

# 1 Alternative hydrodynamic stellar structure equations EXPLAINED

$$\partial_r \bar{m} = -\bar{\rho} \bar{m} \bar{g}_r / \Gamma_1 \bar{P} + 4\pi r^2 \bar{\rho} \quad (1)$$

$$\partial_r \bar{P} = -\bar{\rho} \bar{g}_r \quad (2)$$

$$\partial_r \tilde{L} = -4\pi r^2 \tilde{u}_r \bar{\rho} \bar{g}_r / \Gamma_1 + \tilde{\epsilon}_t \partial_r 4\pi r^2 \bar{\rho} \tilde{u}_r \quad (3)$$

$$\partial_r \bar{T} = -(\Gamma_3 - 1) \bar{\rho} \bar{T} \bar{g}_r / \Gamma_1 \bar{P} \quad (4)$$

$$\partial_t \tilde{X}_i = \tilde{X}_i^{nuc} - (1/\bar{\rho}) \nabla_r f_i - \tilde{u}_r \partial_r \tilde{X}_i \quad (5)$$

Equations (1)-(4) can be derived by four assumptions: (i) adiabatic convection  $\Gamma_1 = \partial \ln P / \partial \ln \rho|_{ad}$ , (ii) hydrostatic equilibrium  $\partial_r = -\rho g_r$ , (iii)  $\epsilon_{nuc} \ll \epsilon_i$ , (iv)  $\epsilon_K \ll \epsilon_i \sim \epsilon_t$ .

## 1.1 Continuity equation

The continuity equation can be derived from first principles for adiabatic convection in hydrostatic equilibrium. We start from the definition of  $\Gamma_1 = \partial \ln P / \partial \ln \rho|_{ad}$ . I will avoid the  $_{ad}$  suffix for clarity reasons and at some point assume hydrostatic equilibrium  $\partial_r P = -\rho g_r$ .

$$\Gamma_1 = \frac{\partial \ln P}{\partial \ln \rho} \quad (6)$$

$$\Gamma_1 = \frac{\rho}{P} \frac{\partial_r P}{\partial_r \rho} \quad (7)$$

$$\Gamma_1 P \partial_r \rho = \rho \partial_r P \quad (8)$$

$$\Gamma_1 P \partial_r \rho = \rho (-\rho g_r) \quad (9)$$

$$\partial_r \rho = -\frac{\rho \rho g_r}{\Gamma_1 P} \quad \backslash .V \text{ (volume)} = 4\pi r^3/3 \quad (10)$$

$$V \partial_r \rho = -\frac{V \rho \rho g_r}{\Gamma_1 P} \quad (11)$$

$$\partial_r V \rho - \rho \partial_r V = -\frac{V \rho \rho g_r}{\Gamma_1 P} \quad (12)$$

$$\partial_r m - \rho 4\pi r^2 = -\frac{m \rho \rho g_r}{\Gamma_1 P} \quad (13)$$

$$\partial_r m = -\frac{\rho m \rho g_r}{\Gamma_1 P} + 4\pi r^2 \rho \quad (14)$$

$$\partial_r m = -\rho m g_r / \Gamma_1 P + 4\pi r^2 \rho \quad (15)$$

## 1.2 Temperature equation

The temperature equation can be derived from first principles for adiabatic convection in hydrostatic equilibrium. We start from the definition of  $\Gamma_1 = \partial \ln P / \partial \ln \rho|_{ad}$ . I will avoid the  $_{ad}$  suffix for clarity reasons and at some point assume hydrostatic equilibrium  $\partial_r P = -\rho g_r$ .

