Tensor calculus in spherical geometry

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BACKGROUND READING:

CONTINUUM MECHANICS (Lecture Notes)
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1 Geometry and scale factors

$$x_1 = r \qquad x_2 = \theta \qquad x_3 = \phi \qquad \text{(coordinates)} \tag{1}$$

$$\mathbf{e_1} = \mathbf{e_r} \qquad \mathbf{e_2} = \mathbf{e}_\theta \qquad \mathbf{e_3} = \mathbf{e}_\phi \qquad \text{(unit base vectors)} \tag{2}$$

$$h_1 = h_r = 1 \qquad h_2 = h_\theta = r \qquad h_3 = h_\phi = r\sin\theta \qquad \text{(scale factors)} \tag{3}$$

2 Christoffel symbols

$$\begin{pmatrix} \Gamma^\theta_{r\theta} = \ 1 & \Gamma^\phi_{r\phi} = \ \sin\theta & \Gamma^\phi_{\theta\phi} = \ \cos\theta \\ \Gamma^r_{\theta\theta} = -1 & \Gamma^r_{\phi\phi} = -\sin\theta & \Gamma^\theta_{\phi\phi} = -\cos\theta \end{pmatrix} \tag{4}$$

3 div V:= ∇.V where $\mathbf{V} = \sum_i V_i \mathbf{e_i}$ is a vector

$$\nabla.\mathbf{V} = \frac{1}{r^2}\frac{\partial(r^2 V_r)}{\partial_r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi} \tag{5}$$

4 div S:= ∇. S where $\mathbf{S} = \sum_{ij} S_{ij}(\mathbf{e_i} \otimes \mathbf{e_i})$ is second order tensor

$$S_r(\mathbf{e_r}) : \qquad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 S_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ S_{\theta r}) + \frac{1}{r\sin\theta}\frac{\partial S_{\phi r}}{\partial\phi} - \frac{S_{\theta\theta}}{r} - \frac{S_{\phi\phi}}{r} \tag{6}$$

$$S_\theta(\mathbf{e}_\theta) : \qquad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 S_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ S_{\theta\theta}) + \frac{1}{r\sin\theta}\frac{\partial S_{\phi\theta}}{\partial\phi} + \frac{S_{\theta r}}{r} - \frac{S_{\phi\phi}\cos\theta}{r\sin\theta} \tag{7}$$

$$S_\phi(\mathbf{e}_\phi) : \qquad \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 S_{r\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ S_{\theta\phi}) + \frac{1}{r\sin\theta}\frac{\partial S_{\phi\phi}}{\partial\phi} + \frac{S_{\phi r}}{r} + \frac{S_{\phi\theta}\cos\theta}{r\sin\theta} \tag{8}$$

5 div T := ∇. T where $\mathbf{T} = \sum_{ijk} T_{ijk}(\mathbf{e_i} \otimes \mathbf{e_j} \otimes \mathbf{e_k})$ is third order tensor

$$T_{rr} : \tag{9}$$
$$T_{r\theta} : \tag{10}$$
$$T_{r\phi} : \tag{11}$$
$$T_{\theta r} : \tag{12}$$
$$T_{\theta\theta} : \tag{13}$$
$$T_{\theta\phi} : \tag{14}$$
$$T_{\phi r} : \tag{15}$$
$$T_{\phi\theta} : \tag{16}$$
$$T_{\phi\phi} : \tag{17}$$

6 Random notes and details

$$y_l = (y_1, y_2, y_3) = \ (x, y, z) \qquad \text{(cartesian coordinates)} \tag{18}$$
$$x_k = (x_1, x_2, x_3) = \ (r, \theta, \phi) \qquad \text{(spherical coordinates)} \tag{19}$$

$$\vec{p} = \sum_k y_k \vec{i}_k \qquad \text{(vector in cartesian coordinates)} \tag{20}$$

6.1 Scale factors

$$h_k = \sqrt{\frac{\partial \vec{p}}{\partial x_k} . \frac{\partial \vec{p}}{\partial x_k}} \tag{21}$$

Scale factors are introduced to satisfy orthonormality condition for directional cosines.

