1 Alternative hydrodynamic stellar structure equations

Below is a set of alternative hydrodynamic stellar structure equations inspired by mean fields from the mean density/temperature/internal energy and pressure flux equations involving gradient term scaled by favrian Reynolds stress $\widetilde{R}_{rr}$ and the other one turbulent dilatation flux $\overline{u'_r d''}$.

$$\partial_r \overline{M} \sim \; - \overline{\rho} \; \overline{M} \;\; \overline{u'_r d''} / \; \widetilde{R}_{rr} + 4\pi r^2 \overline{\rho} \tag{1}$$
$$\partial_r \overline{P} \sim \; - \Gamma_1 \; \overline{\rho} \; \overline{P} \; \overline{u'_r d''} / \; \widetilde{R}_{rr} \tag{2}$$
$$\partial_r \widetilde{L} \sim \; + \widetilde{\epsilon}_t \partial_r 4\pi r^2 \overline{\rho} \widetilde{u}_r - 4\pi r^2 \overline{\rho} \; \overline{P} \; \overline{u'_r d''} / \; \widetilde{R}_{rr} \tag{3}$$
$$\partial_r \overline{T} \sim \; - (\Gamma_3 - 1) \; \overline{\rho} \; \overline{T} \; \overline{u'_r d''} / \; \widetilde{R}_{rr} \tag{4}$$
$$\partial_t \widetilde{X}_i = \widetilde{\dot{X}}_i^{nuc} - (1/\overline{\rho}) \nabla_r f_\alpha - \widetilde{u}_r \partial_r \widetilde{X}_\alpha \tag{5}$$

These equations could be perfectly validated by our ransX framework and are shown on the next page in Figure 1. It appears that there is a universality relation between gradient of a mean thermodynamic variable Q and dilatation flux $\overline{u'_r d''}$, that is:

$$\partial_r \overline{Q} \sim -const \; \overline{\rho} \; \overline{Q} \; \overline{u'_r d''} / \widetilde{R}_{rr} \tag{6}$$

Moreover, we know that due to hydrostatic equilibrium, $\partial_r \overline{P} \sim -\overline{\rho} \; \overline{g}_r$, and based on Equation (2) we can write, that dilatation flux $\overline{u'_r d''}$ is:

$$\overline{u'_r d''} \sim \frac{\widetilde{R}_{rr} \; \overline{g}_r}{\Gamma_1 \; \overline{P}} \tag{7}$$

Also, the expansion velocity $\widetilde{u}_r$ can be replaced by:

$$\widetilde{u}_r = -\partial_t \overline{M} / 4\pi r^2 \overline{\rho} \tag{8}$$

So the alternative hydrodynamic stellar structure equations for hydrostatic convection become a system with only **one unknown the composition flux** f_α for which we need a proper model. See validation of these simplified alternative hydrodynamic stellar structure equation in Figure 2.

$$\partial_r \overline{M} \sim \; - \overline{\rho} \; \overline{M} \; \overline{g}_r / \Gamma_1 \overline{P} + 4\pi r^2 \overline{\rho} \tag{9}$$
$$\partial_r \overline{P} \sim \; - \overline{\rho} \; \overline{g}_r \tag{10}$$
$$\partial_r \widetilde{L} \sim \; - 4\pi r^2 \overline{\rho} \; \overline{g}_r / \Gamma_1 + \widetilde{\epsilon}_t \partial_r 4\pi r^2 \overline{\rho} \widetilde{u}_r \tag{11}$$
$$\partial_r \overline{T} \sim - (\Gamma_3 - 1) \; \overline{\rho} \; \overline{T} \; \overline{g}_r / \Gamma_1 \overline{P} \tag{12}$$
$$\partial_t \widetilde{X}_i = \widetilde{\dot{X}}_i^{nuc} - (1/\overline{\rho}) \nabla_r f_\alpha - \widetilde{u}_r \partial_r \widetilde{X}_\alpha \tag{13}$$

