Welfare-Maximizing Pooled Testing

Edwin Lock
University of Oxford

Francisco Marmolejo-Cossío
Harvard University

Evi Micha
University of Toronto

Ariel Procaccia
Harvard University

Abstract

In an epidemic, how should an organization with limited testing resources safely return to in-person activities after a period of lockdown? We study this question in a setting where the population at hand is heterogeneous in both utility for in-person activities and probability of infection. In such a period of re-integration, tests can be used as a certificate of non-infection, whereby those in negative tests are permitted to return to in-person activities for a designated amount of time. Under the assumption that samples can be pooled, the question of how to allocate the limited testing budget to the population in such a way as to maximize the utility of individuals in negative tests (who subsequently return to in-person activities) is non-trivial, with a large space of potential testing allocations. We show that non-overlapping testing allocations, which are both conceptually and (crucially) logistically more simple to implement, are approximately optimal, and we design an efficient greedy algorithm for finding non-overlapping testing allocations with approximately optimal welfare (overall utility). We also provide empirical evidence in support of the efficacy of our approach.

1 Introduction

Over the course of the ongoing COVID-19 pandemic, it has become abundantly clear that testing is an indispensable tool for combating the virus. At the same time, the pandemic has underscored the fact that this very resource can be prohibitively constrained in multiple ways (e.g. lack of reagents, trained personnel or lab equipment), making it essential to study how to systematically allocate tests for the benefit of the population at hand [Kavanagh et al., 2020, Binnicker, 2020, Dryden-Peterson et al., 2021, De Georgeo et al., 2021, Dhabaan et al., 2020, Abera et al., 2020]. In this regard, pooled testing has emerged as a promising primitive for expanding the reach of limited testing resources. In a pooled test, the samples of multiple individuals are pooled together, and a single test is applied to the mixed sample. If the test is positive, at least one individual in the pool is infected, and if negative, all individuals in the pool are healthy. Indeed, the latter scenario is precisely where savings in applying tests occur, as a single test suffices to determine that multiple individuals are healthy.

Ultimately, testing serves two major purposes: it prevents infections from occurring by identifying infected individuals to be isolated, and it provides a means for individuals in a population to return to in-person activities after being cleared with a negative test result. Our point of departure from prior work is the observation that different individuals have different utilities for resuming in-person activities; in a scientific setting, for example, the benefit an experimentalist derives from being able to work in their lab is typically higher than that of a theoretician who is permitted to go to the office. The goal, therefore, should be to

> *... maximize the expected welfare (overall utility) of individuals who are able to resume in-person activities.*

In more detail, we explore the scenario where a population is currently in lockdown and wishes to begin in-person activities, facilitated by a fixed budget of tests. We assume that the population is heterogeneous: each individual has their own probability of being infected and (as mentioned earlier) their own utility for being able to resume in-person activities. In this setting, tests are allocated to subsets of the population as pooled tests, and an individual is allowed to return to in-person activities (and hence earn their corresponding utility) if any of the tests to which they are assigned results negative (proving that they are healthy). A testing regime thus obtains a certain expected welfare (overall utility) with respect to the randomized realization of infections in the population.

Our problem setting is fundamentally motivated by an ongoing collaboration with a higher education research institution in Mexico aimed at precisely providing safe, resource-optimal alternatives to fully easing a virtual work environment. Indeed, our partner institution has a heterogeneous population of roughly 500 individuals which include students, academic, administrators, security and cleaning staff. Though not the main focus of this work, we are working with our partners to use survey-based methods and input from local epidemiologists to provide a measure of utility of in-person activities as well as probabilities of infection for each member of the population (citation withheld for anonymity). Furthermore, our partners have an in-house qPCR testing facility which has validated pooled testing using saliva samples for pool sizes of up to 5 people. Two testing constraints exhibited by our partners are of crucial importance to our work: there are not sufficient resources available to fully test the population on a regular basis with individual tests, and the laboratory has a strong preference for pooled testing protocols where individuals are not included in more than one test on a regular basis.

To further elucidate the latter constraint, it is important to note that although pooled testing increases resource efficiency with regards to testing reagents, this can come at a significant logistical cost for laboratory personnel when pools are overlapping, especially if the series of tests to be performed is complicated and requires delicate tab-keeping of results. In this regard, pooled testing regimes in which no individual forms part of more than one pooled test are not only conceptually simpler to study but, more importantly, logistically simpler to implement. We call such testing regimes "non-overlapping" due to the fact that pooled tests in the allocation do not overlap, In contrast, the more general overlapping testing regimes allows for the budget of tests to be allocated arbitrarily, subjecting individuals to arbitrarily many tests.

Our results.   Despite these practical considerations that favor non-overlapping testing regimes, our first research question is whether they may be vastly outperformed by overlapping regimes. After all, if that is the case then perhaps supporting overlaps is worth the logistical overhead. However, we show that the worst-case ratio between the welfare of the best overlapping testing regime and the best non-overlapping regime is at most 4, and that in special cases it is even smaller. While a factor of 4 is admittedly non-negligible, the worst example we know of gives a ratio of 7/6. Qualitatively, we interpret these results as suggesting that it is justifiable to focus on non-overlapping testing regimes. Preliminary experimental evidence also suggests that the gain from overlaps is small in practice.

Turning to the algorithmic challenge, even without overlaps, the problem of computing the optimal regime is NP-hard. But we design a greedy polynomial-time algorithm that (roughly speaking) provides a 5-approximation to the optimal non-overlapping testing regime in the worst case. We then compare the performance of our polynomial-time algorithm with optimal non-overlapping testing empirically. In our experiments, we choose a population size, pool size constraints and testing budgets that broadly reflect realities in our ongoing collaboration. In order to compute (approximately) optimal non-overlapping testing regimes, we model the problem as a mixed-integer linear program (MILP). Our results indicate that the greedy algorithm computes testing regimes with close to optimal welfare in our setting when testing budgets are constrained. As the greedy algorithm strongly out-performs the MILP in terms of computational requirements, and testing regimes must be computed daily by our partner institute, we use the former in our implementation.

Related work.   Pooled testing dates back to the seminal work of Dorfman [1943] and has since become a mature field in its own right with a rich literature of protocols typically aimed at solving the following problem: *precisely ascertain the infection status of all individuals in a population with the minimum number of tests.* As mentioned above, our work departs significantly from this objective, as we instead assume that testing resources are initially limited, and with this provide welfare-optimizing testing allocations. For general references and recent results in this theoretical thread of pooled testing, we refer the reader to an in-depth literature review included in Appendix A.

Resource constraints have motivated recent work aimed at optimally utilizing limited testing capabilities to help local communities. Ely et al. [2021] study a model where a policymaker can employ tests of different types, each with differential costs and sensitivities. The policymaker has an overall budget, and testing allocations are measured with respect to the rate at which they correctly classify individuals as infected or healthy. Brault et al. [2021] focus on using limited pooled tests for early screening at a non-diagnostic level with high penalties associated with false negatives. Gollier and Gossner [2020] study pooled tests as a means to estimate infection prevalence and to find healthy individuals in a population. The main differences between their work and ours is that we consider a heterogenous population as well as upper bounds on pool sizes which result from lab limitations.

Most similar to our paper is the work of Lock et al. [2021] and Jonnerby et al. [2020a,b], in which the authors consider a limited testing budget to be used over a heterogeneous population as a means of surveilling and containing viral spread (unlike our work which focuses on testing to find those who are healthy). The testing allocation problem is treated as a multi-objective optimization problem aimed at balancing viral spread with the overall cost of self-isolation. Although not cast as a pooled testing paper, the results of Goldberg and Rudolf [2020] can be interpreted as computing the optimal allocation of a single (arbitrarily large) pooled test to a heterogeneous population as in our model setting. The authors show that computing an optimal single test allocation is NP-hard, and they provide a fully polynomial-time approximation scheme (FPTAS) for finding an approximately optimal single test allocation; we use their FPTAS as a subroutine in our greedy algorithm. Finally, Larremore et al. [2021] and Augenblick et al. [2020] study testing frequency as a crucial factor to limiting viral spread in a pandemic environment, and how pooled testing can increase the reach of a rapid frequency testing regime when test are limited.

2 Model

Let $[n] = \{1, \ldots, n\}$ denote the population and $B \in \mathbb{N}$ be the testing budget, i.e. the number of available tests. Each individual in the population has an independent probability of infection given by $p_i \in [0,1]$ and a utility $u_i \geq 0$ capturing their gain of returning to in-person activities.[1] We also let $q_i = 1 - p_i$ denote the probability that an individual is healthy. A population instance J is given by $(q_1, \ldots q_n, u_1, \ldots, u_n)$.