6.2 Christoffel symbols

$$\Gamma^m_{kl} = \frac{1}{h_k} \frac{\partial h_l}{\partial x_k} \delta_{lm} - \frac{1}{h_m} \frac{\partial h_k}{\partial x_m} \delta_{kl} \tag{22}$$

Unit base vectors $\vec{e_k}$ are function of position and vary in direction as the curvilinear coordinates vary. Hence:

$$\frac{\partial \vec{e_k}}{\partial y_l} = \sum_{m=1}^{3} \Gamma^m_{kl} \ \vec{e_m} \tag{23}$$

6.3 Dyadic product

$$\mathbf{a} = (a_1, a_2, a_3) \tag{24}$$
$$\mathbf{b} = (b_1, b_2, b_3) \tag{25}$$

$$\mathbf{a} \otimes \mathbf{b} = \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{pmatrix} \tag{26}$$

6.4 Definition of a vector V

$$\mathbf{V} = \sum_k V_k \mathbf{e_k} \tag{27}$$

6.5 Derivatives of a vector V

$$V_{k;l} = \frac{\partial V_k}{\partial x_l} + \sum_m \Gamma^k_{ml} V_m \tag{28}$$

$$\begin{pmatrix} & dr & d\theta & d\phi \\ & & & \\ dV_r & \dfrac{\partial V_r}{\partial r} & \left(\dfrac{\partial V_r}{\partial \theta} - V_\theta\right)\dfrac{1}{r} & \left(\dfrac{1}{\sin\theta}\dfrac{\partial V_r}{\partial \phi} - V_\phi\right)\dfrac{1}{r} \\ dV_\theta & \dfrac{\partial V_\theta}{\partial r} & \left(\dfrac{\partial V_\theta}{\partial \theta} + V_r\right)\dfrac{1}{r} & \left(\dfrac{1}{\sin\theta}\dfrac{\partial V_\theta}{\partial \phi} - \dfrac{V_\phi}{\tan\theta}\right)\dfrac{1}{r} \\ dV_\phi & \dfrac{\partial V_\phi}{\partial r} & \dfrac{\partial V_\phi}{\partial \theta}\dfrac{1}{r} & \left(\dfrac{\partial V_\phi}{\partial \phi} + V_r\sin\theta + V_\theta\cos\theta\right)\dfrac{1}{r\sin\theta} \end{pmatrix}$$

6.6 Divergence of a vector V

$$\nabla.\mathbf{V} = \sum_k \frac{\mathbf{e_k}}{h_k}\frac{\partial \mathbf{V}}{\partial x_k} = \sum_k \frac{\mathbf{e_k}}{\partial h_k}\sum_l V_{l;k}\mathbf{e_l} = \sum_k \frac{V_{k;k}}{h_k} = \sum_k \frac{1}{h_k}\left[\frac{\partial V_k}{\partial x_k} + \sum_m \Gamma^k_{mk} V_m\right] \tag{29}$$

6.7 Definition of second order tensor S

$$\mathbf{S} = \sum_{kl} S_{kl}(\mathbf{e_k}\otimes\mathbf{e_l}) = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \tag{30}$$

$$\sum_{kl} S_{kl}(\mathbf{e_k}\otimes\mathbf{e_l}) = \tag{31}$$

$$S_{11}(\mathbf{e_1}\otimes\mathbf{e_1}) + S_{12}(\mathbf{e_1}\otimes\mathbf{e_2}) + S_{13}(\mathbf{e_1}\otimes\mathbf{e_3}) + \tag{32}$$

$$S_{21}(\mathbf{e_2}\otimes\mathbf{e_1}) + S_{22}(\mathbf{e_2}\otimes\mathbf{e_2}) + S_{23}(\mathbf{e_2}\otimes\mathbf{e_3}) + \tag{33}$$

$$S_{31}(\mathbf{e_3}\otimes\mathbf{e_1}) + S_{32}(\mathbf{e_3}\otimes\mathbf{e_2}) + S_{33}(\mathbf{e_3}\otimes\mathbf{e_3}) \tag{34}$$

$$\sum_{kl} S_{kl}(\mathbf{e_k}\otimes\mathbf{e_l}) = S_{11}\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix} + S_{12}\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix} + S_{13}\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix} + \tag{35}$$

$$S_{21}\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix} + S_{22}\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix} + S_{23}\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix} + \tag{36}$$

