Interval Raus criterion for stability analysis of linear systems with dependent coefficients in the characteristic polynomial

Prof. DSc. Lubomir Kolev, Assistant Prof. Simona Petrakieva
lkolev@tu-sofia.bg, PetrakievaS-te@tu-sofia.bg

Abstract

The paper addresses the stability analysis of linear continuous systems under interval uncertainties. An interval generalization of the known Raus criterion is suggested to estimate the stability of the system considered. It is based on obtaining the interval extensions of the elements of the Raus matrix which are nonlinear functions of independent system parameters. The case when these elements are independent intervals is considered. The interval extensions are also determined by using modified affine arithmetic. Two sufficient conditions on stability and instability of the linear system considered are obtained. Numerical example illustrating the applicability of the method suggested is solved in the end of the paper.

Keywords: robust stability analysis of linear systems, interval Raus criterion, interval extension, affine arithmetic.

1. Introduction

Consider the linear system described by the characteristic polynomial

$$q(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n \,. \qquad (1)$$

It is stable if and only if the roots of the respective characteristic equation

$$q(s) = 0 \qquad (2)$$

have negative real part [1]. The necessary condition of system stability is to have positive coefficients in characteristic polynomial (1), i.e.

$$a_i > 0,\ i = 0,1,\ldots,n \,. \qquad (3)$$

It is well known [1] that Raus formulates the necessary and sufficient conditions for the stability of the system considered. He defines the matrix

$$R = \begin{bmatrix} \ldots & \ldots & \ldots & \ldots & \ldots \\ c_1 & c_3 & c_5 & \ldots & c_{2k+1} \\ c_0 & c_2 & c_4 & \ldots & c_{2k} \\ b_1 & b_3 & b_5 & \ldots & b_{2k+1} \\ b_0 & b_2 & b_4 & \ldots & b_{2k} \\ a_1 & a_3 & a_5 & \ldots & a_{2k+1} \\ a_0 & a_2 & a_4 & \ldots & a_{2k} \end{bmatrix} \qquad (4)$$

where

$$b_{2k} = \begin{vmatrix} a_1 & a_{2k+3} \\ a_0 & a_{2k+2} \end{vmatrix}, \qquad b_{2k+1} = \begin{vmatrix} b_0 & b_{2k+2} \\ a_1 & a_{2k+3} \end{vmatrix} \qquad (4a)$$

$$k = 0,1,2,\ldots,\left(\frac{n}{2} - 1\right).$$

c_i, i = 0, 1, 2, … are the same as b_i only the difference that the coefficients a and b in (4a) are substituted by the coefficients b and c, respectively.

The Raus criterion of stability is based on the following theorem:

Theorem 1: A necessary and sufficient condition for stability of the system described by characteristic polynomial (1) is all elements of the first column in Raus matrix (4) to be positive, i.e.

$$\left| \begin{array}{l} a_i > 0 \\ b_i > 0 \\ c_i > 0 \\ \ \ldots \end{array} \right., \ i = 0,1,2,\ldots \,. \qquad (5)$$

In general, all the coefficients $a_i,\ i = 0,1,\ldots,n$ in the characteristic polynomial (1) are nonlinear functions of independent parameters $p_j,\ j = 1,\ldots,m$. Thus, if we evaluate the uncertainty in real systems, each of it takes their values in prescribed independent intervals $\boldsymbol{p}_j, j = 1,2,\ldots,m$. Then the interval form of Raus criterion can be formulated in the following theorem:

Theorem 2: A necessary and sufficient condition for stability of the system described by characteristic polynomial (1) is all elements of the first column in Raus matrix (4) to be positive, when the system parameters $p_j,\ j = 1,\ldots,m$ take their values in prescribed independent intervals $\boldsymbol{p}_j, j = 1,2,\ldots,m$, i.e.

$$\left| \begin{array}{l} a_i(p) > 0 \\ b_i(p) > 0 \\ c_i(p) > 0 \\ \ \ldots \end{array} \right., \ p = [p_j]^{\mathrm{T}}, \qquad . \qquad (6)$$

$$j = 1,2,\ldots,m,\ \ p \in \boldsymbol{p}\ ,\ \ i = 0,1,\ldots,n$$

Let $f(p)$ denote any of the functions involved in Raus matrix (4). Thus, we have to solve for all coefficients $a_i(p),\ b_i(p),\ c_i(p),\ ...\ ,\ i=0,1,...,n$ the following problem:

Problem P1: Check that

$$f(p)>0,\ \ p\in \boldsymbol{p}\ . \tag{7}$$

There are various ways to verify (7). The simplest approach is to use some interval extension $\boldsymbol{F}(\boldsymbol{p})=\left[\underline{F}\quad \overline{F}\right]$ of the function $f(p)$ in $\boldsymbol{p}$. Theorem 1.1 [2] states that the interval extension $\boldsymbol{F}(\boldsymbol{p})$ always contains the range $f(\boldsymbol{p})=[\underline{f}^*,\ \overline{f}^*]$ of the function $f(p)$

$$\boldsymbol{F}(\boldsymbol{p})\supseteq f(\boldsymbol{p})\,. \tag{8}$$

Hence (7) is satisfied if $\underline{F}>0$.

