1 Alternative hydrodynamic stellar structure equations

Below is a set of alternative hydrodynamic stellar structure equations inspired by mean fields from the mean density/temperature/internal energy and pressure flux equations involving gradient term scaled by favrian Reynolds stress $\widetilde{R}_{rr}$ and the other one turbulent dilatation flux $\overline{u'_r d''}$.

$$
\begin{aligned}
\partial_r \overline{m} &= -\, \overline{\rho}\; \overline{m}\;\; \overline{u'_r d''} /\; \widetilde{R}_{rr} + 4\pi r^2 \overline{\rho} && (1)\\
\partial_r \overline{P} &= -\, \Gamma_1\; \overline{\rho}\; \overline{P}\; \overline{u'_r d''} /\; \widetilde{R}_{rr} && (2)\\
\partial_r \widetilde{L} &= +\, \widetilde{\epsilon}_t \partial_r 4\pi r^2 \overline{\rho} \widetilde{u}_r - 4\pi r^2 \overline{\rho}\; \widetilde{u}_r\; \overline{P}\; \overline{u'_r d''} /\; \widetilde{R}_{rr} && (3)\\
\partial_r \overline{T} &= -\, (\Gamma_3 - 1)\; \overline{\rho}\; \overline{T}\; \overline{u'_r d''} /\; \widetilde{R}_{rr} && (4)\\
\partial_t \widetilde{X}_i &= \widetilde{\dot{X}}_i^{nuc} - (1/\overline{\rho}) \nabla_r f_i - \widetilde{u}_r \partial_r \widetilde{X}_i && (5)
\end{aligned}
$$

These equations could be perfectly validated by our ransX framework and are shown on the next page in Figure 1. It appears that there is a universality relation between gradient of a mean thermodynamic variable Q and dilatation flux $\overline{u'_r d''}$, that is:

$$
\partial_r \overline{Q} \sim -\overline{\rho}\; \overline{Q}\; \overline{u'_r d''} / \widetilde{R}_{rr} \tag{6}
$$

Moreover, we know that due to hydrostatic equilibrium, $\partial_r \overline{P} \sim -\overline{\rho}\; \overline{g}_r$, and based on Equation (2) we can write, that dilatation flux $\overline{u'_r d''}$ is:

$$
\overline{u'_r d''} \sim \frac{\widetilde{R}_{rr}\; \overline{g}_r}{\Gamma_1\; \overline{P}} \tag{7}
$$

Also, the expansion velocity $\widetilde{u}_r$ can be replaced by:

$$
\widetilde{u}_r = -\partial_t \overline{M} / 4\pi r^2 \overline{\rho} \tag{8}
$$

So the alternative hydrodynamic stellar structure equations for hydrostatic convection become a system with only **one unknown the composition flux** f_α for which we need a proper model. See validation of these simplified alternative hydrodynamic stellar structure equation in Figure 2.

$$
\begin{aligned}
\partial_r \overline{m} &= -\, \overline{\rho}\; \overline{m}\; \overline{g}_r / \Gamma_1 \overline{P} + 4\pi r^2 \overline{\rho} && (9)\\
\partial_r \overline{P} &= -\, \overline{\rho}\; \overline{g}_r && (10)\\
\partial_r \widetilde{L} &= -\, 4\pi r^2 \widetilde{u}_r\; \overline{\rho}\; \overline{g}_r / \Gamma_1 + \widetilde{\epsilon}_t \partial_r 4\pi r^2 \overline{\rho} \widetilde{u}_r && (11)\\
\partial_r \overline{T} &= -\, (\Gamma_3 - 1)\; \overline{\rho}\; \overline{T}\; \overline{g}_r / \Gamma_1 \overline{P} && (12)\\
\partial_t \widetilde{X}_i &= \widetilde{\dot{X}}_i^{nuc} - (1/\overline{\rho}) \nabla_r f_i - \widetilde{u}_r \partial_r \widetilde{X}_i && (13)
\end{aligned}
$$

1.1 Derivation of the alternative hydrodynamic stellar structure equations from flux evolution equations

From the pressure flux equation (Fig.5) we get, that:

$$+\overline{u'_r u''_r}\ \partial_r \overline{P} = -\Gamma_1\ \overline{u'_r P d} \quad \backslash . \overline{\rho} \tag{14}$$

$$+\overline{\rho}\ \overline{u'_r u''_r}\ \partial_r \overline{P} = -\overline{\rho}\ \Gamma_1\ \overline{u'_r P d} \qquad P = \overline{P} + P' \quad d = \overline{d} + d' \tag{15}$$

$$+\widetilde{R}_{rr}\ \partial_r \overline{P} = -\overline{\rho}\ \Gamma_1\ (\overline{P u'_r}\ \overline{d} + \overline{P}\ \overline{u'_r d'} + \overline{P' u'_r d'}) \qquad \overline{P}\ \overline{u'_r d'} >> \overline{P u'_r}\ \overline{d} + \overline{P' u'_r d'} \ \text{ and } \ \overline{a'b'} = \overline{a'b''} = \overline{a''b'} \tag{16}$$

$$+\widetilde{R}_{rr}\ \partial_r \overline{P} = -\overline{\rho}\ \Gamma_1\ \overline{P}\ \overline{u'_r d''} \tag{17}$$

$$\partial_r \overline{P} = -\overline{\rho}\ \Gamma_1\ \overline{P}\ \overline{u'_r d''}/\widetilde{R}_{rr} \tag{18}$$

$$\partial_r \overline{P} = -\overline{\rho}\widetilde{g}_r \quad \text{hydrostatic equilibrium} \tag{19}$$

$$\overline{u'_r d''} = +\widetilde{R}_{rr}\widetilde{g}_r/\Gamma_1\overline{P} \quad \text{model for dilatation flux used later} \tag{20}$$