$$S_{31}\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix} + S_{32}\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix} + S_{33}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix} \tag{37}$$

$$\sum_m \frac{\mathbf{e_m}}{h_m}.\sum_{kl} S_{kl;m}(\mathbf{e_k}\otimes\mathbf{e_l}) = \sum_m \frac{\mathbf{e_m}}{h_m}.\begin{pmatrix} S_{11;m} & S_{12;m} & S_{13;m} \\ S_{21;m} & S_{22;m} & S_{23;m} \\ S_{31;m} & S_{32;m} & S_{33;m} \end{pmatrix} = \tag{38}$$

$$= \frac{\mathbf{e_1}}{h_1}\begin{pmatrix} S_{11;1} & S_{12;1} & S_{13;1} \\ S_{21,1} & S_{22;1} & S_{23;1} \\ S_{31;1} & S_{32;1} & S_{33;1} \end{pmatrix} + \frac{\mathbf{e_2}}{h_2}\begin{pmatrix} S_{11;2} & S_{12;2} & S_{13;2} \\ S_{21,2} & S_{22;2} & S_{23;2} \\ S_{31;2} & S_{32;2} & S_{33;2} \end{pmatrix} + \frac{\mathbf{e_3}}{h_3}\begin{pmatrix} S_{11;3} & S_{12;3} & S_{13;3} \\ S_{21,3} & S_{22;3} & S_{23;3} \\ S_{31;3} & S_{32;3} & S_{33;3} \end{pmatrix} = \tag{39}$$

$$= \frac{(1\ 0\ 0)}{h_1}\begin{pmatrix} S_{11;1} & S_{12;1} & S_{13;1} \\ S_{21,1} & S_{22;1} & S_{23;1} \\ S_{31;1} & S_{32;1} & S_{33;1} \end{pmatrix} + \frac{(0\ 1\ 0)}{h_2}\begin{pmatrix} S_{11;2} & S_{12;2} & S_{13;2} \\ S_{21,2} & S_{22;2} & S_{23;2} \\ S_{31;2} & S_{32;2} & S_{33;2} \end{pmatrix} + \frac{(0\ 0\ 1)}{h_3}\begin{pmatrix} S_{11;3} & S_{12;3} & S_{13;3} \\ S_{21,3} & S_{22;3} & S_{23;3} \\ S_{31;3} & S_{32;3} & S_{33;3} \end{pmatrix} = \tag{40}$$

$$= \left(\frac{S_{11;1}}{h_1}\ \frac{S_{12;1}}{h_1}\ \frac{S_{13;1}}{h_1}\right) + \left(\frac{S_{21;2}}{h_2}\ \frac{S_{22;2}}{h_2}\ \frac{S_{23;2}}{h_2}\right) + \left(\frac{S_{31;3}}{h_3}\ \frac{S_{32;1}}{h_3}\ \frac{S_{33;3}}{h_3}\right) = \tag{41}$$

$$= \sum_{kl} \frac{S_{kl;k}}{h_k}\mathbf{e_l} \tag{42}$$

6.8 Derivatives of second order tensor S

$$\frac{\partial \mathbf{S}}{\partial x_m} = \sum_{kl} T_{kl;m}(\mathbf{e_k}\otimes\mathbf{e_l}) \tag{43}$$

$$S_{kl;m} = \frac{\partial S_{kl}}{\partial x_m} + \sum_n \Gamma^k_{nm} S_{nl} + \sum_n \Gamma^l_{nm} S_{kn} \tag{44}$$

$$S_{kl;k} = \frac{\partial S_{kl}}{\partial x_k} + \sum_m \Gamma^k_{mk} S_{ml} + \sum_m \Gamma^l_{mk} S_{km} \tag{45}$$

