1 Alternative hydrodynamic stellar structure equations

Below is a set of alternative hydrodynamic stellar structure equations inspired by mean fields from the mean density/temperature/internal energy and pressure flux equations involving gradient term scaled by favrian Reynolds stress $\widetilde{R}_{rr}$ and the other one turbulent dilatation flux $\overline{u'_r d''}$.

$$\partial_r \overline{m} = -\overline{\rho}\ \overline{m}\ \ \overline{u'_r d''}/\ \widetilde{R}_{rr} + 4\pi r^2 \overline{\rho} \tag{1}$$
$$\partial_r \overline{P} = -\Gamma_1\ \overline{\rho}\ \overline{P}\ \overline{u'_r d''}/\ \widetilde{R}_{rr} \tag{2}$$
$$\partial_r \widetilde{L} = +\widetilde{\epsilon}_t \partial_r 4\pi r^2 \overline{\rho} \widetilde{u}_r - 4\pi r^2 \overline{\rho}\ \widetilde{u}_r\ \overline{P}\ \overline{u'_r d''}/\ \widetilde{R}_{rr} \tag{3}$$
$$\partial_r \overline{T} = -(\Gamma_3 - 1)\ \overline{\rho}\ \overline{T}\ \overline{u'_r d''}/\ \widetilde{R}_{rr} \tag{4}$$
$$\partial_t \widetilde{X}_i = \widetilde{\dot{X}}_i^{nuc} - (1/\overline{\rho})\nabla_r f_i - \widetilde{u}_r \partial_r \widetilde{X}_i \tag{5}$$

These equations could be perfectly validated by our ransX framework and are shown on the next page in Figure 1. It appears that there is a universality relation between gradient of a mean thermodynamic variable Q and dilatation flux $\overline{u'_r d''}$, that is:

$$\partial_r \overline{Q} \sim -\overline{\rho}\ \overline{Q}\ \overline{u'_r d''}/\widetilde{R}_{rr} \tag{6}$$

Moreover, we know that due to hydrostatic equilibrium, $\partial_r \overline{P} \sim -\overline{\rho}\ \overline{g}_r$, and based on Equation (2) we can write, that dilatation flux $\overline{u'_r d''}$ is:

$$\overline{u'_r d''} \sim \frac{\widetilde{R}_{rr}\ \overline{g}_r}{\Gamma_1\ \overline{P}} \tag{7}$$

Also, the expansion velocity $\widetilde{u}_r$ can be replaced by:

$$\widetilde{u}_r = -\partial_t \overline{M}/4\pi r^2 \overline{\rho} \tag{8}$$

So the alternative hydrodynamic stellar structure equations for hydrostatic convection become a system with only **one unknown the composition flux** f_α for which we need a proper model. See validation of these simplified alternative hydrodynamic stellar structure equation in Figure 2.

$$\partial_r \overline{m} = -\overline{\rho}\ \overline{m}\ \overline{g}_r/\Gamma_1 \overline{P} + 4\pi r^2 \overline{\rho} \tag{9}$$
$$\partial_r \overline{P} = -\overline{\rho}\ \overline{g}_r \tag{10}$$
$$\partial_r \widetilde{L} = -4\pi r^2 \widetilde{u}_r\ \overline{\rho}\ \overline{g}_r/\Gamma_1 + \widetilde{\epsilon}_t \partial_r 4\pi r^2 \overline{\rho} \widetilde{u}_r \tag{11}$$
$$\partial_r \overline{T} = -(\Gamma_3 - 1)\ \overline{\rho}\ \overline{T}\ \overline{g}_r/\Gamma_1 \overline{P} \tag{12}$$
$$\partial_t \widetilde{X}_i = \widetilde{\dot{X}}_i^{nuc} - (1/\overline{\rho})\nabla_r f_i - \widetilde{u}_r \partial_r \widetilde{X}_i \tag{13}$$

1.1 Derivation of the alternative hydrodynamic stellar structure equations from flux evolution equations

From the pressure flux equation (Fig.5) we get, that:

$$
\begin{aligned}
+\overline{u'_r u''_r}\ \partial_r \overline{P} &= -\Gamma_1\ \overline{u'_r P d} \quad \backslash.\overline{\rho} && (14)\\
+\overline{\rho}\ \overline{u'_r u''_r}\ \partial_r \overline{P} &= -\overline{\rho}\ \Gamma_1\ \overline{u'_r P d} \qquad P = \overline{P} + P' \quad d = \overline{d} + d' && (15)\\
+\widetilde{R}_{rr}\ \partial_r \overline{P} &= -\overline{\rho}\ \Gamma_1\ (\overline{P u'_r}\ \overline{d} + \overline{P}\ \overline{u'_r d'} + \overline{P' u'_r d'}) \qquad \overline{P}\ \overline{u'_r d'} >> \overline{P u'_r}\ \overline{d} + \overline{P' u'_r d'} \ \text{ and } \ \overline{a'b'} = \overline{a'b''} = \overline{a''b'} && (16)\\
+\widetilde{R}_{rr}\ \partial_r \overline{P} &= -\overline{\rho}\ \Gamma_1\ \overline{P}\ \overline{u'_r d''} && (17)\\
\partial_r \overline{P} &= -\overline{\rho}\ \Gamma_1\ \overline{P}\ \overline{u'_r d''}/\widetilde{R}_{rr} && (18)\\
\partial_r \overline{P} &= -\overline{\rho}\widetilde{g}_r \quad \text{hydrostatic equilibrium} && (19)\\
\overline{u'_r d''} &= +\widetilde{R}_{rr}\widetilde{g}_r/\Gamma_1 \overline{P} \quad \text{model for dilatation flux used later} && (20)
\end{aligned}
$$