A single test consists of samples of a subset of the individuals, which we identify with a set $t \subseteq [n]$ of the individuals whose samples are included in the test. Test sizes are bounded by a pool size constraint $G \leq n$, so $|t| \leq G$ for all tests t.[2] For convenience, we introduce the notation $q_S = \prod_{i \in S} q_i$, for any $S \subseteq [n]$, to express the probability that all individuals in S are healthy; hence, q_t is the probability that test t is negative. A *testing regime* $T = (t_1, \ldots, t_B)$ is a collection of B tests t_j satisfying $|t_j| \leq G$ for each $j \in [B]$.

Individuals are allowed to return to in-person activities only if they are included in a negative test. For a given testing regime T, let P_i^T denote the probability that $i \in [n]$ is included in some negative test $t_j \in T$. A testing regime only earns utility from individuals who return to in-person activities as a result of being in a negative test. We let $u(T)$ denote the aggregate expected utility, or *welfare*, earned under testing regime T.[3] Linearity of expectation allows us to express the welfare of T as $u(T) = \sum_{i \in [n]} u_i \cdot P_i^T$. In addition, we let $u(t) := u((t)) = q_t \left(\sum_{i \in t} u_i \right)$ for a single pooled test t. A testing regime T is *optimal* (for a given population) if it maximizes welfare. Without loss of generality, we assume that $B < n$. If this is not the case, testing every person in the population individually is optimal.

Non-overlapping testing regimes. As discussed in Section 1, we are particularly interested in *non-overlapping* testing regimes that include each individual in at most one test. Formally, testing regime T is *non-overlapping* if $t \cap t' = \emptyset$ for all distinct tests $t, t' \in T$. In general, $P_{i,T}$ can be a complicated expression due to correlation between overlapping tests. In non-overlapping testing regimes T, by contrast, test results are independent of one another and the welfare of T is $u(T) = \sum_{t \in T} u(t)$.

---

[1] Utility might reflect people's socioeconomic status, the type of occupation, or mental health considerations.

[2] Pool sizes in pooled tests are limited due to biological constraints. Our partners in Mexico have replicated techniques from Sanghani et al. [2021] to achieve a maximal pool size of 5 with saliva samples.

[3] In the following, we will drop the term 'expected' for brevity, and assume that all welfares and utilities are determined in expectation.

Gain of overlaps. We are interested in quantifying the relative benefit provided by overlapping testing regimes over non-overlapping regimes, because the latter are not only conceptually simpler but also more feasible to implement in practice, as discussed in Section 1. Given a population instance J and budget B, we define the *gain of overlaps* $\mathrm{gain}(B, J)$ as the ratio of the welfare of the optimal testing regime over the welfare of the optimal non-overlapping testing regime. Formally, we let $\mathcal{T}^B$ and $\tilde{\mathcal{T}}^B$ respectively denote the space of all testing regimes and all non-overlapping testing regimes with testing budget B, and write

$$\mathrm{gain}(B, J) = \frac{\max_{T^* \in \mathcal{T}^B} u(T^*)}{\max_{T \in \tilde{\mathcal{T}}^B} u(T)}.$$

The *gain of overlaps given a budget B*, denoted $\mathrm{gain}(B)$, is the worst-case gain over all possible instances J, that is, $\mathrm{gain}(B) = \sup_J gain(B, J)$.

3 Theoretical Results

In this section, our goal is to provide upper bounds for the gain of overlaps. In order to develop intuition for cases in which the gain is greater than 1, we first study, as a warm-up, the case where there are only two available tests ($B = 2$), and show that the gain of overlaps is quite small. More generally, we show that for any value of B, the gain is at most 4. Motivated by this result and the aforementioned practical constraints, we then focus on non-overlapping testing regimes and present a greedy algorithm that achieves a constant-factor approximation with respect to the optimal non-overlapping testing regime.

3.1 Warm-Up: Gain of Overlaps when $B = 2$

We begin by studying the case in which $B = 2$. This case is particularly interesting for two reasons: first, we find the exact value of the gain by providing a lower bound and then showing that it is tight; second, this lower bound is the worst (largest) we know of, for any value of B.

Proposition 1. *For $B = 2$, $gain(B) \geq 7/6$.*

Proof. Consider a population of three individuals $\{1, 2, 3\}$ given by $q_1 = q_2 = 1/2$, $q_3 = 1$ and $u_1 = u_2 = u_3 = 1$. We see that individuals 1 and 2 are identical, and therefore there are only four non-overlapping testing regimes up to symmetries.

- $T^1 = (\{1\}, \{2\})$ yields welfare $u(T^1) = 1$.
- $T^2 = (\{1\}, \{3\})$ yields welfare $u(T^2) = 3/2$.
- $T^3 = (\{1, 2\}, \{3\})$ yields welfare $u(T^3) = 3/2$.
- $T^4 = (\{1, 3\}, \{2\})$ yields welfare $u(T^4) = 3/2$.

Now consider the overlapping testing regime $T^* = (\{1, 3\}, \{2, 3\})$ in which individual 3 is tested twice. Then we have welfare $u(T^*) = 7/4$. $\square$

It is of particular interest in the proof of Proposition 1 that the optimal welfare is achieved by testing the individual who is definitely healthy twice. This is intuitive, as we can test this individual many times without reducing the chances that any test is negative. In fact, in the proof of the upper bound (given in Appendix B), we use the property that the gain of overlaps is maximized when the probability that the individuals who are tested twice are healthy is maximized.

Proposition 2. *For $B = 2$, $gain(B) \leq 7/6$.*

3.2 Upper Bound on Gain of Overlaps for Any $B \geq 2$

In this section, we show that the gain of overlaps is a small constant; not only for the case that there are two tests available, but for any B. Before doing so, we start with some necessary notation. Given a testing regime $T = (t_1, \ldots, t_B)$ and individual $i \in [n]$, we let $T(i) = \{t_j \in T \mid i \in t_j\}$. Furthermore, we denote with $T(i; j)$ the set of tests with index less than j in which i has been tested,

i.e. $T(i; j) = \{t_{j'} \in T(i) : j' < j\}$. We say that test t_j is *pivotal* for individual i if: i is included in t_j, the result of t_j is negative, and all tests in $T(i; j)$ are positive. Equivalently, test t_j is pivotal for individual i if it is the negative test of smallest index in $T(i)$. We let $P_{i,j}^T$ denote the probability that t_j is pivotal for individual i under random infection realizations. In other words:

$$P_{i,j}^T = \begin{cases} \Pr[\,\forall t_{j'} \in T(i; j), t_{j'} \text{ is positive and } t_j \text{ is negative}] & \text{if } t_j \in T(i) \\ 0 & \text{otherwise.} \end{cases}$$

i is in a negative test if and only if a single test $t_j \in T$ is pivotal for i, hence $P_i^T = \sum_{j \in [B]} P_{i,j}^T$. As previously advertised, our main result of this section is that overlapping testing regimes have bounded gain over non-overlapping regimes.

Theorem 1. *For any $B \geq 1$, $gain(B) \leq 4$.*

To prove the theorem, we will show that given an optimal overlapping testing regime T^*, we can find a non-overlapping testing regime T such that $u(T^*)/u(T) \leq 4$ in polynomial time. This does not lead to a polynomial-time algorithm, as it requires access to T^* to begin with.

The proof requires a two lemmas. The first lemma, whose proof appears in Appendix C, is more intuitive: There exists an optimal testing regime $T^* = (t_1^*, \ldots, t_B^*)$ such that if $t_j^* \in T^*(i)$, it must be the case that the probability that t_j^* is pivotal for i is positive.

Lemma 1. *There exists an optimal $T^* = (t_1^*, \ldots, t_B^*)$ such that if $t_j^* \in T^*(i)$, then $P_{i,j}^{T^*} > 0$.*

The second lemma is a non-trivial generalization of Lemma 6 of Goldberg and Rudolf [2020]. At a high level, we show that if a (non-overlapping) testing regime is optimal, no test within this regime can be split into two groups which simultaneously have a "low" probability of being healthy. For the case where the testing regime is non-overlapping, the generalization is straightforward, but for the general case where the testing regime may be overlapping, novel techniques and arguments are used. The lemma's proof is relegated to Appendix D.

Lemma 2. *Suppose that $T^* = (t_1^*, \ldots, t_B^*)$ is an optimal (non-overlapping) testing regime and that $\alpha \in (0, 1)$. For any t_j^* and any $S \subset t_j^*$, if $q_S < \alpha$, then $q_{t_j^* \setminus S} \geq 1 - \alpha$.*

We are now ready to prove the theorem.