6.9 Divergence of second order tensor S

$$\text{div } \mathbf{S} = \nabla.\mathbf{S} = \sum_m \frac{\mathbf{e_m}}{h_m}\frac{\partial \mathbf{S}}{\partial x_m} = \sum_m \frac{\mathbf{e_m}}{h_m}\sum_{kl} S_{kl;m}(\mathbf{e_k}\otimes\mathbf{e_l}) = \sum_{klm}\frac{S_{kl;m}}{h_m}\mathbf{e_m}(\mathbf{e_k}\otimes\mathbf{e_l}) = \tag{46}$$

$$= \sum_{klm}\frac{S_{kl;m}}{h_m}(\mathbf{e_m}\mathbf{e_k})\mathbf{e_l} = \sum_{klm}\frac{S_{kl;m}}{h_m}\delta_{mk}\mathbf{e_l} = \sum_{kl}\frac{S_{kl;k}}{h_k}\mathbf{e_l} = \tag{47}$$

$$= \sum_{kl}\frac{1}{h_k}\left[\frac{\partial S_{kl}}{\partial x_k} + \sum_m \Gamma^k_{mk} S_{ml} + \sum_m \Gamma^l_{mk} S_{km}\right]\mathbf{e_l} \tag{48}$$

6.10 Definition of third order tensor T

$$\mathbf{T} = \sum_{jkl} T_{jkl}(\mathbf{e_j}\otimes\mathbf{e_k}\otimes\mathbf{e_l}) \tag{49}$$

6.11 Derivatives of third order tensor T

$$\frac{\partial \mathbf{T}}{\partial x_m} = \sum_{jkl} T_{jkl;m}(\mathbf{e_j} \otimes \mathbf{e_k} \otimes \mathbf{e_l}) \tag{50}$$

$$T_{jkl;m} = \frac{\partial T_{jkl}}{\partial x_m} + \sum_n \Gamma^j_{nm} T_{nkl} + \sum_n \Gamma^k_{nm} T_{jnl} + \sum_n \Gamma^l_{nm} T_{jkn} \tag{51}$$

$$T_{jkl;j} = \frac{\partial T_{jkl}}{\partial x_j} + \sum_n \Gamma^j_{nj} T_{nkl} + \sum_n \Gamma^k_{nj} T_{jnl} + \sum_n \Gamma^l_{nj} T_{jkn} \tag{52}$$

6.12 Divergence of third order tensor T

$$\nabla.\mathbf{T} = \sum_m \frac{\mathbf{e_m}}{h_m} \frac{\partial T}{\partial x_m} = \sum_m \frac{\mathbf{e_m}}{h_m} \sum_{jkl} T_{jkl;m}(\mathbf{e_j} \otimes \mathbf{e_k} \otimes \mathbf{e_l}) = \sum_{jklm} \frac{T_{jkl;m}}{h_m} \mathbf{e_m}(\mathbf{e_j} \otimes \mathbf{e_k} \otimes \mathbf{e_l}) = \tag{53}$$

$$= \sum_{jklm} \frac{T_{jkl;m}}{h_m} (\mathbf{e_m}\mathbf{e_j})\mathbf{e_k} \otimes \mathbf{e_l} = \sum_{jkl} \frac{T_{jkl;j}}{h_j} (\mathbf{e_k} \otimes \mathbf{e_l}) = \tag{54}$$

$$= \sum_{jkl} \frac{1}{h_j} \left[\frac{\partial T_{jkl}}{\partial x_j} + \sum_n \Gamma^j_{nj} T_{nkl} + \sum_n \Gamma^k_{nj} T_{jnl} + \sum_n \Gamma^l_{nj} T_{jkn} \right] (\mathbf{e_k} \otimes \mathbf{e_l}) \tag{55}$$

6.13 Summary (indices changed to i,j,k)

$$\nabla(.) = \sum_n \frac{\mathbf{e_n}}{h_n} \frac{\partial (.)}{\partial x_n} \tag{56}$$

$$\mathbf{V} = \sum_i V_i \mathbf{e_i} \tag{57}$$

$$\mathbf{S} = \sum_{ij} S_{ij}(\mathbf{e_i} \otimes \mathbf{e_j}) \tag{58}$$

$$\mathbf{T} = \sum_{ijk} T_{ijk}(\mathbf{e_i} \otimes \mathbf{e_j} \otimes \mathbf{e_k}) \tag{59}$$

$$\nabla.\mathbf{V} = \sum_i \frac{1}{h_i} \left[\frac{\partial V_i}{\partial x_i} + \sum_m \Gamma^i_{mi} V_m \right] \tag{60}$$

$$\nabla.\mathbf{S} = \sum_{ij} \frac{1}{h_i} \left[\frac{\partial S_{ij}}{\partial x_i} + \sum_m \Gamma^i_{mi} S_{mj} + \sum_m \Gamma^j_{mi} S_{im} \right] \mathbf{e_j} \tag{61}$$

