1 ransX to PROMPI implementation

1.1 Composition transport equation

$$
\begin{aligned}
\overline{\rho}\widetilde{D}_t\widetilde{X}_i &= -\nabla_r f_i + \overline{\rho}\widetilde{\dot{X}}_i^{\rm nuc} \\
\overline{\rho}\partial_t\widetilde{X}_i + \overline{\rho}\widetilde{u}_r\partial_r\widetilde{X}_i &= -\nabla_r\overline{\rho}\widetilde{X_i''u_r''} + \overline{\rho}\widetilde{\dot{X}}_i^{\rm nuc} \\
\overline{\rho}\partial_t\widetilde{X}_i + \overline{\rho}\widetilde{u}_r\partial_r\widetilde{X}_i &= -\nabla_r\overline{\rho}(\widetilde{X_i u_r} - \widetilde{X}_i\widetilde{u}_r) + \overline{\rho}\widetilde{\dot{X}}_i^{\rm nuc} \\
\overline{\rho}\partial_t\overline{\rho X_i}/\overline{\rho} + \overline{\rho u_r}\partial_r\overline{\rho X_i}/\overline{\rho} &= -\nabla_r\left(\overline{\rho X_i u_r} - \overline{\rho X_i}\,\overline{\rho u_r}/\overline{\rho}\right) + \overline{\rho\dot{X}_i^{\rm nuc}} \\
dd\ \partial_t\ ddxi/dd + ddux\ \partial_r\ ddxi/dd &= -\nabla_r\left(ddxiux - ddxi * ddux/dd\right) + ddxidot
\end{aligned}
\tag{1}
$$

1.2 Composition variance equation

$$
\begin{aligned}
\overline{\rho}\widetilde{D}_t\sigma_i &= -\nabla_r f_i^r - 2f_i\partial_r\widetilde{X}_i + 2\overline{X_i''\rho\dot{X}_i^{\rm nuc}} \\
\overline{\rho}\widetilde{D}_t\widetilde{X_i''X_i''} &= -\nabla_r(\overline{\rho X_i''X_i''u_r''}) - 2\overline{\rho}\widetilde{X_i''u_r''}\partial_r\widetilde{X}_i + 2\overline{X_i''\rho\dot{X}_i^{\rm nuc}} \\
\overline{\rho}\partial_t(\widetilde{X_iX_i} - \widetilde{X}_i\widetilde{X}_i) + \overline{\rho}\widetilde{u}_r\partial_r(\widetilde{X_iX_i} - \widetilde{X}_i\widetilde{X}_i) &= -\nabla_r(\overline{\rho X_iX_iu_r} - 2\widetilde{X}_i\overline{\rho X_iu_r} - \widetilde{u}_r\overline{\rho X_iX_i} + 2\widetilde{X}_i\widetilde{X}_i\overline{\rho u_r}) \\
&\quad - 2\overline{\rho}(\widetilde{X_iu_r} - \widetilde{X}_i\widetilde{u}_r)\partial_r\widetilde{X}_i + (\overline{X_i\rho\dot{X}_i} - \widetilde{X}_i\overline{\rho\dot{X}_i}) \\
&dd\ \partial_t\ (ddxisq/dd - ddxi * ddxi/dd * dd) \\
+\ ddux\ \partial_r\ (ddxisq/dd - ddxi * ddxi/dd * dd) &= -\nabla_r(ddxisqux - 2 * ddxi/dd * ddxiux - ddux/dd * ddxisq + 2 * ddxi * ddxi * ddux/dd * dd) \\
&\quad -\ 2 * dd\ (ddxiux/dd - ddxi * ddux/dd * dd) * \partial_r\ ddxi/dd \\
&\quad + 2 * (ddxixidot - ddxi/dd * ddxidot)
\end{aligned}
\tag{2}
$$