From the turbulent mass flux equation (Fig.5) we get, that:

$$+\widetilde{R}_{rr}/\overline{\rho}\ \partial_r \overline{\rho} = -\overline{\rho}\ \overline{u'_r d''} \tag{21}$$

$$\partial_r \overline{\rho} = -\overline{\rho}\ \overline{\rho}\ \overline{u'_r d''}/\widetilde{R}_{rr} \quad \backslash . V \text{ (volume)} = 4\pi r^3/3 \tag{22}$$

$$V \partial_r \overline{\rho} = -\overline{\rho}\ V\ \overline{\rho}\ \overline{u'_r d''}/\widetilde{R}_{rr} \tag{23}$$

$$\partial_r \overline{\rho} V - \rho \partial_r V = -\overline{\rho}\ V\ \overline{\rho}\ \overline{u'_r d''}/\widetilde{R}_{rr} \tag{24}$$

$$\partial_r \overline{m} - \rho 4\pi r^2 = -\overline{m}\ \overline{\rho}\ \overline{u'_r d''}/\widetilde{R}_{rr} \tag{25}$$

$$\partial_r \overline{m} = +\rho 4\pi r^2 - \overline{m}\ \overline{\rho}\ \overline{u'_r d''}/\widetilde{R}_{rr} \tag{26}$$

$$\partial_r \overline{m} = +\rho 4\pi r^2 - \overline{\rho}\ \overline{m}\ \widetilde{g}_r/\Gamma_1\overline{P} \tag{27}$$

$$\partial_r \overline{m} = -\overline{\rho}\ \overline{m}\ \widetilde{g}_r/\Gamma_1\overline{P} + \rho 4\pi r^2 \tag{28}$$

From the temperature flux equation (Fig.5) we get, that:

$$-\overline{u'_r u''_r}\ \partial_r \overline{T} = +(\Gamma_3 - 1)\ \overline{T}\ \overline{u'_r d''} \quad \backslash . \overline{\rho} \tag{29}$$

$$-\overline{\rho}\ \overline{u'_r u''_r}\ \partial_r \overline{T} = +(\Gamma_3 - 1)\ \overline{\rho}\ \overline{T}\ \overline{u'_r d''} \tag{30}$$

$$-\widetilde{R}_{rr}\ \partial_r \overline{T} = +(\Gamma_3 - 1)\ \overline{\rho}\ \overline{T}\ \overline{u'_r d''} \tag{31}$$

$$\partial_r \overline{T} = -(\Gamma_3 - 1)\ \overline{\rho}\ \overline{T}\ \overline{u'_r d''}/\widetilde{R}_{rr} \tag{32}$$

$$\partial_r \overline{T} = -(\Gamma_3 - 1)\ \overline{\rho}\ \overline{T}\ (\widetilde{R}_{rr}\widetilde{g}_r/\Gamma_1\overline{P})/\widetilde{R}_{rr} \tag{33}$$

$$\partial_r \overline{T} = -(\Gamma_3 - 1)\ \overline{\rho}\ \overline{T}\ \widetilde{g}_r/\Gamma_1\overline{P} \tag{34}$$

From the internal energy flux equation (Fig.5) we get, that:

$$-\widetilde{R}_{rr}\ \partial_r\widetilde{\varepsilon}_i = +\ \overline{u''_r P d} \qquad P = \overline{P} + P' \quad d = \overline{d} + d' \tag{35}$$

$$-\widetilde{R}_{rr}\ \partial_r\widetilde{\varepsilon}_i = +\ \overline{P u''_r}\ \overline{d} + \overline{P}\ \overline{u''_r d'} + \overline{P' u''_r d'} \qquad \overline{P}\ \overline{u''_r d'} >> \overline{P u''_r}\ \overline{d} + \overline{P' u''_r d'} \ \text{ and } \ \overline{a' b'} = \overline{a' b''} = \overline{a'' b'} \tag{36}$$

$$-\widetilde{R}_{rr}\ \partial_r\widetilde{\varepsilon}_i = +\ \overline{P}\ \overline{u''_r d'} \qquad \overline{P} = \overline{\rho}\widetilde{\varepsilon}_i \tag{37}$$

$$-\widetilde{R}_{rr}\ \partial_r\widetilde{\varepsilon}_i = +\ \overline{\rho}\widetilde{\varepsilon}_i\ \overline{u''_r d'} \tag{38}$$

$$\partial_r\widetilde{\varepsilon}_i = -\ \overline{\rho}\widetilde{\varepsilon}_i\ \overline{u''_r d'}/\widetilde{R}_{rr} \tag{39}$$

$$\partial_r\widetilde{\varepsilon}_i = -\ \overline{\rho}\widetilde{\varepsilon}_i\ (\widetilde{R}_{rr}\widetilde{g}_r/\Gamma_1\overline{P})/\widetilde{R}_{rr} \tag{40}$$

$$\partial_r\widetilde{\varepsilon}_i = -\ \overline{\rho}\widetilde{\varepsilon}_i\ \widetilde{g}_r/\Gamma_1\overline{P} \qquad \overline{P} = \overline{\rho}\widetilde{\varepsilon}_i \tag{41}$$

$$\partial_r\widetilde{\varepsilon}_i = -\ \overline{\rho}\widetilde{\varepsilon}_i\ \widetilde{g}_r/\Gamma_1\overline{\rho}\widetilde{\varepsilon}_i \tag{42}$$

$$\partial_r\widetilde{\varepsilon}_i = -\ \widetilde{g}_r/\Gamma_1 \qquad \backslash .4\pi r^2\overline{\rho}\widetilde{u}_r \tag{43}$$

$$4\pi r^2\overline{\rho}\widetilde{u}_r\partial_r\widetilde{\varepsilon}_i = -\ 4\pi r^2\overline{\rho}\widetilde{u}_r\ \widetilde{g}_r/\Gamma_1 \tag{44}$$

$$\partial_r 4\pi r^2\overline{\rho}\widetilde{u}_r\widetilde{\varepsilon}_i - \widetilde{\varepsilon}_i\partial_r 4\pi r^2\overline{\rho}\widetilde{u}_r = -\ 4\pi r^2\overline{\rho}\widetilde{u}_r\ \widetilde{g}_r/\Gamma_1 \quad \varepsilon_K << \varepsilon_i \text{ and } \varepsilon_i \sim \varepsilon_t \tag{45}$$

$$\partial_r 4\pi r^2\overline{\rho}\widetilde{u}_r\widetilde{\varepsilon}_t - \widetilde{\varepsilon}_t\partial_r 4\pi r^2\overline{\rho}\widetilde{u}_r = -\ 4\pi r^2\overline{\rho}\widetilde{u}_r\ \widetilde{g}_r/\Gamma_1 \tag{46}$$

$$\partial_r\widetilde{L} - \widetilde{\varepsilon}_t\partial_r 4\pi r^2\overline{\rho}\widetilde{u}_r = -\ 4\pi r^2\overline{\rho}\widetilde{u}_r\ \widetilde{g}_r/\Gamma_1 \tag{47}$$

$$\partial_r\widetilde{L} = -\ 4\pi r^2\overline{\rho}\widetilde{u}_r\ \widetilde{g}_r/\Gamma_1 + \widetilde{\varepsilon}_t\partial_r 4\pi r^2\overline{\rho}\widetilde{u}_r \tag{48}$$