$$\nabla.\mathbf{T} = \sum_{ijk} \frac{1}{h_i} \left[\frac{\partial T_{ijk}}{\partial x_i} + \sum_m \Gamma^i_{mi} T_{mjk} + \sum_m \Gamma^j_{mi} T_{imk} + \sum_m \Gamma^k_{mi} T_{ijm} \right] (\mathbf{e_j} \otimes \mathbf{e_k}) \tag{62}$$

$$\nabla.\mathbf{V} = \sum_i \frac{1}{h_i}\left[\frac{\partial V_i}{\partial x_i} + \sum_m \Gamma^i_{mi} V_m\right] = \tag{63}$$

$$= \frac{1}{h_r}\left[\frac{\partial V_r}{\partial r} + \sum_m \Gamma^r_{mr} V_m\right] + \frac{1}{h_\theta}\left[\frac{\partial V_\theta}{\partial \theta} + \sum_m \Gamma^\theta_{m\theta} V_m\right] + \frac{1}{h_\phi}\left[\frac{\partial V_\phi}{\partial \phi} + \sum_m \Gamma^\phi_{m\phi} V_m\right] = \tag{64}$$

$$= \frac{1}{h_r}\left[\frac{\partial V_r}{\partial r} + \Gamma^r_{rr} V_r + \Gamma^r_{\theta r} V_\theta + \Gamma^r_{\phi r} V_\phi\right] + \tag{65}$$

$$= \frac{1}{h_\theta}\left[\frac{\partial V_\theta}{\partial \theta} + \Gamma^\theta_{r\theta} V_r + \Gamma^\theta_{\theta\theta} V_\theta + \Gamma^\theta_{\phi\theta} V_\phi\right] + \tag{66}$$

$$= \frac{1}{h_\phi}\left[\frac{\partial V_\phi}{\partial \phi} + \Gamma^\phi_{r\phi} V_r + \Gamma^\phi_{\theta\phi} V_\theta + \Gamma^\phi_{\phi\phi} V_\phi\right] = \tag{67}$$

$$= \frac{1}{1}\left[\frac{\partial V_r}{\partial r} + 0 + 0 + 0\right] + \frac{1}{r}\left[\frac{\partial V_\theta}{\partial \theta} + V_r + 0 + 0\right] + \frac{1}{r\sin\theta}\left[\frac{\partial V_\phi}{\partial \phi} + \sin\theta V_r + \cos\theta V_\theta + 0\right] = \tag{68}$$

$$= \left[\frac{\partial V_r}{\partial r} + \frac{2V_r}{r}\right] + \left[\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\cos\theta V_\theta}{r\sin\theta}\right] + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi} = \tag{69}$$

$$= \frac{1}{r^2}\frac{\partial (r^2 V_r)}{\partial_r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi} \tag{70}$$

$$\nabla.\mathbf{S} = \sum_{ij} \frac{1}{h_i}\left[\frac{\partial S_{ij}}{\partial x_i} + \sum_m \Gamma^i_{mi} S_{mj} + \sum_m \Gamma^j_{mi} S_{im}\right] \mathbf{e_j} \tag{71}$$

$$= \sum_j \frac{1}{h_r}\left[\frac{\partial S_{rj}}{\partial r} + \sum_m \Gamma^r_{mr} S_{mj} + \sum_m \Gamma^j_{mr} S_{rm}\right] \mathbf{e_j} + \tag{72}$$

$$+ \sum_j \frac{1}{h_\theta}\left[\frac{\partial S_{\theta j}}{\partial \theta} + \sum_m \Gamma^\theta_{m\theta} S_{mj} + \sum_m \Gamma^j_{m\theta} S_{\theta m}\right] \mathbf{e_j} + \tag{73}$$

$$+ \sum_j \frac{1}{h_\phi}\left[\frac{\partial S_{\phi j}}{\partial \phi} + \sum_m \Gamma^\phi_{m\phi} S_{mj} + \sum_m \Gamma^j_{m\phi} S_{\phi m}\right] \mathbf{e_j} = \tag{74}$$