Based on Theorem 2 and inclusion property (8) the following results are obvious:

Corollary 1: (Sufficient condition for stability) If for all end-points $\underline{F}_q>0$ of the functions $f_q(p)$

$$\underline{F}_q>0,\ \ q=1,2,3,...\ , \tag{9}$$

then the system considered is stable.

Corollary 2: (Sufficient condition for instability) If at least one of the endpoints

$$\overline{F}_q\le 0,\ \ q=1,2,3,...\ , \tag{10}$$

then the system considered is not stable.

Various interval extensions can be used in implementing Corollaries 1 and 2: natural extension, mean-value form extension, extension using the global optimization methods [2]. The natural extensions are determined using the standard interval arithmetic [3]. Unfortunately this extension is the widest compared to the other types of extensions. The improved interval linearization [5] leads to shorter bounds of the considered extensions. Better results could be obtained if an affine arithmetic is applied to calculate the interval extensions $\boldsymbol{F}(\boldsymbol{p})$ [4]. The shortest interval extensions are obtained using the modified interval arithmetic which will be described briefly in the next section.

The paper is organized as follows. The modified affine arithmetic is described in the next section. The method for obtaining the interval extensions of the elements of Raus matrix using G-intervals is presented in Section 3. Numerical example illustrating the applicability of the new method is solved in Section 4. The paper ends up with concluding remarks in the last Section 5.

2. Modified affine arithmetic

Most often, the functions $f(p)$ are rational functions. Thus, we will define the main mathematical operations for these functions. To maintain completeness we start with the definition of the basic conception, the so-called *generalized interval*.

Definition 1. A generalized (G) interval $\tilde{X}$ of length k is defined as follows:

$$\tilde{X}=x_0+\sum_{i=1}^{k}x_i\boldsymbol{e}_i \tag{11}$$

where $x_i,\ \ i=0,1,...,k$, are real numbers while $\boldsymbol{e}_i$ are unit symmetrical intervals, i.e.

$$\boldsymbol{e}_i=\left[-1,1\right]. \tag{11a}$$

Let

$$\tilde{Y}=y_0+\sum_{i=1}^{k'}y_i\boldsymbol{e}_i \tag{12}$$

be a G-interval of length k'. To simplify presentation, we assume that $k'=k$ where k is the length of $\tilde{X}$ (otherwise, we add zero components either in $\tilde{X}$ or $\tilde{Y}$ depending on whether k is smaller or larger than k').

In general, each of the rational functions can be composed of the simple mathematical operations as follows.

Linear combination. Let $\tilde{X}$ and $\tilde{Y}$ be two G-intervals of length k given by (11) and (12). Also, let $\alpha,\beta\in R$. Then the linear combination of $\tilde{X}$ and $\tilde{Y}$, denoted $\alpha\tilde{X}+\beta\tilde{Y}$, is another G-interval $\tilde{Z}$ of the same length k if its elements z_i are computed as follows:

$$z_i=\alpha x_i+\beta y_i,\quad i=0,1,...,\ k\ . \tag{13}$$

As a corollary we have the definitions of addition of two G-intervals ($\alpha=\beta=1$) and subtraction of two G-intervals ($\alpha=1,\ \beta=-1$).

Now we shall define the operations of multiplication and division of G-intervals. Unlike the linear combination, the operations of multiplication and division of G-intervals result in a G-interval of increased length.

Multiplication. The product $\tilde{X}\tilde{Y}$ of two G-intervals $\tilde{X}$ and $\tilde{Y}$ of length k is a G-interval $\tilde{Z}$ of length k + 1 if the components z_i of $\tilde{Z}$ are computed as follows:

$$u=\sum_{i=1}^{m}\left|x_i\right|,v=\sum_{i=1}^{m}\left|y_i\right|,c=0.5\sum_{i=1}^{m}x_iy_i\ , \tag{14a}$$

$$z_0=x_0y_0+c\,,z_i=x_0y_i+y_0x_i\,,i=1,...,k\ , \tag{14b}$$

$$z_{m+1}=u\,v-\left|c\right|\,. \tag{14c}$$

It has to be noted that the multiplication (14) leads to smaller overestimation as compared with the multiplication used the standard affine arithmetic in [6] because of the "correction" introduced by the additional term c.

