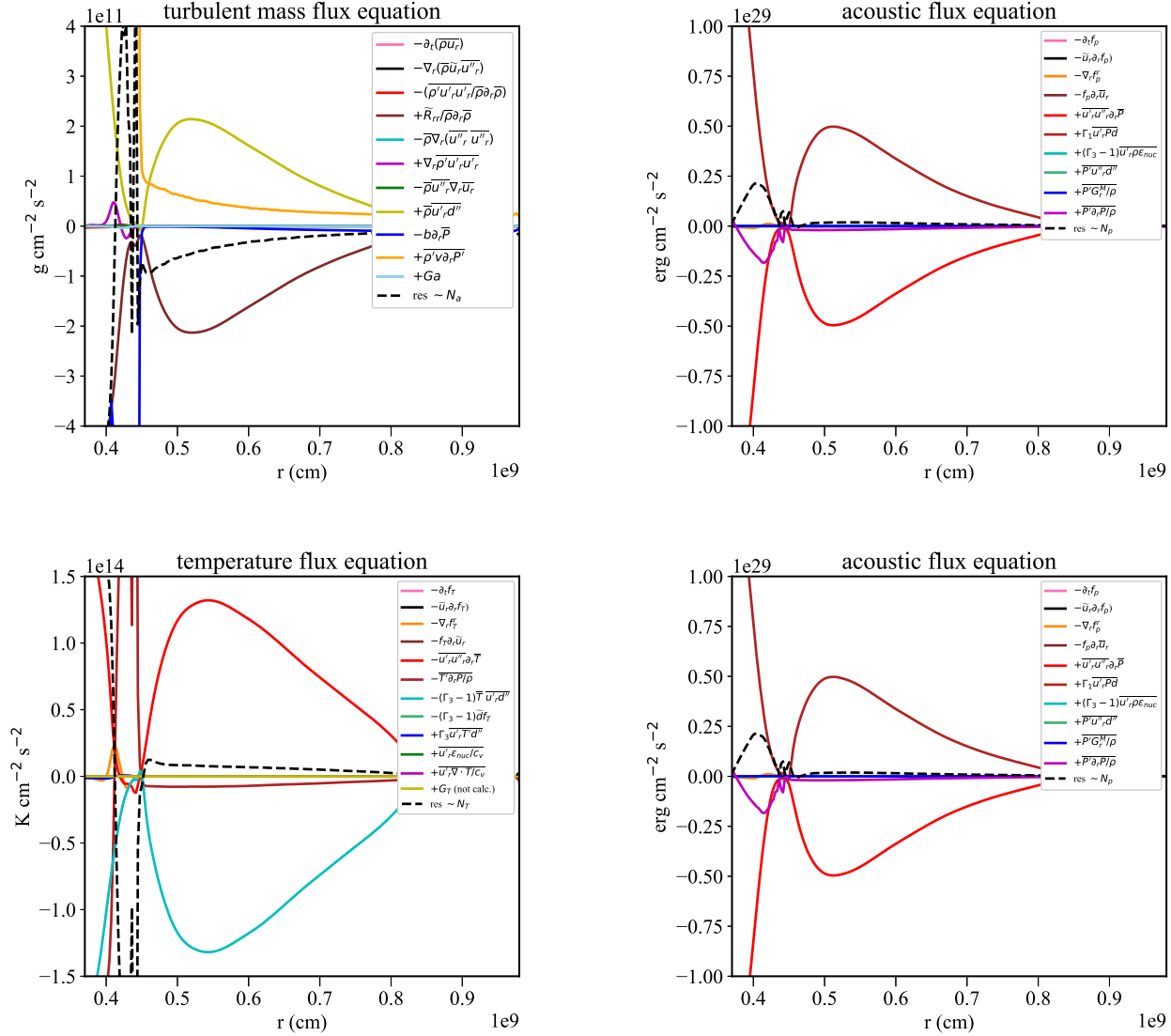


Figure 5: Flux evolution equations based on low-res oxygen burning model.