$$= \frac{1}{h_r}\left[\frac{\partial S_{rr}}{\partial r} + \sum_m \Gamma^r_{mr} S_{mr} + \sum_m \Gamma^r_{mr} S_{rm}\right] \mathbf{e_r} + \tag{75}$$

$$+ \frac{1}{h_r}\left[\frac{\partial S_{r\theta}}{\partial r} + \sum_m \Gamma^r_{mr} S_{m\theta} + \sum_m \Gamma^\theta_{mr} S_{rm}\right] \mathbf{e}_\theta + \tag{76}$$

$$+ \frac{1}{h_r}\left[\frac{\partial S_{r\phi}}{\partial r} + \sum_m \Gamma^r_{mr} S_{m\phi} + \sum_m \Gamma^\phi_{mr} S_{rm}\right] \mathbf{e}_\phi + \tag{77}$$

$$+ \frac{1}{h_\theta}\left[\frac{\partial S_{\theta r}}{\partial \theta} + \sum_m \Gamma^\theta_{m\theta} S_{mr} + \sum_m \Gamma^r_{m\theta} S_{\theta m}\right] \mathbf{e_r} + \tag{78}$$

$$+ \frac{1}{h_\theta}\left[\frac{\partial S_{\theta\theta}}{\partial \theta} + \sum_m \Gamma^\theta_{m\theta} S_{m\theta} + \sum_m \Gamma^\theta_{m\theta} S_{\theta m}\right] \mathbf{e}_\theta + \tag{79}$$

$$+ \frac{1}{h_\theta}\left[\frac{\partial S_{\theta\phi}}{\partial \theta} + \sum_m \Gamma^\theta_{m\theta} S_{m\phi} + \sum_m \Gamma^\phi_{m\theta} S_{\theta m}\right] \mathbf{e}_\phi + \tag{80}$$

$$+\frac{1}{h_\phi}\left[\frac{\partial S_{\phi r}}{\partial \phi}+\sum_m \Gamma^{\phi}_{m\phi}S_{mr}+\sum_m \Gamma^{r}_{m\phi}S_{\phi m}\right]\mathbf{e_r}+ \tag{81}$$

$$+\frac{1}{h_\phi}\left[\frac{\partial S_{\phi \theta}}{\partial \phi}+\sum_m \Gamma^{\phi}_{m\phi}S_{m\theta}+\sum_m \Gamma^{\theta}_{m\phi}S_{\phi m}\right]\mathbf{e}_\theta+ \tag{82}$$

$$+\frac{1}{h_\phi}\left[\frac{\partial S_{\phi \phi}}{\partial \phi}+\sum_m \Gamma^{\phi}_{m\phi}S_{m\phi}+\sum_m \Gamma^{\phi}_{m\phi}S_{\phi m}\right]\mathbf{e}_\phi= \tag{83}$$

$$=\frac{1}{h_r}\left\{\left[\frac{\partial S_{rr}}{\partial r}+\left(\Gamma^r_{rr}S_{rr}+\Gamma^r_{\theta r}S_{\theta r}+\Gamma^r_{\phi r}S_{\phi r}\right)+\left(\Gamma^r_{rr}S_{rr}+\Gamma^r_{\theta r}S_{r\theta}+\Gamma^r_{\phi r}S_{r\phi}\right)\right]\right\}\mathbf{e_r}+ \tag{84}$$

$$+\frac{1}{h_r}\left\{\left[\frac{\partial S_{r\theta}}{\partial r}+\left(\Gamma^r_{rr}S_{r\theta}+\Gamma^r_{\theta r}S_{\theta\theta}+\Gamma^r_{\phi r}S_{\phi\theta}\right)+\left(\Gamma^\theta_{rr}S_{rr}+\Gamma^\theta_{\theta r}S_{r\theta}+\Gamma^\theta_{\phi r}S_{r\phi}\right)\right]\right\}\mathbf{e}_\theta+ \tag{85}$$

$$+\frac{1}{h_r}\left\{\left[\frac{\partial S_{r\phi}}{\partial r}+\left(\Gamma^r_{rr}S_{r\phi}+\Gamma^r_{\theta r}S_{\theta\phi}+\Gamma^r_{\phi r}S_{\phi\phi}\right)+\left(\Gamma^\phi_{rr}S_{rr}+\Gamma^\phi_{\theta r}S_{r\theta}+\Gamma^\phi_{\phi r}S_{r\phi}\right)\right]\right\}\mathbf{e}_\phi+ \tag{86}$$