From the turbulent mass flux equation (Fig.5) we get, that:

$$
\begin{aligned}
+\widetilde{R}_{rr}/\overline{\rho}\ \partial_r \overline{\rho} &= -\overline{\rho}\ \overline{u'_r d''} && (21)\\
\partial_r \overline{\rho} &= -\overline{\rho}\ \overline{\rho}\ \overline{u'_r d''}/\widetilde{R}_{rr} \quad \backslash.V \ (\text{volume}) = 4\pi r^3/3 && (22)\\
V \partial_r \overline{\rho} &= -\overline{\rho}\ V\ \overline{\rho}\ \overline{u'_r d''}/\widetilde{R}_{rr} && (23)\\
\partial_r \overline{\rho} V - \rho \partial_r V &= -\overline{\rho}\ V\ \overline{\rho}\ \overline{u'_r d''}/\widetilde{R}_{rr} && (24)\\
\partial_r \overline{m} - \rho 4\pi r^2 &= -\overline{m}\ \overline{\rho}\ \overline{u'_r d''}/\widetilde{R}_{rr} && (25)\\
\partial_r \overline{m} &= +\rho 4\pi r^2 - \overline{m}\ \overline{\rho}\ \overline{u'_r d''}/\widetilde{R}_{rr} && (26)\\
\partial_r \overline{m} &= +\rho 4\pi r^2 - \overline{\rho}\ \overline{m}\ \widetilde{g}_r/\Gamma_1 \overline{P} && (27)\\
\partial_r \overline{m} &= -\overline{\rho}\ \overline{m}\ \widetilde{g}_r/\Gamma_1 \overline{P} + \rho 4\pi r^2 && (28)
\end{aligned}
$$

From the temperature flux equation (Fig.5) we get, that:

$$
\begin{aligned}
-\overline{u'_r u''_r}\ \partial_r \overline{T} &= +(\Gamma_3 - 1)\ \overline{T}\ \overline{u'_r d''} \quad \backslash.\overline{\rho} && (29)\\
-\overline{\rho}\ \overline{u'_r u''_r}\ \partial_r \overline{T} &= +(\Gamma_3 - 1)\ \overline{\rho}\ \overline{T}\ \overline{u'_r d''} && (30)\\
-\widetilde{R}_{rr}\ \partial_r \overline{T} &= +(\Gamma_3 - 1)\ \overline{\rho}\ \overline{T}\ \overline{u'_r d''} && (31)\\
\partial_r \overline{T} &= -(\Gamma_3 - 1)\ \overline{\rho}\ \overline{T}\ \overline{u'_r d''}/\widetilde{R}_{rr} && (32)\\
\partial_r \overline{T} &= -(\Gamma_3 - 1)\ \overline{\rho}\ \overline{T}\ (\widetilde{R}_{rr}\widetilde{g}_r/\Gamma_1 \overline{P})/\widetilde{R}_{rr} && (33)\\
\partial_r \overline{T} &= -(\Gamma_3 - 1)\ \overline{\rho}\ \overline{T}\ \widetilde{g}_r/\Gamma_1 \overline{P} && (34)
\end{aligned}
$$

From the internal energy flux equation (Fig.5) we get, that:

$$
\begin{aligned}
-\widetilde{R}_{rr}\ \partial_r \widetilde{\varepsilon}_i &= +\ \overline{u''_r P d} \qquad P = \overline{P} + P' \quad d = \overline{d} + d' && (35)\\
-\widetilde{R}_{rr}\ \partial_r \widetilde{\varepsilon}_i &= +\ \overline{P u''_r}\ \overline{d} + \overline{P}\ \overline{u''_r d'} + \overline{P' u''_r d'} \qquad \overline{P}\ \overline{u''_r d'} >> \overline{P u''_r}\ \overline{d} + \overline{P' u''_r d'} \ \text{ and } \ \overline{a'b'} = \overline{a'b''} = \overline{a''b'} && (36)\\
-\widetilde{R}_{rr}\ \partial_r \widetilde{\varepsilon}_i &= +\ \overline{P}\ \overline{u''_r d'} \qquad \overline{P} = \overline{\rho}\widetilde{\varepsilon}_i && (37)\\
-\widetilde{R}_{rr}\ \partial_r \widetilde{\varepsilon}_i &= +\ \overline{\rho}\widetilde{\varepsilon}_i\ \overline{u''_r d'} && (38)\\
\partial_r \widetilde{\varepsilon}_i &= -\ \overline{\rho}\widetilde{\varepsilon}_i\ \overline{u''_r d'}/\widetilde{R}_{rr} && (39)\\
\partial_r \widetilde{\varepsilon}_i &= -\ \overline{\rho}\widetilde{\varepsilon}_i\ (\widetilde{R}_{rr}\widetilde{g}_r/\Gamma_1 \overline{P})/\widetilde{R}_{rr} && (40)\\
\partial_r \widetilde{\varepsilon}_i &= -\ \overline{\rho}\widetilde{\varepsilon}_i\ \widetilde{g}_r/\Gamma_1 \overline{P} \qquad \overline{P} = \overline{\rho}\widetilde{\varepsilon}_i && (41)\\
\partial_r \widetilde{\varepsilon}_i &= -\ \overline{\rho}\widetilde{\varepsilon}_i\ \widetilde{g}_r/\Gamma_1 \overline{\rho}\widetilde{\varepsilon}_i && (42)\\
\partial_r \widetilde{\varepsilon}_i &= -\ \widetilde{g}_r/\Gamma_1 \qquad \backslash .4\pi r^2 \overline{\rho}\widetilde{u}_r && (43)\\
4\pi r^2 \overline{\rho}\widetilde{u}_r \partial_r \widetilde{\varepsilon}_i &= -\ 4\pi r^2 \overline{\rho}\widetilde{u}_r\ \widetilde{g}_r/\Gamma_1 && (44)\\
\partial_r 4\pi r^2 \overline{\rho}\widetilde{u}_r \widetilde{\varepsilon}_i - \widetilde{\varepsilon}_i \partial_r 4\pi r^2 \overline{\rho}\widetilde{u}_r &= -\ 4\pi r^2 \overline{\rho}\widetilde{u}_r\ \widetilde{g}_r/\Gamma_1 \quad \varepsilon_K << \varepsilon_i \text{ and } \varepsilon_i \sim \varepsilon_t && (45)\\
\partial_r 4\pi r^2 \overline{\rho}\widetilde{u}_r \widetilde{\varepsilon}_t - \widetilde{\varepsilon}_t \partial_r 4\pi r^2 \overline{\rho}\widetilde{u}_r &= -\ 4\pi r^2 \overline{\rho}\widetilde{u}_r\ \widetilde{g}_r/\Gamma_1 && (46)\\
\partial_r \widetilde{L} - \widetilde{\varepsilon}_t \partial_r 4\pi r^2 \overline{\rho}\widetilde{u}_r &= -\ 4\pi r^2 \overline{\rho}\widetilde{u}_r\ \widetilde{g}_r/\Gamma_1 && (47)\\
\partial_r \widetilde{L} &= -\ 4\pi r^2 \overline{\rho}\widetilde{u}_r\ \widetilde{g}_r/\Gamma_1 + \widetilde{\varepsilon}_t \partial_r 4\pi r^2 \overline{\rho}\widetilde{u}_r && (48)
\end{aligned}
$$