Proof of Theorem 1. We begin by constructing an intermediate non-overlapping testing regime T from T^* by simply assigning each individual to only a single test chosen arbitrary among all tests to which that individual is assigned to in T^*. Thus, we have that for each $j \in [B]$ $t_j \subseteq t_j^*$ which meana that $q_{t_j} \geq q_{t_j^*}$. Now, for each t_j, we let S_j be the smallest subset of t_j such that $q_{S_j} < 1/2$ (if $q_{t_j} \geq 1/2$, then $S_j = \emptyset$). Note that for each $i \in S_j$, we have $q_{S_j \setminus \{i\}} \geq 1/2$, as S_j is the smallest possible set that has probability less than $1/2$ to be negative. In addition, we can show that $q_{t_j \setminus S_j} \geq 1/2$. To see this, notice that $S_j \subseteq t_j \subseteq t_j^*$ and $q_{S_j} < 1/2$. From Lemma 2, we know that $q_{t_j^* \setminus S_j} \geq 1/2$, but since $t_j \setminus S_j \subseteq t_j^* \setminus S_j$, it follows that $q_{t_j \setminus S_j} \geq q_{t_j^* \setminus S_j} \geq 1/2$ as desired.

Next, consider two disjoint testing regimes given by T^1 where $t_j^1 = S_j$ and T^2 where $t_j^2 = t_j \setminus S_j$. Using the properties of T from the previous paragraph, we wish to show that for each $i \in t_j^\ell$ where $\ell \in \{1, 2\}$, we have that $P_i^{T^\ell} \geq q_i \cdot 1/2$. To that end, in the case of $\ell = 1$, we get $P_i^{T^1} = q_{t_j^1} = q_{S_j} < 1/2$. However, we also know that $q_{t_j^1 \setminus \{i\}} = q_{S_j \setminus \{i\}} \geq 1/2$, and hence $q_{t_j^1} = q_i \cdot q_{t_j^1 \setminus \{i\}} \geq q_i \cdot 1/2$ as desired. As for the case where $\ell = 2$, we get $P_i^{T^2} = q_{t_j^2} = q_{t_j \setminus S_j}$. The right hand side can be decomposed as $q_{t_j \setminus S_j} = q_i \cdot q_{t_j^2 \setminus \{i\}}$. However, as we have shown above, the choice of S_j ensures that $q_{t_j \setminus S_j} \geq 1/2$, and it follows that $q_i \cdot q_{t_j^2 \setminus \{i\}} \geq 1/2 \geq q_i \cdot 1/2$. We conclude that $P_i^{T^2} \geq q_i \cdot 1/2$, as desired.

Without loss of generality, assume that $\bigcup_{j \in [B]} t_j^* = [n']$ for some $n' \leq n$, i.e., the first n' individuals are included in some test under T^*. Notice that $\bigcup_{j \in [B]} t_j = [n']$, as each individual who is included in some test under T^* is also included in some test under T, and no individual who is not included in

5

some test under T^* is included in T. Thus, we get that

$$u(T^*) = \sum_{i \in [n']} P_i^{T^*} \cdot u_i = \sum_{j \in [B]} \left(\sum_{i \in S_j} P_i^{T^*} \cdot u_i + \sum_{i \in T_j \setminus S_j} P_i^{T^*} \cdot u_i \right).$$

We now consider two cases, depending on whether $\sum_{j \in [B]} \sum_{i \in S_j} P_i^{T^*} \cdot u_i \geq \sum_{j \in [B]} \sum_{i \in t_j \setminus S_j} P_i^{T^*} \cdot u_i$. If this is the case, we get that

$$u(T^*) \leq 2 \cdot \sum_{j \in [B]} \sum_{i \in S_j} P_i^{T^*} \cdot u_i \leq 2 \cdot \sum_{j \in [B]} \sum_{i \in S_j} q_i \cdot u_i.$$

It also holds that

$$u(T^1) = \sum_{j \in [B]} \sum_{i \in S_j} P_i^{T^1} \cdot u_i \geq \sum_{j \in [B]} \sum_{i \in S_j} 1/2 \cdot q_i \cdot u_i.$$

Thus, we conclude that

$$\frac{u(T^*)}{u(T^1)} \leq \frac{2 \sum_{i \in S_j} q_i \cdot u_i}{\sum_{i \in S_j} 1/2 \cdot q_i \cdot u_i} \leq 4.$$

Using similar arguments, we can show that $u(T^*)/u(T^2) \leq 4$ when $\sum_{j \in [B]} \sum_{i \in S_j} P_i^{T^*} \cdot u_i < \sum_{j \in [B]} \sum_{i \in T_j \setminus S_j} P_i^{T^*} \cdot u_i$, and the theorem follows. $\square$

While this proves that the gain of overlaps cannot be larger than 4, the worst known example is the one illustrated in Proposition 1, providing a lower bound of $7/6$. Interestingly, after running many simulations we were not able to find a better lower bound, and we believe that the gain of overlaps is less than 4. Moreover, it is possible to show that $gain(3) \leq 7/3$ and $gain(4) \leq 15/4$; the details are in Appendix E.

3.3   Greedy Algorithm for the Non-Overlapping Testing Regime

In light of the previous result, hereinafter we focus on non-overlapping testing regimes. Consider the case where $B = 1$. If G is a constant, we can efficiently enumerate all $O(n^G)$ potential pooled tests and find the optimal test t^*. On the other hand, when $G = n$, it follows from the work of Goldberg and Rudolf [2020] that even when there is only one test, it is NP-hard to find the subset of individuals that maximizes the expected welfare of the test. On the positive side, they provide a fully polynomial-time approximation scheme (FPTAS) for the same case. Here, we show that we can adjust the main ideas of their algorithm to obtain an FPTAS for the case where there is one test with up to $G \in [n]$ individuals in it; the proof is relegated to Appendix F.

Lemma 3. *When $B = 1$, there is an FPTAS for computing approximately optimal $(t^*) \in \mathcal{T}^1$.*

Our goal is to approximate the optimal non-overlapping testing regime when there are B available tests. Given the FPTAS of Lemma 3 for the single test case, a natural greedy approach is the following: design a test in each step by applying the FPTAS for the single test case to the remaining individuals. In other words, in each iteration we greedily find the test that approximates the optimal test over the available individuals using the FPTAS, add this test to the testing regime, disregard all individuals that are included in this test and continue to create greedy tests in the same fashion for the remaining individuals until the budget is exhausted. The above procedure results in an non-overlapping testing regime with at most B tests, as we never consider individuals that have already been included in a test. We refer this algorithm as ϵ-*Greedy*, where ϵ is the error tolerance used in the FPTAS algorithm[4]. We show that this algorithm gives a $5/(1-\epsilon)$ approximation to the optimal non-overlapping testing regime. The non-trivial proof is relegated to Appendix G.

Theorem 2. *ϵ-Greedy returns a $5/(1-\epsilon)$-approximate non-overlapping testing regime.*

---

[4]When G is constant, we can efficiently compute optimal t^* at each step of the approach above via brute force. We call this algorithm *Greedy*

Note that one can combine Theorem 1 and Theorem 2 to see that ϵ-Greedy gives a constant-factor approximation to the optimal *overlapping* testing regime. Furthermore, we can also show that Greedy is optimal in instances where testing budgets are low and individuals can only take utilities and probabilities from a finite set of values. Specifically, assume that the population at hand can be partitioned into C clusters, where the i-th cluster has n_i individuals with identical utility u_i and probability of infection p_i[5]. Moreover, suppose that $(t^*) \in \mathcal{T}^1$ is an optimal single pooled test for this clustered population. In Appendix I we prove the following result, which identifies a rather natural setting where greedy is optimal — even with respect to the optimal *overlapping* testing regime.

Proposition 3. *For testing budget $B > 0$, if $B \cdot |t^* \cap C_i| \leq n_i$, Greedy returns an optimal allocation that applies B distinct copies (in terms of composition) of t^*.*

Finally, in Appendix H, we also consider the case where all the individuals have the same utility. First, we show that when B is a constant we can find the optimal testing regime. For general B, we design a variant of Greedy which sorts the individuals in decreasing order with respect to the probability of being healthy and in each step adds individuals to the current test as long as the expected utility of the test decreases. We show that this algorithm returns an e-approximate non-overlapping testing regime and this result is tight.

4 Implementation and Experiments

The pilot study. Together with additional collaborators, we are working closely with a higher education research institute in Mexico to implement our testing and reopening strategy in practice. Our partner institute has a heterogeneous population of approximately 500 individuals, which include students, academics, administrators, security and cleaning staff. The institute also has an in-house qPCR testing facility which has validated pooled testing using saliva samples for pool sizes up to 5. The testing facility has expressed a wish to perform non-overlapping testing only. In order to study the effectiveness of our reopening strategy in practice, we will launch a month-long randomized controlled trial in the coming weeks (as of May 2022). Our pilot study divides the population into a treatment and a control group, and subjects the treatment group of approximately 250 individuals to our testing strategy while only allowing individuals in the control group access to the institute with special permission. In our pilot study, we compute an optimal testing regime every day, and invite pooled individuals to submit their samples for testing. We emphasize that the collection and aggregation of utilities and health probabilities must be undertaken with care, as incorrect data may lead to biases and sub-optimal allocations. We are working with medical experts and epidemiologists to mitigate this risk. For more details about our experiment, we refer to the pre-analysis plan (citation redacted for anonymity).