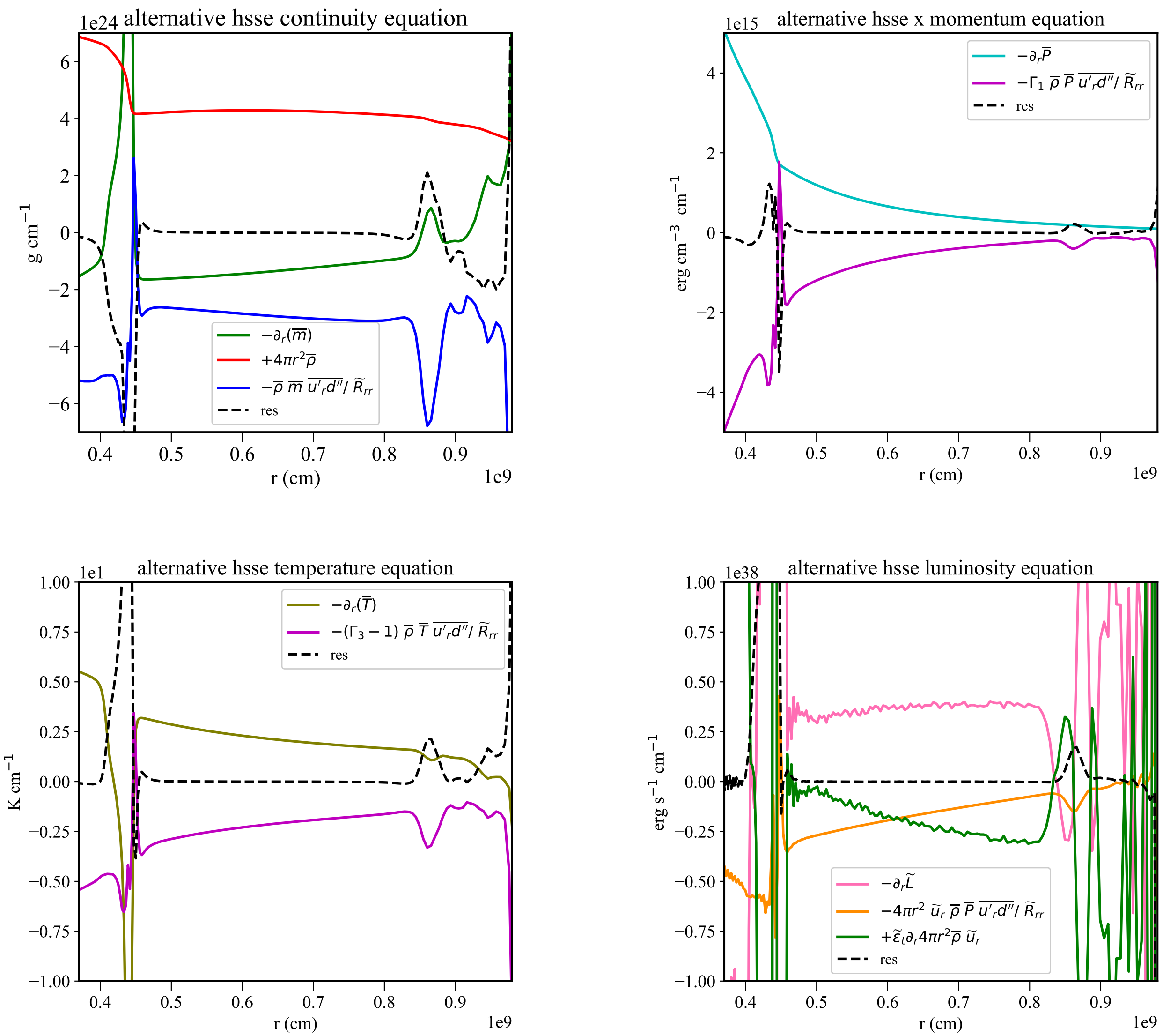

Figure 1: Alternative hydrodynamic stellar structure continuity.