Table 1: Definitions:

ρ density	g_r radial gravitational acceleration
$m = \rho V = \rho \frac{4}{3} \pi r^3$ mass	$M = \int \rho(r) dV = \int \rho(r) 4\pi r^2 dr$ integrated mass
T temperature	$\mathcal{S} = \rho \epsilon_{\text{nuc}}(q)$ nuclear energy production (cooling function)
P pressure	$\tau_{ij} = 2\mu S_{ij}$ viscous stress tensor (μ kinematic viscosity)
u_r, u_θ, u_ϕ velocity components	$S_{ij} = (1/2)(\partial_i u_j + \partial_j u_i)$ strain rate
$\mathbf{u} = u(u_r, u_\theta, u_\phi)$ velocity	$\tilde{R}_{ij} = \overline{\rho u_i'' u_j''}$ Reynolds stress tensor
$j_z = r \sin \theta u_\phi$ specific angular momentum	$F_T = \chi \partial_r T$ heat flux
$d = \nabla \cdot \mathbf{u}$ dilatation	$\Gamma_1 = (d \ln P / d \ln \rho) _s$
ϵ_I specific internal energy	$\Gamma_2 / (\Gamma_2 - 1) = (d \ln P / d \ln T) _s$
h specific enthalpy	$\Gamma_3 - 1 = (d \ln T / d \ln \rho) _s$
$k = (1/2) \overline{u_i'' u_i''}$ turbulent kinetic energy	$\tilde{k}^r = (1/2) \overline{u_r'' u_r''} = (1/2) \tilde{R}_{rr} / \bar{\rho}$ radial turbulent kinetic energy
ϵ_k specific kinetic energy	$\tilde{k}^\theta = (1/2) \overline{u_\theta'' u_\theta''} = (1/2) \tilde{R}_{\theta\theta} / \bar{\rho}$ angular turbulent kinetic energy
ϵ_t specific total energy	$\tilde{k}^\phi = (1/2) \overline{u_\phi'' u_\phi''} = (1/2) \tilde{R}_{\phi\phi} / \bar{\rho}$ angular turbulent kinetic energy
s specific entropy	$\tilde{k}^h = \tilde{k}^\theta + \tilde{k}^\phi$ horizontal turbulent kinetic energy
$v = 1/\rho$ specific volume	$f_k = (1/2) \overline{\rho u_i'' u_i''}$ turbulent kinetic energy flux
X_α mass fraction of isotope α	$f_k^r = (1/2) \overline{\rho u_r'' u_r''}$ radial turbulent kinetic energy flux
$\dot{X}_\alpha^{\text{nuc}}$ rate of change of X_α	$f_k^\theta = (1/2) \overline{\rho u_\theta'' u_\theta''}$ angular turbulent kinetic energy flux
A_α number of nucleons in isotope α	$f_k^\phi = (1/2) \overline{\rho u_\phi'' u_\phi''}$ angular turbulent kinetic energy flux
Z_α charge of isotope α	$f_k^h = f_k^\theta + f_k^\phi$ horizontal turbulent kinetic energy flux
A mean number of nucleons per isotope	$W_p = \overline{P' d''}$ turbulent pressure dilatation
Z mean charge per isotope	$W_b = \overline{\rho u_r'' g_r}$ buoyancy
$f_P = \overline{P' u_r'}$ acoustic flux	$f_T = -\overline{\chi \partial_r T}$ heat flux (χ thermal conductivity)

Table 2: Definitions (continued):