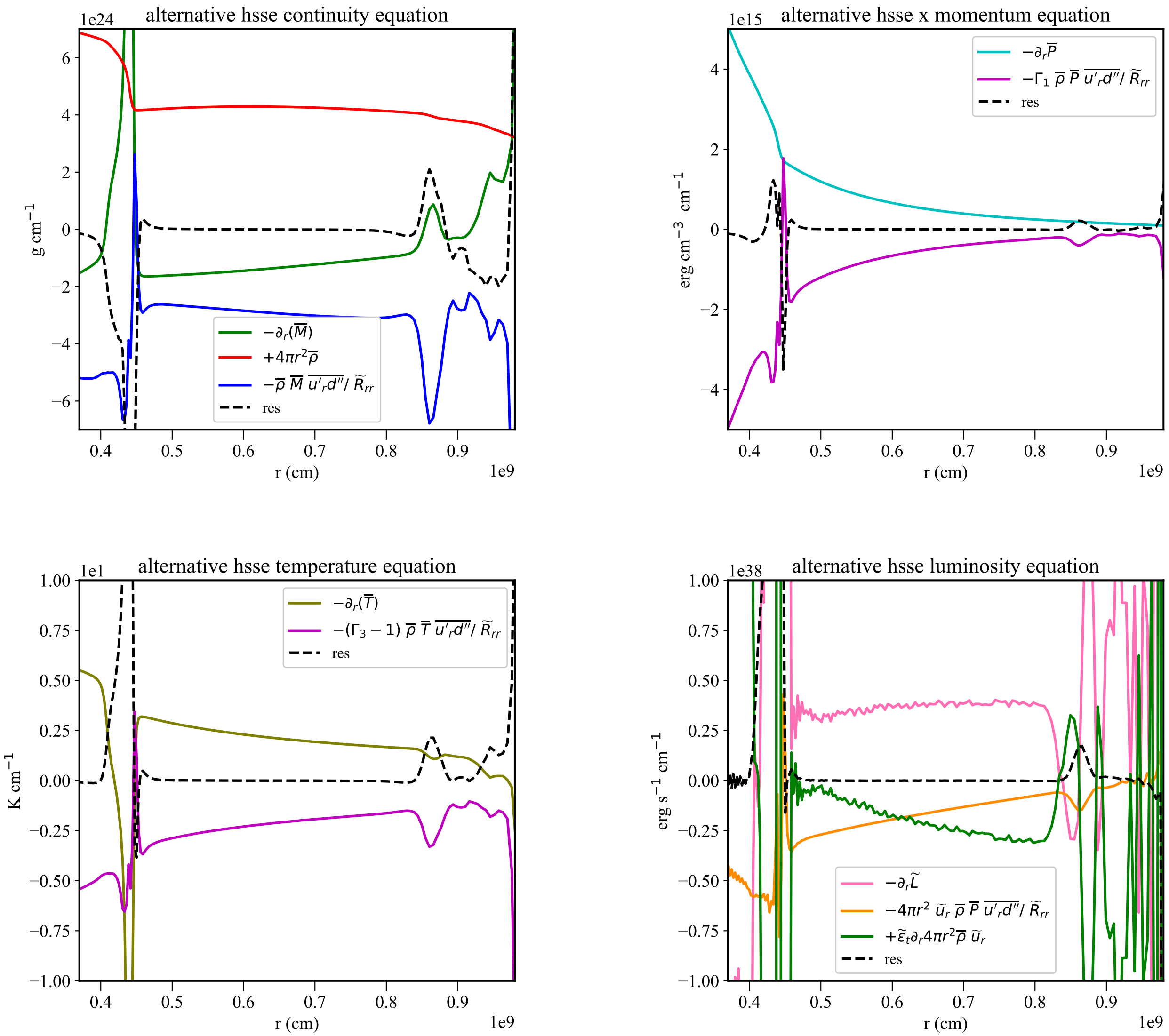

Figure 1: Alternative hydrodynamic stellar structure continuity.