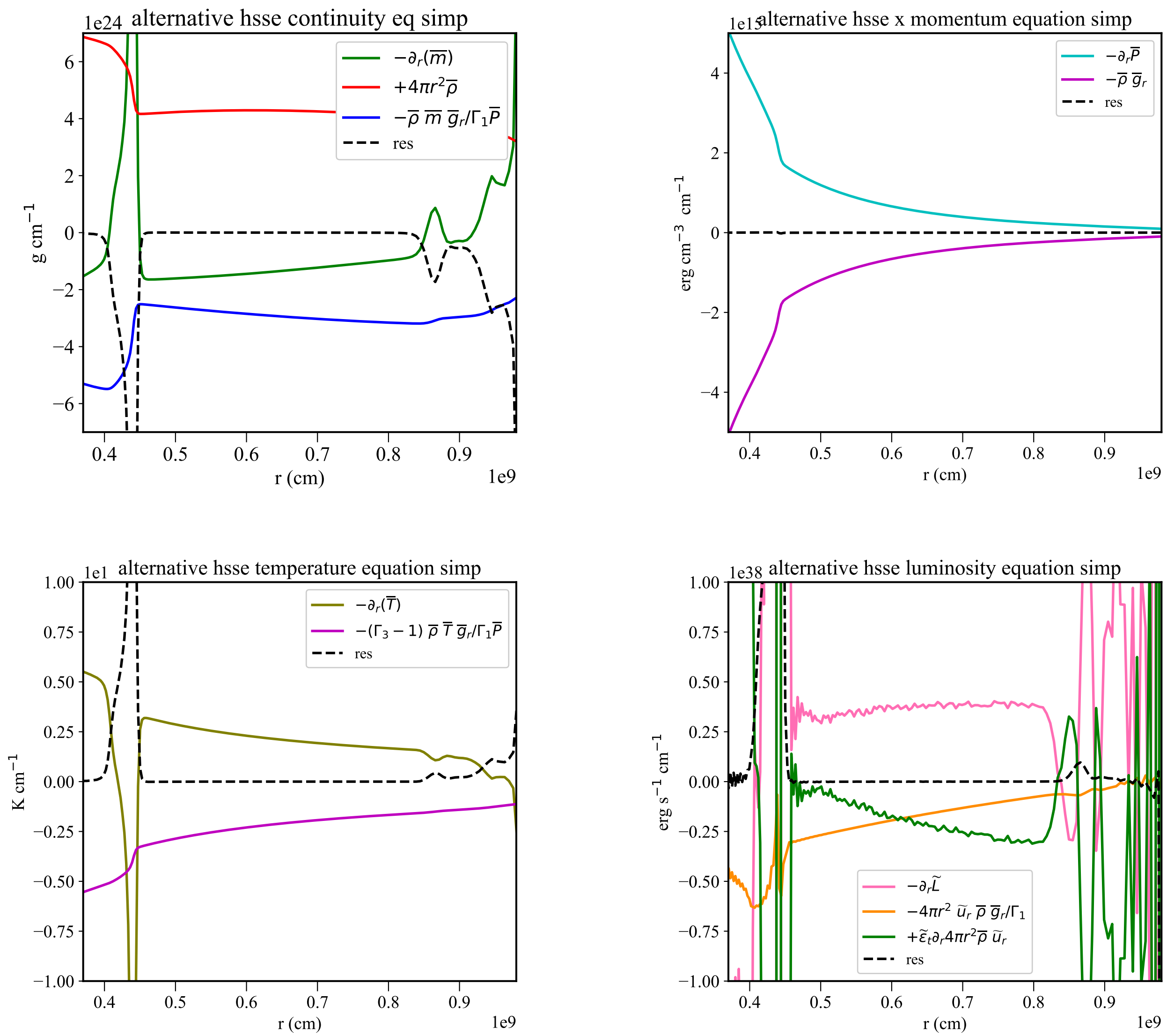

Figure 2: Simplified alternative hydrodynamic stellar structure continuity.

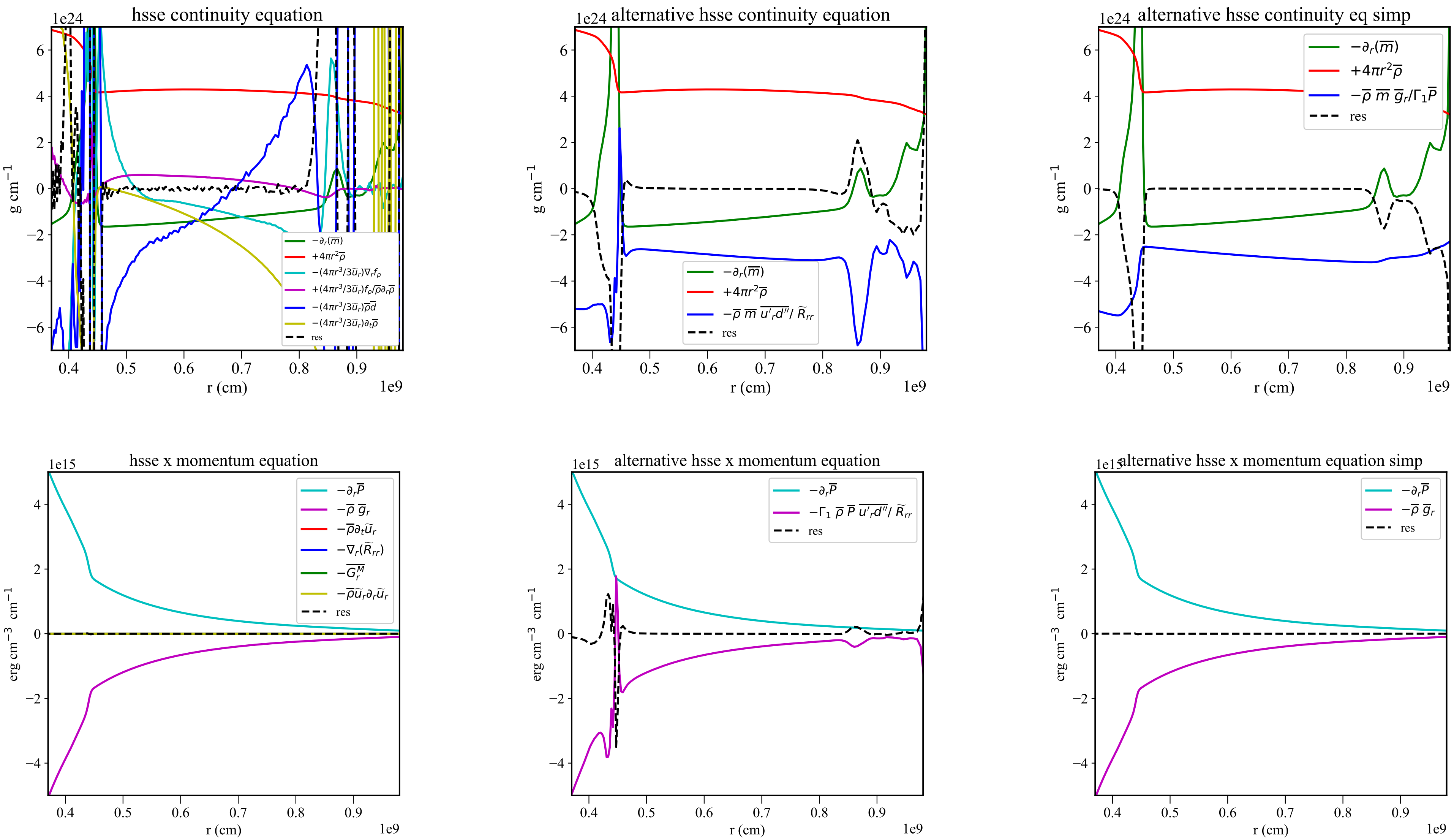

Figure 3: Comparison of all versions of continuity and momentum hydrodynamic stellar structure equations.

