1 Hydrodynamic stellar structure equations (non-local and time-dependent)

Below is a set of hydrodynamic stellar structure equations derived from RANS (viscosity explicitly neglected), where red terms are the ones used in classical approach:

$$\partial_r \overline{M} = 4\pi r^2 \overline{\rho} + (4\pi r^3/3\widetilde{u}_r)\left[-\nabla_r f_\rho + (f_\rho/\overline{\rho})\partial_r \overline{\rho} - \overline{\rho}\overline{d} - \partial_t \overline{\rho}\right] \tag{1}$$

$$\partial_r \overline{P} = \overline{\rho}\widetilde{g} - \overline{\rho}\partial_t \widetilde{u}_r - \nabla_r \widetilde{R}_{rr} - \overline{G}^M_r - \overline{\rho}\widetilde{u}_r \partial_r \widetilde{u}_r \tag{2}$$

$$\partial_r \widetilde{L} = 4\pi r^2 \overline{\rho}\widetilde{\epsilon}_{nuc} + 4\pi r^2\left[-\nabla_r(f_i + f_{th} + f_K + f_p) - \overline{P d} - \widetilde{R}_{ir}\partial_r \widetilde{u}_i + W_b + \overline{\rho}\widetilde{D}_t \widetilde{u}_i \widetilde{u}_i/2 - \overline{\rho}\partial_t \widetilde{\epsilon}_t\right] + \widetilde{\epsilon}_t \partial_r 4\pi r^2 \overline{\rho}\widetilde{u}_r \tag{3}$$

$$\partial_r \overline{T} = (1/\overline{u}_r)\left[-\nabla_r f_T + (1-\Gamma_3)\overline{T}\;\overline{d} + (2-\Gamma_3)\overline{T'd'} + \epsilon_{nuc}/c_v + \nabla\cdot f_{th}/(\rho c_v) - \partial_t T\right] \tag{4}$$

$$\partial_t \widetilde{X}_i = \widetilde{\dot{X}}_i^{nuc} - (1/\overline{\rho})\nabla_r f_\alpha - \widetilde{u}_r \partial_r \widetilde{X}_\alpha \tag{5}$$

1.1 Continuity Equation

Derivation

Using full 3D hydrodynamic continuity equation, we derive its mean field counterpart in the following way:

$$\begin{aligned}
\partial_t \rho + \nabla \rho \mathbf{u} &= 0 \\
\partial_t \overline{\rho} + \overline{u}_r \partial_r \overline{\rho} &= -\nabla_r \overline{\rho' u'_r} - \overline{\rho}\overline{d} \\
\partial_t \overline{\rho} + \overline{u''}_r \partial_r \overline{\rho} + \widetilde{u}_r \partial_r \overline{\rho} &= -\nabla_r \overline{\rho' u'_r} - \overline{\rho}\overline{d} \\
\partial_t \overline{\rho} + \widetilde{u}_r \partial_r \overline{\rho} &= -\nabla_r \overline{\rho' u'_r} + (\overline{\rho' u'_r}/\overline{\rho})\partial_r \overline{\rho} - \overline{\rho}\overline{d} \\
\partial_t \overline{\rho} + \widetilde{u}_r \partial_r \overline{\rho} &= -\nabla_r f_\rho + (f_\rho/\overline{\rho})\partial_r \overline{\rho} - \overline{\rho}\overline{d} \\
\widetilde{D}_t \overline{\rho} &= -\nabla_r f_\rho + (f_\rho/\overline{\rho})\partial_r \overline{\rho} - \overline{\rho}\overline{d}
\end{aligned}$$

For the derivation, we used the following identities: $\overline{\rho u_r} = \overline{\rho' u'_r} + \overline{\rho u_r}$ and $\overline{u''}_r = \overline{u}_r - \widetilde{u}_r$ and $\overline{\rho}\overline{u''}_r = -f_\rho$ and $f_\rho = \overline{\rho' u'_r}$ (turbulent mass flux)

From there, let us now express gradient of mean density $\partial_r \overline{\rho}$. We get:

$$\partial_r \overline{\rho} = -(1/\widetilde{u}_r)\left(\nabla_r f_\rho + (f_\rho/\overline{\rho})\partial_r \overline{\rho} - \overline{\rho}\overline{d} - \partial_t \overline{\rho}\right) \tag{6}$$

We want to derive hydrodynamic continuity equation into the form familiar from its classical form. Therefore, let us now find relation between $\partial_r \overline{\rho}$ and $\partial_r \overline{M}$ using total differentials of ρ and M. We know, that $\rho = \rho(r,t)$ and $M = M(r,t)$ and:

$$\begin{aligned} d\rho =& \partial_r \rho \ dr + \partial_t \rho \ dt \\ dM =& \partial_r M dr + \partial_t M dt \end{aligned}$$

Let us now transform the $d\rho$ equation to dM equation by multiplying it by volume $V = 4\pi r^3/3$ and few algebraic modifications:

$$\begin{aligned} V d\rho &= V \partial_r \rho \ dr + V \partial_t \rho \ dt \\ dM - \rho dV &= V \partial_r \rho \ dr + V \partial_t \rho \ dt \\ dM - 4\pi r^2 \rho dr &= V \partial_r \rho \ dr + V \partial_t \rho \ dt \\ dM &= 4\pi r^2 \rho dr + V \partial_r \rho \ dr + V \partial_t \rho \ dt \\ dM &= (4\pi r^2 \rho + V \partial_r \rho) dr + V \partial_t \rho \ dt \end{aligned}$$

By comparing this result to $dM = \partial_r M dr + \partial_t M dt$ we get:

$$(4\pi r^2 \rho + V \partial_r \rho) dr + V \partial_t \rho \ dt = \partial_r M dr + \partial_t M dt$$

and

$$\partial_r M = V \partial_r \rho + 4\pi r^2 \rho \tag{7}$$

By space-time averaging and using equation 6 we get the desired form of the continuity equation for stellar evolution.