Computation in practice. As discussed above, even the problem of allocating a single test is computationally hard. In order to make the computation of testing regimes tractable for our pilot study, we formulate the problem of non-overlapping testing as optimization problems that we solve using commercial solvers. As described in Appendix J, the problem of allocating a single test can be formulated as a mixed-integer conic optimization program (MICP), and solved using the commercial conic solver MOSEK (https://mosek.com). This implementation is used by our Greedy algorithm in our simulations. When multiple tests are to be allocated, we formulate a mixed-integer linear program (MILP) that approximates an optimal non-overlapping solution and can solved by any MILP solver. The main idea behind the MILP formulation is to approximate existing exponential constraints with piecewise-linear functions that can be formulated as a collection of mixed integer linear constraints. In the implementation, the accuracy of this approximation can be adjusted by tuning the number K of segments of the piecewise-linear function, at the cost of introducing more (integer) variables, thereby increasing the time to solve the program. We provide practical (additive) approximation guarantees for this approach, which we refer to as the *Approx algorithm* and show in experiments that these guarantees are competitive in our pilot regime. A detailed description of the optimization formulations can be found in Appendix J.

Re-pooling submitted samples. Our testing strategy invites everyone for testing who is part of a pooled test in the optimal solution. In practice, our partner institute has observed — in an independent

---

[5]We highlight that clusters are a very natural constraint in terms of population structure which we intend to use in our pilot with our partner institutions in Mexico.
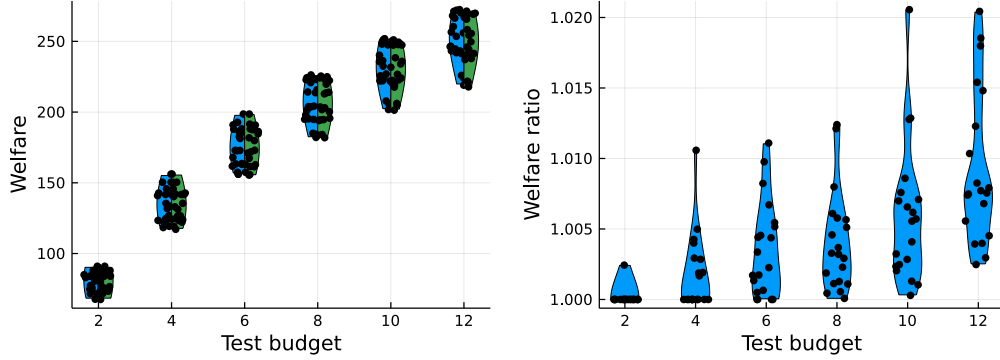
Figure 1: Outcomes of Greedy and Approx with population size $n = 250$, pool size constraint $G = 5$ and testing budgets $B \in \{2, 4, \dots, 12\}$. Left: Welfares achieved by Approx (left regions, blue) and Greedy (right regions, red). Right: Ratio between the welfares of Approx and Greedy. In both figures, each black dot corresponds to one of the 20 randomly generated populations.

pooled testing trial — that a small fraction of invited participants fail to submit samples. In order to optimize pooling, we propose a second optimization round, in which we compute an (approximately) optimal pooling among the samples that have been submitted. It is immediate that the second optimization round cannot decrease the expected welfare achieved.

4.1 Experiments

In order to evaluate the accuracy and running times of the Greedy and Approx algorithms, we run computational experiments in scenarios that broadly reflect the parameters in our upcoming pilot study. In our experiments, we have a population of $n = 250$ individuals and compute testing regimes with budgets $B \in \{2, 4, 6, 8, 10, 12\}$. While our partnering testing laboratory has committed to collecting saliva samples, which lead to a pool size limit of $G = 5$, we also study regimes in which the pool sizes are limited to $G = 10$ (the limit for nasopharyngeal swabs) or unbounded ($G = n$).

We study the average-case behavior of both algorithms for each configuration by first generating 20 random populations of size $n = 250$. The utility of each individual is drawn independently and uniformly at random from the integers $\{1, 2, \dots, 10\}$, while the probability of being healthy is drawn independently and uniformly at random from $\{0, 0.1, \dots, 1\}$. We then run Greedy and Approx on each population for each pool size $G \in \{5, 10, n\}$, recording the welfare achieved for both algorithms, as well as their running times (in milliseconds).[6] Note that we record the true welfares that we compute from the testing regimes returned by both algorithms, and not the objective values of the underlying MILP and conic optimization problems, as the latter will be an approximation of the true welfare. For Approx, we tune the parameter K governing the quality of the approximation so that the additive approximation guarantee is small.[7] The code used to run these experiments can be found at (redacted for anonymity).

We also refer to Appendix L for preliminary experiments that compare non-overlapping with overlapping testing on small populations; these give additional evidence that the average-case gain from overlaps is small.

Results. Figure 1 shows the welfares achieved by both algorithms for pool size constraint $G = 5$, as well as the welfare ratios. For the latter, we divide the welfare achieved by Approx by the welfare achieved by Greedy for each population, and depict the resulting ratios as black dots. The analogous Figs. 2 and 3 for pool size constraints $G = 10$ and $G = 250$ can be found in Appendix K. In Table 1, we list the mean welfares and running times of both algorithms, as well as the approximation guarantee of Approx; Appendix K contains analogous tables for $G = 10$ and $G = 250$.

---

[6]The experiments were run on a 2021 MacBook Air. Gurobi 9.5.1 was used to solve the MILP, and MOSEK 9.3.13 was used to solve the MICP.

[7]K was set to $K = 18$ for the regime $G = 5$, $K = 21$ for the regime $G = 10$, and $K = 65$ for the regime $G = 250$.

$n = 250, G = 5$	Approx			Greedy	
Budget	Welfare*	Guarantee (add.)	Time*	Welfare*	Time*
2	80.7739	0.15676	78 ms	80.7657	26 ms
4	134.919	0.313521	320 ms	134.696	55 ms
6	175.108	0.470281	1249 ms	174.493	87 ms
8	206.689	0.627041	12534 ms	205.859	115 ms
10	231.305	0.783801	41558 ms	229.944	143 ms
12	251.142	0.940562	116942 ms	248.869	172 ms

Table 1: Summary showing welfare and computation time for Approx and Greedy on populations of size $n = 250$ and pool size constraint $G = 5$, with testing budgets $B \in \{2, 4, \ldots, 12\}$. Starred columns show mean values computed over the 20 random populations. We also provide additive approximation guarantee of Approx (compared to optimal non-overlapping welfare).

Overall, we observe that Approx computes near-optimal non-overlapping testing regimes in a reasonable[8] amount of time. Greedy performs as well as Approx when testing budgets are low and remains competitive also for the higher budgets we considered. While the running time of Approx appears to increase exponentially with the testing budget B, Greedy scales linearly and is extremely fast (even for larger populations and testing budgets). In combination with the competitive welfare results, this makes Greedy attractive for practical implementations of our testing strategy that rely on a quick turnaround, run on 'budget hardware' or wish to avoid costly cloud computing services. Comparing the scenarios with different pool sizes $G \in \{5, 10, n\}$, we see that increasing pool sizes from 5 to 10 increases mean welfare by approximately 12.4 when the testing budget is set at $B = 12$, and unbounded pool sizes further increase mean welfare by approximately 8. This suggests that the pool size limit imposed by saliva sampling, as opposed to the limit of 10 for nasopharyngeal samples, may be considered a limitation. For this reason, institutions may wish to weigh the positives and negatives of saliva and nasopharyngeal sampling carefully.

5   Discussion

This work introduces a novel utility-based approach to pooled testing in resource-constrained environments. In this setting, we provide strong theoretical and empirical performance guarantees that further justify the implementation of non-overlapping testing regimes beyond their essential logistical simplicity. As mentioned throughout the paper, our work is heavily informed by an ongoing collaboration in Mexico with the immediate goal of implementing a forthcoming pilot of the overall methodology in a resource-constrained setting. In this regard, the efficient algorithms we have provided for computing approximately optimal non-overlapping testing regimes are of key importance going forward.

There are many directions for future work. On a theoretical level, there is a gap between our upper bound of 4 and lower bound of 7/6 on the gain of overlaps. On a more practical level, the overall testing and re-integration policy we propose is static in nature, as we consider the one-shot setting where a testing budget is to be fully utilized by a policymaker. But testing can be dynamic, with allocations chosen adaptively as a function of previous test results, and it is valuable to understand what potential benefits this extended functionality can bring. Additionally, policymakers (including our partners in Mexico) potentially have access to different types of tests, each with different associated costs and performance (i.e., pool size and sensitivity), and providing optimal budget-constrained allocations in this heterogeneous test setting is a key open question. Most importantly, as our collaboration in Mexico progresses, we hope to obtain valuable insight regarding end-to-end implementations and implications of applying welfare-maximizing testing regimes so that resource-constrained communities can be better prepared for future COVID-19 outbreaks.