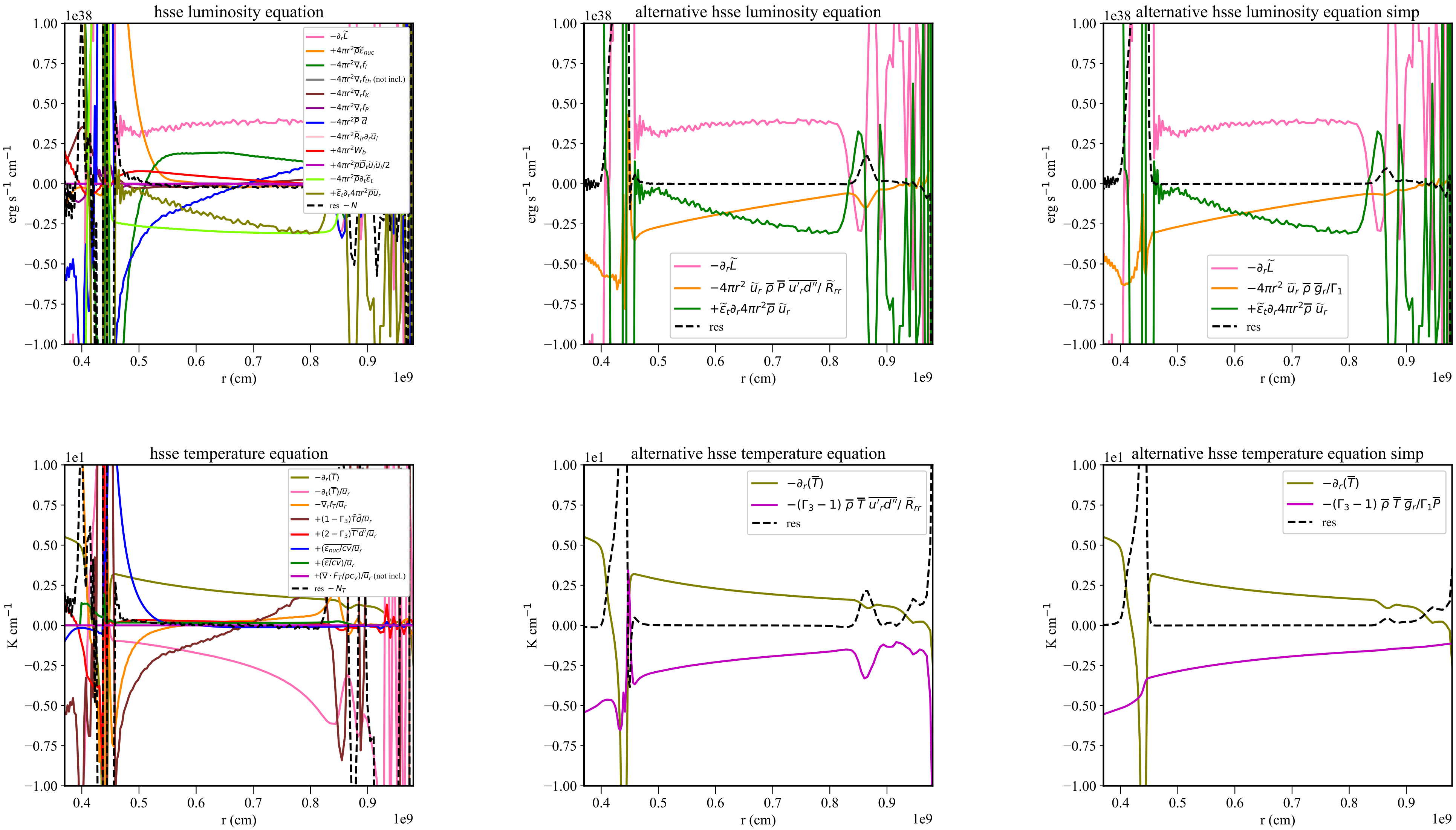

Figure 4: Comparison of all version of luminosity and temperature hydrodynamic stellar structure equations.