$f_I = \overline{\rho \epsilon_I'' u_r''}$ internal energy flux	$f_\alpha = \overline{\rho X_\alpha'' u_r''}$ X_α flux
$f_s = \overline{\rho s'' u_r''}$ entropy flux	$f_{jz} = \overline{\rho j_z'' u_r''}$ angular momentum flux
$f_T = \overline{u_r' T'}$ turbulent heat flux	$f_A = \overline{\rho A'' u_r''}$ A (mean number of nucleons per isotope) flux
$f_h = \overline{\rho h'' u_r''}$ enthalpy flux	$f_Z = \overline{\rho Z'' u_r''}$ Z (mean charge per isotope) flux
$b = \overline{v' \rho'}$ density-specific volume covariance	$\mathcal{N}_\rho, \mathcal{N}_{ur}, \mathcal{N}_{u\theta}, \mathcal{N}_{u\phi}, \mathcal{N}_{jz}, \mathcal{N}_\alpha, \mathcal{N}_A, \mathcal{N}_Z$ numerical effect
$f_\tau = f_\tau^r + f_\tau^\theta + f_\tau^\phi$ viscous flux	$\mathcal{N}_{\epsilon I} = -\nabla_r f_\tau + \varepsilon_k$ numerical effect
$f_\tau^r = -\overline{\tau_{rr}' u_r'}$ viscous flux	$\mathcal{N}_{\epsilon k} = -\varepsilon_k$ numerical effect
$f_\tau^\theta = -\overline{\tau_{\theta r}' u_\theta'}$ viscous flux	$\mathcal{N}_{\epsilon t} = -\nabla_r f_\tau$ numerical effect
$f_\tau^\phi = -\overline{\tau_{\phi r}' u_\phi'}$ viscous flux	$\mathcal{N}_s = \overline{-\varepsilon_k / T}$ numerical effect
$f_\tau^h = f_\tau^\theta + f_\tau^\phi$ viscous flux	$\mathcal{N}_h = -\nabla_r f_\tau + (\Gamma_3 - 1)\varepsilon_k$ numerical effect
$f_I^r = \overline{\rho \epsilon_I'' u_r'' u_r''}$ radial flux of f_I	$\mathcal{N}_P = +(\Gamma_3 - 1)\varepsilon_k$ numerical effect
$f_s^r = \overline{\rho s'' u_r'' u_r''}$ radial flux of f_s	$\mathcal{N}_T = +(\tau_{ij} \partial_j u_i) / (c_v \rho)$ numerical effect
$f_h^r = \overline{\rho h'' u_r'' u_r''}$ radial flux of f_h	$\mathcal{N}_{Rrr} = -2\nabla_r f_\tau^r - 2\varepsilon_k^r$ numerical effect
$f_T^r = \overline{T' u_r' u_r'}$ radial flux of f_T	$\mathcal{N}_{R\theta\theta} = -2\nabla_r f_\tau^\theta - 2\varepsilon_k^\theta$ numerical effect
$f_{jz}^r = \overline{\rho j_z'' u_r'' u_r''}$ radial flux of f_{jz}	$\mathcal{N}_{R\phi\phi} = -2\nabla_r f_\tau^\phi - 2\varepsilon_k^\phi$ numerical effect
$f_\alpha^r = \overline{\rho X_\alpha'' u_r'' u_r''}$ radial flux of f_α	$\mathcal{N}_k = -\nabla_r f_\tau - \varepsilon_k$ numerical effect
$f_A^r = \overline{\rho A'' u_r'' u_r''}$ radial flux of f_A	$\mathcal{N}_{kr} = -\nabla_r f_\tau^r - \varepsilon_k^r$ numerical effect
$f_Z^r = \overline{\rho Z'' u_r'' u_r''}$ radial flux of f_Z	$\mathcal{N}_{kh} = -\nabla_r f_\tau^h - \varepsilon_k^h$ numerical effect
$\mathcal{G}_k^r = -(1/2)\overline{G_{rr}^R} - \overline{u_r'' G_r^M}$	$\mathcal{N}_a = -\varepsilon_a$ numerical effect

Table 3: Definitions (continued):