$$\partial_r \overline{M} = \ 4\pi r^2 \overline{\rho} + (4\pi r^3 / 3\widetilde{u}_r) \left(-\nabla_r f_\rho + (f_\rho/\overline{\rho}) \partial_r \overline{\rho} - \overline{\rho} \overline{d} - \partial_t \overline{\rho} \right) \tag{8}$$

Term description:

$$\partial_r \overline{M} = \underbrace{4\pi r^2 \overline{\rho}}_{\text{density distribution}} + (4\pi r^3 / 3\widetilde{u}_r) (\underbrace{-\nabla_r f_\rho}_{\text{transport of turbulent density field}} + \underbrace{(f_\rho/\overline{\rho}) \partial_r \overline{\rho}}_{\text{down-gradient density source/sink term}} - \underbrace{\overline{\rho} \overline{d}}_{\text{compressibility effects}} - \underbrace{\partial_t \overline{\rho}}_{\text{time-dependence}}) \tag{9}$$

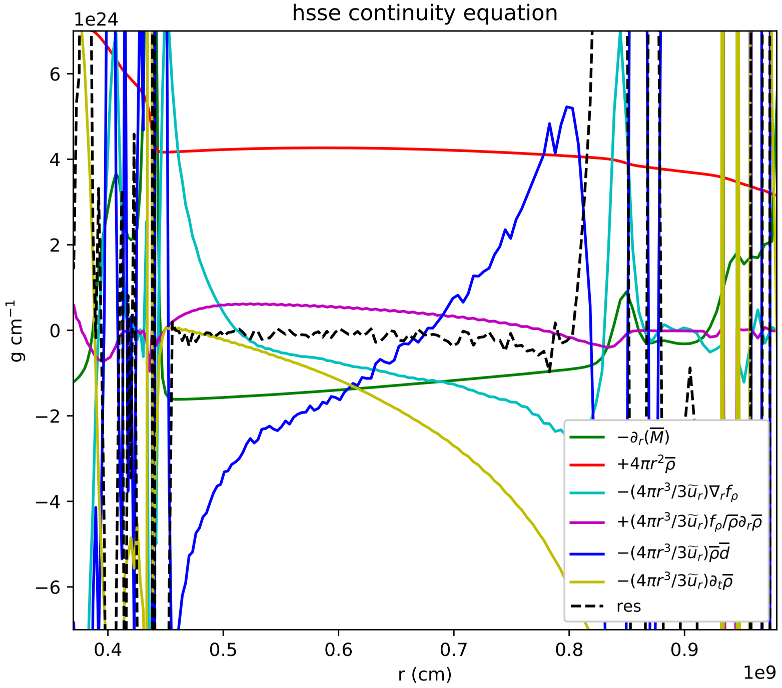

Figure 1: Hydrodynamic stellar structure continuity equation.

1.2 Momentum Equation

Derivation

We start from the RANS equation for X momentum and modify it into form familiar from classical stellar evolution theory.

$$\overline{\rho}\widetilde{D}_t\widetilde{u}_r = -\nabla_r\widetilde{R}_{rr} - \overline{G_r^M} - \partial_r\overline{P} + \overline{\rho}\widetilde{g}_r \tag{10}$$

$$\overline{\rho}\partial_t\widetilde{u}_r + \overline{\rho}\widetilde{u}_r\partial_r\widetilde{u}_r = -\nabla_r\widetilde{R}_{rr} - \overline{G_r^M} - \partial_r\overline{P} + \overline{\rho}\widetilde{g}_r \tag{11}$$

$$\partial_r\overline{P} = \overline{\rho}\widetilde{g}_r - \overline{\rho}\partial_t\widetilde{u}_r - \nabla_r\widetilde{R}_{rr} - \overline{G_r^M} - \overline{\rho}\widetilde{u}_r\partial_r\widetilde{u}_r \tag{12}$$

Term description:

$$\partial_r \overline{P} = \underbrace{\overline{\rho}\widetilde{g}_r}_{\text{gravity}} - \underbrace{\overline{\rho}\partial_t \widetilde{u}_r}_{\text{acceleration due to expansion}} - \underbrace{\nabla_r \widetilde{R}_{rr}}_{\text{transport of turbulent velocity field}} - \underbrace{\overline{G_r^M}}_{\text{centrifugal forces}} - \underbrace{\overline{\rho}\widetilde{u}_r \partial_r \widetilde{u}_r}_{\text{advection due to expansion}} \tag{13}$$

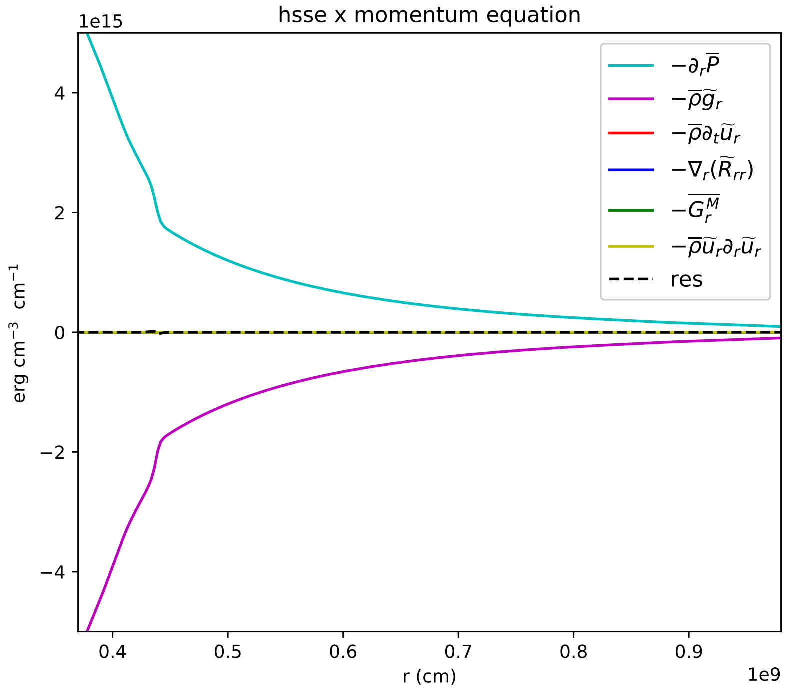

Figure 2: Hydrodynamic stellar structure momentum equation.

