Divergence of tensors in spherical geometry up to third order

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BACKGROUND READING:

CONTINUUM MECHANICS (Lecture Notes)
Zdenek Martinec, Department of Geophysics, Faculty of Mathematics and Physics, Charles University in Prague

$$\nabla(.) = \sum_n \frac{\mathbf{e_n}}{h_n}\frac{\partial(.)}{\partial x_n} \quad : \text{nabla operator}$$

$$\mathbf{V} = \sum_i V_i \mathbf{e_i} \quad : \text{tensor of first order (vector)} \tag{1}$$

$$\mathbf{S} = \sum_{ij} S_{ij}(\mathbf{e_i} \otimes \mathbf{e_j}) \quad : \text{tensor of second order} \tag{2}$$

$$\mathbf{T} = \sum_{ijk} T_{ijk}(\mathbf{e_i} \otimes \mathbf{e_j} \otimes \mathbf{e_k}) \quad : \text{tensor of third order} \tag{3}$$

$$\nabla \cdot \mathbf{V} = \sum_i \frac{1}{h_i}\left[\frac{\partial V_i}{\partial x_i} + \sum_m \Gamma^i_{mi} V_m\right] \quad : \text{div of first order tensor (vector)} \tag{4}$$

$$\nabla \cdot \mathbf{S} = \sum_{ij} \frac{1}{h_i}\left[\frac{\partial S_{ij}}{\partial x_i} + \sum_m \Gamma^i_{mi} S_{mj} + \sum_m \Gamma^j_{mi} S_{im}\right]\mathbf{e_j} \quad : \text{div of second order tensor} \tag{5}$$

$$\nabla \cdot \mathbf{T} = \sum_{ijk} \frac{1}{h_i}\left[\frac{\partial T_{ijk}}{\partial x_i} + \sum_m \Gamma^i_{mi} T_{mjk} + \sum_m \Gamma^j_{mi} T_{imk} + \sum_m \Gamma^k_{mi} T_{ijm}\right](\mathbf{e_j} \otimes \mathbf{e_k}) \quad : \text{div of third order tensor} \tag{6}$$

Divergence of first order tensor $\nabla \cdot \mathbf{V}$

$$\frac{1}{r^2}\frac{\partial (r^2 V_r)}{\partial_r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi} \tag{7}$$

Divergence of second order tensor $\nabla \cdot \mathbf{S}$

$$S_r(\mathbf{e_r}): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 S_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ S_{\theta r}) + \frac{1}{r\sin\theta}\frac{\partial S_{\phi r}}{\partial\phi} - \frac{S_{\theta\theta}}{r} - \frac{S_{\phi\phi}}{r} \tag{8}$$

$$S_\theta(\mathbf{e}_\theta): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 S_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ S_{\theta\theta}) + \frac{1}{r\sin\theta}\frac{\partial S_{\phi\theta}}{\partial\phi} + \frac{S_{\theta r}}{r} - \frac{S_{\phi\phi}\cos\theta}{r\sin\theta} \tag{9}$$

$$S_\phi(\mathbf{e}_\phi): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 S_{r\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ S_{\theta\phi}) + \frac{1}{r\sin\theta}\frac{\partial S_{\phi\phi}}{\partial\phi} + \frac{S_{\phi r}}{r} + \frac{S_{\phi\theta}\cos\theta}{r\sin\theta} \tag{10}$$

Divergence of third order tensor $\nabla \cdot \mathbf{T}$

$$T_{rr}\ (\mathbf{e_r}\otimes\mathbf{e_r}): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{rrr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta rr}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi rr}}{\partial\phi} - \frac{T_{\theta\theta r}}{r} - \frac{T_{\theta r\theta}}{r} - \frac{T_{\phi\phi r}}{r} - \frac{T_{\phi r\phi}}{r} \tag{11}$$

$$T_{r\theta}\ (\mathbf{e_r}\otimes\mathbf{e}_\theta): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{rr\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta r\theta}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi r\theta}}{\partial\phi} - \frac{T_{\theta\theta\theta}}{r} + \frac{T_{\theta rr}}{r} - \frac{T_{\phi\phi\theta}}{r} - \frac{T_{\phi r\phi}\cos\theta}{r\sin\theta} \tag{12}$$

$$T_{r\phi}\ (\mathbf{e_r}\otimes\mathbf{e}_\phi): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{rr\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta r\phi}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi r\phi}}{\partial\phi} + \frac{T_{\theta\theta\phi}}{r} - \frac{T_{\phi\phi\phi}}{r} + \frac{T_{\phi r\phi}\cos\theta}{r\sin\theta} \tag{13}$$

$$T_{\theta r}\ (\mathbf{e}_\theta\otimes\mathbf{e_r}): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\theta r}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\theta r}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta r}}{\partial\phi} + \frac{T_{\theta rr}}{r} - \frac{T_{\theta\theta\theta}}{r} - \frac{T_{\phi\phi r}\cos\theta}{r\sin\theta} - \frac{T_{\phi\theta\phi}}{r} \tag{14}$$

$$T_{\theta\theta}\ (\mathbf{e}_\theta\otimes\mathbf{e}_\theta): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\theta\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\theta\theta}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta\theta}}{\partial\phi} + \frac{T_{\theta r\theta}}{r} + \frac{T_{\theta\theta r}}{r} - \frac{T_{\phi\phi\theta}\cos\theta}{r\sin\theta} - \frac{T_{\phi\theta\phi}\cos\theta}{r\sin\theta} \tag{15}$$

$$T_{\theta\phi}\ (\mathbf{e}_\theta\otimes\mathbf{e}_\phi): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\theta\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\theta\phi}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta\phi}}{\partial\phi} + \frac{T_{\theta r\phi}}{r} + \frac{T_{\phi\theta r}}{r} + \frac{T_{\phi\theta\theta}\cos\theta}{r\sin\theta} \tag{16}$$

$$T_{\phi r}\ (\mathbf{e}_\phi\otimes\mathbf{e_r}): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\phi r}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\phi r}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi r}}{\partial\phi} - \frac{T_{\theta\phi\theta}}{r} + \frac{T_{\phi rr}}{r} + \frac{T_{\phi\theta r}\cos\theta}{r\sin\theta} - \frac{T_{\phi\phi\phi}}{r} \tag{17}$$

$$T_{\phi\theta}\ (\mathbf{e}_\phi\otimes\mathbf{e}_\theta): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\phi\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\phi\theta}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi\theta}}{\partial\phi} + \frac{T_{\theta\phi r}}{r} + \frac{T_{\phi r\theta}}{r} + \frac{T_{\phi\theta\theta}\cos\theta}{r\sin\theta} - \frac{T_{\phi\theta\phi}\cos\theta}{r\sin\theta} \tag{18}$$

$$T_{\phi\phi}\ (\mathbf{e}_\phi\otimes\mathbf{e}_\phi): \quad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\phi\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ T_{\theta\phi\phi}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi\phi}}{\partial\phi} + \frac{T_{\phi r\phi}}{r} + \frac{T_{\phi\theta\phi}\cos\theta}{r\sin\theta} + \frac{T_{\phi\phi r}}{r} + \frac{T_{\phi\phi\theta}\cos\theta}{r\sin\theta} \tag{19}$$