---

[8]In our context, a reasonable amount of time is anything below 15 minutes on a modern desktop PC. After computing the testing regime, pooled individuals must be invited to submit samples, samples must be collected and recorded, a final pooling of submitted samples must be computed, and samples must be processed and tested — all within a day.

References

A. Abera, H. Belay, A. Zewude, B. Gidey, D. Nega, B. Dufera, A. Abebe, T. Endriyas, B. Getachew, H. Birhanu, et al. Establishment of COVID-19 testing laboratory in resource-limited settings: challenges and prospects reported from ethiopia. *Global Health Action*, 13(1):1841963, 2020.

S. Ahn, W.-N. Chen, and A. Özgür. Adaptive group testing on networks with community structure. In *2021 IEEE International Symposium on Information Theory (ISIT)*, pages 1242–1247. IEEE, 2021.

M. Aldridge, O. Johnson, and J. Scarlett. Group Testing: An Information Theory Perspective. *Foundations and Trends® in Communications and Information Theory*, 15(3-4):196–392, 2019. ISSN 1567-2190. doi: 10.1561/0100000099. URL http://dx.doi.org/10.1561/0100000099.

N. Augenblick, J. T. Kolstad, Z. Obermeyer, and A. Wang. Group testing in a pandemic: The role of frequent testing, correlated risk, and machine learning. Working Paper 27457, National Bureau of Economic Research, July 2020. URL http://www.nber.org/papers/w27457.

M. J. Binnicker. Challenges and controversies to testing for COVID-19. *Journal of clinical microbiology*, 58(11):e01695–20, 2020.

V. Brault, B. Mallein, and J.-F. Rupprecht. Group testing as a strategy for COVID-19 epidemiological monitoring and community surveillance. *PLoS computational biology*, 17(3):e1008726, 2021.

B. Cleary, J. A. Hay, B. Blumenstiel, M. Harden, M. Cipicchio, J. Bezney, B. Simonton, D. Hong, M. Senghore, A. K. Sesay, et al. Using viral load and epidemic dynamics to optimize pooled testing in resource-constrained settings. *Science translational medicine*, 13(589):eabf1568, 2021.

A. Cohen, N. Shlezinger, A. Solomon, Y. C. Eldar, and M. Médard. Multi-level group testing with application to one-shot pooled COVID-19 tests. In *ICASSP 2021-2021 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 1030–1034. IEEE, 2021.

M. Cuturi, O. Teboul, Q. Berthet, A. Doucet, and J.-P. Vert. Noisy adaptive group testing using bayesian sequential experimental design. *arXiv preprint arXiv:2004.12508*, 2020.

M. R. De Georgeo, J. M. De Georgeo, T. M. Egan, K. P. Klee, M. S. Schwemm, H. Bye-Kollbaum, and A. J. Kinser. Containing sars-cov-2 in hospitals facing finite ppe, limited testing, and physical space variability: Navigating resource constrained enhanced traffic control bundling. *Journal of Microbiology, Immunology and Infection*, 54(1):4–11, 2021.

G. N. Dhabaan, W. A. Al-Soneidar, and N. N. Al-Hebshi. Challenges to testing COVID-19 in conflict zones: Yemen as an example. *Journal of global health*, 10(1), 2020.

R. Dorfman. The detection of defective members of large populations. *The Annals of Mathematical Statistics*, 14(4):436–440, 1943.

S. Dryden-Peterson, G. E. Velásquez, T. J. Stopka, S. Davey, S. Lockman, and B. O. Ojikutu. Disparities in SARS-CoV-2 testing in massachusetts during the COVID-19 pandemic. *JAMA network open*, 4(2):e2037067–e2037067, 2021.

D. Du, F. K. Hwang, and F. Hwang. *Combinatorial group testing and its applications*, volume 12. World Scientific, 2000.

J. Ely, A. Galeotti, O. Jann, and J. Steiner. Optimal test allocation. *Journal of Economic Theory*, 193: 105236, 2021.

J. Emmanuel, M. Bassett, H. Smith, and J. Jacobs. Pooling of sera for human immunodeficiency virus (hiv) testing: an economical method for use in developing countries. *Journal of clinical pathology*, 41(5):582–585, 1988.

S. Ghosh, A. Rajwade, S. Krishna, N. Gopalkrishnan, T. E. Schaus, A. Chakravarthy, S. Varahan, V. Appu, R. Ramakrishnan, S. Ch, et al. Tapestry: A single-round smart pooling technique for covid-19 testing. *MedRxiv*, 2020.

N. Goldberg and G. Rudolf. On the complexity and approximation of the maximum expected value all-or-nothing subset. *Discrete Applied Mathematics*, 283:1–10, 2020.

C. Gollier and O. Gossner. Group testing against Covid-19. Technical report, EconPol Policy Brief, 2020.

D. Hong, R. Dey, X. Lin, B. Cleary, and E. Dobriban. Group testing via hypergraph factorization applied to COVID-19. *Nature Communications*, 13(1):1–13, 2022.

F. K. Hwang. A method for detecting all defective members in a population by group testing. *Journal of the American Statistical Association*, 67(339):605–608, 1972.

J. Jonnerby, P. Lazos, E. Lock, F. Marmolejo-Cossío, C. B. Ramsey, M. Shukla, and D. Sridhar. Maximising the Benefits of an Acutely Limited Number of COVID-19 Tests. *ArXiv (preprint)*, pages 1–12, 2020a. URL http://arxiv.org/abs/2004.13650.

J. Jonnerby, P. Lazos, E. Lock, F. Marmolejo-Cossío, C. B. Ramsey, and D. Sridhar. Test and Contain: A Resource-Optimal Testing Strategy for COVID-19. *AI for Social Good Workshop*, (July), 2020b.

M. M. Kavanagh, N. A. Erondu, O. Tomori, V. J. Dzau, E. A. Okiro, A. Maleche, I. C. Aniebo, U. Rugege, C. B. Holmes, and L. O. Gostin. Access to lifesaving medical resources for african countries: COVID-19 testing and response, ethics, and politics. *The Lancet*, 395(10238):1735–1738, 2020.

D. B. Larremore, B. Wilder, E. Lester, S. Shehata, J. M. Burke, J. A. Hay, M. Tambe, M. J. Mina, and R. Parker. Test sensitivity is secondary to frequency and turnaround time for COVID-19 screening. *Science advances*, 7(1):eabd5393, 2021.

C. H. Li. A sequential method for screening experimental variables. *Journal of the American Statistical Association*, 57(298):455–477, 1962.

E. Lock, F. J. Marmolejo-Cossío, J. Jonnerby, N. Rajgopal, H. A. Guzmán-Gutiérrez, L. A. Benavides-Vázquez, J. R. Tello-Ayala, and P. Lazos. Optimal testing and containment strategies for universities in mexico amid covid-19. In *Equity and Access in Algorithms, Mechanisms, and Optimization*, pages 1–9. 2021.

L. Mutesa, P. Ndishimye, Y. Butera, J. Souopgui, A. Uwineza, R. Rutayisire, E. L. Ndoricimpaye, E. Musoni, N. Rujeni, T. Nyatanyi, et al. A pooled testing strategy for identifying sars-cov-2 at low prevalence. *Nature*, 589(7841):276–280, 2021.

O. U. Nalbantoglu. Group testing performance evaluation for sars-cov-2 massive scale screening and testing. *BMC Medical Research Methodology*, 20(1):1–11, 2020.

P. Nikolopoulos, T. Guo, S. R. Srinivasavaradhan, C. Fragouli, and S. Diggavi. Community aware group testing. *arXiv preprint arXiv:2007.08111*, 2020.

H. B. Petersen, B. Bah, and P. Jung. Practical high-throughput, non-adaptive and noise-robust sars-cov-2 testing. *arXiv preprint arXiv:2007.09171*, 2020.

H. R. Sanghani, D. A. Nawrot, F. Marmolejo-Cossío, J. M. Taylor, J. Craft, E. Kalimeris, M. I. Andersson, and S. R. Vasudevan. Concentrating Pooled COVID-19 Patient Lysates to Improve Reverse Transcription Quantitative PCR Sensitivity and Efficiency. *Clinical Chemistry*, 67(5): 797–798, 2021. ISSN 0009-9147. doi: 10.1093/clinchem/hvab035.

X. M. Tu, E. Litvak, and M. Pagano. On the informativeness and accuracy of pooled testing in estimating prevalence of a rare disease: application to hiv screening. *Biometrika*, 82(2):287–297, 1995.

L. M. Wein and S. A. Zenios. Pooled testing for hiv screening: capturing the dilution effect. *Operations Research*, 44(4):543–569, 1996.