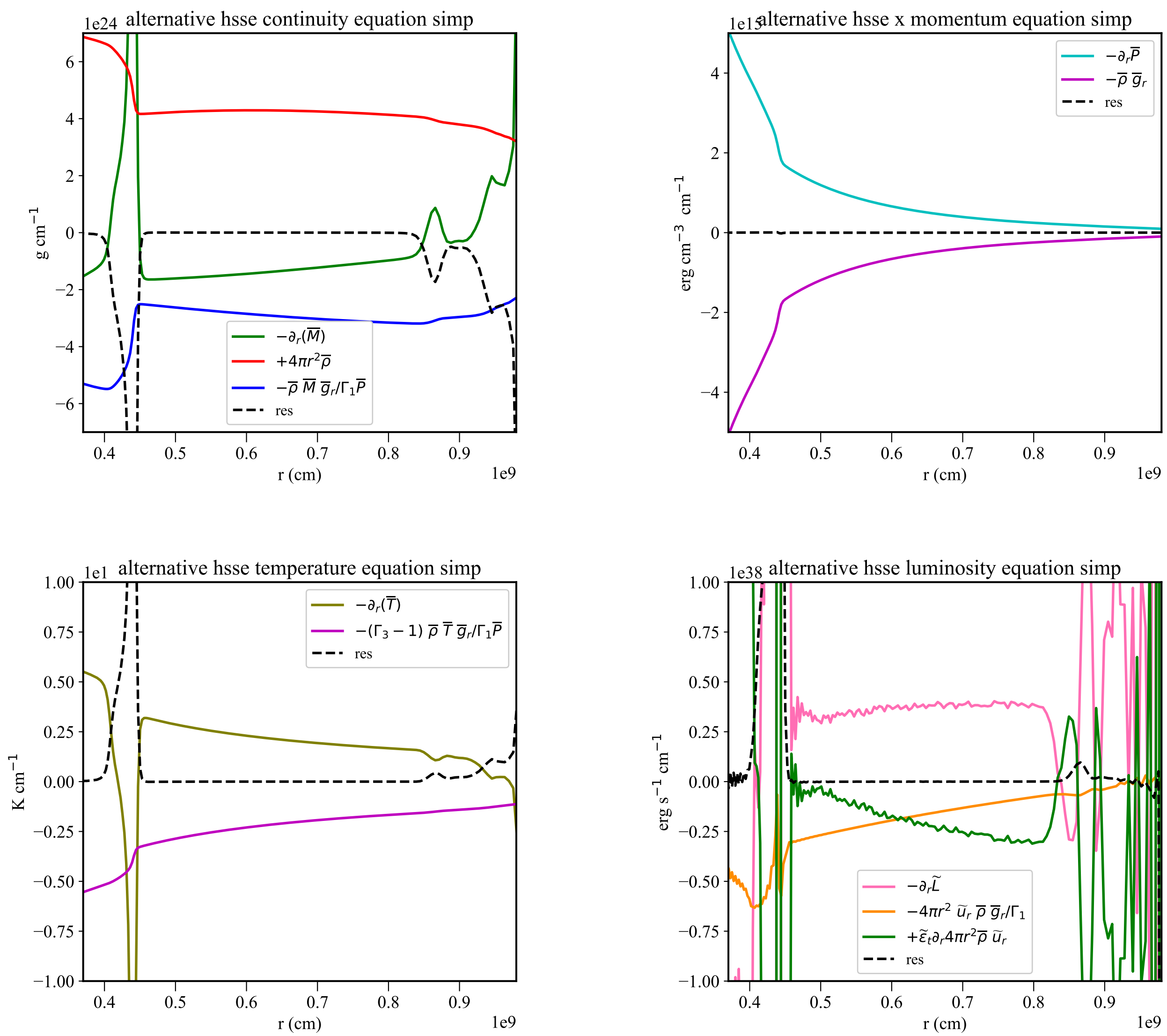

Figure 2: Simplified alternative hydrodynamic stellar structure continuity.