To define the operation of division, we have to consider the operation reciprocal $1/\tilde{Y}$ of a G-interval. To do this we need some definitions. The G-interval $\tilde{X}$ is reduced to the corresponding (ordinary) interval $\boldsymbol{x} = [\underline{x}, \overline{x}]$ if the summation operations in (11) are carried out. By abuse of language, we shall also say that $\tilde{X}$ does not contain zero (is positive or negative) if the corresponding reduced interval $\boldsymbol{x}$ does not contain zero (is positive or negative).

Reciprocal. Let $\tilde{Y}$ be a G-interval of length k that does not contain zero. Then the reciprocal $\tilde{Z} = 1/\tilde{Y}$ is another G-interval of length k+1 if its components z_i are computed as follows:

$$s = -1/(\underline{y}\,\overline{y}),\ y_1 = -\sqrt{-1/s},\ y_2 = -y_1\,, \tag{15a}$$

$$y_s = \begin{cases} y_2\,, & if\ \underline{y} > 0 \\ y_1\,, & if\ \overline{y} < 0 \end{cases}, \tag{15b}$$

$$\underline{f} = 1/y_s - s\,y_s\,, \quad \overline{f} = 1/\underline{y} - s\,\underline{y}\,, \tag{15c}$$

$$f_0 = 0.5\left(\underline{f} + \overline{f}\right), \quad r_f = \overline{f} - f_0\,, \tag{15d}$$

$$z_0 = sy_0 + f_0\,, \quad z_i = s\,y_i\,, i = 1,...,k\,, \tag{15e}$$

$$z_{m+1} = r_f \tag{15f}$$

when $\underline{y}$ and $\overline{y}$ are the endpoints of the reduced interval $\boldsymbol{y}$.

The above formulae follow directly from the general approach for enclosing univariate functions [7]-[10] by a linear interval form.

The division rule given below is based on the expression

$$\tilde{X}/\tilde{Y} = \frac{x_0}{y_0} + \frac{\sum_{i=1}^{m}(y_0 x_i - x_0 y_i)e_i}{y_0(y_0 + \sum_{i=1}^{m} y_j e_j)} = c + \frac{1}{\tilde{Y}}\left[\sum_{i=1}^{m}(x_i - c\,y_i)e_i\right] \tag{16}$$

if ($0 \notin \tilde{Y}$).

Division. Let $\tilde{X}$ and $\tilde{Y}$ be G-intervals of length k and $0 \notin \tilde{Y}$. Then the division $\tilde{X}/\tilde{Y}$ is a G-interval $\tilde{Z}$ of length k+2 whose components z_i are computed as follows:

$$\tilde{Q} = 1/\tilde{Y}\,, \tag{17a}$$

$$c = x_0 / y_0\,, \quad p_0 = 0\,, \quad p_i = x_i - c\,y_i\,, \quad i = 1,...,k\,, \tag{17b}$$

$$\tilde{P} = \sum_{i=1}^{m} p_i e_i\,, \tag{17c}$$

$$\tilde{V} = \tilde{Q} \cdot \tilde{P}\,, \tag{17d}$$

$$z_0 = c + v_0\,, z_i = v_i\,, i = 1,...,k+2 \tag{17e}$$

It is seen that the division increases the length of the resulting interval $\tilde{Z}$ by two because of the reciprocal (17a) and multiplication (17d), each operation adding one more element to the initial k elements of $\tilde{X}$ or $\tilde{Y}$.

3. Interval Raus criterion with G-intervals

In this section, we are interested in solving the problem P1 for all the elements of the first column in Raus matrix (4). A method capable of finding the interval extensions $\boldsymbol{F}(\boldsymbol{p})$ of functions $f(p),\ p \in \boldsymbol{p}$ that uses affine arithmetic will be suggested here. This method consists of the following:

1) The nonlinear functions $a_i(p),\ i = 0,1,...,n$ are given in explicit form of the vector of system parameters p.

2) The nonlinear functions $b_i(p),\ i = 0,1,...,n$ are dependent on the vector of system parameters p in explicit form. For this reason we apply the relations (4a) and get the expressions of the nonlinear functions $b_i(p),\ i = 0,1$ in implicit form of independent parameters $p_j,\ j = 1,2,...,m$.