$$\Gamma_1 = \frac{\partial \ln P}{\partial \ln \rho} \quad (16)$$

$$\Gamma_1 = \frac{\rho}{P} \frac{\partial_r P}{\partial_r \rho} \quad \backslash \cdot \partial_r \ln T \quad (17)$$

$$\partial_r \ln T \Gamma_1 = \frac{\rho}{P} \frac{\partial_r P}{\partial_r \rho} \partial_r \ln T \quad (18)$$

$$P \partial_r \ln T \Gamma_1 = \rho \frac{\partial_r P}{\partial_r \rho} \partial_r \ln T \quad (19)$$

$$P \partial_r \ln T \Gamma_1 = \frac{\partial_r P}{\partial_r \ln \rho} \partial_r \ln T \quad (20)$$

$$P \partial_r \ln T \Gamma_1 = \frac{-\rho g_r}{\partial_r \ln \rho} \partial_r \ln T \quad (21)$$

$$\frac{P \partial_r T \Gamma_1}{T} = - \frac{\rho g_r}{\partial_r \ln \rho} \partial_r \ln T \quad (22)$$

$$- \frac{P \partial_r T \Gamma_1}{\rho g_r T} = \frac{\partial_r \ln T}{\partial_r \ln \rho} \quad (23)$$

$$- \frac{P \partial_r T \Gamma_1}{\rho g_r T} = \Gamma_3 - 1 \quad (24)$$

$$\partial_r T = - \frac{(\Gamma_3 - 1) \rho g_r T}{P \Gamma_1} \quad (25)$$

$$\partial_r T = - (\Gamma_3 - 1) \rho T g_r / \Gamma_1 P \quad (26)$$

## 1.3 Luminosity equation

The luminosity equation can be derived from first principles for adiabatic convection in hydrostatic equilibrium. We start from the definition of  $\Gamma_1 = \partial \ln P / \partial \ln \rho|_{ad}$ . I will avoid the  $_{ad}$  suffix for clarity reasons and at some point assume hydrostatic equilibrium  $\partial_r P = -\rho g_r$ , nuclear energy production and kinetic energy being negligible compared to internal energy i.e.  $\epsilon_{nuc} \ll \epsilon_i$  and  $\epsilon_K \ll \epsilon_i$ .

$$\Gamma_1 = \frac{\partial \ln P}{\partial \ln \rho} \quad (27)$$

$$\Gamma_1 = \frac{\rho}{P} \frac{\partial_r P}{\partial_r \rho} \quad (28)$$

$$\Gamma_1 = \frac{\rho}{P} \frac{-\rho g_r}{\partial_r \rho} = -\frac{\rho^2}{P} \frac{g_r}{\partial_r \rho} \quad (29)$$

$$\Gamma_1 = -\frac{g_r}{P \partial_r \rho / \rho^2} \quad (30)$$

$$dq = du + Pdv \quad dq = \epsilon_{nuc} \ll du = d\epsilon_i \quad v = 1/\rho \quad \text{1st thermodynamic law, e.g. Kippenhahn and Weigert, 1994, p.19} \quad (31)$$

$$d\epsilon_i = -Pdv = -Pd(1/\rho) = Pd\rho/\rho^2 \quad (32)$$

$$\Gamma_1 = -\frac{g_r}{\partial_r \epsilon_i} \quad (33)$$

$$\partial_r \epsilon_i = -\frac{g_r}{\Gamma_1} \quad \epsilon_K \ll \epsilon_i \text{ and } \epsilon_i \sim \epsilon_t \quad (34)$$

$$\partial_r \epsilon_t = -\frac{g_r}{\Gamma_1} \quad \backslash .4\pi r^2 \rho u_r \quad (35)$$

$$4\pi r^2 \rho u_r \partial_r \epsilon_t = -\frac{4\pi r^2 \rho u_r g_r}{\Gamma_1} \quad (36)$$

$$\partial_r 4\pi r^2 \rho u_r \epsilon_t - \epsilon_t \partial_r 4\pi r^2 \rho u_r = -\frac{4\pi r^2 \rho u_r g_r}{\Gamma_1} \quad (37)$$

$$\partial_r L - \epsilon_t \partial_r 4\pi r^2 \rho u_r = -\frac{4\pi r^2 \rho u_r g_r}{\Gamma_1} \quad (38)$$

$$\partial_r L = -\frac{4\pi r^2 \rho u_r g_r}{\Gamma_1} + \epsilon_t \partial_r 4\pi r^2 \rho u_r \quad (39)$$

$$\partial_r L = -4\pi r^2 u_r \rho g_r / \Gamma_1 + \epsilon_t \partial_r 4\pi r^2 \rho u_r \quad (40)$$

