CPSC-354 Report

Ethan Tapia
Chapman University

September 15, 2025

Abstract

Contents

1 Introduction

2 Week by Week

2.1 Week 1

Lecture Summary

We introduced *formal systems* and worked with Hofstadter's MIU-system as a rule–based rewriting game. Alphabet: $\Sigma = \{M, I, U\}$. Axiom (start string): MI. Production rules:

(R1) If a string ends in I, append U: $xI \Rightarrow xIU$.

(R2) If a string is Mx, duplicate x: $Mx \Rightarrow Mxx$.

(R3) Replace any III by U: $xIIIy \Rightarrow xUy$.

(R4) Delete any UU: $xUUy \Rightarrow xy$.

Key idea: reason about *invariants* that rules preserve, instead of searching blindly through derivations.

Homework: The MU-puzzle

Definition 2.1 (I–count and residue)**.** For a string w, let $\#_I(w)$ be the number of I's in w, and define the residue

$$\varphi(w) \;=\; \#_I(w) \bmod 3 \in \{0, 1, 2\}.$$

Lemma 2.2 (Effect of each rule on $\#_I$)**.** *For any string w:*

1. ***(R1)** and **(R4)** do not change $\#_I$.*
2. ***(R2)** doubles the number of I's* after *the initial M, so φ is multiplied by 2 modulo 3.*
3. ***(R3)** decreases $\#_I$ by 3, so φ is unchanged.*

Proposition 2.3 (Invariant modulo 3)**.** *Every string derivable from MI has $\varphi \in \{1, 2\}$. In particular, no derivable string has $\varphi = 0$.*

Proof. We use induction on the length of a derivation from MI.

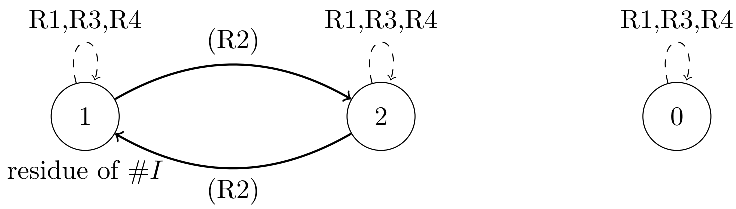
Base. $\varphi(MI) = 1$.

Step. Assume $\varphi \in \{1, 2\}$ for some derivable w. By Lemma 2.2, rules (R1), (R3), and (R4) keep φ unchanged, and rule (R2) maps $1 \leftrightarrow 2$ modulo 3. None of these operations yields 0 from a value in $\{1, 2\}$. Therefore the next string also has $\varphi \in \{1, 2\}$. □

Theorem 2.4 (MU is unreachable)**.** *MU cannot be derived from MI in the MIU-system.*

Proof. MU contains zero I's, hence $\varphi(MU) = 0$. By Proposition 2.3, every derivable string has residue 1 or 2. Thus MU is not derivable. □

Conclusion. Starting from MI we can toggle the residue $1 \leftrightarrow 2$ with (R2) and otherwise keep it fixed with (R1), (R3), (R4). We never reach residue 0, so no sequence of legal rule applications yields MU.



Question: If the MU-puzzle shows that some goals are unreachable due to invariants (like the mod-3 property of I's), how does this idea connect to undecidability in programming languages?

2.2 Week 2

Lecture Summary

We introduced *Abstract Reduction Systems (ARS)*: a pair (A, R) with one-step reduction $R \subseteq A \times A$. Key notions: reducible/normal form, joinability, confluence, termination, and unique normal forms.

Homework Part 2: The 8 Combinations

We provide an example ARS for each combination of (confluent, terminating, unique NFs). If a row is impossible, we explain why.