1.3 Composition flux equation

$$\overline{\rho}\widetilde{D}_t(f_i/\overline{\rho}) = -\nabla_r f_i^r - f_i\partial_r\widetilde{u}_r - \widetilde{R}_{rr}\partial_r\widetilde{X}_i - \overline{X_i''}\partial_r\overline{P} - \overline{X_i''\partial_r P'} + \overline{u_r''\rho\dot{X}_i^{\rm nuc}} + \mathcal{G}_i \tag{3}$$

$$\overline{\rho}\partial_t\widetilde{X_i''u_r''} + \overline{\rho}\widetilde{u}_r\partial_r\widetilde{X_i''u_r''} = -\nabla_r\overline{\rho}\widetilde{X_i''u_r''u_r''} - \overline{\rho}\widetilde{X_i''u_r''}\partial_r\widetilde{u}_r - \overline{\rho}\widetilde{u_r''u_r''}\partial_r\widetilde{X}_i - \overline{X_i''}\partial_r\overline{P} - \overline{X_i''\partial_r P'} + \overline{u_r''\rho\dot{X}_i^{\rm nuc}} + \overline{G_r^i} - \overline{X_i''G_r^M}$$

$$\begin{aligned}
\overline{\rho}\partial_t\widetilde{X_i''u_r''} + \overline{\rho}\widetilde{u}_r\partial_r\widetilde{X_i''u_r''} = &-\nabla_r\overline{\rho}\widetilde{X_i''u_r''u_r''} - \overline{\rho}\widetilde{X_i''u_r''}\partial_r\widetilde{u}_r - \overline{\rho}\widetilde{u_r''u_r''}\partial_r\widetilde{X}_i - \overline{X_i''}\partial_r\overline{P} - \overline{X_i''\partial_r P'} + \overline{u_r''\rho\dot{X}_i^{\rm nuc}} \\
&- \overline{\rho X_i''u_\theta''u_\theta''/r} - \overline{\rho X_i''u_\phi''u_\phi''/r} + \overline{\rho X_i''u_\theta u_\theta/r} + \overline{\rho X_i''u_\phi u_\phi/r}
\end{aligned}$$

$$\begin{aligned}
\overline{\rho}\partial_t(\widetilde{X_iu_r} - \widetilde{X}_i\widetilde{u}_r) + \overline{\rho}\widetilde{u}_r\partial_r(\widetilde{X_iu_r} - \widetilde{X}_i\widetilde{u}_r) = &-\nabla_r(\overline{\rho X_iu_ru_r} - \widetilde{X}_i\overline{\rho u_ru_r} - 2\widetilde{u}_r\overline{\rho X_iu_r} + 2\overline{\rho}\widetilde{X}_i\widetilde{u}_r\widetilde{u}_r) \\
&- \overline{\rho}(\widetilde{X_iu_r} - \widetilde{X}_i\widetilde{u}_r)\partial_r\widetilde{u}_r - \overline{\rho}(\widetilde{u_ru_r} - \widetilde{u}_r\widetilde{u}_r)\partial_r\widetilde{X}_i \\
&- (\overline{X_i}\partial_r\overline{P} - \widetilde{X}_i\partial_r\overline{P}) - (\overline{X_i\partial_r P} - \overline{X_i}\partial_r\overline{P}) + (\overline{u_r\rho\dot{X}_i^{\rm nuc}} - \widetilde{u}_r\overline{\rho\dot{X}_i^{\rm nuc}}) \\
&- (\overline{\rho X_iu_\theta u_\theta} - \widetilde{X}_i\overline{\rho u_\theta u_\theta} - 2\widetilde{u}_\theta\overline{\rho X_iu_\theta} + 2\overline{\rho}\widetilde{X}_i\widetilde{u}_\theta\widetilde{u}_\theta)/r \\
&- (\overline{\rho X_iu_\phi u_\phi} - \widetilde{X}_i\overline{\rho u_\phi u_\phi} - 2\widetilde{u}_\phi\overline{\rho X_iu_\phi} + 2\overline{\rho}\widetilde{X}_\phi\widetilde{u}_\phi\widetilde{u}_\phi)/r \\
&+ (\overline{\rho X_iu_\theta u_\theta} - \widetilde{X}_i\overline{\rho u_\theta u_\theta})/r \\
&+ (\overline{\rho X_iu_\phi u_\phi} - \widetilde{X}_i\overline{\rho u_\phi u_\phi})/r
\end{aligned}$$