$$+\frac{1}{h_\theta}\left\{\left[\frac{\partial S_{\theta r}}{\partial \theta}+\left(\Gamma^\theta_{r\theta}S_{rr}+\Gamma^\theta_{\theta\theta}S_{\theta r}+\Gamma^\theta_{\phi\theta}S_{\phi r}\right)+\left(\Gamma^r_{r\theta}S_{\theta r}+\Gamma^r_{\theta\theta}S_{\theta\theta}+\Gamma^r_{\phi\theta}S_{\theta\phi}\right)\right]\right\}\mathbf{e_r}+ \tag{87}$$

$$+\frac{1}{h_\theta}\left\{\left[\frac{\partial S_{\theta\theta}}{\partial \theta}+\left(\Gamma^\theta_{r\theta}S_{r\theta}+\Gamma^\theta_{\theta\theta}S_{\theta\theta}+\Gamma^\theta_{\phi\theta}S_{\phi\theta}\right)+\left(\Gamma^\theta_{r\theta}S_{\theta r}+\Gamma^\theta_{\theta\theta}S_{\theta\theta}+\Gamma^\theta_{\phi\theta}S_{\theta\phi}\right)\right]\right\}\mathbf{e}_\theta+ \tag{88}$$

$$+\frac{1}{h_\theta}\left\{\left[\frac{\partial S_{\theta\phi}}{\partial \theta}+\left(\Gamma^\theta_{r\theta}S_{r\phi}+\Gamma^\theta_{\theta\theta}S_{\theta\phi}+\Gamma^\theta_{\phi\theta}S_{\phi\phi}\right)+\left(\Gamma^\phi_{r\theta}S_{\theta r}+\Gamma^\phi_{\theta\theta}S_{\theta\theta}+\Gamma^\phi_{\phi\theta}S_{\theta\phi}\right)\right]\right\}\mathbf{e}_\phi+ \tag{89}$$

$$+\frac{1}{h_\phi}\left\{\left[\frac{\partial S_{\phi r}}{\partial \phi}+\left(\Gamma^\phi_{r\phi}S_{rr}+\Gamma^\phi_{\theta\phi}S_{\theta r}+\Gamma^\phi_{\phi\phi}S_{\phi r}\right)+\left(\Gamma^r_{r\phi}S_{\phi r}+\Gamma^r_{\theta\phi}S_{\phi\theta}+\Gamma^r_{\phi\phi}S_{\phi\phi}\right)\right]\right\}\mathbf{e_r}+ \tag{90}$$

$$+\frac{1}{h_\phi}\left\{\left[\frac{\partial S_{\phi\theta}}{\partial \phi}+\left(\Gamma^\phi_{r\phi}S_{r\theta}+\Gamma^\phi_{\theta\phi}S_{\theta\theta}+\Gamma^\phi_{\phi\phi}S_{\phi\theta}\right)+\left(\Gamma^\theta_{r\phi}S_{\phi r}+\Gamma^\theta_{\theta\phi}S_{\phi\theta}+\Gamma^\theta_{\phi\phi}S_{\phi\phi}\right)\right]\right\}\mathbf{e}_\theta+ \tag{91}$$

$$+\frac{1}{h_\phi}\left\{\left[\frac{\partial S_{\phi\phi}}{\partial \phi}+\left(\Gamma^\phi_{r\phi}S_{r\phi}+\Gamma^\phi_{\theta\phi}S_{\theta\phi}+\Gamma^\phi_{\phi\phi}S_{\phi\phi}\right)+\left(\Gamma^\phi_{r\phi}S_{\phi r}+\Gamma^\phi_{\theta\phi}S_{\phi\theta}+\Gamma^\phi_{\phi\phi}S_{\phi\phi}\right)\right]\right\}\mathbf{e}_\phi= \tag{92}$$

$$=\frac{\partial S_{rr}}{\partial r}\mathbf{e_r}+\left[\frac{1}{r}\left(\frac{\partial S_{\theta r}}{\partial\theta}+S_{rr}+S_{\theta\theta}\right)\right]\mathbf{e_r}+\left[\frac{1}{r\sin\theta}\left(\frac{\partial S_{\phi r}}{\partial\phi}+\sin\theta S_{rr}+\cos\theta S_{\theta r}-\sin\theta S_{\phi\phi}\right)\right]\mathbf{e_r}+ \tag{93}$$