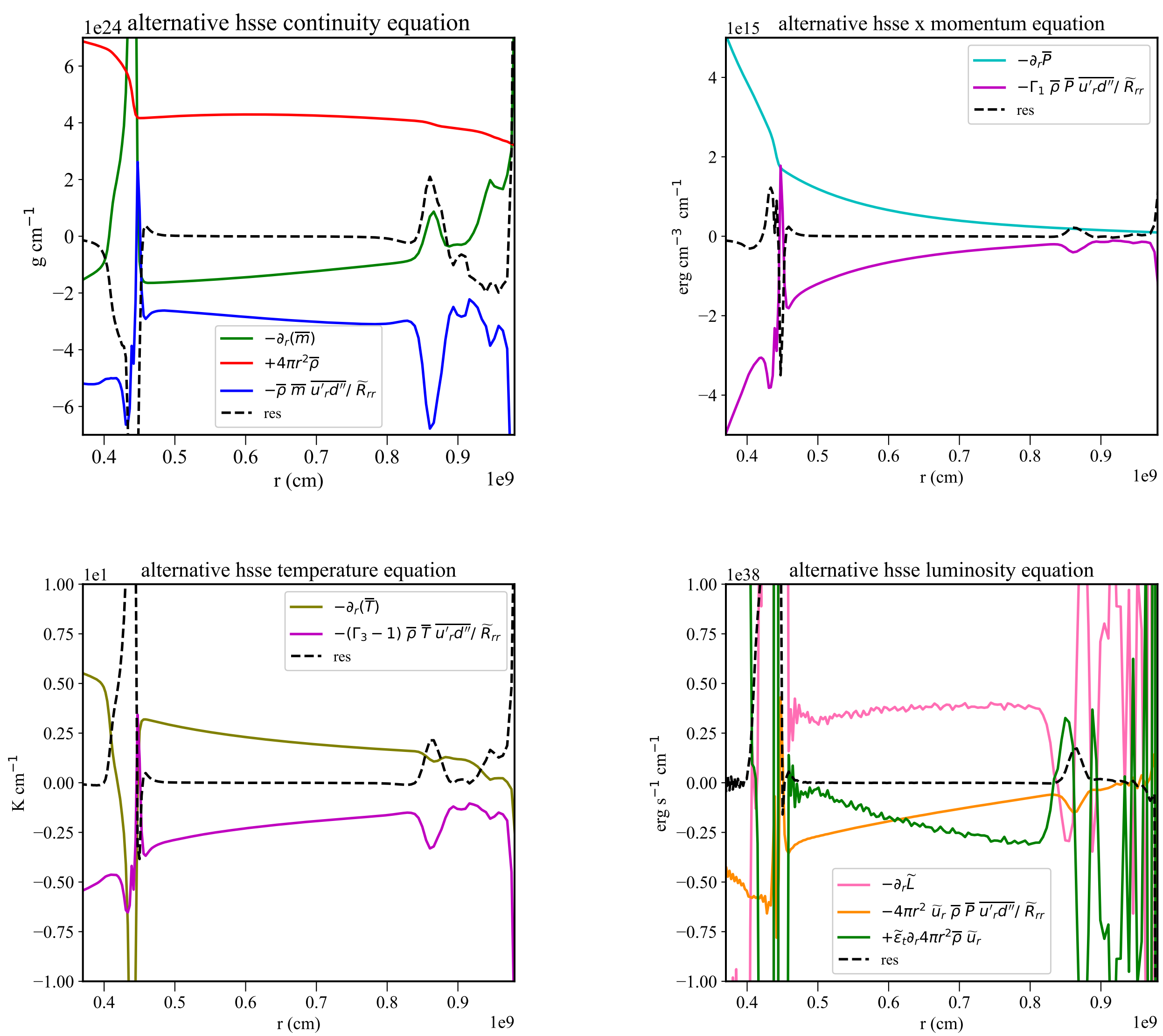

Figure 1: Alternative hydrodynamic stellar structure continuity.