$\mathcal{G}_k^\theta = -(1/2)\overline{G_{\theta\theta}^R} - \overline{u_\theta'' G_\theta^M}$	\mathcal{N}_b numerical effect
$\mathcal{G}_k^\phi = -(1/2)\overline{G_{\phi\phi}^R} - \overline{u_\phi'' G_\phi^M}$	$\mathcal{N}_{fI} = -\nabla_r(\overline{\epsilon_I'' \tau_{rr}'}) + \overline{u_r'' \tau_{ij} \partial_i u_j} - \varepsilon_I$ numerical effect
$\mathcal{G}_k^h = +\mathcal{G}_k^\theta + \mathcal{G}_k^\phi$	$\mathcal{N}_{fh} = -\nabla_r(\overline{h'' \tau_{rr}'}) + \overline{u_r'' (\Gamma_3 - 1) \tau_{ij} \partial_i u_j} - \overline{u_r'' \nabla_i u_i \tau_{ji}} - \varepsilon_h$ numerical effect
$\mathcal{G}_a = +\overline{\rho' v G_r^M}$	$\mathcal{N}_{fs} = -\nabla_r(\overline{s'' \tau_{rr}'}) + \overline{u_r'' \tau_{ij} \partial_i u_j / T} - \varepsilon_s$ numerical effect
$\mathcal{G}_I = -\overline{G_r^I} - \overline{\epsilon_I'' G_r^M}$	$\mathcal{N}_{fA} = -\nabla_r(\overline{A'' \tau_{rr}'}) - \varepsilon_A$ numerical effect
$\mathcal{G}_\alpha = -\overline{G_r^\alpha} - \overline{X_\alpha'' G_r^M}$	$\mathcal{N}_{fZ} = -\nabla_r(\overline{Z'' \tau_{rr}'}) - \varepsilon_Z$ numerical effect
$\mathcal{G}_A = -\overline{G_r^A} - \overline{A'' G_r^M}$	$\mathcal{N}_{f\alpha} = -\nabla_r(\overline{\alpha'' \tau_{rr}'}) - \varepsilon_\alpha$ numerical effect
$\mathcal{G}_Z = -\overline{G_r^Z} - \overline{Z'' G_r^M}$	$\mathcal{N}_{fjz} = -\nabla_r(\overline{j_z'' \tau_{rr}'}) - \varepsilon_{jz}$ numerical effect
$\mathcal{G}_h = -\overline{G_r^h} - \overline{h'' G_r^M}$	$\mathcal{N}_{fT} = +\overline{T' \partial_i \tau_{ri} / \rho} + \overline{u_r' \tau_{ij} \partial_i u_j / \rho c_v}$ numerical effect
$\mathcal{G}_T = -\overline{G_r^T} - \overline{T' G_r^M}$	
$\mathcal{G}_s = -\overline{G_r^s} - \overline{s'' G_r^M}$	
$\mathcal{G}_{jz} = -\overline{G_r^{jz}} - \overline{j_z'' G_r^M}$	
$\sigma_\rho = \overline{\rho' \rho'}$	$\mathcal{N}_{\sigma_\rho}$ numerical effect
$\sigma_P = \overline{P' P'}$	$\mathcal{N}_{\sigma_P} = +2(\Gamma_3 - 1) \overline{P' \tau_{ij} \partial_i u_j}$ numerical effect
$\sigma_T = \overline{T' T'}$	$\mathcal{N}_{\sigma_T} = +2 \overline{T' \tau_{ij} \partial_i u_j / \rho c_v}$ numerical effect
$\sigma_{ur} = \overline{u_r'' u_r''}$	$\mathcal{N}_{\sigma_{ur}} = +2 \nabla_r f_\tau^r - 2 \varepsilon_k^r$ numerical effect
$\sigma_s = \overline{s'' s''}$	$\mathcal{N}_{\sigma_s} = +2 \overline{s'' \tau_{ij} \partial_j u_i / T}$ numerical effect
$\sigma_\alpha = \overline{X_\alpha'' X_\alpha''}$	$\mathcal{N}_{\sigma_\alpha}$ numerical effect numerical effect
$\sigma_{\epsilon I} = \overline{\epsilon_I'' \epsilon_I''}$	$\mathcal{N}_{\sigma_{\epsilon I}} = +2 \overline{\epsilon_I'' \tau_{ij} \partial_j u_i}$ numerical effect

Table 4: Definitions (continued):