3) The nonlinear functions $c_i(p),\ i = 0,1,...,n$ are also dependent on the vector of system parameters p in implicit form. Thus, we do the same operations as described in the previous item for $b_i(p),\ i = 0,1,...,n$.

We repeat these actions until we get the expressions of all elements of the first column in Raus matrix (4) as nonlinear functions of system parameters in implicit form $f(p)$. To find the interval extensions considered we do the following: first, we present the components of parameter vector p by generalized intervals

$$p_j = p_j^0 + \sum_{s=1}^{m} p_{js}\boldsymbol{e}_s\,,\ \boldsymbol{e}_s = [-1,\ \ 1]\,. \tag{18}$$

Then we apply the necessary simple mathematical operations of modified affine arithmetic (described in previous Section 2 to make a linearization of the resulting functions $f(p)$. Thus, we get the interval extensions in the following form:

$$\boldsymbol{F}(\boldsymbol{p}) = f_0 + \sum_{j=1}^{n_f} f_j \boldsymbol{e}_j,\ \boldsymbol{e}_j = [-1,\quad 1] \tag{19}$$

where the lengths n_f of the respective G-intervals depend on the type of nonlinearity of the functions $a_i(p),\ b_i(p),\ c_i(p),\ \dots\ ,\ i = 0,1,\dots,n$ with respect to the independent parameters $p_j,\ \ j = 1,2,\dots,m$.

The G-intervals (19) reduce to the corresponding (ordinary) intervals

$$\boldsymbol{F}(\boldsymbol{p}) = f_0 + [-r_f,\quad r_f] = [\underline{F}\quad \overline{F}] \tag{20}$$

where

$$r_f = \sum_{j=1}^{n_f} |f_j|, \tag{20a}$$

$$\underline{F} = 0.5(f_0 - r_f), \tag{20b}$$

$$\overline{F} = 0.5(f_0 + r_f) \tag{20c}$$

if the operations in (19) are carried out.

At the end, we make the following conclusions based on the Theorem 2 corollaries:

1) If all $\underline{F}_q > 0,\ q = 1,2,3,\dots$, then the system considered is stable.

2) If at least one of the endpoints $\overline{F}_q \le 0,\ q = 1,2,3,\dots$, then the system considered is not stable.

Remark 1: If the expressions of $b_i(p),\ c_i(p),\ \dots\ ,\ i = 0,1$ are complex functions of system parameters $p_j,\ \ j = 1,2,\dots,m$ it is sufficient to consider them as an independent intervals with respect to these parameters. Then the elements of Raus matrix (4) are calculated consecutively as follows: $\boldsymbol{A}_0,\ \boldsymbol{A}_1,\dots,\ \boldsymbol{A}_n$, $\boldsymbol{B}_0,\ \boldsymbol{B}_1,\dots,\ \boldsymbol{B}_n$, $\boldsymbol{C}_0,\ \boldsymbol{C}_1,\dots,\ \boldsymbol{C}_n$ and etc. In this case if none of the interval extensions $\boldsymbol{A}_0,\ \boldsymbol{A}_1,\ \boldsymbol{B}_0,\ \boldsymbol{A}_1,\ \boldsymbol{C}_0,\ \boldsymbol{C}_1\ \dots$contain zero we can make the same conclusions about the system as in Corollaries 1 and 2 of Theorem 2. In this case the number of calculations is decreased.

If one or more of the interval extensions $\boldsymbol{B}_i,\ \boldsymbol{C}_i,\dots,\ i = 0,1$ contain zero then they can be obtained using the fact that they are nonlinear functions of system parameters $p_j,\ \ j = 1,2,\dots,m$. In such a way we get shorter estimations of the interval extensions. Then we can make a final decision about the stability of the system studied.

4. Numerical example

The applicability of the above method will be illustrated by an example assessing the stability of the linear interval parameter system described by characteristic polynomial (1). In this example the order n of the associate characteristic polynomial is 5, i.e. $n = 5$ and

$$q(s) = a_0 s^5 + a_1 s^4 + a_2 s^3 + a_3 s^2 + a_4 s + a_5 \tag{21}$$

where

$$\begin{aligned}
&a_0(p) = 187 \\
&a_1(p) = 80869 + 0.044 p_3 \\
&a_2(p) = 0.044 p_1 + 2.556 p_2 + 10.5 p_3 + 4.064*10^7 \\
&a_3(p) = 10.5 p_1 + 2389.7 p_2 + 340 p_3 + 10^{-4} p_2 p_3 + \\
&\qquad + 3.638*10^9 \\
&a_4(p) = 340 p_1 + 2.139*10^5 p_2 + 10^{-4} p_1 p_2 + \\
&\qquad + 0.02 p_2 p_3 + 88.74*10^9 \\
&a_5(p) = 0.02 p_1 p_2 + 5.218*10^6 p_2
\end{aligned} \tag{21a}$$