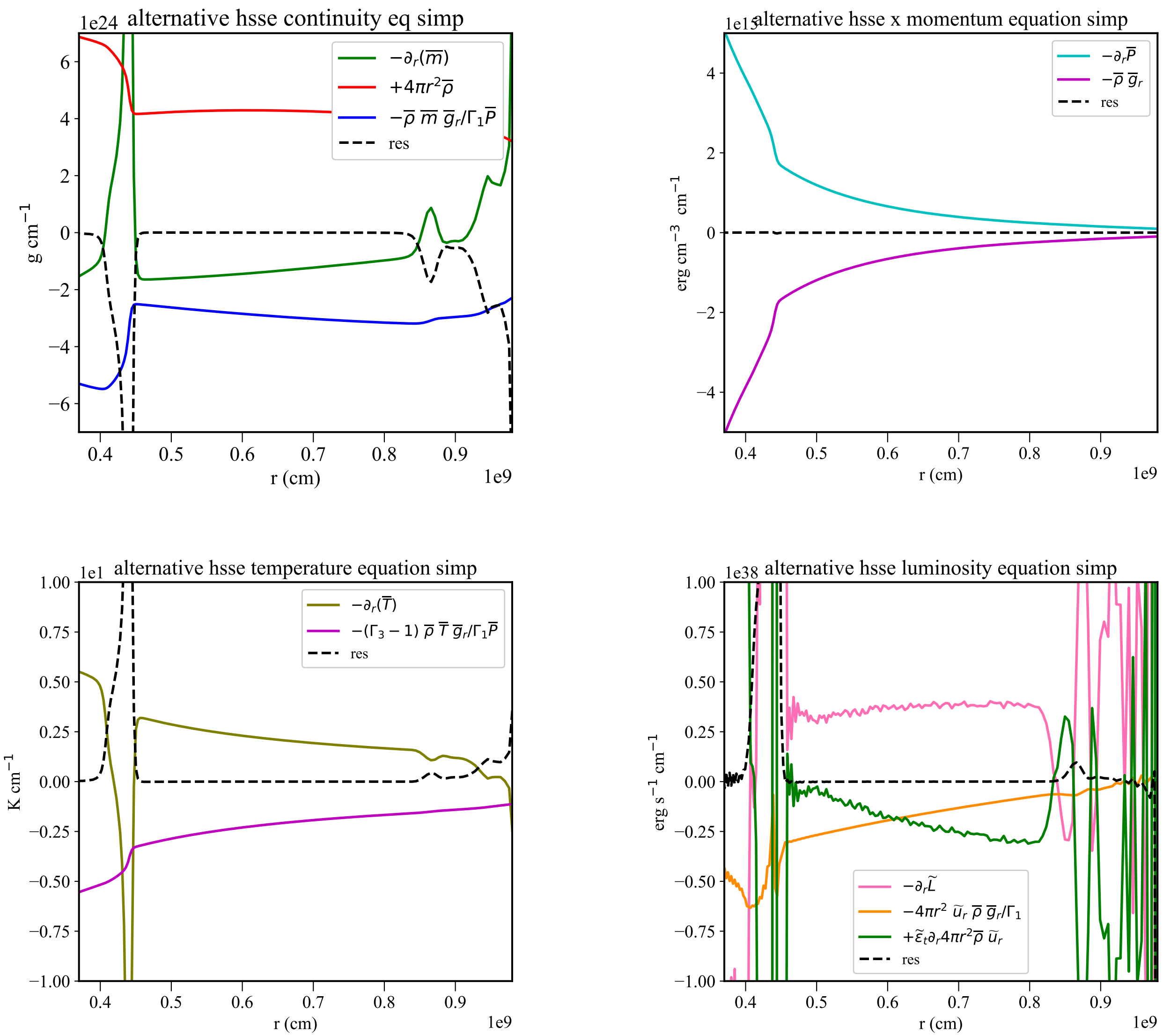

Figure 2: Simplified alternative hydrodynamic stellar structure continuity.

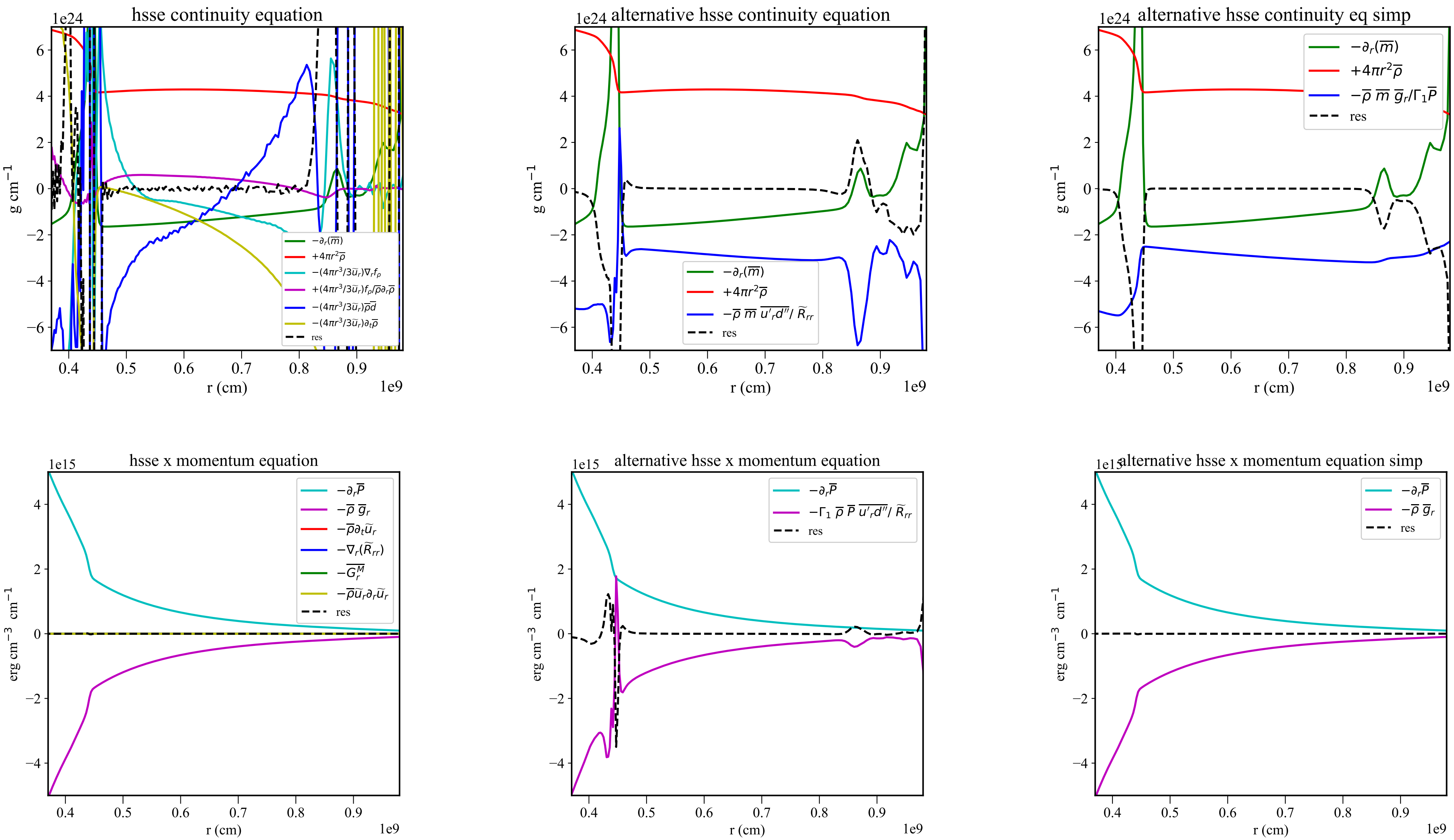

Figure 3: Comparison of all versions of continuity and momentum hydrodynamic stellar structure equations.