A Supplementary Literature on Pooled Testing

Pooled testing dates back to the seminal work of Dorfman [1943] who sought to facilitate syphilis diagnostics during World War II. Dorfman's protocol proceeds in two stages: The first stage tests individuals in disjoint groups of a fixed size. Given the results of the first stage, the second stage involves individually testing individuals from positive groups to precisely find who is infected in the population. Theoretical guarantees are provided when the number of individuals who are infected in the population is known beforehand, which permits computing an optimal pool size in the first stage of the protocol in such a way as to minimize the overall number of tests which are needed to precisely ascertain the health status of all individuals in a population. Pooled testing has since become a mature field in its own right with a rich literature of protocols aimed at solving the same objective: *precisely ascertaining the infection status of all individuals in a population with the minimum number of tests.*

In this vein, pooled testing protocols are typically categorized with respect to two axes: assumptions on infection rates, and whether protocol allows for adaptive testing allocations. For the former, two leading regimes are that of combinatorial pooled testing, where a fixed number of infections are known to exist within the population, however the identity of those infected is unknown, and probabilistic pooled testing, where infections occur according to a well-defined probability distribution. With respect to the latter axis, testing regimes can either be adaptive, where they occur in rounds and the allocation of a given round can depend on the results of tests from previous rounds, or non-adaptive, where all tests are allocated at once. For example, Dorfman's protocol operates in the combinatorial adaptive regime, and it has since been significantly improved starting with the s-stage algorithm of Li [1962] and continuing with the asymptotically optimal generalized binary splitting approach of Hwang [1972]. More recent theoretical results include: adaptive methods for combinatorial testing which make use of hypergraph factorization in early stages of testing [Hong et al., 2022], using compressed sensing for non-adaptive combinatorial pooled testing [Cohen et al., 2021, Petersen et al., 2020, Ghosh et al., 2020], pooled testing under network-based (non-i.i.d.) infection models [Ahn et al., 2021, Nikolopoulos et al., 2020], Bayesian infection inference in the adaptive noisy test result regime [Cuturi et al., 2020]. For a general references to pooled testing, we refer the reader to Du et al. [2000] as well as Aldridge et al. [2019] for an information-theoretic focus on the subject.

Beyond theory, pooled testing has been applied to combat various diseases in the past, especially HIV/AIDS [Tu et al., 1995, Wein and Zenios, 1996, Emmanuel et al., 1988]. From the outset of the COVID-19 pandemic it became clear that testing resource constraints would be a large issue for multiple countries, and hence pooled testing became a viable option for combatting the virus, especially as it was shown that qPCR tests can be sensitive enough to pool samples in a pooled test [Sanghani et al., 2021, Mutesa et al., 2021, Nalbantoglu, 2020]. Although having access to a pooled testing primitive is a necessary condition for implementing existing protocols, practical constraints often render these approaches unfeasible. On one hand, complicated pooled testing regimes can be difficult to implement logistically at scale with limited laboratory personnel and workflow infrastructure [Cleary et al., 2021]. On the other hand, the unfortunate reality is that many resource-constrained populations are in a situation where their testing budget falls far below the information theoretic lower bounds required to precisely ascertain the health profile of all individuals as per traditional pooled testing objectives.

B Proof of Proposition 2

Proof. Suppose that given an instance, the optimal testing regime is T^*, where $t_1^* \cap t_2^* \neq \emptyset$. Let $A = t_1^* \setminus t_2^*$, $B = t_2^* \setminus t_1^*$ and $C = t_1^* \cap t_2^*$. Note that $t_1^* \cup t_2^* = A \cup B \cup C$. Without loss of generality, we assume that $q_A \geq q_B$ and with a slight abuse of notation, we denote $u_A = \sum_{i \in A} u_i$, $u_B = \sum_{i \in B} u_i$, and $u_C = \sum_{i \in C} u_i$. Moreover, we define the following four different testing regimes:

- T^1 with $t_1^1 = A \cup C$ and $t_2^1 = B$

- T^2 with $t_1^2 = A$ and $t_2^2 = C$

- T^3 with $t_1^3 = B$ and $t_2^3 = C$

- T^4 with $t_1^4 = A \cup B$ and $t_2^4 = C$

We start with the following necessary lemma.

Lemma 4. *For any $\hat{T} \in \{T^1, T^2, T^3, T^4\}$, the ratio $u(T^*)/u(\hat{T})$ is maximized when $q_C = 1$.*

Proof. We first show that the statement is true for $\hat{T} = T^1$. We need to show that

$$\frac{q_C\left(q_A \cdot u_A + q_B \cdot u_B + (q_A + (1 - q_A) \cdot q_B) \cdot u_C\right)}{q_C \cdot q_A \cdot u_A + q_B \cdot u_B + q_C \cdot q_A \cdot u_C}$$
$$\leq \frac{q_A \cdot u_A + q_B \cdot u_B + (q_A + (1 - q_A) \cdot q_B) \cdot u_C}{q_A \cdot u_A + q_B \cdot u_B + q_A \cdot u_C}$$
$$\Rightarrow q_C \leq \frac{q_C \cdot q_A \cdot u_A + q_B \cdot u_B + q_C \cdot q_A \cdot u_C}{q_A \cdot u_A + q_B \cdot u_B + q_A \cdot u_C}$$

which is true since

$$q_C = q_C \cdot \frac{q_A \cdot u_A + q_B \cdot u_B + q_A \cdot u_C}{q_A \cdot u_A + q_B \cdot u_B + q_A \cdot u_C} \leq \frac{q_C \cdot q_A \cdot u_A + q_B \cdot u_B + q_C \cdot q_A \cdot u_C}{q_A \cdot u_A + q_B \cdot u_B + q_A \cdot u_C}.$$

Now, we consider the case that $\hat{T} = T^2$. Here, we need to show that

$$\frac{q_C\left(q_A \cdot u_A + q_B \cdot u_B + (q_A + (1 - q_A) \cdot q_B) \cdot u_C\right)}{q_A \cdot u_A + q_C \cdot u_C}$$
$$\leq \frac{q_A \cdot u_A + q_B \cdot u_B + (q_A + (1 - q_A) \cdot q_B) \cdot u_C}{q_A \cdot u_A + u_C}$$
$$\Rightarrow q_C \leq \frac{q_A \cdot u_A + q_C \cdot u_C}{q_A \cdot u_A + u_C},$$

which is true since

$$q_C = q_C \cdot \frac{q_A \cdot u_A + u_C}{q_A \cdot u_A + u_C} \leq \frac{q_A \cdot u_A + q_C \cdot u_C}{q_A \cdot u_A + u_C}.$$

With similar arguments as above, we can show that the ratio $u(T^*)/u(T^3)$ is maximized when $q_C = 1$.

Lastly for $\hat{T} = T^4$, we need to show that

$$\frac{q_C\left(q_A \cdot u_A + q_B \cdot u_B + (q_A + (1 - q_A) \cdot q_B) \cdot u_C\right)}{q_A \cdot q_B \cdot (u_A + u_B) + q_C \cdot u_C}$$
$$\leq \frac{q_A \cdot u_A + q_B \cdot u_B + (q_A + (1 - q_A) \cdot q_B) \cdot u_C}{q_A \cdot q_B \cdot (u_A + u_B) + u_C}$$
$$\Rightarrow q_C \leq \frac{q_A \cdot q_B \cdot (u_A + u_B) + q_C \cdot u_C}{q_A \cdot q_B \cdot (u_A + u_B) + u_C},$$

which is true since

$$q_C = q_C \cdot \frac{q_A \cdot q_B \cdot (u_A + u_B) + u_C}{q_A \cdot q_B \cdot (u_A + u_B) + u_C} \leq \frac{q_A \cdot q_B \cdot (u_A + u_B) + q_C \cdot u_C}{q_A \cdot q_B \cdot (u_A + u_B) + u_C}.$$

$\square$

Using Lemma 4, hereinafter, we consider the case that $q_C = 1$.

We distinguish into two cases.

Case I: $q_A \geq 5/6$. In this case, note that

$$\frac{(q_A + (1 - q_A) \cdot q_B) \cdot u_C}{q_A \cdot u_C} \leq \frac{(q_A + (1 - q_A) \cdot q_A) \cdot u_C}{q_A \cdot u_C} = 2 - q_A \leq \frac{7}{6}.$$

where the second transition follows since $q_A \geq q_B$ and the last transition follows since $q_A \geq 5/6$. Hence, we see that

$$\frac{q_A \cdot u_A + q_B \cdot u_B + (q_A + (1 - q_A) \cdot q_A) \cdot u_C}{q_A \cdot u_A + q_B \cdot u_B + q_A \cdot u_C} \leq \frac{7}{6},$$

meaning that $u(T^*)/u(T^1) \leq 7/6$.

13

Case II: $q_A < 5/6$. Here, for the sake of contradiction, suppose that for any testing without overlaps the approximation ratio is more that $7/6$. Then, we have that $u(T^*)/u(\hat{T}) > 7/6$, for any $\hat{T} \in \{T^1, T^2, T^3, T^4\}$.