1.3 Luminosity Equation

$$\overline{\rho}\widetilde{D}_t\widetilde{\epsilon}_t = -\nabla_r(f_i + f_{th} + f_K + f_p) - \overline{P}\ \overline{d} - \widetilde{R}_{ir}\partial_r\widetilde{u}_r + W_b + \overline{\rho}\widetilde{\epsilon}_{nuc} + \overline{\rho}\widetilde{D}_t\widetilde{u}_i\widetilde{u}_i/2 \quad (14)$$

$$\overline{\rho}\partial_t\widetilde{\epsilon}_t + \overline{\rho}\widetilde{u}_r\partial_r\widetilde{\epsilon}_t = -\nabla_r(f_i + f_{th} + f_K + f_p) - \overline{P}\ \overline{d} - \widetilde{R}_{ir}\partial_r\widetilde{u}_r + W_b + \overline{\rho}\widetilde{\epsilon}_{nuc} + \overline{\rho}\widetilde{D}_t\widetilde{u}_i\widetilde{u}_i/2 \quad (15)$$

$$4\pi r^2\overline{\rho}\partial_t\widetilde{\epsilon}_t + 4\pi r^2\overline{\rho}\widetilde{u}_r\partial_r\widetilde{\epsilon}_t = 4\pi r^2\left[-\nabla_r(f_i + f_{th} + f_K + f_p) - \overline{P}\ \overline{d} - \widetilde{R}_{ir}\partial_r\widetilde{u}_r + W_b + \overline{\rho}\widetilde{\epsilon}_{nuc} + \overline{\rho}\widetilde{D}_t\widetilde{u}_i\widetilde{u}_i/2\right] \quad (16)$$

$$4\pi r^2\overline{\rho}\partial_t\widetilde{\epsilon}_t + \underbrace{\partial_r 4\pi r^2\overline{\rho}\widetilde{u}_r\widetilde{\epsilon}_t}_{\partial_r\widetilde{L}} - \widetilde{\epsilon}_t\partial_r 4\pi r^2\overline{\rho}\widetilde{u}_r = 4\pi r^2\left[-\nabla_r(f_i + f_{th} + f_K + f_p) - \overline{P}\ \overline{d} - \widetilde{R}_{ir}\partial_r\widetilde{u}_r + W_b + \overline{\rho}\widetilde{\epsilon}_{nuc} + \overline{\rho}\widetilde{D}_t\widetilde{u}_i\widetilde{u}_i/2\right] \quad (17)$$

$$\partial_r\widetilde{L} = 4\pi r^2\left[-\nabla_r(f_i + f_{th} + f_K + f_p) - \overline{P}\ \overline{d} - \widetilde{R}_{ir}\partial_r\widetilde{u}_r + W_b + \overline{\rho}\widetilde{\epsilon}_{nuc} + \overline{\rho}\widetilde{D}_t\widetilde{u}_i\widetilde{u}_i/2 - \overline{\rho}\partial_t\widetilde{\epsilon}_t\right] + \widetilde{\epsilon}_t\partial_r 4\pi r^2\overline{\rho}\widetilde{u}_r \quad (18)$$

Or

$$\partial_r\widetilde{L} = 4\pi r^2\overline{\rho}\widetilde{\epsilon}_{nuc} + 4\pi r^2\left[-\nabla_r(f_i + f_{th} + f_K + f_p) - \overline{P}\ \overline{d} - \widetilde{R}_{ir}\partial_r\widetilde{u}_r + W_b + \overline{\rho}\widetilde{D}_t\widetilde{u}_i\widetilde{u}_i/2 - \overline{\rho}\partial_t\widetilde{\epsilon}_t\right] + \widetilde{\epsilon}_t\partial_r 4\pi r^2\overline{\rho}\widetilde{u}_r \quad (19)$$

Some term description:

$$\partial_r\widetilde{L} = \underbrace{4\pi r^2\overline{\rho}\widetilde{\epsilon}_{nuc}}_{\text{nuclear}} + 4\pi r^2[\underbrace{-\nabla_r(f_i + f_K + f_p + f_{th})}_{\text{transport of internal energy, kinetic energy, pressure and heat due to conduction and radiation}} -$$
$$\underbrace{\overline{P}\ \overline{d}}_{\text{compressibility}} - \underbrace{\widetilde{R}_{ir}\partial_r\widetilde{u}_r}_{\text{down-gradient source/sink term}} + \underbrace{W_b}_{\text{buoyancy work}} + \underbrace{\overline{\rho}\widetilde{D}_t\widetilde{u}_i\widetilde{u}_i/2 - \overline{\rho}\partial_t\widetilde{\epsilon}_t}_{\text{time-dependence}}] + \widetilde{\epsilon}_t\partial_r 4\pi r^2\overline{\rho}\widetilde{u}_r \quad (20)$$