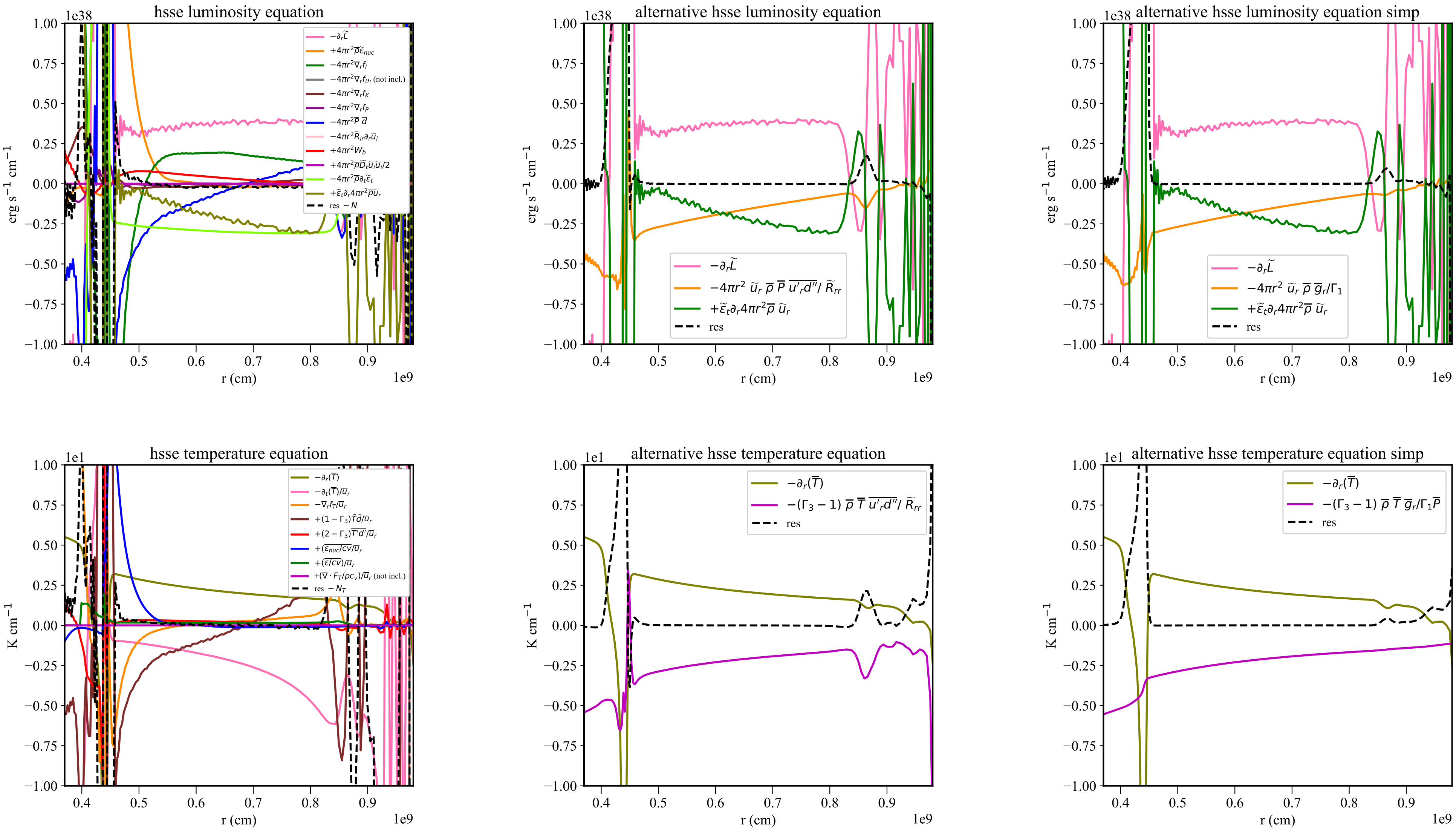

Figure 4: Comparison of all version of luminosity and temperature hydrodynamic stellar structure equations.