$$\begin{aligned}
 \varepsilon_k^r &= \overline{\tau'_{rr} \partial_r u''_r} + \overline{\tau'_{r\theta} (1/r) \partial_\theta u''_r} + \overline{\tau'_{r\phi} (1/r \sin \theta) \partial_\phi u''_r} \\
 \varepsilon_k^\theta &= \overline{\tau'_{\theta r} \partial_r u''_\theta} + \overline{\tau'_{\theta\theta} (1/r) \partial_\theta u''_\theta} + \overline{\tau'_{\theta\phi} (1/r \sin \theta) \partial_\phi u''_\theta} \\
 \varepsilon_k^\phi &= \overline{\tau'_{\phi r} \partial_r u''_\phi} + \overline{\tau'_{\phi\theta} (1/r) \partial_\theta u''_\phi} + \overline{\tau'_{\phi\phi} (1/r \sin \theta) \partial_\phi u''_\phi} \\
 \varepsilon_k &= \varepsilon_k^r + \varepsilon_k^\theta + \varepsilon_k^\phi \\
 \varepsilon_k^h &= \varepsilon_k^\theta + \varepsilon_k^\phi \\
 \varepsilon_a &= \overline{\rho' v \nabla_r \tau'_{rr}} \\
 \varepsilon_I &= \overline{\tau'_{rr} \partial_r \epsilon''_I} + \overline{\tau'_{r\theta} (1/r) \partial_\theta \epsilon''_I} + \overline{\tau'_{r\phi} (1/r \sin \theta) \partial_\phi \epsilon''_I} \\
 \varepsilon_s &= \overline{\tau'_{rr} \partial_r s''} + \overline{\tau'_{r\theta} (1/r) \partial_\theta s''} + \overline{\tau'_{r\phi} (1/r \sin \theta) \partial_\phi s''} \\
 \varepsilon_\alpha &= \overline{\tau'_{rr} \partial_r X''_\alpha} + \overline{\tau'_{r\theta} (1/r) \partial_\theta X''_\alpha} + \overline{\tau'_{r\phi} (1/r \sin \theta) \partial_\phi X''_\alpha} \\
 \varepsilon_A &= \overline{\tau'_{rr} \partial_r A''} + \overline{\tau'_{r\theta} (1/r) \partial_\theta A''} + \overline{\tau'_{r\phi} (1/r \sin \theta) \partial_\phi A''} \\
 \varepsilon_Z &= \overline{\tau'_{rr} \partial_r Z''} + \overline{\tau'_{r\theta} (1/r) \partial_\theta Z''} + \overline{\tau'_{r\phi} (1/r \sin \theta) \partial_\phi Z''} \\
 \varepsilon_h &= \overline{\tau'_{rr} \partial_r h''} + \overline{\tau'_{r\theta} (1/r) \partial_\theta h''} + \overline{\tau'_{r\phi} (1/r \sin \theta) \partial_\phi h''} \\
 \varepsilon_{jz} &= \overline{\tau'_{rr} \partial_r j''_z} + \overline{\tau'_{r\theta} (1/r) \partial_\theta j''_z} + \overline{\tau'_{r\phi} (1/r \sin \theta) \partial_\phi j''_z} \\
 G_r^M &= -\overline{\rho u_\theta u_\theta / r} - \overline{\rho u_\phi u_\phi / r} \\
 G_\theta^M &= +\overline{\rho u_\theta u_r / r} - \overline{\rho u_\phi u_\phi / (r \tan \theta)} \\
 G_\phi^M &= +\overline{\rho u_\phi u_r / r} + \overline{\rho u_\phi u_\theta / (r \tan \theta)} \\
 G_{rr}^R &= -\overline{\rho u''_\theta u''_\theta u''_r / r} - \overline{\rho u''_\theta u''_r u''_\theta / r} - \overline{\rho u''_\phi u''_\phi u''_r / r} - \overline{\rho u''_\phi u''_r u''_\phi / r} \\
 G_{\theta\theta}^R &= +\overline{\rho u''_\theta u''_r u''_\theta / r} + \overline{\rho u''_\theta u''_\theta u''_r / r} - \overline{\rho u''_\phi u''_\phi u''_\theta / (r \tan \theta)} - \overline{u''_\phi u''_\theta u''_\phi / (r \tan \theta)} \\
 G_{\phi\phi}^R &= +\overline{\rho u''_\phi u''_r u''_\phi / r} + \overline{\rho u''_\phi u''_\theta u''_\phi / (r \tan \theta)} + \overline{\rho u''_\phi u''_\phi u''_r / r} + \overline{\rho u''_\phi u''_\theta u''_\phi / (r \tan \theta)} \\
 G_r^I &= -\overline{\rho \epsilon''_I u''_\theta u''_\theta / r} - \overline{\rho \epsilon''_I u''_\phi u''_\phi / r} \\
 G_r^s &= -\overline{\rho s'' u''_\theta u''_\theta / r} - \overline{\rho s'' u''_\phi u''_\phi / r} \\
 G_r^\alpha &= -\overline{\rho X''_\alpha u''_\theta u''_\theta / r} - \overline{\rho X''_\alpha u''_\phi u''_\phi / r} \\
 G_r^A &= -\overline{\rho A'' u''_\theta u''_\theta / r} - \overline{\rho A'' u''_\phi u''_\phi / r} \\
 G_r^Z &= -\overline{\rho Z'' u''_\theta u''_\theta / r} - \overline{\rho Z'' u''_\phi u''_\phi / r} \\
 G_r^h &= -\overline{\rho h'' u''_\theta u''_\theta / r} - \overline{\rho h'' u''_\phi u''_\phi / r} \\
 G_r^T &= -\overline{\rho T' u''_\theta u''_\theta / r} - \overline{\rho T' u''_\phi u''_\phi / r} \\
 G_r^{jz} &= -\overline{\rho j''_z u''_\theta u''_\theta / r} - \overline{\rho j''_z u''_\phi u''_\phi / r}
 \end{aligned}$$

$$\nabla(\cdot) = \nabla_r(\cdot) + \nabla_\theta(\cdot) + \nabla_\phi(\cdot) = \frac{1}{r^2} \partial_r(r^2 \cdot) + \frac{1}{r \sin \theta} \partial_\theta(\sin \theta \cdot) + \frac{1}{r \sin \theta} \partial_\phi(\cdot)$$