Confluent	**Terminating**	**Unique NFs**	**Example**
True	True	True	$A = \{a\},\ R = \emptyset$ (Fig. 1)
True	True	False	*Impossible*
True	False	True	$A = \{a, b\},\ R = \{(a, a), (a, b)\}$ (Fig. 2)
True	False	False	$A = \{a\},\ R = \{(a, a)\}$ (Fig. 3)
False	True	True	*Impossible*
False	True	False	$A = \{a, b, c\},\ R = \{(a, b), (a, c)\}$ (Fig. 4)
False	False	True	*Impossible*
False	False	False	$A = \{a, b, c\},\ R = \{(a, b), (a, c), (b, b), (c, c)\}$ (Fig. 5)

Why some rows are impossible. If an ARS has unique normal forms, it must be confluent. If an ARS is both confluent and terminating, then every element reduces to a unique normal form. Therefore the rows $(\mathrm{T}, \mathrm{T}, \mathrm{F})$, $(\mathrm{F}, \mathrm{T}, \mathrm{T})$, and $(\mathrm{F}, \mathrm{F}, \mathrm{T})$ cannot occur.

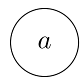


Figure 1: Combination (True, True, True). Terminating, confluent, unique NF.

Figure 2: Combination (True, False, True). Non-terminating, confluent, unique NF b.

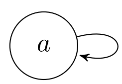


Figure 3: Combination (True, False, False). Non-terminating, confluent, no normal form.

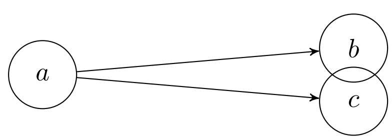


Figure 4: Combination (False, True, False). Terminating, not confluent; two distinct normal forms b, c are not joinable.

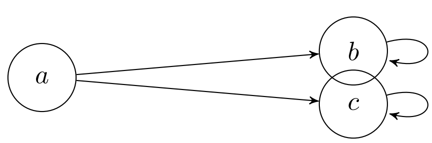


Figure 5: Combination (False, False, False). Non-terminating (loops), not confluent, no unique normal forms.

Conclusion. The MU-puzzle illustrates how invariants prove impossibility in a formal system. The ARS framework provides the general language to study rewrite systems via termination, confluence, and normal forms. The 8-combination analysis shows which behaviors are possible and which are structurally impossible.

Question: Could there be a general framework that unifies invariants with confluence and termination, so that impossibility and determinism appear as two sides of the same rewriting theory?

2.3 Week 3

Lecture Summary
TBD

Homework 3

Exercise 5 Consider an ARS with [A = a,b* $= \varepsilon, a, b, aa, ab, ba, bb, aaa, \ldots,]andrewriterules[ab \to ba, \quad ba \to ab, \quad aa \to \varepsilon, \quad b \to \varepsilon.]$

Reduce some example strings such as $abba$ and $bababa$.

$$abba \to aa \to \varepsilon,\ bababa \qquad\qquad \to aaa \to a. \tag{1}$$

Find two strings that are not equivalent. How many non-equivalent strings can you find?ε a

These have different normal forms and cannot be transformed into each other.

How many equivalence classes does $\overset{*}{\longleftrightarrow}$ have? What are the normal forms? There are two equivalence classes:

(a) Strings whose normal form is ε,

(b) Strings whose normal form is a.

The class is determined by the parity of the number of a's in the string.

Can you modify the ARS so that it becomes terminating without changing its equivalence classes? Yes. Remove one of the first two rules. They only permute a and b and do not affect equivalence classes, but having both makes the system non-terminating.

Question:
If I remove all the b's from a string, does the remaining word reduce to a or to ε?" This can be answered using the ARS because $b \to \varepsilon$ always deletes b's, and the final result depends only on whether the number of a's left is odd or even. Odd $\mapsto a$, even $\mapsto \varepsilon$.

Exercise 5b Now replace the rule $aa \to \varepsilon$ with $aa \to a$.

1. **Reduce some example strings such as $abba$ and $bababa$.**

$$abba \to aa \to a,\ bababa \qquad\qquad \to aaa \to aa \to a. \tag{2}$$

2. **Find two strings that are not equivalent.**

   - ε
   - a

How many equivalence classes are there? What are the normal forms? There are two equivalence classes:

(a) Strings with no a's ; $\rightarrow$; normal form ε,

(b) Strings with at least one a ; $\rightarrow$; normal form a.

Modify the ARS to make it terminating. As above, remove one of the two swapping rules $ab \leftrightarrow ba$.

Question: Is the system confluent? That is, if a string can be reduced in two different ways, do the reductions always lead to the same normal form?

3 Evidence of Participation

4 Conclusion

References

[BLA] Author, Title, Publisher, Year.