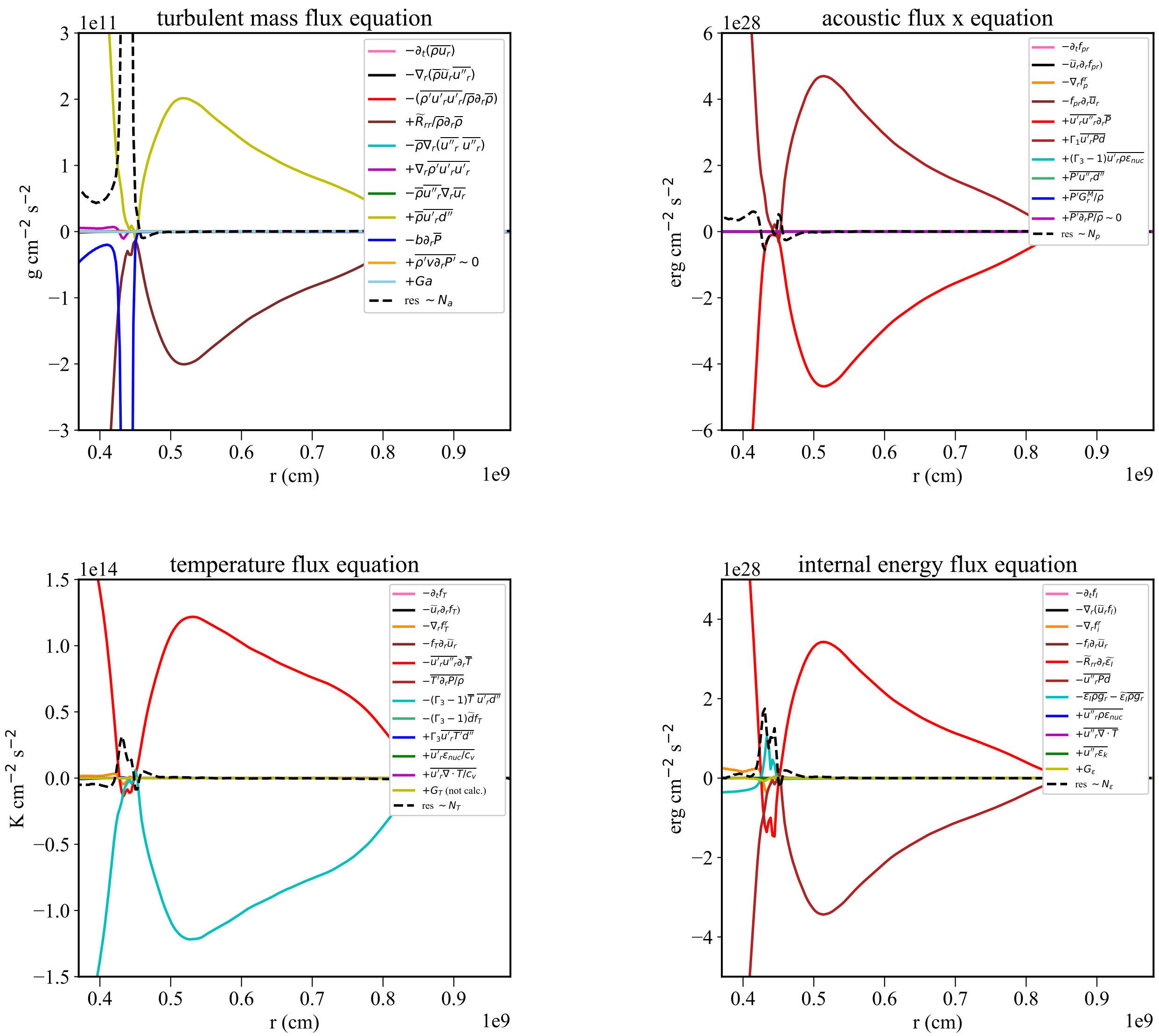

Figure 5: Flux evolution equations based on low-res oxygen burning model.

Table 1: Definitions:

ρ density

$m = \rho V = \rho \frac{4}{3} \pi r^3$ mass

T temperature

P pressure

u_r, u_θ, u_ϕ velocity components

$\mathbf{u} = u(u_r, u_\theta, u_\phi)$ velocity

$j_z = r \sin\theta \, u_\phi$ specific angular momentum

$d = \nabla \cdot \mathbf{u}$ dilatation

ϵ_I specific internal energy

h specific enthalpy

$k = (1/2)\widetilde{u''_i u''_i}$ turbulent kinetic energy

ϵ_k specific kinetic energy

ϵ_t specific total energy

s specific entropy

$v = 1/\rho$ specific volume

X_α mass fraction of isotope α

$\dot{X}_\alpha^{nuc}$ rate of change of X_α

A_α number of nucleons in isotope α

Z_α charge of isotope α

A mean number of nucleons per isotope

Z mean charge per isotope

$f_P = \overline{P' u'_r}$ acoustic flux

g_r radial gravitational acceleration

$M = \int \rho(r) dV = \int \rho(r) 4\pi r^2 dr$ integrated mass

$\mathcal{S} = \rho \epsilon_{\rm nuc}(q)$ nuclear energy production (cooling function)

$\tau_{ij} = 2\mu S_{ij}$ viscous stress tensor (μ kinematic viscosity)

$S_{ij} = (1/2)(\partial_i u_j + \partial_j u_i)$ strain rate

$\widetilde{R}_{ij} = \overline{\rho} \widetilde{u''_i u''_j}$ Reynolds stress tensor

$F_T = \chi \partial_r T$ heat flux

$\Gamma_1 = (d \; ln \; P / d \; ln \; \rho)|_s$

$\Gamma_2 / (\Gamma_2 - 1) = (d \; ln \; P / d \; ln \; T)|_s$

$\Gamma_3 - 1 = (d \; ln \; T / d \; ln \; \rho)|_s$

$\widetilde{k}^r = (1/2)\widetilde{u''_r u''_r} = (1/2)\widetilde{R}_{rr}/\overline{\rho}$ radial turbulent kinetic energy

$\widetilde{k}^\theta = (1/2)\widetilde{u''_\theta u''_\theta} = (1/2)\widetilde{R}_{\theta\theta}/\overline{\rho}$ angular turbulent kinetic energy

$\widetilde{k}^\phi = (1/2)\widetilde{u''_\phi u''_\phi} = (1/2)\widetilde{R}_{\phi\phi}/\overline{\rho}$ angular turbulent kinetic energy

$\widetilde{k}^h = \widetilde{k}^\theta + \widetilde{k}^\phi$ horizontal turbulent kinetic energy

$f_k = (1/2)\overline{\rho} \widetilde{u''_i u''_i u''_r}$ turbulent kinetic energy flux

$f_k^r = (1/2)\overline{\rho} \widetilde{u''_r u''_r u''_r}$ radial turbulent kinetic energy flux

$f_k^\theta = (1/2)\overline{\rho} \widetilde{u''_\theta u''_\theta u''_r}$ angular turbulent kinetic energy flux

$f_k^\phi = (1/2)\overline{\rho} \widetilde{u''_\phi u''_\phi u''_r}$ angular turbulent kinetic energy flux

$f_k^h = f_k^\theta + f_k^\phi$ horizontal turbulent kinetic energy flux

$W_p = \overline{P' d''}$ turbulent pressure dilatation

$W_b = \overline{\rho} \overline{u''_r} \widetilde{g}_r$ buoyancy

$f_T = -\overline{\chi \partial_r T}$ heat flux (χ thermal conductivity)

Table 2: Definitions (continued):