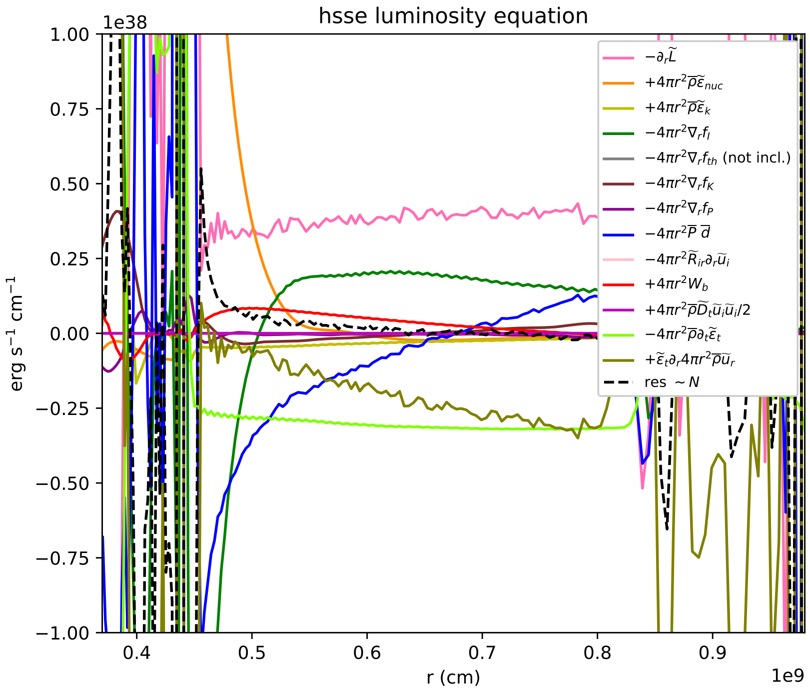

Figure 3: Hydrodynamic stellar structure luminosity equation.

1.4 Temperature Equation

Derivation

We start from the RANS equation for temperature evolution. At the end, there will be no resemblance to the temperature equation of the classical stellar evolution theory.

$$\overline{D}_t \overline{T} = -\nabla_r f_T + (1-\Gamma_3)\overline{T}\;\overline{d} + (2-\Gamma_3)\overline{T'd'} + \overline{(\nabla \cdot f_{th})/\rho c_v} + \overline{\epsilon_{\rm nuc}/c_v}$$

$$\partial_t \overline{T} + \overline{u}_r \partial_r \overline{T} = -\nabla_r f_T + (1-\Gamma_3)\overline{T}\;\overline{d} + (2-\Gamma_3)\overline{T'd'} + \overline{(\nabla \cdot f_{th})/\rho c_v} + \overline{\epsilon_{\rm nuc}/c_v}$$

$$\overline{u}_r \partial_r \overline{T} = -\nabla_r f_T + (1-\Gamma_3)\overline{T}\;\overline{d} + (2-\Gamma_3)\overline{T'd'} + \overline{(\nabla \cdot f_{th})/\rho c_v} + \overline{\epsilon_{\rm nuc}/c_v} - \partial_t \overline{T}$$

$$\partial_r \overline{T} = -\left(1/\overline{u}_r\right)\left(\nabla_r f_T + (1-\Gamma_3)\overline{T}\ \overline{d} + (2-\Gamma_3)\overline{T'd'} + \overline{(\nabla \cdot f_{th})/\rho c_v} + \overline{\epsilon_{\rm nuc}/c_v} - \partial_t \overline{T}\right)$$

Term description:

$$\partial_r \overline{T} = -\left(1/\overline{u}_r\right)\Big(\underbrace{\nabla_r f_T}_{\text{transport of turbulent temperature field}} + \underbrace{(1-\Gamma_3)\overline{T}\ \overline{d} + (2-\Gamma_3)\overline{T'd'}}_{\text{compressibility effects}} + \underbrace{\overline{(\nabla \cdot f_{th})/\rho c_v}}_{\text{source/sink term due to thermal transport}} + \underbrace{\overline{\epsilon_{\rm nuc}/c_v}}_{\text{source/sink due to nuclear burning}} - \underbrace{\partial_t \overline{T}}_{\text{time-dependence}} \Big) \tag{21}$$

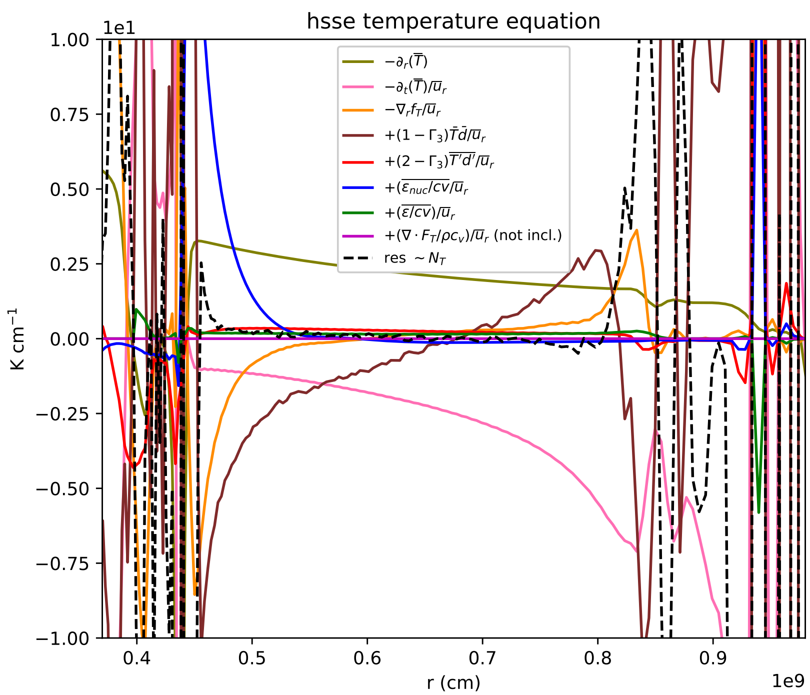

Figure 4: Hydrodynamic stellar structure temperature equation.