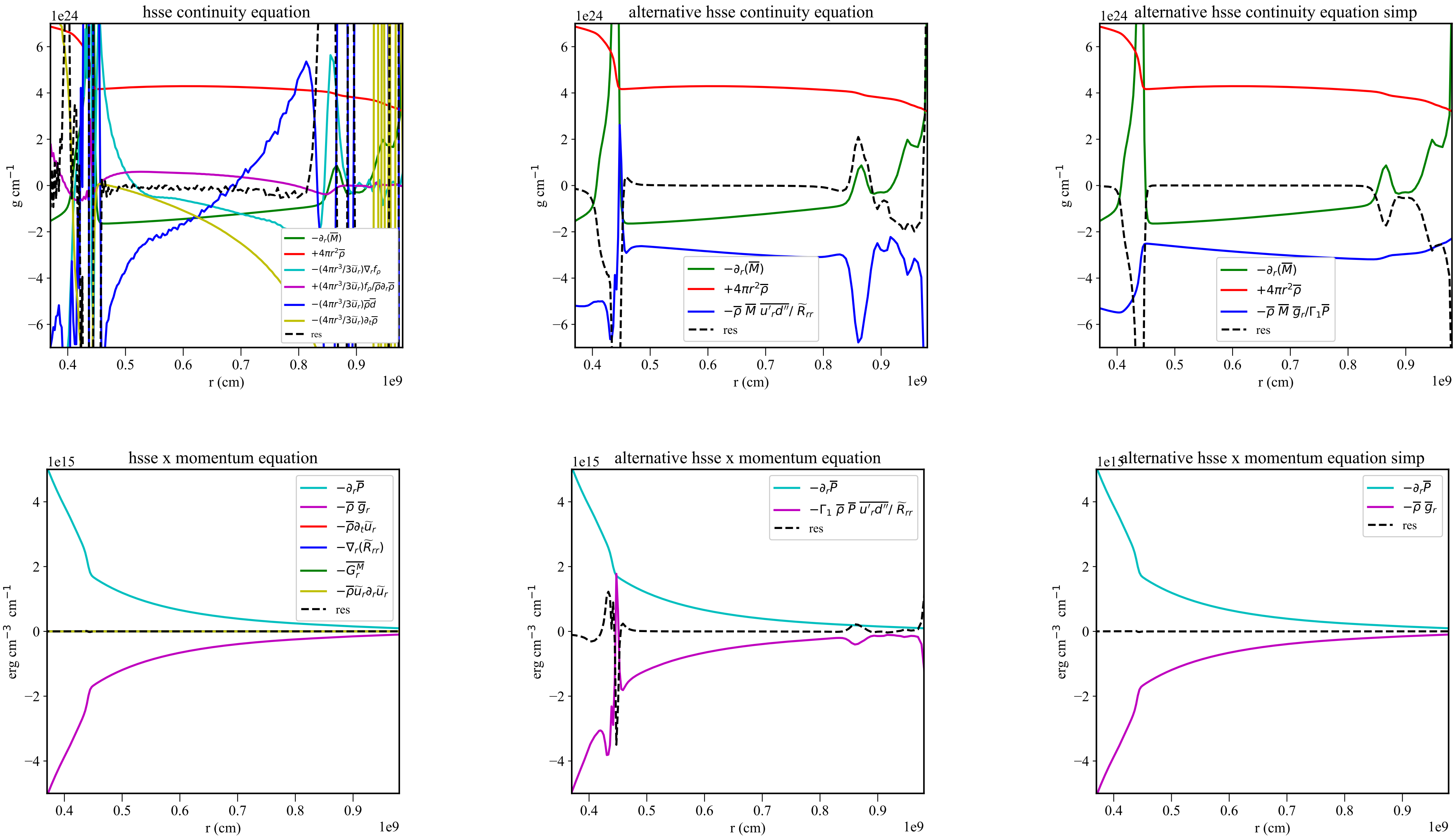

Figure 3: Comparison of all versions of continuity and momentum hydrodynamic stellar structure equations.