Table 1: Definitions:

$\rho$ density	$g_r$ radial gravitational acceleration
$m = \rho V = \rho \frac{4}{3} \pi r^3$ mass	$M = \int \rho(r) V = \int \rho(r) dV = \int \rho(r) 4\pi r^2 dr$ integrated mass
$T$ temperature	$\mathcal{S} = \rho \epsilon_{\text{nuc}}(q)$ nuclear energy production (cooling function)
$P$ pressure	$\tau_{ij} = 2\mu S_{ij}$ viscous stress tensor ( $\mu$ kinematic viscosity)
$u_r, u_\theta, u_\phi$ velocity components	$S_{ij} = (1/2)(\partial_i u_j + \partial_j u_i)$ strain rate
$\mathbf{u} = u(u_r, u_\theta, u_\phi)$ velocity	$\tilde{R}_{ij} = \overline{\rho u_i'' u_j''}$ Reynolds stress tensor
$j_z = r \sin \theta u_\phi$ specific angular momentum	$F_T = \chi \partial_r T$ heat flux
$d = \nabla \cdot \mathbf{u}$ dilatation	$\Gamma_1 = (d \ln P / d \ln \rho) _s$
$\epsilon_I$ specific internal energy	$\Gamma_2 / (\Gamma_2 - 1) = (d \ln P / d \ln T) _s$
$h$ specific enthalpy	$\Gamma_3 - 1 = (d \ln T / d \ln \rho) _s$
$k = (1/2) \overline{u_i'' u_i''}$ turbulent kinetic energy	$\tilde{k}^r = (1/2) \overline{u_r'' u_r''} = (1/2) \tilde{R}_{rr} / \bar{\rho}$ radial turbulent kinetic energy
$\epsilon_k$ specific kinetic energy	$\tilde{k}^\theta = (1/2) \overline{u_\theta'' u_\theta''} = (1/2) \tilde{R}_{\theta\theta} / \bar{\rho}$ angular turbulent kinetic energy
$\epsilon_t$ specific total energy	$\tilde{k}^\phi = (1/2) \overline{u_\phi'' u_\phi''} = (1/2) \tilde{R}_{\phi\phi} / \bar{\rho}$ angular turbulent kinetic energy
$s$ specific entropy	$\tilde{k}^h = \tilde{k}^\theta + \tilde{k}^\phi$ horizontal turbulent kinetic energy
$v = 1/\rho$ specific volume	$f_k = (1/2) \overline{\rho u_i'' u_i''}$ turbulent kinetic energy flux
$X_\alpha$ mass fraction of isotope $\alpha$	$f_k^r = (1/2) \overline{\rho u_r'' u_r''}$ radial turbulent kinetic energy flux
$\dot{X}_\alpha^{\text{nuc}}$ rate of change of $X_\alpha$	$f_k^\theta = (1/2) \overline{\rho u_\theta'' u_\theta''}$ angular turbulent kinetic energy flux
$A_\alpha$ number of nucleons in isotope $\alpha$	$f_k^\phi = (1/2) \overline{\rho u_\phi'' u_\phi''}$ angular turbulent kinetic energy flux
$Z_\alpha$ charge of isotope $\alpha$	$f_k^h = f_k^\theta + f_k^\phi$ horizontal turbulent kinetic energy flux
$A$ mean number of nucleons per isotope	$W_p = \overline{P' d''}$ turbulent pressure dilatation
$Z$ mean charge per isotope	$W_b = \overline{\rho u_r'' g_r}$ buoyancy
$f_P = \overline{P' u_r'}$ acoustic flux	$f_T = -\overline{\chi \partial_r T}$ heat flux ( $\chi$ thermal conductivity)