1.5 Composition Equation

Derivation

We start from the RANS equation for compostion and modify it into form familiar from classical stellar evolution theory.

$$
\begin{aligned}
\overline{\rho}\widetilde{D}_t\widetilde{X}_\alpha &= -\nabla_r f_\alpha + \overline{\rho}\widetilde{\dot{X}_\alpha^{nuc}} \\
\overline{\rho}\partial_t\widetilde{X}_\alpha + \overline{\rho}\widetilde{u}_r\partial_r\widetilde{X}_\alpha &= -\nabla_r f_\alpha + \overline{\rho}\widetilde{\dot{X}_\alpha^{nuc}} \\
\overline{\rho}\partial_t\widetilde{X}_\alpha &= -\nabla_r f_\alpha + \overline{\rho}\widetilde{\dot{X}_\alpha^{nuc}} - \overline{\rho}\widetilde{u}_r\partial_r\widetilde{X}_\alpha \\
\partial_t\widetilde{X}_\alpha &= \widetilde{\dot{X}}_\alpha^{nuc} - (1/\overline{\rho})\nabla_r f_\alpha - \widetilde{u}_r\partial_r\widetilde{X}_\alpha
\end{aligned}
$$

Term description:

$$
\partial_t\widetilde{X}_\alpha = \underbrace{\widetilde{\dot{X}}_\alpha^{nuc}}_{\text{nuclear burning}} - \underbrace{(1/\overline{\rho})\nabla_r f_\alpha}_{\text{transport of turbulent composition field}} - \underbrace{\widetilde{u}_r\partial_r\widetilde{X}_\alpha}_{\text{advection due to expansion}} \tag{22}
$$

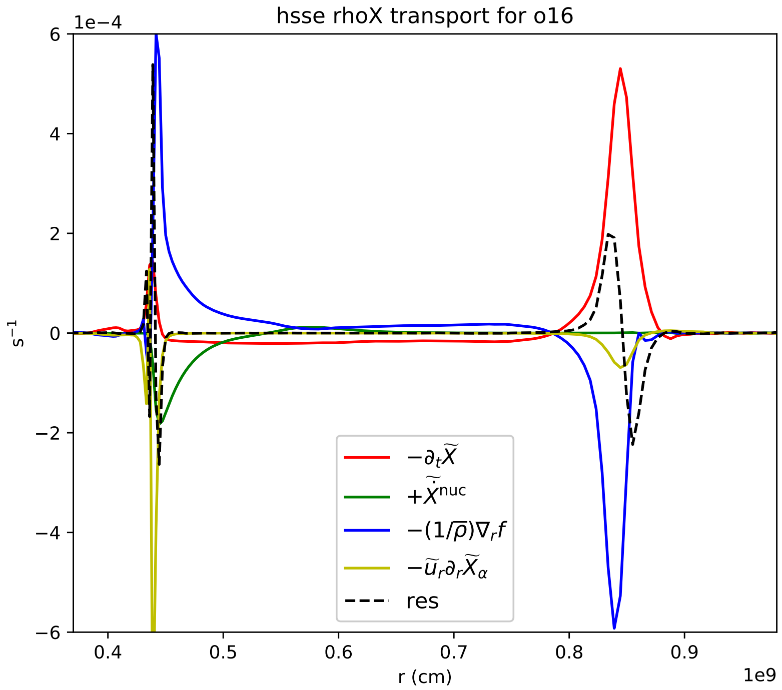

Figure 5: Hydrodynamic stellar structure composition transport equation.

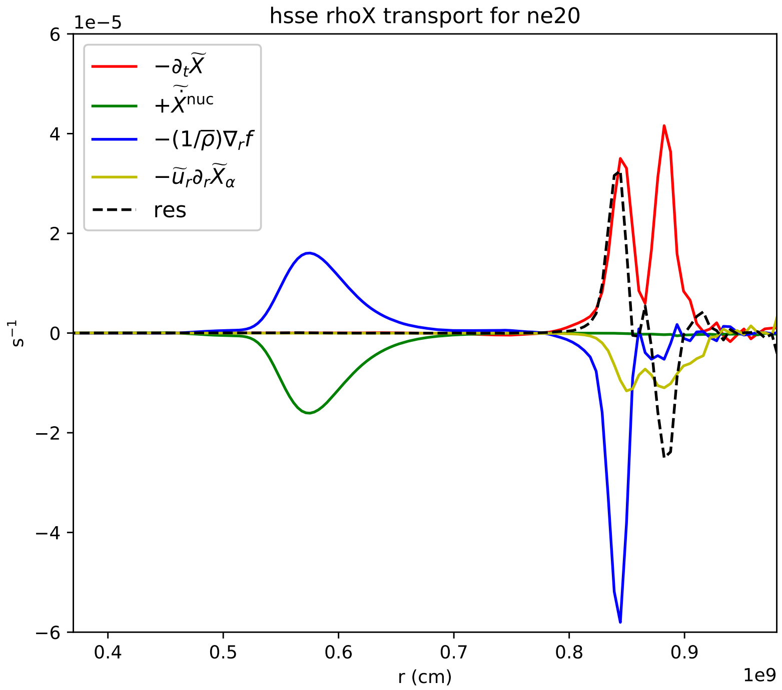

Figure 6: Hydrodynamic stellar structure composition transport equation.