$f_I = \overline{\rho}\widetilde{\epsilon_I'' u_r''}$	internal energy flux	$f_\alpha = \overline{\rho}\widetilde{X_\alpha'' u_r''}$	X_α flux
$f_s = \overline{\rho}\widetilde{s'' u_r''}$	entropy flux	$f_{jz} = \overline{\rho}\widetilde{j_z'' u_r''}$	angular momentum flux
$f_T = \overline{u_r' T'}$	turbulent heat flux	$f_A = \overline{\rho}\widetilde{A'' u_r''}$	A (mean number of nucleons per isotope) flux
$f_h = \overline{\rho}\widetilde{h'' u_r''}$	enthalpy flux	$f_Z = \overline{\rho}\widetilde{Z'' u_r''}$	Z (mean charge per isotope) flux
$b = \overline{v'\rho'}$	density-specific volume covariance	$\mathcal{N}_\rho, \mathcal{N}_{ur}, \mathcal{N}_{u\theta}, \mathcal{N}_{u\phi}, \mathcal{N}_{jz}, \mathcal{N}_\alpha, \mathcal{N}_A, \mathcal{N}_Z$	numerical effect
$f_\tau = f_\tau^r + f_\tau^\theta + f_\tau^\phi$	viscous flux	$\mathcal{N}_{\epsilon I} = -\nabla_r f_\tau + \varepsilon_k$	numerical effect
$f_\tau^r = -\overline{\tau'_{rr} u'_r}$	viscous flux	$\mathcal{N}_{\epsilon k} = -\varepsilon_k$	numerical effect
$f_\tau^\theta = -\overline{\tau'_{\theta r} u'_\theta}$	viscous flux	$\mathcal{N}_{\epsilon t} = -\nabla_r f_\tau$	numerical effect
$f_\tau^\phi = -\overline{\tau'_{\phi r} u'_\phi}$	viscous flux	$\mathcal{N}_s = \overline{-\varepsilon_k / T}$	numerical effect
$f_\tau^h = f_\tau^\theta + f_\tau^\phi$	viscous flux	$\mathcal{N}_h = -\nabla_r f_\tau + (\Gamma_3 - 1)\varepsilon_k$	numerical effect
$f_I^r = \overline{\rho}\epsilon_I''\widetilde{u_r'' u_r''}$	radial flux of f_I	$\mathcal{N}_P = +(\Gamma_3 - 1)\varepsilon_k$	numerical effect
$f_s^r = \overline{\rho}s''\widetilde{u_r'' u_r''}$	radial flux of f_s	$\mathcal{N}_T = +\overline{(\tau_{ij}\partial_j u_i)/(c_v \rho)}$	numerical effect
$f_h^r = \overline{\rho}h''\widetilde{u_r'' u_r''}$	radial flux of f_h	$\mathcal{N}_{Rrr} = -2\nabla_r f_\tau^r - 2\varepsilon_k^r$	numerical effect
$f_T^r = \overline{T' u_r' u_r'}$	radial flux of f_T	$\mathcal{N}_{R\theta\theta} = -2\nabla_r f_\tau^\theta - 2\varepsilon_k^\theta$	numerical effect
$f_{jz}^r = \overline{\rho}j_z''\widetilde{u_r'' u_r''}$	radial flux of f_{jz}	$\mathcal{N}_{R\phi\phi} = -2\nabla_r f_\tau^\phi - 2\varepsilon_k^\phi$	numerical effect
$f_\alpha^r = \overline{\rho}X_\alpha''\widetilde{u_r'' u_r''}$	radial flux of f_α	$\mathcal{N}_k = -\nabla_r f_\tau - \varepsilon_k$	numerical effect
$f_A^r = \overline{\rho}A''\widetilde{u_r'' u_r''}$	radial flux of f_A	$\mathcal{N}_{kr} = -\nabla_r f_\tau^r - \varepsilon_k^r$	numerical effect
$f_Z^r = \overline{\rho}Z''\widetilde{u_r'' u_r''}$	radial flux of f_Z	$\mathcal{N}_{kh} = -\nabla_r f_\tau^h - \varepsilon_k^h$	numerical effect
$\mathcal{G}_k^r = -(1/2)\overline{G_{rr}^R} - \overline{u_r'' G_r^M}$		$\mathcal{N}_a = -\varepsilon_a$	numerical effect

Table 3: Definitions (continued):

$\mathcal{G}_k^\theta = -(1/2)\overline{G_{\theta\theta}^R} - \overline{u''_\theta G_\theta^M}$	$\mathcal{N}_b$ numerical effect
$\mathcal{G}_k^\phi = -(1/2)\overline{G_{\phi\phi}^R} - \overline{u''_\phi G_\phi^M}$	$\mathcal{N}_{fI} = -\nabla_r(\overline{\epsilon''_I \tau'_{rr}}) + \overline{u''_r \tau_{ij}\partial_i u_j} - \varepsilon_I$ numerical effect
$\mathcal{G}_k^h = +\mathcal{G}_k^\theta + \mathcal{G}_k^\phi$	$\mathcal{N}_{fh} = -\nabla_r(\overline{h''\tau'_{rr}}) + \overline{u''_r(\Gamma_3 - 1)\tau_{ij}\partial_i u_j} - \overline{u''_r \nabla_i u_i \tau_{ji}} - \varepsilon_h$ numerical effect
$\mathcal{G}_a = +\overline{\rho' v G_r^M}$	$\mathcal{N}_{fs} = -\nabla_r(\overline{s''\tau'_{rr}}) + \overline{u''_r \tau_{ij}\partial_i u_j / T} - \varepsilon_s$ numerical effect
$\mathcal{G}_I = -\overline{G_r^I} - \overline{\epsilon''_I G_r^M}$	$\mathcal{N}_{fA} = -\nabla_r(\overline{A''\tau'_{rr}}) - \varepsilon_A$ numerical effect
$\mathcal{G}_\alpha = -\overline{G_r^\alpha} - \overline{X''_\alpha G_r^M}$	$\mathcal{N}_{fZ} = -\nabla_r(\overline{Z''\tau'_{rr}}) - \varepsilon_Z$ numerical effect
$\mathcal{G}_A = -\overline{G_r^A} - \overline{A'' G_r^M}$	$\mathcal{N}_{f\alpha} = -\nabla_r(\overline{\alpha''\tau'_{rr}}) - \varepsilon_\alpha$ numerical effect
$\mathcal{G}_Z = -\overline{G_r^Z} - \overline{Z'' G_r^M}$	$\mathcal{N}_{fjz} = -\nabla_r(\overline{j''_z \tau'_{rr}}) - \varepsilon_{jz}$ numerical effect
$\mathcal{G}_h = -\overline{G_r^h} - \overline{h'' G_r^M}$	$\mathcal{N}_{fT} = +\overline{T'\partial_i \tau_{ri}/\rho} + \overline{u'_r \tau_{ij}\partial_i u_j / \rho c_v}$ numerical effect
$\mathcal{G}_T = -\overline{G_r^T} - \overline{T' G_r^M}$	
$\mathcal{G}_s = -\overline{G_r^s} - \overline{s'' G_r^M}$	
$\mathcal{G}_{jz} = -\overline{G_r^{jz}} - \overline{j''_z G_r^M}$	
$\sigma_\rho = \overline{\rho'\rho'}$	$\mathcal{N}_{\sigma_\rho}$ numerical effect
$\sigma_P = \overline{P'P'}$	$\mathcal{N}_{\sigma_P} = +2(\Gamma_3 - 1)\overline{P'\tau_{ij}\partial_i u_j}$ numerical effect
$\sigma_T = \overline{T'T'}$	$\mathcal{N}_{\sigma_T} = +\overline{2T'\tau_{ij}\partial_i u_j / \rho c_v}$ numerical effect
$\sigma_{ur} = \widetilde{u''_r u''_r}$	$\mathcal{N}_{\sigma_{ur}} = +2\nabla_r f_\tau^r - 2\varepsilon_k^r$ numerical effect
$\sigma_s = \widetilde{s''s''}$	$\mathcal{N}_{\sigma_s} = +2\overline{s''\tau_{ij}\partial_j u_i / T}$ numerical effect
$\sigma_\alpha = \widetilde{X''_\alpha X''_\alpha}$	$\mathcal{N}_{\sigma_\alpha}$ numerical effect numerical effect
$\sigma_{\epsilon I} = \widetilde{\epsilon''_I \epsilon''_I}$	$N_{\sigma\epsilon_I} = +2\overline{\epsilon''_I \tau_{ij}\partial_j u_i}$ numerical effect