Table 2: Definitions (continued):

$f_I = \overline{\rho \epsilon_I'' u_r''}$ internal energy flux	$f_\alpha = \overline{\rho X_\alpha'' u_r''}$ $X_\alpha$ flux
$f_s = \overline{\rho s'' u_r''}$ entropy flux	$f_{jz} = \overline{\rho j_z'' u_r''}$ angular momentum flux
$f_T = \overline{u_r' T'}$ turbulent heat flux	$f_A = \overline{\rho A'' u_r''}$ $A$ (mean number of nucleons per isotope) flux
$f_h = \overline{\rho h'' u_r''}$ enthalpy flux	$f_Z = \overline{\rho Z'' u_r''}$ $Z$ (mean charge per isotope) flux
$b = \overline{v' \rho'}$ density-specific volume covariance	$\mathcal{N}_\rho, \mathcal{N}_{ur}, \mathcal{N}_{u\theta}, \mathcal{N}_{u\phi}, \mathcal{N}_{jz}, \mathcal{N}_\alpha, \mathcal{N}_A, \mathcal{N}_Z$ numerical effect
$f_\tau = f_\tau^r + f_\tau^\theta + f_\tau^\phi$ viscous flux	$\mathcal{N}_{\epsilon I} = -\nabla_r f_\tau + \varepsilon_k$ numerical effect
$f_\tau^r = -\overline{\tau_{rr}' u_r'}$ viscous flux	$\mathcal{N}_{\epsilon k} = -\varepsilon_k$ numerical effect
$f_\tau^\theta = -\overline{\tau_{\theta r}' u_\theta'}$ viscous flux	$\mathcal{N}_{\epsilon t} = -\nabla_r f_\tau$ numerical effect
$f_\tau^\phi = -\overline{\tau_{\phi r}' u_\phi'}$ viscous flux	$\mathcal{N}_s = \overline{-\varepsilon_k / T}$ numerical effect
$f_\tau^h = f_\tau^\theta + f_\tau^\phi$ viscous flux	$\mathcal{N}_h = -\nabla_r f_\tau + (\Gamma_3 - 1)\varepsilon_k$ numerical effect
$f_I^r = \overline{\rho \epsilon_I'' u_r'' u_r''}$ radial flux of $f_I$	$\mathcal{N}_P = +(\Gamma_3 - 1)\varepsilon_k$ numerical effect
$f_s^r = \overline{\rho s'' u_r'' u_r''}$ radial flux of $f_s$	$\mathcal{N}_T = +(\tau_{ij} \partial_j u_i) / (c_v \rho)$ numerical effect
$f_h^r = \overline{\rho h'' u_r'' u_r''}$ radial flux of $f_h$	$\mathcal{N}_{Rrr} = -2\nabla_r f_\tau^r - 2\varepsilon_k^r$ numerical effect
$f_T^r = \overline{T' u_r' u_r'}$ radial flux of $f_T$	$\mathcal{N}_{R\theta\theta} = -2\nabla_r f_\tau^\theta - 2\varepsilon_k^\theta$ numerical effect
$f_{jz}^r = \overline{\rho j_z'' u_r'' u_r''}$ radial flux of $f_{jz}$	$\mathcal{N}_{R\phi\phi} = -2\nabla_r f_\tau^\phi - 2\varepsilon_k^\phi$ numerical effect
$f_\alpha^r = \overline{\rho X_\alpha'' u_r'' u_r''}$ radial flux of $f_\alpha$	$\mathcal{N}_k = -\nabla_r f_\tau - \varepsilon_k$ numerical effect
$f_A^r = \overline{\rho A'' u_r'' u_r''}$ radial flux of $f_A$	$\mathcal{N}_{kr} = -\nabla_r f_\tau^r - \varepsilon_k^r$ numerical effect
$f_Z^r = \overline{\rho Z'' u_r'' u_r''}$ radial flux of $f_Z$	$\mathcal{N}_{kh} = -\nabla_r f_\tau^h - \varepsilon_k^h$ numerical effect
$\mathcal{G}_k^r = -(1/2)\overline{G_{rr}^R} - \overline{u_r'' G_r^M}$	$\mathcal{N}_a = -\varepsilon_a$ numerical effect