Table 1: Definitions:

ρ density	g_r radial gravitational acceleration
T temperature	$\mathcal{S} = \rho\epsilon_{\rm nuc}(q)$ nuclear energy production (cooling function)
P pressure	$\tau_{ij} = 2\mu S_{ij}$ viscous stress tensor (μ kinematic viscosity)
u_r, u_θ, u_ϕ velocity components	$S_{ij} = (1/2)(\partial_i u_j + \partial_j u_i)$ strain rate
$\mathbf{u} = u(u_r, u_\theta, u_\phi)$ velocity	$\widetilde{R}_{ij} = \overline{\rho}\widetilde{u''_i u''_j}$ Reynolds stress tensor
$j_z = r\sin\theta\ u_\phi$ specific angular momentum	$F_T = \chi\partial_r T$ heat flux
$d = \nabla\cdot\mathbf{u}$ dilatation	$\Gamma_1 = (d\ ln\ P/d\ ln\ \rho)\vert_s$
ϵ_I specific internal energy	$\Gamma_2/(\Gamma_2 - 1) = (d\ ln\ P/d\ ln\ T)\vert_s$
h specific enthalpy	$\Gamma_3 - 1 = (d\ ln\ T/d\ ln\ \rho)\vert_s$
$k = (1/2)\widetilde{u''_i u''_i}$ turbulent kinetic energy	$\widetilde{k}^r = (1/2)\widetilde{u''_r u''_r} = (1/2)\widetilde{R}_{rr}/\overline{\rho}$ radial turbulent kinetic energy
ϵ_k specific kinetic energy	$\widetilde{k}^\theta = (1/2)\widetilde{u''_\theta u''_\theta} = (1/2)\widetilde{R}_{\theta\theta}/\overline{\rho}$ angular turbulent kinetic energy
ϵ_t specific total energy	$\widetilde{k}^\phi = (1/2)\widetilde{u''_\phi u''_\phi} = (1/2)\widetilde{R}_{\phi\phi}/\overline{\rho}$ angular turbulent kinetic energy
s specific entropy	$\widetilde{k}^h = \widetilde{k}^\theta + \widetilde{k}^\phi$ horizontal turbulent kinetic energy
$v = 1/\rho$ specific volume	$f_k = (1/2)\overline{\rho}\widetilde{u''_i u''_i u''_r}$ turbulent kinetic energy flux
X_α mass fraction of isotope α	$f_k^r = (1/2)\overline{\rho}\widetilde{u''_r u''_r u''_r}$ radial turbulent kinetic energy flux
$\dot{X}_\alpha^{\rm nuc}$ rate of change of X_α	$f_k^\theta = (1/2)\overline{\rho}\widetilde{u''_\theta u''_\theta u''_r}$ angular turbulent kinetic energy flux
A_α number of nucleons in isotope α	$f_k^\phi = (1/2)\overline{\rho}\widetilde{u''_\phi u''_\phi u''_r}$ angular turbulent kinetic energy flux
Z_α charge of isotope α	$f_k^h = f_k^\theta + f_k^\phi$ horizontal turbulent kinetic energy flux
A mean number of nucleons per isotope	$W_p = \overline{P'd''}$ turbulent pressure dilatation
Z mean charge per isotope	$W_b = \overline{\rho}\overline{u''_r}\widetilde{g}_r$ buoyancy
$f_P = \overline{P'u'_r}$ acoustic flux	$f_T = -\overline{\chi\partial_r T}$ heat flux (χ thermal conductivity)

Table 2: Definitions (continued):

$f_I = \overline{\rho}\widetilde{\epsilon''_I u''_r}$ internal energy flux

$f_s = \overline{\rho}\widetilde{s'' u''_r}$ entropy flux

$f_T = \overline{u'_r T'}$ turbulent heat flux

$f_h = \overline{\rho}\widetilde{h'' u''_r}$ enthalpy flux

$b = \overline{v' \rho'}$ density-specific volume covariance

$f_\tau = f_\tau^r + f_\tau^\theta + f_\tau^\phi$ viscous flux

$f_\tau^r = -\overline{\tau'_{rr} u'_r}$ viscous flux

$f_\tau^\theta = -\overline{\tau'_{\theta r} u'_\theta}$ viscous flux

$f_\tau^\phi = -\overline{\tau'_{\phi r} u'_\phi}$ viscous flux

$f_\tau^h = f_\tau^\theta + f_\tau^\phi$ viscous flux

$f_I^r = \overline{\rho}\epsilon''_I \widetilde{u''_r u''_r}$ radial flux of f_I

$f_s^r = \overline{\rho} s'' \widetilde{u''_r u''_r}$ radial flux of f_s

$f_h^r = \overline{\rho} h'' \widetilde{u''_r u''_r}$ radial flux of f_h

$f_T^r = \overline{T' u'_r u'_r}$ radial flux of f_T

$f_{jz}^r = \overline{\rho} j''_z \widetilde{u''_r u''_r}$ radial flux of f_{jz}

$f_\alpha^r = \overline{\rho} X''_\alpha \widetilde{u''_r u''_r}$ radial flux of f_α

$f_A^r = \overline{\rho} A'' \widetilde{u''_r u''_r}$ radial flux of f_A

$f_Z^r = \overline{\rho} Z'' \widetilde{u''_r u''_r}$ radial flux of f_Z

$\mathcal{G}_k^r = -(1/2)\overline{G_{rr}^R} - \overline{u''_r G_r^M}$

$f_\alpha = \overline{\rho}\widetilde{X''_\alpha u''_r}$ X_α flux

$f_{jz} = \overline{\rho}\widetilde{j''_z u''_r}$ angular momentum flux

$f_A = \overline{\rho}\widetilde{A'' u''_r}$ A (mean number of nucleons per isotope) flux

$f_Z = \overline{\rho}\widetilde{Z'' u''_r}$ Z (mean charge per isotope) flux

$\mathcal{N}_\rho, \mathcal{N}_{ur}, \mathcal{N}_{u\theta}, \mathcal{N}_{u\phi}, \mathcal{N}_{jz}, \mathcal{N}_\alpha, \mathcal{N}_A, \mathcal{N}_Z$ numerical effect