Table 4: Definitions (continued):

$$
\begin{aligned}
\varepsilon_k^r &= \overline{\tau'_{rr}\partial_r u''_r} + \overline{\tau'_{r\theta}(1/r)\partial_\theta u''_r} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi u''_r} \\
\varepsilon_k^\theta &= \overline{\tau'_{\theta r}\partial_r u''_\theta} + \overline{\tau'_{\theta\theta}(1/r)\partial_\theta u''_\theta} + \overline{\tau'_{\theta\phi}(1/r\sin\theta)\partial_\phi u''_\theta} \\
\varepsilon_k^\phi &= \overline{\tau'_{\phi r}\partial_r u''_\phi} + \overline{\tau'_{\phi\theta}(1/r)\partial_\theta u''_\phi} + \overline{\tau'_{\phi\phi}(1/r\sin\theta)\partial_\phi u''_\phi} \\
\varepsilon_k &= \varepsilon_k^r + \varepsilon_k^\theta + \varepsilon_k^\phi \\
\varepsilon_k^h &= \varepsilon_k^\theta + \varepsilon_k^\phi \\
\varepsilon_a &= \overline{\rho' v \nabla_r \tau'_{rr}} \\
\varepsilon_I &= \overline{\tau'_{rr}\partial_r \epsilon''_I} + \overline{\tau'_{r\theta}(1/r)\partial_\theta \epsilon''_I} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi \epsilon''_I} \\
\varepsilon_s &= \overline{\tau'_{rr}\partial_r s''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta s''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi s''} \\
\varepsilon_\alpha &= \overline{\tau'_{rr}\partial_r X''_\alpha} + \overline{\tau'_{r\theta}(1/r)\partial_\theta X''_\alpha} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi X''_\alpha} \\
\varepsilon_A &= \overline{\tau'_{rr}\partial_r A''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta A''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi A''} \\
\varepsilon_Z &= \overline{\tau'_{rr}\partial_r Z''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta Z''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi Z''} \\
\varepsilon_h &= \overline{\tau'_{rr}\partial_r h''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta h''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi h''} \\
\varepsilon_{jz} &= \overline{\tau'_{rr}\partial_r j''_z} + \overline{\tau'_{r\theta}(1/r)\partial_\theta j''_z} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi j''_z}
\end{aligned}
$$

$$
\begin{aligned}
\overline{G_r^M} &= -\overline{\rho u_\theta u_\theta / r} - \overline{\rho u_\phi u_\phi / r} \\
\overline{G_\theta^M} &= +\overline{\rho u_\theta u_r / r} - \overline{\rho u_\phi u_\phi / (r\tan\theta)} \\
\overline{G_\phi^M} &= +\overline{\rho u_\phi u_r / r} + \overline{\rho u_\phi u_\theta / (r\tan\theta)} \\
\overline{G_{rr}^R} &= -\overline{\rho u''_\theta u''_\theta u''_r / r} - \overline{\rho u''_\theta u''_r u''_\theta / r} - \overline{\rho u''_\phi u''_\phi u''_r / r} - \overline{\rho u''_\phi u''_r u''_\phi / r} \\
\overline{G_{\theta\theta}^R} &= +\overline{\rho u''_\theta u''_r u''_\theta / r} + \overline{\rho u''_\theta u''_\theta u''_r / r} - \overline{\rho u''_\phi u''_\phi u''_\theta / (r\tan\theta)} - \overline{u''_\phi u''_\theta u''_\phi / (r\tan\theta)} \\
\overline{G_{\phi\phi}^R} &= +\overline{\rho u''_\phi u''_r r_\phi / r} + \overline{\rho u''_\phi u''_\theta u''_\phi / (r\tan\theta)} + \overline{\rho u''_\phi u''_\phi u''_r / r} + \overline{\rho u''_\phi u''_\phi u''_\theta / (r\tan\theta)} \\
\overline{G_r^I} &= -\overline{\rho \epsilon''_I u''_\theta u''_\theta / r} - \overline{\rho \epsilon''_I u''_\phi u''_\phi / r} \\
\overline{G_r^s} &= -\overline{\rho s'' u''_\theta u''_\theta / r} - \overline{\rho s'' u''_\phi u''_\phi / r} \\
\overline{G_r^\alpha} &= -\overline{\rho X''_\alpha u''_\theta u''_\theta / r} - \overline{\rho X''_\alpha u''_\phi u''_\phi / r} \\
\overline{G_r^A} &= -\overline{\rho A'' u''_\theta u''_\theta / r} - \overline{\rho A'' u''_\phi u''_\phi / r} \\
\overline{G_r^Z} &= -\overline{\rho Z'' u''_\theta u''_\theta / r} - \overline{\rho Z'' u''_\phi u''_\phi / r} \\
\overline{G_r^h} &= -\overline{\rho h'' u''_\theta u''_\theta / r} - \overline{\rho h'' u''_\phi u''_\phi / r} \\
\overline{G_r^T} &= -\overline{\rho T' u'_\theta u'_\theta / r} - \overline{\rho T' u'_\phi u'_\phi / r} \\
\overline{G_r^{jz}} &= -\overline{\rho j''_z u''_\theta u''_\theta / r} - \overline{\rho j''_z u''_\phi u''_\phi / r}
\end{aligned}
$$

$$
\nabla(.) = \nabla_r(.) + \nabla_\theta(.) + \nabla_\phi(.) = \frac{1}{r^2}\partial_r(r^2 .) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta .) + \frac{1}{r\sin\theta}\partial_\phi(.)
$$