Table 3: Definitions (continued):

$\mathcal{G}_k^\theta = -(1/2)\overline{G_{\theta\theta}^R} - \overline{u_\theta'' G_\theta^M}$	$\mathcal{N}_b$ numerical effect
$\mathcal{G}_k^\phi = -(1/2)\overline{G_{\phi\phi}^R} - \overline{u_\phi'' G_\phi^M}$	$\mathcal{N}_{fI} = -\nabla_r(\overline{\epsilon_I'' \tau_{rr}'}) + \overline{u_r'' \tau_{ij} \partial_i u_j} - \varepsilon_I$ numerical effect
$\mathcal{G}_k^h = +\mathcal{G}_k^\theta + \mathcal{G}_k^\phi$	$\mathcal{N}_{fh} = -\nabla_r(\overline{h'' \tau_{rr}'}) + \overline{u_r'' (\Gamma_3 - 1) \tau_{ij} \partial_i u_j} - \overline{u_r'' \nabla_i u_i \tau_{ji}} - \varepsilon_h$ numerical effect
$\mathcal{G}_a = +\overline{\rho' v G_r^M}$	$\mathcal{N}_{fs} = -\nabla_r(\overline{s'' \tau_{rr}'}) + \overline{u_r'' \tau_{ij} \partial_i u_j / T} - \varepsilon_s$ numerical effect
$\mathcal{G}_I = -\overline{G_r^I} - \overline{\epsilon_I'' G_r^M}$	$\mathcal{N}_{fA} = -\nabla_r(\overline{A'' \tau_{rr}'}) - \varepsilon_A$ numerical effect
$\mathcal{G}_\alpha = -\overline{G_r^\alpha} - \overline{X_\alpha'' G_r^M}$	$\mathcal{N}_{fZ} = -\nabla_r(\overline{Z'' \tau_{rr}'}) - \varepsilon_Z$ numerical effect
$\mathcal{G}_A = -\overline{G_r^A} - \overline{A'' G_r^M}$	$\mathcal{N}_{f\alpha} = -\nabla_r(\overline{\alpha'' \tau_{rr}'}) - \varepsilon_\alpha$ numerical effect
$\mathcal{G}_Z = -\overline{G_r^Z} - \overline{Z'' G_r^M}$	$\mathcal{N}_{fjz} = -\nabla_r(\overline{j_z'' \tau_{rr}'}) - \varepsilon_{jz}$ numerical effect
$\mathcal{G}_h = -\overline{G_r^h} - \overline{h'' G_r^M}$	$\mathcal{N}_{fT} = +\overline{T' \partial_i \tau_{ri} / \rho} + \overline{u_r' \tau_{ij} \partial_i u_j / \rho c_v}$ numerical effect
$\mathcal{G}_T = -\overline{G_r^T} - \overline{T' G_r^M}$	
$\mathcal{G}_s = -\overline{G_r^s} - \overline{s'' G_r^M}$	
$\mathcal{G}_{jz} = -\overline{G_r^{jz}} - \overline{j_z'' G_r^M}$	
$\sigma_\rho = \overline{\rho' \rho'}$	$\mathcal{N}_{\sigma_\rho}$ numerical effect
$\sigma_P = \overline{P' P'}$	$\mathcal{N}_{\sigma_P} = +2(\Gamma_3 - 1) \overline{P' \tau_{ij} \partial_i u_j}$ numerical effect
$\sigma_T = \overline{T' T'}$	$\mathcal{N}_{\sigma_T} = +2 \overline{T' \tau_{ij} \partial_i u_j / \rho c_v}$ numerical effect
$\sigma_{ur} = \overline{u_r'' u_r''}$	$\mathcal{N}_{\sigma_{ur}} = +2 \nabla_r f_\tau^r - 2 \varepsilon_k^r$ numerical effect
$\sigma_s = \overline{s'' s''}$	$\mathcal{N}_{\sigma_s} = +2 \overline{s'' \tau_{ij} \partial_j u_i / T}$ numerical effect
$\sigma_\alpha = \overline{X_\alpha'' X_\alpha''}$	$\mathcal{N}_{\sigma_\alpha}$ numerical effect    numerical effect
$\sigma_{\epsilon I} = \overline{\epsilon_I'' \epsilon_I''}$	$\mathcal{N}_{\sigma_{\epsilon I}} = +2 \overline{\epsilon_I'' \tau_{ij} \partial_j u_i}$ numerical effect