$\mathcal{N}_{\epsilon I} = -\nabla_r f_\tau + \varepsilon_k$ numerical effect

$\mathcal{N}_{\epsilon k} = -\varepsilon_k$ numerical effect

$\mathcal{N}_{\epsilon t} = -\nabla_r f_\tau$ numerical effect

$\mathcal{N}_s = \overline{-\varepsilon_k / T}$ numerical effect

$\mathcal{N}_h = -\nabla_r f_\tau + (\Gamma_3 - 1)\varepsilon_k$ numerical effect

$\mathcal{N}_P = +(\Gamma_3 - 1)\varepsilon_k$ numerical effect

$\mathcal{N}_T = +\overline{(\tau_{ij}\partial_j u_i)/(c_v \rho)}$ numerical effect

$\mathcal{N}_{Rrr} = -2\nabla_r f_\tau^r - 2\varepsilon_k^r$ numerical effect

$\mathcal{N}_{R\theta\theta} = -2\nabla_r f_\tau^\theta - 2\varepsilon_k^\theta$ numerical effect

$\mathcal{N}_{R\phi\phi} = -2\nabla_r f_\tau^\phi - 2\varepsilon_k^\phi$ numerical effect

$\mathcal{N}_k = -\nabla_r f_\tau - \varepsilon_k$ numerical effect

$\mathcal{N}_{kr} = -\nabla_r f_\tau^r - \varepsilon_k^r$ numerical effect

$\mathcal{N}_{kh} = -\nabla_r f_\tau^h - \varepsilon_k^h$ numerical effect

$\mathcal{N}_a = -\varepsilon_a$ numerical effect

Table 3: Definitions (continued):

$\mathcal{G}_k^\theta = -(1/2)\overline{G_{\theta\theta}^R} - \overline{u''_\theta G_\theta^M}$	$\mathcal{N}_b$ numerical effect
$\mathcal{G}_k^\phi = -(1/2)\overline{G_{\phi\phi}^R} - \overline{u''_\phi G_\phi^M}$	$\mathcal{N}_{fI} = -\nabla_r(\overline{\epsilon''_I \tau'_{rr}}) + \overline{u''_r \tau_{ij}\partial_i u_j} - \varepsilon_I$ numerical effect
$\mathcal{G}_k^h = +\mathcal{G}_k^\theta + \mathcal{G}_k^\phi$	$\mathcal{N}_{fh} = -\nabla_r(\overline{h''\tau'_{rr}}) + \overline{u''_r(\Gamma_3 - 1)\tau_{ij}\partial_i u_j} - \overline{u''_r \nabla_i u_i \tau_{ji}} - \varepsilon_h$ numerical effect
$\mathcal{G}_a = +\overline{\rho' v G_r^M}$	$\mathcal{N}_{fs} = -\nabla_r(\overline{s''\tau'_{rr}}) + \overline{u''_r \tau_{ij}\partial_i u_j / T} - \varepsilon_s$ numerical effect
$\mathcal{G}_I = -\overline{G_r^I} - \overline{\epsilon''_I G_r^M}$	$\mathcal{N}_{fA} = -\nabla_r(\overline{A''\tau'_{rr}}) - \varepsilon_A$ numerical effect
$\mathcal{G}_\alpha = -\overline{G_r^\alpha} - \overline{X''_\alpha G_r^M}$	$\mathcal{N}_{fZ} = -\nabla_r(\overline{Z''\tau'_{rr}}) - \varepsilon_Z$ numerical effect
$\mathcal{G}_A = -\overline{G_r^A} - \overline{A'' G_r^M}$	$\mathcal{N}_{f\alpha} = -\nabla_r(\overline{\alpha''\tau'_{rr}}) - \varepsilon_\alpha$ numerical effect
$\mathcal{G}_Z = -\overline{G_r^Z} - \overline{Z'' G_r^M}$	$\mathcal{N}_{fjz} = -\nabla_r(\overline{j''_z \tau'_{rr}}) - \varepsilon_{jz}$ numerical effect
$\mathcal{G}_h = -\overline{G_r^h} - \overline{h'' G_r^M}$	$\mathcal{N}_{fT} = +\overline{T'\partial_i \tau_{ri}/\rho} + \overline{u'_r \tau_{ij}\partial_i u_j / \rho c_v}$ numerical effect
$\mathcal{G}_T = -\overline{G_r^T} - \overline{T' G_r^M}$	
$\mathcal{G}_s = -\overline{G_r^s} - \overline{s'' G_r^M}$	
$\mathcal{G}_{jz} = -\overline{G_r^{jz}} - \overline{j''_z G_r^M}$	
$\sigma_\rho = \overline{\rho'\rho'}$	$\mathcal{N}_{\sigma_\rho}$ numerical effect
$\sigma_P = \overline{P'P'}$	$\mathcal{N}_{\sigma_P} = +2(\Gamma_3 - 1)\overline{P'\tau_{ij}\partial_i u_j}$ numerical effect
$\sigma_T = \overline{T'T'}$	$\mathcal{N}_{\sigma_T} = +\overline{2T'\tau_{ij}\partial_i u_j / \rho c_v}$ numerical effect
$\sigma_{ur} = \widetilde{u''_r u''_r}$	$\mathcal{N}_{\sigma_{ur}} = +2\nabla_r f_\tau^r - 2\varepsilon_k^r$ numerical effect
$\sigma_s = \widetilde{s'' s''}$	$\mathcal{N}_{\sigma_s} = +2\overline{s''\tau_{ij}\partial_j u_i / T}$ numerical effect
$\sigma_\alpha = \widetilde{X''_\alpha X''_\alpha}$	$\mathcal{N}_{\sigma_\alpha}$ numerical effect numerical effect
$\sigma_{\epsilon I} = \widetilde{\epsilon''_I \epsilon''_I}$	$N_{\sigma\epsilon_I} = +2\overline{\epsilon''_I \tau_{ij}\partial_j u_i}$ numerical effect