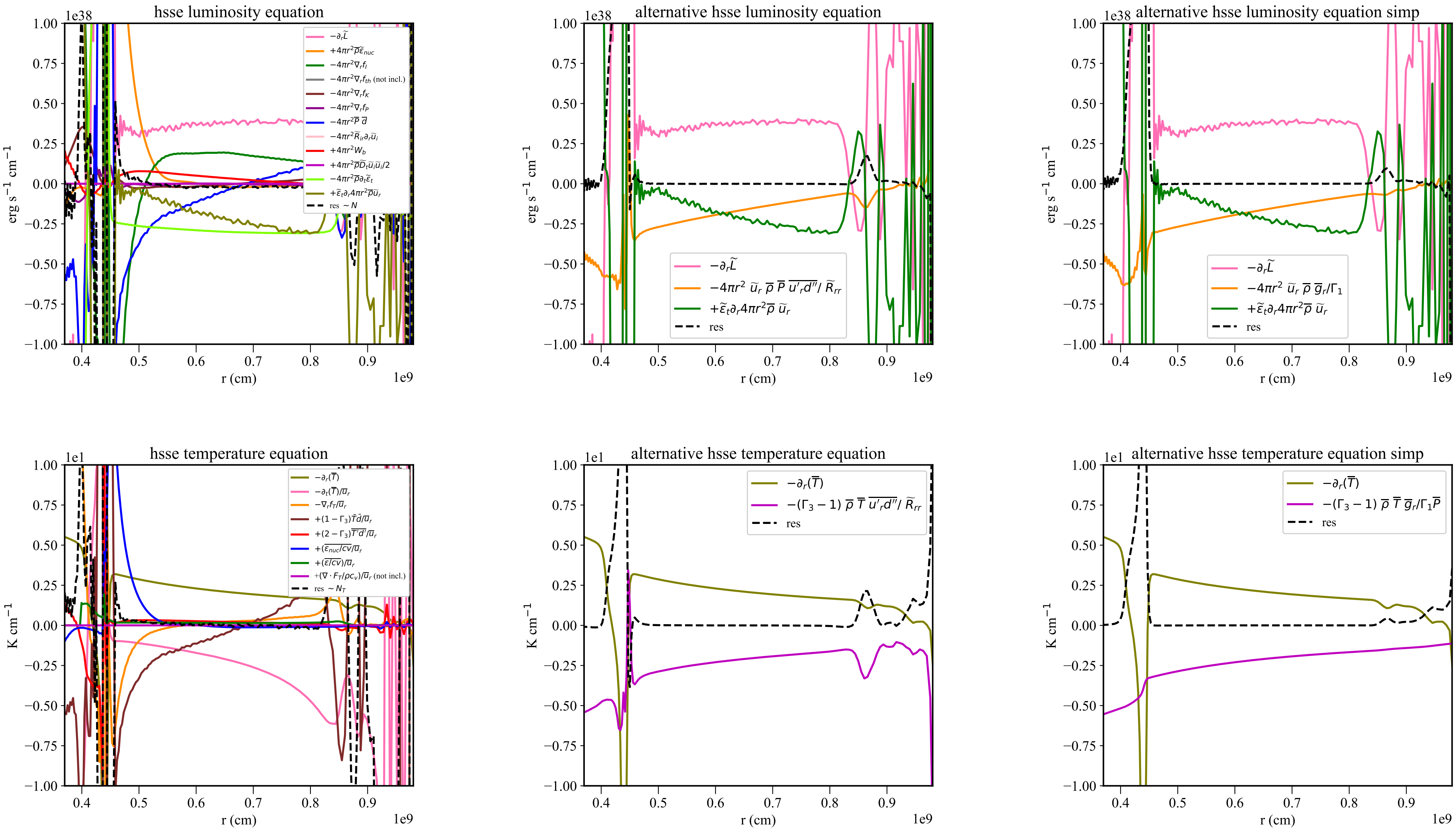

Figure 4: Comparison of all version of luminosity and temperature hydrodynamic stellar structure equations.

Table 1: Definitions:

ρ density	g_r radial gravitational acceleration
T temperature	$\mathcal{S} = \rho\epsilon_{\rm nuc}(q)$ nuclear energy production (cooling function)
P pressure	$\tau_{ij} = 2\mu S_{ij}$ viscous stress tensor (μ kinematic viscosity)
u_r, u_θ, u_ϕ velocity components	$S_{ij} = (1/2)(\partial_i u_j + \partial_j u_i)$ strain rate
$\mathbf{u} = u(u_r, u_\theta, u_\phi)$ velocity	$\widetilde{R}_{ij} = \overline{\rho}\widetilde{u''_i u''_j}$ Reynolds stress tensor
$j_z = r\sin\theta\ u_\phi$ specific angular momentum	$F_T = \chi\partial_r T$ heat flux
$d = \nabla\cdot\mathbf{u}$ dilatation	$\Gamma_1 = (d\ ln\ P/d\ ln\ \rho)\vert_s$
ϵ_I specific internal energy	$\Gamma_2/(\Gamma_2 - 1) = (d\ ln\ P/d\ ln\ T)\vert_s$
h specific enthalpy	$\Gamma_3 - 1 = (d\ ln\ T/d\ ln\ \rho)\vert_s$
$k = (1/2)\widetilde{u''_i u''_i}$ turbulent kinetic energy	$\widetilde{k}^r = (1/2)\widetilde{u''_r u''_r} = (1/2)\widetilde{R}_{rr}/\overline{\rho}$ radial turbulent kinetic energy
ϵ_k specific kinetic energy	$\widetilde{k}^\theta = (1/2)\widetilde{u''_\theta u''_\theta} = (1/2)\widetilde{R}_{\theta\theta}/\overline{\rho}$ angular turbulent kinetic energy
ϵ_t specific total energy	$\widetilde{k}^\phi = (1/2)\widetilde{u''_\phi u''_\phi} = (1/2)\widetilde{R}_{\phi\phi}/\overline{\rho}$ angular turbulent kinetic energy
s specific entropy	$\widetilde{k}^h = \widetilde{k}^\theta + \widetilde{k}^\phi$ horizontal turbulent kinetic energy
$v = 1/\rho$ specific volume	$f_k = (1/2)\overline{\rho}u''_i\widetilde{u''_i u''_r}$ turbulent kinetic energy flux
X_α mass fraction of isotope α	$f_k^r = (1/2)\overline{\rho}u''_r\widetilde{u''_r u''_r}$ radial turbulent kinetic energy flux
$\dot{X}_\alpha^{\rm nuc}$ rate of change of X_α	$f_k^\theta = (1/2)\overline{\rho}u''_\theta\widetilde{u''_\theta u''_r}$ angular turbulent kinetic energy flux
A_α number of nucleons in isotope α	$f_k^\phi = (1/2)\overline{\rho}u''_\phi\widetilde{u''_\phi u''_r}$ angular turbulent kinetic energy flux
Z_α charge of isotope α	$f_k^h = f_k^\theta + f_k^\phi$ horizontal turbulent kinetic energy flux
A mean number of nucleons per isotope	$W_p = \overline{P'd''}$ turbulent pressure dilatation
Z mean charge per isotope	$W_b = \overline{\rho}\overline{u''_r}\widetilde{g}_r$ buoyancy
$f_P = \overline{P'u'_r}$ acoustic flux	$f_T = -\overline{\chi\partial_r T}$ heat flux (χ thermal conductivity)

Table 2: Definitions (continued):