Thus,

$$\frac{u(T^*)}{u(T^1)} > \frac{7}{6}$$

$$\Rightarrow \frac{q_A \cdot u_A + q_B \cdot u_B + (q_A + (1 - q_A) \cdot q_B) \cdot u_C}{q_A \cdot u_A + q_B \cdot u_B + q_A \cdot u_C} > \frac{7}{6}$$

$$\Rightarrow (6 \cdot (q_A + (1 - q_A) \cdot q_B) - 7q_A) \cdot u_C > q_A \cdot u_A + q_B \cdot u_B. \tag{1}$$

Now, from the fact that $u(T^*)/u(T^2) > 7/6$, we get that

$$\Rightarrow \frac{q_A \cdot u_A + q_B \cdot u_B + (q_A + (1 - q_A) \cdot q_B) \cdot u_C}{q_A \cdot u_A + u_C} > \frac{7}{6}$$

$$\Rightarrow q_B \cdot u_B > \frac{1}{6} q_A \cdot u_A + (\frac{7}{6} - (q_A + (1 - q_A) \cdot q_B)) \cdot u_C$$

$$\Rightarrow q_B \cdot u_B > \frac{1}{6} q_A \cdot u_A + (\frac{7}{6} - (q_A + (1 - q_A) \cdot q_A)) \cdot u_C \tag{2}$$

where the last inequality follows from the fact that $q_A \geq q_B$. With a very similar argument we can conclude that

$$q_A \cdot u_A > \frac{1}{6} q_B \cdot u_B + (\frac{7}{6} - (q_A + (1 - q_A) \cdot q_A)) \cdot u_C \tag{3}$$

from the fact that $u(T^*)/u(T^3) > 7/6$. From Equation (2) and Equation (3), we get that

$$q_A \cdot u_A + q_B \cdot u_B > \frac{6}{5} \cdot 2 \cdot (\frac{7}{6} - (q_A + (1 - q_A) \cdot q_A)) \cdot u_C$$

and from Equation (1), we conclude that

$$(6 \cdot (q_A + (1 - q_A) \cdot q_B) - 7q_A) > \frac{6}{5} \cdot 2 \cdot (\frac{7}{6} - (q_A + (1 - q_A) \cdot q_A))$$

which is true when $1/2 < q_A < 2/3$. Hence, from now one we assume that q_A lies in the interval $(1/2, 2/3)$.

Moreover, we have that

$$\frac{u(T^*)}{u(T^2)} > \frac{7}{6}$$

$$\Rightarrow \frac{q_A \cdot u_A + q_B \cdot u_B + (q_A + (1 - q_A) \cdot q_B) \cdot u_C}{q_A \cdot u_A + u_C} > \frac{7}{6}$$

$$\Rightarrow \frac{(6 \cdot (q_A + (1 - q_A) \cdot q_B) - 7q_A) \cdot u_C + (q_A + (1 - q_A) \cdot q_B) \cdot u_C}{q_A \cdot u_A + u_C} > \frac{7}{6}$$

$$\Rightarrow \frac{7 \cdot (1 - q_A) \cdot q_B \cdot u_C}{q_A \cdot u_A + u_C} > \frac{7}{6}$$

$$\Rightarrow (6 \cdot (1 - q_A) \cdot q_B - 1) \cdot u_C > q_A \cdot u_A \tag{4}$$

where the third inequality follows from Equation (1), and similarly using the fact that $u(T^*)/u(T^3)$, we get

$$(6 \cdot (1 - q_A) \cdot q_B - 1) \cdot u_C > q_B \cdot u_B \tag{5}$$

Lastly,

$$\frac{u(T^*)}{u(T^4)} > \frac{7}{6}$$

14

$$\Rightarrow \frac{q_A \cdot u_A + q_B \cdot u_B + (q_A + (1 - q_A) \cdot q_B) \cdot u_C}{q_A \cdot q_B \cdot (u_A + u_B) + u_C} > \frac{7}{6}$$

$$\Rightarrow q_A \cdot u_A + q_B \cdot u_B > \frac{(7 - 6(q_A + (1 - q_A) \cdot q_B)) \cdot u_C}{6 - 7 \cdot q_B} \tag{6}$$

where the last inequality follows from the fact that $q_B < 6/7$ since $q_B \leq q_A$ and $q_A < 5/6$.

Now, from Equation (4), Equation (5) and Equation (6), we get that

$$\frac{u(T^*)}{u(T^1)} = \frac{q_A \cdot u_A + q_B \cdot u_B + (q_A + (1 - q_A) \cdot q_B) \cdot u_C}{q_A \cdot u_A + q_B \cdot u_B + q_A \cdot u_C}$$

$$\leq \frac{2 \cdot (6 \cdot (1 - q_A) \cdot q_B - 1) \cdot u_C + (q_A + (1 - q_A) \cdot q_B) \cdot u_C}{\frac{7 - 6(q_A + (1 - q_A) \cdot q_B) \cdot u_C}{6 - 7 \cdot q_B} + q_A \cdot u_C}$$

which is maximized when $q_A = q_B = 1/2$, given that $1/2 < q_A < 2/3$ and then we get that $u(T^*)/u(T^1) \leq 7/6$ and reach a contradiction. $\square$

C   Proof of Lemma 1

Proof. We re-write $P_{i,j}^{T^*}$ for $i \in t_j^*$ using conditional probabilities:

$$P_{i,j}^{T^*} = \Pr[\forall t_{j'}^* \in T^*(i; j), t_{j'}^* \text{ is positive and } t_j^* \text{ is negative}].$$
$$= \Pr[t_j^* \text{ negative}] \cdot \Pr[\forall t_{j'}^* \in T^*(i; j), t_{j'}^* \text{ positive} \mid t_j^* \text{ negative}]$$
$$= q_{t_j^*} \cdot \Pr[\forall t_{j'}^* \in T^*(i; j) \, \exists i' \in t_{j'}^* \setminus t_j^* \mid i \text{ infected}].$$

First of all, we notice that it must be the case that $q_{t_j^*} = \prod_{i \in t_j^*} q_i > 0$. If this is not so then there must be some individual $i \in t_j^*$ such that $p_i = 1$, and it is straightforward to see that it is always sub-optimal to include such an individual in any testing regime. As for the second term, to show that it is non-zero, we begin by using the fact that T^* is optimal to show that without loss of generality, we can assume that for each $t_{j'}^* \in T^*(i; j)$ it must be the case that there exists an $i \in t_{j'}^* \setminus t_j^*$ such that $p_i > 0$.

Suppose that this is not the case and that there exist $t_{j'}^*, t_j^* \in T^*$ such that every $i \in t_{j'}^* \setminus t_j^*$ has $p_i = 0$ ($q_i = 1$). We show that either T^* is sub-optimal, or we can construct an optimal testing regime where this is no longer the case. To do so, we assume that without loss of generality $j' = 1$ and $j = 2$ (we can arbitrarily re-order test indices), and write the expected utility of T^* as follows:

$$u(T^*) = \sum_{i \in [n]} u_i \cdot P_i^{T^*}$$

$$= \sum_{i \in [n]} \sum_{j \in [B]} u_i \cdot P_{i,j}^{T^*}$$

$$= \sum_{j \in [B]} \sum_{i \in t_j^*} u_i \cdot P_{i,j}^{T^*}$$

$$= \left(\sum_{i \in t_1^*} u_i \cdot P_{i,1}^{T^*} \right) + \left(\sum_{i \in t_2^*} u_i \cdot P_{i,2}^{T^*} \right) + \sum_{j=3}^{B} \sum_{i \in t_j^*} u_i \cdot P_{i,j}^{T^*}$$

$$= u(t_1^*) + \left(\sum_{i \in t_1^*} u_i \cdot P_{i,2}^{T^*} + \sum_{i \in t_2^* \setminus t_1^*} u_i \cdot P_{i,2}^{T^*} \right) + \sum_{j=3}^{B} \sum_{i \in t_j^*} u_i \cdot P_{i,j}^{T^*}$$

$$= u(t_1^*) + \left(\sum_{i \in t_2^* \setminus t_1^*} u_i \cdot P_{i,2}^{T^*} \right) + \sum_{j=3}^{B} \sum_{i \in t_j^*} u_i \cdot P_{i,j}^{T^*}$$

$$= u(t_1^*) + q_{t_2^*} \sum_{i \in t_2^* \setminus t_1^*} u_i + \sum_{j=3}^{B} \sum_{i \in t_j^*} u_i \cdot P_{i,j}^{T^*}$$

$$\le u(t_1^*) + u(t_2^* \setminus t_1^*) + \sum_{j=3}^{B} \sum_{i \in t_j^*} u_i \cdot P_{i,j}^{T^*}$$

The first 4 equalities follow from re-ordering terms in the sums. In the fifth equality, we make use of the fact that if $i \in t_1^*$, it must be the case that $P_{i,2}^{T^*} = 0$, for if t_2^* is pivotal for them, then that test must be negative, which implies that t_1^* is negative (for individuals in $t^* \setminus t_2^*$ are guaranteed to be healthy by assumption), contradicting the pivotal nature of t_2^*. If $i \in t_2^* \setminus t_1^*$, then t_2^* is pivotal only if it is negative, hence $P_{i,2}^{T^*} = q_{t_2^*}$, which in turn justifies the following equality. Finally, we know that $q_{t_2^*} \le q_{t_2^* \setminus t_1^*}$, and that $u(t_2^* \setminus t_1^*) = q_{t_2^* \setminus t_1^*} \sum_{i \in t_2^* \setminus t_1^*} u_i$, from which the final equality holds. Putting everything together, let us consider T' where $t_j' = t_j^*$ for $j \ne 2$ and $t_2' = t_2^* \setminus t_1^*$. From the above, it follows that either T^* is sub-optimal (as T' achieves more welfare), or we can replace T^* with T', and in either case, we can ensure that our desired property holds.