Table 4: Definitions (continued):

$\varepsilon_k^r = \overline{\tau'_{rr} \partial_r u_r''} + \overline{\tau'_{r\theta} (1/r) \partial_\theta u_r''} + \overline{\tau'_{r\phi} (1/r \sin \theta) \partial_\phi u_r''}$	$\overline{G_r^M} = -\overline{\rho u_\theta u_\theta / r} - \overline{\rho u_\phi u_\phi / r}$
$\varepsilon_k^\theta = \overline{\tau'_{\theta r} \partial_r u_\theta''} + \overline{\tau'_{\theta\theta} (1/r) \partial_\theta u_\theta''} + \overline{\tau'_{\theta\phi} (1/r \sin \theta) \partial_\phi u_\theta''}$	$\overline{G_\theta^M} = +\overline{\rho u_\theta u_r / r} - \overline{\rho u_\phi u_\phi / (r \tan \theta)}$
$\varepsilon_k^\phi = \overline{\tau'_{\phi r} \partial_r u_\phi''} + \overline{\tau'_{\phi\theta} (1/r) \partial_\theta u_\phi''} + \overline{\tau'_{\phi\phi} (1/r \sin \theta) \partial_\phi u_\phi''}$	$\overline{G_\phi^M} = +\overline{\rho u_\phi u_r / r} + \overline{\rho u_\phi u_\theta / (r \tan \theta)}$
$\varepsilon_k = \varepsilon_k^r + \varepsilon_k^\theta + \varepsilon_k^\phi$	$\overline{G_{rr}^R} = -\overline{\rho u_\theta'' u_\theta'' u_r'' / r} - \overline{\rho u_\theta'' u_r'' u_\theta'' / r} - \overline{\rho u_\phi'' u_\phi'' u_r'' / r} - \overline{\rho u_\phi'' u_r'' u_\phi'' / r}$
$\varepsilon_k^h = \varepsilon_k^\theta + \varepsilon_k^\phi$	$\overline{G_{\theta\theta}^R} = +\overline{\rho u_\theta'' u_r'' u_\theta'' / r} + \overline{\rho u_\theta'' u_\theta'' u_r'' / r} - \overline{\rho u_\phi'' u_\phi'' u_\theta'' / (r \tan \theta)} - \overline{u_\phi'' u_\theta'' u_\phi'' / (r \tan \theta)}$
$\varepsilon_a = \overline{\rho' v \nabla_r \tau'_{rr}}$	$\overline{G_{\phi\phi}^R} = +\overline{\rho u_\phi'' u_r'' r_\phi / r} + \overline{\rho u_\phi'' u_\theta'' u_\phi'' / (r \tan \theta)} + \overline{\rho u_\phi'' u_\phi'' u_r'' / r} + \overline{\rho u_\phi'' u_\phi'' u_\theta'' / (r \tan \theta)}$
$\varepsilon_I = \overline{\tau'_{rr} \partial_r \epsilon_I''} + \overline{\tau'_{r\theta} (1/r) \partial_\theta \epsilon_I''} + \overline{\tau'_{r\phi} (1/r \sin \theta) \partial_\phi \epsilon_I''}$	$\overline{G_r^I} = -\overline{\rho \epsilon_I'' u_\theta'' u_\theta'' / r} - \overline{\rho \epsilon_I'' u_\phi'' u_\phi'' / r}$