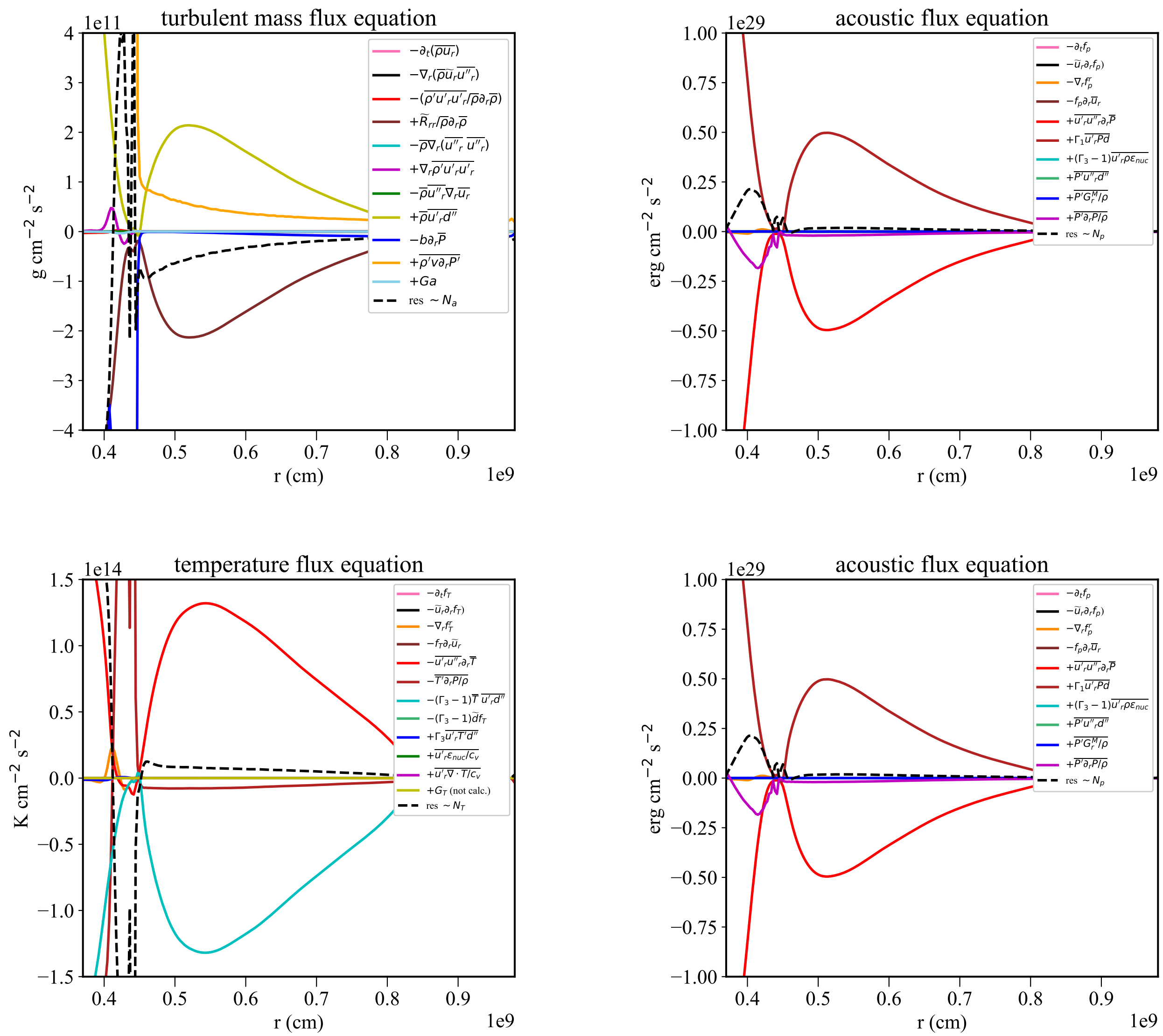

Figure 5: Flux evolution equations based on low-res oxygen burning model.

Table 1: Definitions:

ρ density	g_r radial gravitational acceleration
$m = \rho V = \rho \frac{4}{3}\pi r^3$ mass	$M = \int \rho(r) V = \int \rho(r) dV = \int \rho(r) 4\pi r^2 dr$ integrated mass
T temperature	$\mathcal{S} = \rho\epsilon_{\rm nuc}(q)$ nuclear energy production (cooling function)
P pressure	$\tau_{ij} = 2\mu S_{ij}$ viscous stress tensor (μ kinematic viscosity)
u_r, u_θ, u_ϕ velocity components	$S_{ij} = (1/2)(\partial_i u_j + \partial_j u_i)$ strain rate
$\mathbf{u} = u(u_r, u_\theta, u_\phi)$ velocity	$\widetilde{R}_{ij} = \overline{\rho}\widetilde{u''_i u''_j}$ Reynolds stress tensor
$j_z = r \sin\theta\ u_\phi$ specific angular momentum	$F_T = \chi\partial_r T$ heat flux
$d = \nabla \cdot \mathbf{u}$ dilatation	$\Gamma_1 = (d\ ln\ P/d\ ln\ \rho)\|_s$
ϵ_I specific internal energy	$\Gamma_2/(\Gamma_2 - 1) = (d\ ln\ P/d\ ln\ T)\|_s$
h specific enthalpy	$\Gamma_3 - 1 = (d\ ln\ T/d\ ln\ \rho)\|_s$
$k = (1/2)\widetilde{u''_i u''_i}$ turbulent kinetic energy	$\widetilde{k}^r = (1/2)\widetilde{u''_r u''_r} = (1/2)\widetilde{R}_{rr}/\overline{\rho}$ radial turbulent kinetic energy
ϵ_k specific kinetic energy	$\widetilde{k}^\theta = (1/2)\widetilde{u''_\theta u''_\theta} = (1/2)\widetilde{R}_{\theta\theta}/\overline{\rho}$ angular turbulent kinetic energy
ϵ_t specific total energy	$\widetilde{k}^\phi = (1/2)\widetilde{u''_\phi u''_\phi} = (1/2)\widetilde{R}_{\phi\phi}/\overline{\rho}$ angular turbulent kinetic energy
s specific entropy	$\widetilde{k}^h = \widetilde{k}^\theta + \widetilde{k}^\phi$ horizontal turbulent kinetic energy
$v = 1/\rho$ specific volume	$f_k = (1/2)\overline{\rho}\widetilde{u''_i u''_i u''_r}$ turbulent kinetic energy flux
X_α mass fraction of isotope α	$f_k^r = (1/2)\overline{\rho}\widetilde{u''_r u''_r u''_r}$ radial turbulent kinetic energy flux
$\dot{X}_\alpha^{nuc}$ rate of change of X_α	$f_k^\theta = (1/2)\overline{\rho}\widetilde{u''_\theta u''_\theta u''_r}$ angular turbulent kinetic energy flux
A_α number of nucleons in isotope α	$f_k^\phi = (1/2)\overline{\rho}\widetilde{u''_\phi u''_\phi u''_r}$ angular turbulent kinetic energy flux
Z_α charge of isotope α	$f_k^h = f_k^\theta + f_k^\phi$ horizontal turbulent kinetic energy flux
A mean number of nucleons per isotope	$W_p = \overline{P'd''}$ turbulent pressure dilatation
Z mean charge per isotope	$W_b = \overline{\rho}\overline{u''_r}\widetilde{g}_r$ buoyancy
$f_P = \overline{P'u'_r}$ acoustic flux	$f_T = -\overline{\chi\partial_r T}$ heat flux (χ thermal conductivity)

Table 2: Definitions (continued):