Table 4: Definitions (continued):

$$
\begin{aligned}
\varepsilon_k^r &= \overline{\tau'_{rr}\partial_r u''_r} + \overline{\tau'_{r\theta}(1/r)\partial_\theta u''_r} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi u''_r} \\
\varepsilon_k^\theta &= \overline{\tau'_{\theta r}\partial_r u''_\theta} + \overline{\tau'_{\theta\theta}(1/r)\partial_\theta u''_\theta} + \overline{\tau'_{\theta\phi}(1/r\sin\theta)\partial_\phi u''_\theta} \\
\varepsilon_k^\phi &= \overline{\tau'_{\phi r}\partial_r u''_\phi} + \overline{\tau'_{\phi\theta}(1/r)\partial_\theta u''_\phi} + \overline{\tau'_{\phi\phi}(1/r\sin\theta)\partial_\phi u''_\phi} \\
\varepsilon_k &= \varepsilon_k^r + \varepsilon_k^\theta + \varepsilon_k^\phi \\
\varepsilon_k^h &= \varepsilon_k^\theta + \varepsilon_k^\phi \\
\varepsilon_a &= \overline{\rho' v \nabla_r \tau'_{rr}} \\
\varepsilon_I &= \overline{\tau'_{rr}\partial_r \epsilon''_I} + \overline{\tau'_{r\theta}(1/r)\partial_\theta \epsilon''_I} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi \epsilon''_I} \\
\varepsilon_s &= \overline{\tau'_{rr}\partial_r s''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta s''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi s''} \\
\varepsilon_\alpha &= \overline{\tau'_{rr}\partial_r X''_\alpha} + \overline{\tau'_{r\theta}(1/r)\partial_\theta X''_\alpha} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi X''_\alpha} \\
\varepsilon_A &= \overline{\tau'_{rr}\partial_r A''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta A''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi A''} \\
\varepsilon_Z &= \overline{\tau'_{rr}\partial_r Z''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta Z''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi Z''} \\
\varepsilon_h &= \overline{\tau'_{rr}\partial_r h''} + \overline{\tau'_{r\theta}(1/r)\partial_\theta h''} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi h''} \\
\varepsilon_{jz} &= \overline{\tau'_{rr}\partial_r j''_z} + \overline{\tau'_{r\theta}(1/r)\partial_\theta j''_z} + \overline{\tau'_{r\phi}(1/r\sin\theta)\partial_\phi j''_z}
\end{aligned}
$$

$$
\begin{aligned}
\overline{G^M_r} &= -\overline{\rho u_\theta u_\theta/r} - \overline{\rho u_\phi u_\phi/r} \\
\overline{G^M_\theta} &= +\overline{\rho u_\theta u_r/r} - \overline{\rho u_\phi u_\phi/(r\tan\theta)} \\
\overline{G^M_\phi} &= +\overline{\rho u_\phi u_r/r} + \overline{\rho u_\phi u_\theta/(r\tan\theta)} \\
\overline{G^R_{rr}} &= -\overline{\rho u''_\theta u''_\theta u''_r/r} - \overline{\rho u''_\theta u''_r u''_\theta/r} - \overline{\rho u''_\phi u''_\phi u''_r/r} - \overline{\rho u''_\phi u''_r u''_\phi/r} \\
\overline{G^R_{\theta\theta}} &= +\overline{\rho u''_\theta u''_r u''_\theta/r} + \overline{\rho u''_\theta u''_\theta u''_r/r} - \overline{\rho u''_\phi u''_\phi u''_\theta/(r\tan\theta)} - \overline{u''_\phi u''_\theta u''_\phi/(r\tan\theta)} \\
\overline{G^R_{\phi\phi}} &= +\overline{\rho u''_\phi u''_r r_\phi/r} + \overline{\rho u''_\phi u''_\theta u''_\phi/(r\tan\theta)} + \overline{\rho u''_\phi u''_\phi u''_r/r} + \overline{\rho u''_\phi u''_\phi u''_\theta/(r\tan\theta)} \\
\overline{G^I_r} &= -\overline{\rho \epsilon''_I u''_\theta u''_\theta/r} - \overline{\rho \epsilon''_I u''_\phi u''_\phi/r} \\
\overline{G^s_r} &= -\overline{\rho s'' u''_\theta u''_\theta/r} - \overline{\rho s'' u''_\phi u''_\phi/r} \\
\overline{G^\alpha_r} &= -\overline{\rho X''_\alpha u''_\theta u''_\theta/r} - \overline{\rho X''_\alpha u''_\phi u''_\phi/r} \\
\overline{G^A_r} &= -\overline{\rho A'' u''_\theta u''_\theta/r} - \overline{\rho A'' u''_\phi u''_\phi/r} \\
\overline{G^Z_r} &= -\overline{\rho Z'' u''_\theta u''_\theta/r} - \overline{\rho Z'' u''_\phi u''_\phi/r} \\
\overline{G^h_r} &= -\overline{\rho h'' u''_\theta u''_\theta/r} - \overline{\rho h'' u''_\phi u''_\phi/r} \\
\overline{G^T_r} &= -\overline{\rho T' u'_\theta u'_\theta/r} - \overline{\rho T' u'_\phi u'_\phi/r} \\
\overline{G^{jz}_r} &= -\overline{\rho j''_z u''_\theta u''_\theta/r} - \overline{\rho j''_z u''_\phi u''_\phi/r}
\end{aligned}
$$

$$
\nabla(.) = \nabla_r(.) + \nabla_\theta(.) + \nabla_\phi(.) = \frac{1}{r^2}\partial_r(r^2.) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta.) + \frac{1}{r\sin\theta}\partial_\phi(.)
$$