$\varepsilon_s = \overline{\tau'_{rr} \partial_r s''} + \overline{\tau'_{r\theta} (1/r) \partial_\theta s''} + \overline{\tau'_{r\phi} (1/r \sin \theta) \partial_\phi s''}$	$\overline{G_r^s} = -\overline{\rho s'' u_\theta'' u_\theta'' / r} - \overline{\rho s'' u_\phi'' u_\phi'' / r}$
$\varepsilon_\alpha = \overline{\tau'_{rr} \partial_r X_\alpha''} + \overline{\tau'_{r\theta} (1/r) \partial_\theta X_\alpha''} + \overline{\tau'_{r\phi} (1/r \sin \theta) \partial_\phi X_\alpha''}$	$\overline{G_r^\alpha} = -\overline{\rho X_\alpha'' u_\theta'' u_\theta'' / r} - \overline{\rho X_\alpha'' u_\phi'' u_\phi'' / r}$
$\varepsilon_A = \overline{\tau'_{rr} \partial_r A''} + \overline{\tau'_{r\theta} (1/r) \partial_\theta A''} + \overline{\tau'_{r\phi} (1/r \sin \theta) \partial_\phi A''}$	$\overline{G_r^A} = -\overline{\rho A'' u_\theta'' u_\theta'' / r} - \overline{\rho A'' u_\phi'' u_\phi'' / r}$
$\varepsilon_Z = \overline{\tau'_{rr} \partial_r Z''} + \overline{\tau'_{r\theta} (1/r) \partial_\theta Z''} + \overline{\tau'_{r\phi} (1/r \sin \theta) \partial_\phi Z''}$	$\overline{G_r^Z} = -\overline{\rho Z'' u_\theta'' u_\theta'' / r} - \overline{\rho Z'' u_\phi'' u_\phi'' / r}$
$\varepsilon_h = \overline{\tau'_{rr} \partial_r h''} + \overline{\tau'_{r\theta} (1/r) \partial_\theta h''} + \overline{\tau'_{r\phi} (1/r \sin \theta) \partial_\phi h''}$	$\overline{G_r^h} = -\overline{\rho h'' u_\theta'' u_\theta'' / r} - \overline{\rho h'' u_\phi'' u_\phi'' / r}$
$\varepsilon_{jz} = \overline{\tau'_{rr} \partial_r j_z''} + \overline{\tau'_{r\theta} (1/r) \partial_\theta j_z''} + \overline{\tau'_{r\phi} (1/r \sin \theta) \partial_\phi j_z''}$	$\overline{G_r^T} = -\overline{\rho T' u_\theta' u_\theta' / r} - \overline{\rho T' u_\phi' u_\phi' / r}$
	$\overline{G_r^{jz}} = -\overline{\rho j_z'' u_\theta'' u_\theta'' / r} - \overline{\rho j_z'' u_\phi'' u_\phi'' / r}$

$$\nabla(\cdot) = \nabla_r(\cdot) + \nabla_\theta(\cdot) + \nabla_\phi(\cdot) = \frac{1}{r^2} \partial_r(r^2 \cdot) + \frac{1}{r \sin \theta} \partial_\theta(\sin \theta \cdot) + \frac{1}{r \sin \theta} \partial_\phi(\cdot)$$