$f_I = \overline{\rho}\widetilde{\epsilon_I'' u_r''}$ internal energy flux	$f_\alpha = \overline{\rho}\widetilde{X_\alpha'' u_r''}$ X_α flux
$f_s = \overline{\rho}\widetilde{s'' u_r''}$ entropy flux	$f_{jz} = \overline{\rho}\widetilde{j_z'' u_r''}$ angular momentum flux
$f_T = \overline{u_r' T'}$ turbulent heat flux	$f_A = \overline{\rho}\widetilde{A'' u_r''}$ A (mean number of nucleons per isotope) flux
$f_h = \overline{\rho}\widetilde{h'' u_r''}$ enthalpy flux	$f_Z = \overline{\rho}\widetilde{Z'' u_r''}$ Z (mean charge per isotope) flux
$b = \overline{v'\rho'}$ density-specific volume covariance	$\mathcal{N}_\rho, \mathcal{N}_{ur}, \mathcal{N}_{u\theta}, \mathcal{N}_{u\phi}, \mathcal{N}_{jz}, \mathcal{N}_\alpha, \mathcal{N}_A, \mathcal{N}_Z$ numerical effect
$f_\tau = f_\tau^r + f_\tau^\theta + f_\tau^\phi$ viscous flux	$\mathcal{N}_{\epsilon I} = -\nabla_r f_\tau + \varepsilon_k$ numerical effect
$f_\tau^r = -\overline{\tau_{rr}' u_r'}$ viscous flux	$\mathcal{N}_{\epsilon k} = -\varepsilon_k$ numerical effect
$f_\tau^\theta = -\overline{\tau_{\theta r}' u_\theta'}$ viscous flux	$\mathcal{N}_{\epsilon t} = -\nabla_r f_\tau$ numerical effect
$f_\tau^\phi = -\overline{\tau_{\phi r}' u_\phi'}$ viscous flux	$\mathcal{N}_s = \overline{-\varepsilon_k / T}$ numerical effect
$f_\tau^h = f_\tau^\theta + f_\tau^\phi$ viscous flux	$\mathcal{N}_h = -\nabla_r f_\tau + (\Gamma_3 - 1)\varepsilon_k$ numerical effect
$f_I^r = \overline{\rho}\widetilde{\epsilon_I'' u_r'' u_r''}$ radial flux of f_I	$\mathcal{N}_P = +(\Gamma_3 - 1)\varepsilon_k$ numerical effect
$f_s^r = \overline{\rho}\widetilde{s'' u_r'' u_r''}$ radial flux of f_s	$\mathcal{N}_T = +\overline{(\tau_{ij}\partial_j u_i)/(c_v \rho)}$ numerical effect
$f_h^r = \overline{\rho}\widetilde{h'' u_r'' u_r''}$ radial flux of f_h	$\mathcal{N}_{Rrr} = -2\nabla_r f_\tau^r - 2\varepsilon_k^r$ numerical effect
$f_T^r = \overline{T' u_r' u_r'}$ radial flux of f_T	$\mathcal{N}_{R\theta\theta} = -2\nabla_r f_\tau^\theta - 2\varepsilon_k^\theta$ numerical effect
$f_{jz}^r = \overline{\rho}\widetilde{j_z'' u_r'' u_r''}$ radial flux of f_{jz}	$\mathcal{N}_{R\phi\phi} = -2\nabla_r f_\tau^\phi - 2\varepsilon_k^\phi$ numerical effect
$f_\alpha^r = \overline{\rho}\widetilde{X_\alpha'' u_r'' u_r''}$ radial flux of f_α	$\mathcal{N}_k = -\nabla_r f_\tau - \varepsilon_k$ numerical effect
$f_A^r = \overline{\rho}\widetilde{A'' u_r'' u_r''}$ radial flux of f_A	$\mathcal{N}_{kr} = -\nabla_r f_\tau^r - \varepsilon_k^r$ numerical effect
$f_Z^r = \overline{\rho}\widetilde{Z'' u_r'' u_r''}$ radial flux of f_Z	$\mathcal{N}_{kh} = -\nabla_r f_\tau^h - \varepsilon_k^h$ numerical effect
$\mathcal{G}_k^r = -(1/2)\overline{G_{rr}^R} - \overline{u_r'' G_r^M}$	$\mathcal{N}_a = -\varepsilon_a$ numerical effect

Table 3: Definitions (continued):

$\mathcal{G}_k^\theta = -(1/2)\overline{G_{\theta\theta}^R} - \overline{u_\theta'' G_\theta^M}$	$\mathcal{N}_b$ numerical effect
$\mathcal{G}_k^\phi = -(1/2)\overline{G_{\phi\phi}^R} - \overline{u_\phi'' G_\phi^M}$	$\mathcal{N}_{fI} = -\nabla_r(\overline{\epsilon_I'' \tau_{rr}'}) + \overline{u_r'' \tau_{ij} \partial_i u_j} - \varepsilon_I$ numerical effect
$\mathcal{G}_k^h = +\mathcal{G}_k^\theta + \mathcal{G}_k^\phi$	$\mathcal{N}_{fh} = -\nabla_r(\overline{h'' \tau_{rr}'}) + \overline{u_r''(\Gamma_3 - 1)\tau_{ij}\partial_i u_j} - \overline{u_r'' \nabla_i u_i \tau_{ji}} - \varepsilon_h$ numerical effect
$\mathcal{G}_a = +\overline{\rho' v G_r^M}$	$\mathcal{N}_{fs} = -\nabla_r(\overline{s'' \tau_{rr}'}) + \overline{u_r'' \tau_{ij} \partial_i u_j / T} - \varepsilon_s$ numerical effect
$\mathcal{G}_I = -\overline{G_r^I} - \overline{\epsilon_I'' G_r^M}$	$\mathcal{N}_{fA} = -\nabla_r(\overline{A'' \tau_{rr}'}) - \varepsilon_A$ numerical effect
$\mathcal{G}_\alpha = -\overline{G_r^\alpha} - \overline{X_\alpha'' G_r^M}$	$\mathcal{N}_{fZ} = -\nabla_r(\overline{Z'' \tau_{rr}'}) - \varepsilon_Z$ numerical effect
$\mathcal{G}_A = -\overline{G_r^A} - \overline{A'' G_r^M}$	$\mathcal{N}_{f\alpha} = -\nabla_r(\overline{\alpha'' \tau_{rr}'}) - \varepsilon_\alpha$ numerical effect
$\mathcal{G}_Z = -\overline{G_r^Z} - \overline{Z'' G_r^M}$	$\mathcal{N}_{fjz} = -\nabla_r(\overline{j_z'' \tau_{rr}'}) - \varepsilon_{jz}$ numerical effect
$\mathcal{G}_h = -\overline{G_r^h} - \overline{h'' G_r^M}$	$\mathcal{N}_{fT} = +\overline{T' \partial_i \tau_{ri} / \rho} + \overline{u_r' \tau_{ij} \partial_i u_j / \rho c_v}$ numerical effect
$\mathcal{G}_T = -\overline{G_r^T} - \overline{T' G_r^M}$	
$\mathcal{G}_s = -\overline{G_r^s} - \overline{s'' G_r^M}$	
$\mathcal{G}_{jz} = -\overline{G_r^{jz}} - \overline{j_z'' G_r^M}$	
$\sigma_\rho = \overline{\rho' \rho'}$	$\mathcal{N}_{\sigma_\rho}$ numerical effect
$\sigma_P = \overline{P' P'}$	$\mathcal{N}_{\sigma_P} = +2(\Gamma_3 - 1)\overline{P' \tau_{ij} \partial_i u_j}$ numerical effect
$\sigma_T = \overline{T' T'}$	$\mathcal{N}_{\sigma_T} = +\overline{2T' \tau_{ij} \partial_i u_j / \rho c_v}$ numerical effect
$\sigma_{ur} = \widetilde{u_r'' u_r''}$	$\mathcal{N}_{\sigma_{ur}} = +2\nabla_r f_\tau^r - 2\varepsilon_k^r$ numerical effect
$\sigma_s = \widetilde{s'' s''}$	$\mathcal{N}_{\sigma_s} = +2\overline{s'' \tau_{ij} \partial_j u_i / T}$ numerical effect
$\sigma_\alpha = \widetilde{X_\alpha'' X_\alpha''}$	$\mathcal{N}_{\sigma_\alpha}$ numerical effect numerical effect
$\sigma_{\epsilon I} = \widetilde{\epsilon_I'' \epsilon_I''}$	$N_{\sigma \epsilon_I} = +2\overline{\epsilon_I'' \tau_{ij} \partial_j u_i}$ numerical effect