$$\begin{aligned}
dd\ \partial_t(ddxiux/dd - ddxi*ddux/dd*dd) + ddux\ \partial_r(ddxiux/dd - ddxi*ddux/dd*dd) = \\
-\nabla_r(ddxiuxux - ddxi/dd*dduxux - 2*ddux/dd*ddxiux + 2*ddxi*ddux*ddux/dd*dd) \\
-(ddxiux - ddxi*ddux/dd)*\partial_r ddux/dd - (dduxux - ddux*ddux/dd)*\partial_r ddxi/dd \\
-(xi\ \partial_r\ pp - ddxi/dd\ \partial_r\ pp) - (xigradxpp - xi\ \partial_r\ pp) + (ddxidotux - ddux/dd*ddxidot) \\
-(ddxiuyuy - ddxi/dd*dduyuy - 2*dduy/dd*ddxiuy + 2*ddxi*dduy*dduy/dd*dd)/r \\
-(ddxiuzuz - ddxi/dd*dduzuz - 2*dduz/dd*ddxiuz + 2*ddxi*dduz*dduz/dd*dd)/r \\
+(ddxiuyuy - ddxi/dd*dduyuy)/r \\
+(ddxiuzuz - ddxi/dd*dduzuz)/r
\end{aligned}$$

1.4 Density-specific volume covariance

$$
\begin{aligned}
\overline{D}_t b &= +\,\overline{v}\nabla_r\overline{\rho}\,\overline{u''_r} - \overline{\rho}\nabla_r(\overline{u'_r v'}) + 2\overline{\rho}\,\overline{v'd'} && (4)\\
\partial_t b + \overline{u}_r\partial_r b &= \overline{v}\nabla_r\overline{\rho}(\overline{u}_r - \widetilde{u}_r) - \overline{\rho}\nabla_r(\overline{u_r v} - \overline{u}_r\overline{v}) + 2\overline{\rho}(\overline{vd} - \overline{v}\overline{d}) && (5)\\
\partial_t\overline{v'\rho'} + \overline{u}_r\partial_r(\overline{v'\rho'}) &= \overline{v}\nabla_r\overline{\rho}(\overline{u}_r - \widetilde{u}_r) - \overline{\rho}\nabla_r(\overline{u_r v} - \overline{u}_r\overline{v}) + 2\overline{\rho}(\overline{vd} - \overline{v}\overline{d}) && (6)\\
\partial_t(\underbrace{\overline{v\rho}}_{1} - \overline{v}\ \overline{\rho}) + \overline{u}_r\partial_r(\underbrace{\overline{v\rho}}_{1} - \overline{v}\ \overline{\rho}) &= \overline{v}\nabla_r\overline{\rho}(\overline{u}_r - \widetilde{u}_r) - \overline{\rho}\nabla_r(\overline{u_r v} - \overline{u}_r\overline{v}) + 2\overline{\rho}(\overline{vd} - \overline{v}\overline{d}) && (7)\\
-\partial_t(\overline{v}\ \overline{\rho}) - \overline{u}_r\partial_r(\overline{v}\ \overline{\rho}) &= \overline{v}\nabla_r\overline{\rho}(\overline{u}_r - \widetilde{u}_r) - \overline{\rho}\nabla_r(\overline{u_r v} - \overline{u}_r\overline{v}) + 2\overline{\rho}(\overline{vd} - \overline{v}\overline{d}) && (8)\\
-\partial_t(sv*dd) - ux\partial_r(sv*dd) &= sv*\nabla_r(dd*ux - ddux/dd) - dd\nabla_r(svux - sv*ux) + 2*dd(svdivu - sv*divu) && (9)
\end{aligned}
$$