It is seen from (21a) the vector of parameters p is 3-dimensional, i.e.

$$p = [p_1\quad p_2\quad p_3]^{\mathrm{T}}. \tag{22}$$

The respective vectors of centers and radii are

$$p^0 = [14000\quad 10000\quad 480]^{\mathrm{T}} \tag{22a}$$

and

$$R(p) = [3000\quad 3000\quad 100]^{\mathrm{T}}. \tag{22b}$$

We formulate the Raus matrix (defined by (4)) as follows:

$$R = \begin{bmatrix} c_1 & c_3 & c_5 \\ c_0 & c_2 & c_4 \\ b_1 & b_3 & b_5 \\ b_0 & b_2 & b_4 \\ a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \end{bmatrix} \tag{23}$$

where

$$b_0 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = a_1 a_2 - a_0 a_3$$

$$b_2 = \begin{vmatrix} a_1 & a_5 \\ a_0 & a_4 \end{vmatrix} = a_1 a_4 - a_0 a_5$$

$$b_1 = \begin{vmatrix} b_0 & b_2 \\ a_1 & a_3 \end{vmatrix} = b_0 a_3 - a_1 b_2$$

$$b_3 = \begin{vmatrix} b_0 & b_4 \\ a_1 & a_5 \end{vmatrix} = b_0 a_5 \qquad (23a)$$

$$b_4 = b_5 = 0$$

$$c_0 = \begin{vmatrix} c_1 & c_3 \\ c_0 & c_2 \end{vmatrix} = b_1 b_2 - b_0 b_3$$

$$c_1 = \begin{vmatrix} c_0 & c_2 \\ b_1 & b_3 \end{vmatrix} = c_0 b_3 - b_1 c_2$$

$$c_2 = c_3 = c_4 = c_5 = 0$$

To decrease the number of calculations we have in mind Remark 1 and consider $\boldsymbol{B}_i$ and $\boldsymbol{C}_i$, $i = 0,1$ as independent intervals with respect to the system parameters p_j, $j = 1,2,...,m$. In fact, they are dependent on the coefficients a_i, $i = 0,1,...,5$ (21).

To find the interval extensions of functions $a_i(p)$, $b_i(p)$, $c_i(p)$, ... , $i = 0,1$ we consider the relationships (23a) and (21a). Thus, we apply the simple mathematical operations "multiplication" and "linear combination" (in two cases – "addition" and "substraction") and get the following left bounds of interval extensions:

$$\begin{array}{lll} \underline{a}_0 = 187 & \underline{b}_0 = 2.6*10^{12} & \underline{c}_0 = 8.5*10^{44} \\ \underline{a}_1 = 8.1*10^{4} & \underline{b}_1 = 8.9*10^{21} & \underline{c}_1 = 6.6*10^{67} \end{array} \qquad (24)$$

As it is seen from (24) the left bounds of all determined interval extensions are positive. Based on the Corollary 1 of Theorem 2 the system (21) is stable.

5. Conclusion

A new interval technique for stability analysis of linear interval systems described by the characteristic polynomial (1) has been suggested. It is based on computing the interval extensions of the elements of Raus matrix (4) when the coefficients a_i, $i = 0,1,...,n$ in the characteristic polynomial (1) are nonlinear functions of independent system parameters p_j, $j = 1,...,m$ which take their values in prescribed intervals $\boldsymbol{p}_j, j = 1,2,...,m$. The interval extensions considered are determined using modified affine arithmetic which provides the shortest outer bounds of the ranges studied. Two sufficient conditions for stability of the system considered are defined. A numerical example is solved at the end of the paper. To simplify the calculation process we determine the interval extensions in the case when the elements $b_i(p)$, $c_i(p)$, ... , $i = 0,1,...,5$ in the Raus matrix are dependent intervals with respect to the coefficients $a_i(p)$, $i = 0,1,...,5$ (21). They are wider than those which are calculated when the elements of Raus matrix (4) are implicit functions of system parameters p_j, $j = 1,...,m$. Thus, these estimations are sufficient to make a conclusion about the stability of the system. In the example, all the interval extensions of the elements of the first column in Raus matrix (23) are non-negative. Hence the system under consideration is stable.

The technique of this method can be applied to the other criteria of stability of linear systems (Frezer-Duncan, Nyquist etc.) which will be reported in future publications.

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