Table 4: Definitions (continued):

$$
\begin{aligned}
\varepsilon_k^r &= \overline{\tau'_{rr}\partial_r u''_r} + \overline{\tau'_{r\theta}(1/r)\partial_\theta u''_r} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi u''_r} \\
\varepsilon_k^\theta &= \overline{\tau'_{\theta r}\partial_r u''_\theta} + \overline{\tau'_{\theta\theta}(1/r)\partial_\theta u''_\theta} + \overline{\tau'_{\theta\phi}(1/r\sin\theta)\partial_\phi u''_\theta} \\
\varepsilon_k^\phi &= \overline{\tau'_{\phi r}\partial_r u''_\phi} + \overline{\tau'_{\phi\theta}(1/r)\partial_\theta u''_\phi} + \overline{\tau'_{\phi\phi}(1/r\sin\theta)\partial_\phi u''_\phi} \\
\varepsilon_k &= \varepsilon_k^r + \varepsilon_k^\theta + \varepsilon_k^\phi \\
\varepsilon_k^h &= \varepsilon_k^\theta + \varepsilon_k^\phi \\
\varepsilon_a &= \overline{\rho' v \nabla_r \tau'_{rr}} \\
\varepsilon_I &= \overline{\tau'_{rr}\partial_r \epsilon''_I} + \overline{\tau'_{r\theta}(1/r)\partial_\theta \epsilon''_I} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi \epsilon''_I} \\
\varepsilon_s &= \overline{\tau'_{rr}\partial_r s''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta s''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi s''} \\
\varepsilon_\alpha &= \overline{\tau'_{rr}\partial_r X''_\alpha} + \overline{\tau'_{r\theta}(1/r)\partial_\theta X''_\alpha} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi X''_\alpha} \\
\varepsilon_A &= \overline{\tau'_{rr}\partial_r A''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta A''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi A''} \\
\varepsilon_Z &= \overline{\tau'_{rr}\partial_r Z''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta Z''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi Z''} \\
\varepsilon_h &= \overline{\tau'_{rr}\partial_r h''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta h''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi h''} \\
\varepsilon_{jz} &= \overline{\tau'_{rr}\partial_r j''_z} + \overline{\tau'_{r\theta}(1/r)\partial_\theta j''_z} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi j''_z}
\end{aligned}
$$

$$
\begin{aligned}
\overline{G_r^M} &= -\overline{\rho u_\theta u_\theta/r} - \overline{\rho u_\phi u_\phi/r} \\
\overline{G_\theta^M} &= +\overline{\rho u_\theta u_r/r} - \overline{\rho u_\phi u_\phi/(r\tan\theta)} \\
\overline{G_\phi^M} &= +\overline{\rho u_\phi u_r/r} + \overline{\rho u_\phi u_\theta/(r\tan\theta)} \\
\overline{G_{rr}^R} &= -\overline{\rho u''_\theta u''_\theta u''_r/r} - \overline{\rho u''_\theta u''_r u''_\theta/r} - \overline{\rho u''_\phi u''_\phi u''_r/r} - \overline{\rho u''_\phi u''_r u''_\phi/r} \\
\overline{G_{\theta\theta}^R} &= +\overline{\rho u''_\theta u''_r u''_\theta/r} + \overline{\rho u''_\theta u''_\theta u''_r/r} - \overline{\rho u''_\phi u''_\theta u''_\phi/(r\tan\theta)} - \overline{u''_\phi u''_\theta u''_\phi/(r\tan\theta)} \\
\overline{G_{\phi\phi}^R} &= +\overline{\rho u''_\phi u''_r r_\phi/r} + \overline{\rho u''_\phi u''_\theta u''_\phi/(r\tan\theta)} + \overline{\rho u''_\phi u''_\phi u''_r/r} + \overline{\rho u''_\phi u''_\phi u''_\theta/(r\tan\theta)} \\
\overline{G_r^I} &= -\overline{\rho\epsilon''_I u''_\theta u''_\theta/r} - \overline{\rho\epsilon''_I u''_\phi u''_\phi/r} \\
\overline{G_r^s} &= -\overline{\rho s'' u''_\theta u''_\theta/r} - \overline{\rho s'' u''_\phi u''_\phi/r} \\
\overline{G_r^\alpha} &= -\overline{\rho X''_\alpha u''_\theta u''_\theta/r} - \overline{\rho X''_\alpha u''_\phi u''_\phi/r} \\
\overline{G_r^A} &= -\overline{\rho A'' u''_\theta u''_\theta/r} - \overline{\rho A'' u''_\phi u''_\phi/r} \\
\overline{G_r^Z} &= -\overline{\rho Z'' u''_\theta u''_\theta/r} - \overline{\rho Z'' u''_\phi u''_\phi/r} \\
\overline{G_r^h} &= -\overline{\rho h'' u''_\theta u''_\theta/r} - \overline{\rho h'' u''_\phi u''_\phi/r} \\
\overline{G_r^T} &= -\overline{\rho T' u'_\theta u'_\theta/r} - \overline{\rho T' u'_\phi u'_\phi/r} \\
\overline{G_r^{jz}} &= -\overline{\rho j''_z u''_\theta u''_\theta/r} - \overline{\rho j''_z u''_\phi u''_\phi/r}
\end{aligned}
$$

$$
\nabla(.) = \nabla_r(.) + \nabla_\theta(.) + \nabla_\phi(.) = \frac{1}{r^2}\partial_r(r^2.) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta.) + \frac{1}{r\sin\theta}\partial_\phi(.)
$$