$$+\frac{\partial S_{r\theta}}{\partial r}\mathbf{e}_\theta+\left[\frac{1}{r}\left(\frac{\partial S_{\theta\theta}}{\partial\theta}+S_{r\theta}+S_{\theta r}\right)\right]\mathbf{e}_\theta+\left[\frac{1}{r\sin\theta}\left(\frac{\partial S_{\phi\theta}}{\partial\phi}+\sin\theta S_{r\phi}+\cos\theta S_{\theta\theta}-\cos\theta S_{\phi\phi}\right)\right]\mathbf{e}_\theta+ \tag{94}$$

$$+\frac{\partial S_{r\phi}}{\partial r}\mathbf{e}_\phi+\left[\frac{1}{r}\left(\frac{\partial S_{\theta\phi}}{\partial\theta}+S_{r\phi}\right)\right]\mathbf{e}_\phi+\left[\frac{1}{r\sin\theta}\left(\frac{\partial S_{\phi\phi}}{\partial\phi}+\sin\theta S_{r\phi}+\cos\theta S_{\theta\phi}+\sin\theta S_{\phi r}+\cos\theta S_{\phi\theta}\right)\right]\mathbf{e}_\phi= \tag{95}$$

$$=\left[\left(\frac{\partial S_{rr}}{\partial r}+\frac{2S_{rr}}{r}\right)+\left(\frac{1}{r}\frac{\partial S_{\theta r}}{\partial\theta}+\frac{\cos\theta S_{\theta r}}{r\sin\theta}\right)+\left(\frac{1}{r\sin\theta}\frac{\partial S_{\phi r}}{\partial\phi}\right)-\frac{S_{\theta\theta}}{r}-\frac{S_{\phi\phi}}{r}\right]\mathbf{e_r}+ \tag{96}$$

$$+\left[\left(\frac{\partial S_{r\theta}}{\partial r}+\frac{2S_{r\theta}}{r}\right)+\left(\frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial\theta}+\frac{\cos\theta S_{\theta\theta}}{r\sin\theta}\right)+\left(\frac{1}{r\sin\theta}\frac{\partial S_{\phi\theta}}{\partial\phi}\right)+\frac{S_{\theta r}}{r}-\frac{\cos\theta S_{\phi\phi}}{r\sin\theta}\right]\mathbf{e}_\theta+ \tag{97}$$

$$+\left[\left(\frac{\partial S_{r\phi}}{\partial r}+\frac{2S_{r\phi}}{r}\right)+\left(\frac{1}{r}\frac{\partial S_{\theta\phi}}{\partial\theta}+\frac{\cos\theta S_{\theta\phi}}{r\sin\theta}\right)+\left(\frac{1}{r\sin\theta}\frac{\partial S_{\phi\phi}}{\partial\phi}\right)+\frac{S_{\phi r}}{r}+\frac{\cos\theta S_{\phi\theta}}{r\sin\theta}\right]\mathbf{e}_\phi= \tag{98}$$

$$=\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2S_{rr})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ S_{\theta r})+\frac{1}{r\sin\theta}\frac{\partial S_{\phi r}}{\partial\phi}-\frac{S_{\theta\theta}}{r}-\frac{S_{\phi\phi}}{r}\right]\mathbf{e_r}+ \tag{99}$$

$$+\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2S_{r\theta})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ S_{\theta\theta})+\frac{1}{r\sin\theta}\frac{\partial S_{\phi\theta}}{\partial\phi}+\frac{S_{\theta r}}{r}-\frac{S_{\phi\phi}\cos\theta}{r\sin\theta}\right]\mathbf{e}_\theta+ \tag{100}$$

$$+\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2S_{r\phi})+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ S_{\theta\phi})+\frac{1}{r\sin\theta}\frac{\partial S_{\phi\phi}}{\partial\phi}+\frac{S_{\phi r}}{r}+\frac{S_{\phi\theta}\cos\theta}{r\sin\theta}\right]\mathbf{e}_\phi \tag{101}$$