1.5 Density variance equation

$$
\begin{aligned}
\widetilde{D}_t\sigma_\rho &= -\,\nabla_r\overline{(\rho'\rho'u''_r)} - 2\overline{\rho}\ \overline{\rho'd''} - 2\overline{\rho'u''_r}\partial_r\overline{\rho} - 2\widetilde{d}\ \sigma_\rho - \overline{\rho'\rho'd''} && (10)\\
\partial_t\overline{\rho'\rho'} + \widetilde{u}_r\partial_r\overline{\rho'\rho'} &= -\,\nabla_r\overline{(\rho'\rho'u''_r)} - 2\overline{\rho}\ \overline{\rho'd''} - 2\overline{\rho'u''_r}\partial_r\overline{\rho} - 2\widetilde{d}\overline{\rho'\rho'} - \overline{\rho'\rho'd''} && (11)\\
\partial_t(\overline{\rho\rho} - \overline{\rho}\ \overline{\rho}) + \widetilde{u}_r\partial_r(\overline{\rho\rho} - \overline{\rho}\ \overline{\rho}) &= -\,\nabla_r(\overline{\rho\rho u_r} - 2\overline{\rho u_r}\ \overline{\rho} + \overline{\rho\rho}\ \overline{u_r} - \overline{\rho\rho}\widetilde{u}_r + \overline{\rho}\ \overline{\rho}\widetilde{d}) && (12)\\
&\quad -\,2\overline{\rho}(\overline{\rho d} - \overline{\rho}\widetilde{d} - \overline{\rho}\ \overline{d} + \overline{\rho}\widetilde{d}) && (13)\\
&\quad -\,2(\overline{\rho u_r} - \overline{\rho}\widetilde{u}_r - \overline{\rho}\ \overline{u}_r + \overline{\rho}\widetilde{u}_r)\partial_r\overline{\rho} && (14)\\
&\quad -\,2\widetilde{d}(\overline{\rho\rho} - \overline{\rho}\ \overline{\rho}) - (\overline{\rho\rho u_r} - 2\overline{\rho u_r}\ \overline{\rho} + \overline{\rho}\ \overline{\rho u_r} - \overline{\rho\rho}\widetilde{u}_r + \overline{\rho}\ \overline{\rho}\widetilde{d}) && (15)\\
\partial_t(ddsq - dd*dd) & && (16)\\
+ddux/dd\partial_r(ddsq - dd*dd) &= \\
&\quad -\,\nabla_r(ddddux - 2*ddux*dd + ddsq*ux - ddsq*ddux/dd + dd*dd*ddux/dd)\\
&\quad -\,2*dd*(-dd*divu + dddivu) - 2*(-dd*ux + ddux)\partial_r dd\\
&\quad -\,2*dddivu/dd*(ddsq - dd*dd)\\
&\quad -\,(dddddivu - 2*dddivu*dd + ddsq*divu - ddsq*dddivu/dd + dd*dddivu)
\end{aligned}
$$

1.6 Internal energy variance equation

$$\overline{\rho}\widetilde{D}_t\sigma_{\epsilon I} = -\nabla_r(\overline{\rho\epsilon''_I\epsilon''_I u''_r}) - 2f_I\partial_r\widetilde{\epsilon}_I - 2\overline{\epsilon''_I}\ \overline{P}\ \widetilde{d} - 2\overline{P}\ \overline{\epsilon''_I d''} - 2\widetilde{d}\ \overline{\epsilon''_I P'} - 2\overline{\epsilon''_I P' d''} + 2\overline{\epsilon''_I \mathcal{S}} \tag{17}$$

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon''_I\epsilon''_I} = -\nabla_r(\overline{\rho\epsilon''_I\epsilon''_I u''_r}) - 2\overline{\rho}\widetilde{\epsilon''_I u''_r}\partial_r\widetilde{\epsilon}_I - 2\overline{\epsilon''_I}\ \overline{P}\ \widetilde{d} - 2\overline{P}\ \overline{\epsilon''_I d''} - 2\widetilde{d}\ \overline{\epsilon''_I P'} - 2\overline{\epsilon''_I P' d''} + 2\overline{\epsilon''_I\rho\varepsilon_{nuc}} \tag{18}$$

$$\overline{\rho}\partial_t\widetilde{\epsilon''_I\epsilon''_I} + \overline{\rho}\widetilde{u}_r\nabla_r(\widetilde{\epsilon''_I\epsilon''_I}) = -\nabla_r(\overline{\rho\epsilon''_I\epsilon''_I u''_r}) - 2\overline{\rho}\widetilde{\epsilon''_I u''_r}\partial_r\widetilde{\epsilon}_I - 2\overline{\epsilon''_I}\ \overline{P}\ \widetilde{d} - 2\overline{P}\ \overline{\epsilon''_I d''} - 2\widetilde{d}\ \overline{\epsilon''_I P'} - 2\overline{\epsilon''_I P' d''} + 2\overline{\epsilon''_I\rho\varepsilon_{nuc}} \tag{19}$$