$f_I = \overline{\rho}\widetilde{\epsilon_I'' u_r''}$	internal energy flux	$f_\alpha = \overline{\rho}\widetilde{X_\alpha'' u_r''}$	X_α flux
$f_s = \overline{\rho}\widetilde{s'' u_r''}$	entropy flux	$f_{jz} = \overline{\rho}\widetilde{j_z'' u_r''}$	angular momentum flux
$f_T = \overline{u_r' T'}$	turbulent heat flux	$f_A = \overline{\rho}\widetilde{A'' u_r''}$	A (mean number of nucleons per isotope) flux
$f_h = \overline{\rho}\widetilde{h'' u_r''}$	enthalpy flux	$f_Z = \overline{\rho}\widetilde{Z'' u_r''}$	Z (mean charge per isotope) flux
$b = \overline{v'\rho'}$	density-specific volume covariance	$\mathcal{N}_\rho, \mathcal{N}_{ur}, \mathcal{N}_{u\theta}, \mathcal{N}_{u\phi}, \mathcal{N}_{jz}, \mathcal{N}_\alpha, \mathcal{N}_A, \mathcal{N}_Z$	numerical effect
$f_\tau = f_\tau^r + f_\tau^\theta + f_\tau^\phi$	viscous flux	$\mathcal{N}_{\epsilon I} = -\nabla_r f_\tau + \varepsilon_k$	numerical effect
$f_\tau^r = -\overline{\tau_{rr}' u_r'}$	viscous flux	$\mathcal{N}_{\epsilon k} = -\varepsilon_k$	numerical effect
$f_\tau^\theta = -\overline{\tau_{\theta r}' u_\theta'}$	viscous flux	$\mathcal{N}_{\epsilon t} = -\nabla_r f_\tau$	numerical effect
$f_\tau^\phi = -\overline{\tau_{\phi r}' u_\phi'}$	viscous flux	$\mathcal{N}_s = \overline{-\varepsilon_k / T}$	numerical effect
$f_\tau^h = f_\tau^\theta + f_\tau^\phi$	viscous flux	$\mathcal{N}_h = -\nabla_r f_\tau + (\Gamma_3 - 1)\varepsilon_k$	numerical effect
$f_I^r = \overline{\rho}\widetilde{\epsilon_I'' u_r'' u_r''}$	radial flux of f_I	$\mathcal{N}_P = +(\Gamma_3 - 1)\varepsilon_k$	numerical effect
$f_s^r = \overline{\rho}\widetilde{s'' u_r'' u_r''}$	radial flux of f_s	$\mathcal{N}_T = +\overline{(\tau_{ij}\partial_j u_i)/(c_v \rho)}$	numerical effect
$f_h^r = \overline{\rho}\widetilde{h'' u_r'' u_r''}$	radial flux of f_h	$\mathcal{N}_{Rrr} = -2\nabla_r f_\tau^r - 2\varepsilon_k^r$	numerical effect
$f_T^r = \overline{T' u_r' u_r'}$	radial flux of f_T	$\mathcal{N}_{R\theta\theta} = -2\nabla_r f_\tau^\theta - 2\varepsilon_k^\theta$	numerical effect
$f_{jz}^r = \overline{\rho}\widetilde{j_z'' u_r'' u_r''}$	radial flux of f_{jz}	$\mathcal{N}_{R\phi\phi} = -2\nabla_r f_\tau^\phi - 2\varepsilon_k^\phi$	numerical effect
$f_\alpha^r = \overline{\rho}\widetilde{X_\alpha'' u_r'' u_r''}$	radial flux of f_α	$\mathcal{N}_k = -\nabla_r f_\tau - \varepsilon_k$	numerical effect
$f_A^r = \overline{\rho}\widetilde{A'' u_r'' u_r''}$	radial flux of f_A	$\mathcal{N}_{kr} = -\nabla_r f_\tau^r - \varepsilon_k^r$	numerical effect
$f_Z^r = \overline{\rho}\widetilde{Z'' u_r'' u_r''}$	radial flux of f_Z	$\mathcal{N}_{kh} = -\nabla_r f_\tau^h - \varepsilon_k^h$	numerical effect
$\mathcal{G}_k^r = -(1/2)\overline{G_{rr}^R} - \overline{u_r'' G_r^M}$		$\mathcal{N}_a = -\varepsilon_a$	numerical effect

Table 3: Definitions (continued):

$\mathcal{G}_k^\theta = -(1/2)\overline{G_{\theta\theta}^R} - \overline{u''_\theta G_\theta^M}$	$\mathcal{N}_b$ numerical effect
$\mathcal{G}_k^\phi = -(1/2)\overline{G_{\phi\phi}^R} - \overline{u''_\phi G_\phi^M}$	$\mathcal{N}_{fI} = -\nabla_r(\overline{\epsilon''_I \tau'_{rr}}) + \overline{u''_r \tau_{ij}\partial_i u_j} - \varepsilon_I$ numerical effect
$\mathcal{G}_k^h = +\mathcal{G}_k^\theta + \mathcal{G}_k^\phi$	$\mathcal{N}_{fh} = -\nabla_r(\overline{h''\tau'_{rr}}) + \overline{u''_r(\Gamma_3-1)\tau_{ij}\partial_i u_j} - \overline{u''_r \nabla_i u_i \tau_{ji}} - \varepsilon_h$ numerical effect
$\mathcal{G}_a = +\overline{\rho' v G_r^M}$	$\mathcal{N}_{fs} = -\nabla_r(\overline{s''\tau'_{rr}}) + \overline{u''_r \tau_{ij}\partial_i u_j / T} - \varepsilon_s$ numerical effect
$\mathcal{G}_I = -\overline{G_r^I} - \overline{\epsilon''_I G_r^M}$	$\mathcal{N}_{fA} = -\nabla_r(\overline{A''\tau'_{rr}}) - \varepsilon_A$ numerical effect
$\mathcal{G}_\alpha = -\overline{G_r^\alpha} - \overline{X''_\alpha G_r^M}$	$\mathcal{N}_{fZ} = -\nabla_r(\overline{Z''\tau'_{rr}}) - \varepsilon_Z$ numerical effect
$\mathcal{G}_A = -\overline{G_r^A} - \overline{A'' G_r^M}$	$\mathcal{N}_{f\alpha} = -\nabla_r(\overline{\alpha''\tau'_{rr}}) - \varepsilon_\alpha$ numerical effect
$\mathcal{G}_Z = -\overline{G_r^Z} - \overline{Z'' G_r^M}$	$\mathcal{N}_{fjz} = -\nabla_r(\overline{j''_z \tau'_{rr}}) - \varepsilon_{jz}$ numerical effect
$\mathcal{G}_h = -\overline{G_r^h} - \overline{h'' G_r^M}$	$\mathcal{N}_{fT} = +\overline{T'\partial_i \tau_{ri}/\rho} + \overline{u'_r \tau_{ij}\partial_i u_j / \rho c_v}$ numerical effect
$\mathcal{G}_T = -\overline{G_r^T} - \overline{T' G_r^M}$	
$\mathcal{G}_s = -\overline{G_r^s} - \overline{s'' G_r^M}$	
$\mathcal{G}_{jz} = -\overline{G_r^{jz}} - \overline{j''_z G_r^M}$	
$\sigma_\rho = \overline{\rho'\rho'}$	$\mathcal{N}_{\sigma_\rho}$ numerical effect
$\sigma_P = \overline{P'P'}$	$\mathcal{N}_{\sigma_P} = +2(\Gamma_3-1)\overline{P'\tau_{ij}\partial_i u_j}$ numerical effect
$\sigma_T = \overline{T'T'}$	$\mathcal{N}_{\sigma_T} = +\overline{2T'\tau_{ij}\partial_i u_j / \rho c_v}$ numerical effect
$\sigma_{ur} = \widetilde{u''_r u''_r}$	$\mathcal{N}_{\sigma_{ur}} = +2\nabla_r f_\tau^r - 2\varepsilon_k^r$ numerical effect
$\sigma_s = \widetilde{s''s''}$	$\mathcal{N}_{\sigma_s} = +2\overline{s''\tau_{ij}\partial_j u_i / T}$ numerical effect
$\sigma_\alpha = \widetilde{X''_\alpha X''_\alpha}$	$\mathcal{N}_{\sigma_\alpha}$ numerical effect numerical effect
$\sigma_{\epsilon I} = \widetilde{\epsilon''_I \epsilon''_I}$	$N_{\sigma\epsilon_I} = +2\overline{\epsilon''_I \tau_{ij}\partial_j u_i}$ numerical effect

Table 4: Definitions (continued):