$$\begin{aligned}
&dd * \partial_t(ddeiei/dd - ddei * ddei/(dd * dd)) + \\
&ddux * \nabla_r(ddeiei/dd - ddei * ddei/(dd * dd)) = \\
&\qquad - \nabla_r(ddeieiux/dd - 2 * ddei/dd * ddeiux/dd - ddux/dd * ddeiei/dd \\
&\qquad + 2 * ddei * ddei * ddux/(dd * dd * dd)) \\
&\qquad - 2 * dd * (ddeiux/dd - ddei * ddux/(dd * dd))\partial_r ddei/dd \\
&\qquad - 2 * (ei - ddei/dd) * pp * dddivu/dd \\
&\qquad - 2 * pp * (eidd - ei * dddivu/dd - ddei/dd * divu + ddei * dddivu/dd) \\
&\qquad - 2 * dddivu/dd * (eippdivu - eidivu * pp - ddei/dd * ppdivu \\
&\qquad + ddei/dd * pp * dd - eipp * dddivu/dd + ei * pp * dddivu/dd) \\
&\qquad + 2 * (eiddenuc - ddei/dd * (ddenuc1 + ddenuc2))
\end{aligned} \tag{20}$$

1.7 MLT velocity

$$u_{MLT} \equiv (u'_{rms}) = \frac{F_c}{\alpha_E c_P (T'_{rms})} = \frac{\overline{\rho}\widetilde{h''u''_r}}{\alpha_E\widetilde{c_P}(\widetilde{TT} - \widetilde{T}\widetilde{T})^{1/2}} \sim \frac{\overline{\rho}\overline{h'u'_r}}{\alpha_E\overline{c_P}(\overline{TT} - \overline{T}\ \overline{T})^{1/2}}? \tag{21}$$

$$u_{MLT} \equiv (u'_{rms}) = \frac{\overline{\rho}(\widetilde{hu_r} - \widetilde{h}\widetilde{u_r})}{\alpha_E\widetilde{c_P}(\widetilde{TT} - \widetilde{T}\widetilde{T})^{1/2}} \sim \frac{\overline{\rho}(\overline{hu_r} - \overline{h}\overline{u_r})}{\alpha_E\overline{c_P}(\overline{TT} - \overline{T}\ \overline{T})^{1/2}}$$

$$u_{MLT} \equiv (u'_{rms}) = \frac{ddhhux - ddhh * ddux/dd}{\alpha_E * ddcp/dd\ (ddttsq/dd - ddtt * ddtt/dd * dd)^{1/2}} \sim \frac{dd * hhux - dd * hh * ux}{\alpha_E * cp\ (ttsq - tt * tt)^{1/2}}$$

1.8 Usefull identities

$$\overline{a''} = \overline{a - \widetilde{a}} = \overline{a} - \widetilde{a} \tag{22}$$

$$\widetilde{a''b''} = \widetilde{(a - \widetilde{a}) * (b - \widetilde{b})} = \widetilde{ab} - \widetilde{a}\widetilde{b} \tag{23}$$

$$\overline{a'b'} = \overline{(a - \overline{a}) * (b - \overline{b})} = \overline{ab} - \overline{a}\overline{b} = \overline{a'b''} \tag{24}$$

$$\widetilde{a''b''c''} = \widetilde{(a - \widetilde{a}) * (b - \widetilde{b}) * (c - \widetilde{c})} = \widetilde{abc} - \widetilde{a}\widetilde{bc} - \widetilde{b}\widetilde{ac} - \widetilde{c}\widetilde{ab} + 2\widetilde{a}\widetilde{b}\widetilde{c} \tag{25}$$

$$\overline{a'b'c''} = \overline{(a - \overline{a}) * (b - \overline{b}) * (c - \widetilde{c})} = \overline{abc} - \overline{ac}\ \overline{b} - \overline{a}\ \overline{bc} + \overline{a}\ \overline{b}\ \overline{c} - \overline{ab}\ \widetilde{c} + \overline{a}\ \overline{b}\widetilde{c} \tag{26}$$

$$\overline{a''b'c''} = \overline{(a - \widetilde{a}) * (b - \overline{b}) * (c - \widetilde{c})} = \overline{abc} - \overline{ac}\overline{b} - \widetilde{a}\overline{bc} + \widetilde{a}\overline{b}\overline{c} - \overline{ab}\widetilde{c} + \overline{a}\overline{b}\widetilde{c} \tag{27}$$

$$\overline{a''bc} = \overline{(a - \widetilde{a})bc} = \overline{abc} - \widetilde{a}\overline{bc} \tag{28}$$

$$\overline{a''\partial_r b'} = \overline{(a - \widetilde{a})\partial_r b'} = \overline{a\partial_r b'} - \cancelto{0}{\widetilde{a}\overline{\partial_r b'}} = \overline{a\partial_r b} - \overline{a}\partial_r \overline{b} \tag{29}$$