$$
\begin{aligned}
\varepsilon_k^r &= \overline{\tau'_{rr}\partial_r u''_r} + \overline{\tau'_{r\theta}(1/r)\partial_\theta u''_r} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi u''_r} \\
\varepsilon_k^\theta &= \overline{\tau'_{\theta r}\partial_r u''_\theta} + \overline{\tau'_{\theta\theta}(1/r)\partial_\theta u''_\theta} + \overline{\tau'_{\theta\phi}(1/r\sin\theta)\partial_\phi u''_\theta} \\
\varepsilon_k^\phi &= \overline{\tau'_{\phi r}\partial_r u''_\phi} + \overline{\tau'_{\phi\theta}(1/r)\partial_\theta u''_\phi} + \overline{\tau'_{\phi\phi}(1/r\sin\theta)\partial_\phi u''_\phi} \\
\varepsilon_k &= \varepsilon_k^r + \varepsilon_k^\theta + \varepsilon_k^\phi \\
\varepsilon_k^h &= \varepsilon_k^\theta + \varepsilon_k^\phi \\
\varepsilon_a &= \overline{\rho' v \nabla_r \tau'_{rr}} \\
\varepsilon_I &= \overline{\tau'_{rr}\partial_r \epsilon''_I} + \overline{\tau'_{r\theta}(1/r)\partial_\theta \epsilon''_I} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi \epsilon''_I} \\
\varepsilon_s &= \overline{\tau'_{rr}\partial_r s''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta s''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi s''} \\
\varepsilon_\alpha &= \overline{\tau'_{rr}\partial_r X''_\alpha} + \overline{\tau'_{r\theta}(1/r)\partial_\theta X''_\alpha} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi X''_\alpha} \\
\varepsilon_A &= \overline{\tau'_{rr}\partial_r A''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta A''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi A''} \\
\varepsilon_Z &= \overline{\tau'_{rr}\partial_r Z''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta Z''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi Z''} \\
\varepsilon_h &= \overline{\tau'_{rr}\partial_r h''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta h''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi h''} \\
\varepsilon_{jz} &= \overline{\tau'_{rr}\partial_r j''_z} + \overline{\tau'_{r\theta}(1/r)\partial_\theta j''_z} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi j''_z}
\end{aligned}
$$

$$
\begin{aligned}
\overline{G_r^M} &= -\overline{\rho u_\theta u_\theta/r} - \overline{\rho u_\phi u_\phi/r} \\
\overline{G_\theta^M} &= +\overline{\rho u_\theta u_r/r} - \overline{\rho u_\phi u_\phi/(r\tan\theta)} \\
\overline{G_\phi^M} &= +\overline{\rho u_\phi u_r/r} + \overline{\rho u_\phi u_\theta/(r\tan\theta)} \\
\overline{G_{rr}^R} &= -\overline{\rho u''_\theta u''_\theta u''_r/r} - \overline{\rho u''_\theta u''_r u''_\theta/r} - \overline{\rho u''_\phi u''_\phi u''_r/r} - \overline{\rho u''_\phi u''_r u''_\phi/r} \\
\overline{G_{\theta\theta}^R} &= +\overline{\rho u''_\theta u''_r u''_\theta/r} + \overline{\rho u''_\theta u''_\theta u''_r/r} - \overline{\rho u''_\phi u''_\phi u''_\theta/(r\tan\theta)} - \overline{u''_\phi u''_\theta u''_\phi/(r\tan\theta)} \\
\overline{G_{\phi\phi}^R} &= +\overline{\rho u''_\phi u''_r r_\phi/r} + \overline{\rho u''_\phi u''_\theta u''_\phi/(r\tan\theta)} + \overline{\rho u''_\phi u''_\phi u''_r/r} + \overline{\rho u''_\phi u''_\phi u''_\theta/(r\tan\theta)} \\
\overline{G_r^I} &= -\overline{\rho\epsilon''_I u''_\theta u''_\theta/r} - \overline{\rho\epsilon''_I u''_\phi u''_\phi/r} \\
\overline{G_r^s} &= -\overline{\rho s'' u''_\theta u''_\theta/r} - \overline{\rho s'' u''_\phi u''_\phi/r} \\
\overline{G_r^\alpha} &= -\overline{\rho X''_\alpha u''_\theta u''_\theta/r} - \overline{\rho X''_\alpha u''_\phi u''_\phi/r} \\
\overline{G_r^A} &= -\overline{\rho A'' u''_\theta u''_\theta/r} - \overline{\rho A'' u''_\phi u''_\phi/r} \\
\overline{G_r^Z} &= -\overline{\rho Z'' u''_\theta u''_\theta/r} - \overline{\rho Z'' u''_\phi u''_\phi/r} \\
\overline{G_r^h} &= -\overline{\rho h'' u''_\theta u''_\theta/r} - \overline{\rho h'' u''_\phi u''_\phi/r} \\
\overline{G_r^T} &= -\overline{\rho T' u'_\theta u'_\theta/r} - \overline{\rho T' u'_\phi u'_\phi/r} \\
\overline{G_r^{jz}} &= -\overline{\rho j''_z u''_\theta u''_\theta/r} - \overline{\rho j''_z u''_\phi u''_\phi/r}
\end{aligned}
$$

$$
\nabla(.) = \nabla_r(.) + \nabla_\theta(.) + \nabla_\phi(.) = \frac{1}{r^2}\partial_r(r^2 .) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta .) + \frac{1}{r\sin\theta}\partial_\phi(.)
$$