With this in hand, this means if T^* is optimal, and we consider $i \in t_j^*$ as in the beginning of the proof, we can construct a pool of individuals S by picking one individual from the set $T_j^* \setminus T_{j'}^*$ for each $T_{j'}^* \in T(i; j)$ such that the individual has non-zero probability of infection. From the above, we are guaranteed to be able to construct such an S which is non-empty. Furthermore, if all individuals in S are infected, it follows that each $T_{j'} \in T(i; j)$ is positive without compromising a negative test on T_j^*. This in turn implies that $\Pr[\forall t_{j'}^* \in T^*(i; j) \ \exists i' \in t_{j'}^* \setminus t_j^* \mid i \text{ infected}] \ge \Pr[1 - q_S] > 0$. Putting this together with the fact that $q_{t_j^*} > 0$ completes the proof.

$\square$

D Proof of Lemma 2

Proof. First note that for any $i \in t_j^*$,

$$
\begin{aligned}
P_{i,j}^{T^*} &= \Pr[\, \forall t_{j'}^* \in T^*(i; j), t_{j'}^* \text{ is positive and } t_j^* \text{ is negative}] \\
&= \Pr[\, \forall t_{j'}^* \in T^*(i; j), t_{j'}^* \text{ is positive } \mid t_j^* \text{ is negative}] \cdot \Pr[t_j^* \text{ is negative}] \\
&= \Pr[\, \forall t_{j'}^* \in T^*(i; j), t_{j'}^* \setminus t_j^* \text{ is positive } \mid t_j^* \text{ is negative}] \cdot \Pr[t_j^* \text{ is negative}] \\
&= \Pr[\, \forall t_{j'}^* \in T^*(i; j), t_{j'}^* \setminus t_j^* \text{ is positive }] \cdot \Pr[t_j^* \text{ is negative}] \\
&= \Pr[\, \forall t_{j'}^* \in T^*(i; j), t_{j'}^* \setminus t_j^* \text{ is positive }] \cdot q_{t_j^*}
\end{aligned}
$$

where the third equality follows since $t_{j'}^*$ can be positive if and only if some individual in $t_{j'}^* \setminus t_j^*$ is infected as given that t_j^* is negative we conclude that any individual in t_j^* is healthy, and the fourth inequality follows since each individual has an independent probability to be infected.

Assume for the sake of contradiction that for some $j \in [B]$, there exists $S \subset T_{j^*}^*$ such that $q_S < \alpha$ and $q_{T_{j^*}^* \setminus S} < 1 - \alpha$. Without loss of generality, we assume that $j^* = B$. Then, since from Lemma 1, we know that $P_{i,j}^{t^*} > 0$ for any individual i such that $i \in T_j^*$, we have that

$$u(T^*) = \sum_{i \in [n]} P_i^T \cdot u_i$$

$$= \sum_{i \in [n]} \sum_{j \in [B]} P_{i,j}^{T^*} \cdot u_i$$

$$= \sum_{j \in [B-1]} \sum_{i \in [n]} P_{i,j}^{T^*} \cdot u_i + \sum_{i \in [n]} I_{i,B}^{T^*} \cdot q_{t_B^*} \cdot \Pr[\forall t_{j'}^* \in T^*(i; B), t_{j'}^* \setminus t_B^* \text{ is positive}] \cdot u_i$$

$$= \sum_{j \in [B-1]} \sum_{i \in [n]} P_{i,j}^{T^*} \cdot u_i + q_S \cdot q_{t_B^* \setminus S} \cdot \sum_{i \in t_B^*} \Pr[\forall t_{j'}^* \in T^*(i; B), t_{j'}^* \setminus t_B^* \text{ is positive}] \cdot u_i$$

16

$$= \sum_{j \in [B-1]} \sum_{i \in [n]} P_{i,j}^{T^*} \cdot u_i + q_{t_B^* \setminus S} \cdot \left(q_S \cdot \sum_{i \in S} \Pr[\forall t_{j'}^* \in T^*(i; B), t_{j'}^* \setminus t_B^* \text{ is positive}] \cdot u_i \right)$$

$$+ q_S \left(q_{t_B^* \setminus S} \cdot \sum_{i \in t_B^* \setminus S} \Pr[\forall t_{j'}^* \in T^*(i; B), t_{j'}^* \setminus t_B^* \text{ is positive}] \cdot u_i \right)$$

$$\leq \sum_{j \in [B-1]} \sum_{i \in [n]} P_{i,j}^{T^*} \cdot u_i + q_{t_B^* \setminus S} \cdot \left(q_S \cdot \sum_{i \in S} \Pr[\forall t_{j'}^* \in T^*(i; B), t_{j'}^* \setminus S \text{ is positive}] \cdot u_i \right)$$

$$+ q_S \left(q_{t_B^* \setminus S} \cdot \sum_{i \in t_B^* \setminus S} \Pr[\forall t_{j'}^* \in T^*(i; B), t_{j'}^* \setminus (t_B^* \setminus S) \text{ is positive}] \cdot u_i \right)$$

$$< \sum_{j \in [B-1]} \sum_{i \in [n]} P_{i,j}^{T^*} \cdot u_i + \max \left\{ q_S \cdot \sum_{i \in S} \Pr[\forall t_{j'}^* \in T^*(i; B), t_{j'}^* \setminus S \text{ is positive}] \cdot u_i, \right.$$

$$\left. q_{t_B^* \setminus S} \cdot \sum_{i \in t_j^* \setminus S} \Pr[\forall t_{j'}^* \in T^*(i; B), t_{j'}^* \setminus (t_B^* \setminus S) \text{ is positive}] \cdot u_i \right\}$$

To justify the first inequality, we begin by showing that

$$\sum_{i \in S} \Pr[\forall t_{j'}^* \in T^*(i; j), \ t_{j'}^* \setminus t_B^* \text{ is positive}] \leq \sum_{i \in S} \Pr[\forall t_{j'}^* \in T^*(i; j), \ t_{j'}^* \setminus S \text{ is positive}].$$

Since $S \subseteq t_B^*$, it follows that for any $t_{j'}^* \in T^*(i; j)$ we have $t_{j'}^* \setminus t_B^* \subseteq t_{j'}^* \setminus S$. This in turn implies that if $t_{j'}^* \setminus t_B^*$ is positive, it must be the case that $t_{j'}^* \setminus S$ is positive. It follows that the event where $\forall t_{j'}^* \in T^*(i : j), \ t_{j'}^* \setminus t_B^*$ is positive implies that $\forall t_{j'}^* \in T^*(i : j), \ t_{j'}^* \setminus S$ is positive, hence the inequality follows. Furthermore, the argumentation above can be replicated with $t_B^* \setminus S$ rather than S to fully justify the inequality from the main derivation. As for the second inequality, in the derivation, this follows from the assumption that $q_S < \alpha$ and $q_{t_B^* \setminus S} < 1 - \alpha$.

To reach a contradiction, let T be a testing regime such that $t_j = t_j^*$ for any $j \in [B-1]$ and $t_B = S$ if

$$q_S \cdot \sum_{i \in S} \Pr[\forall t_{j'}^* \in T^*(i; B), t_{j'}^* \setminus S \text{ is positive}] \cdot u_i$$

$$\geq q_{T_B^* \setminus S} \cdot \sum_{i \in T_B^* \setminus S} \Pr[\forall t_{j'}^* \in T^*(i; B), t_{j'}^* \setminus (t_B^* \setminus S) \text{ is positive}] \cdot u_i$$

and $t_B = t_B^* \setminus S$, otherwise. Then,

$$u(T) = \sum_{i \in [n]} P_i^T \cdot u_i$$

$$= \sum_{i \in [n]} \sum_{j \in [B]} P_{i,j}^T \cdot u_i$$

$$= \sum_{j \in [B-1]} \sum_{i \in [n]} P_{i,j}^{T^*} \cdot u_i + \sum_{i \in t_B} q_{t_B} \cdot \Pr[\forall t_{j'}^* \in T^*(i; B), t_{j'}^* \setminus t_B \text{ is positive}] \cdot u_i$$

$$= \sum_{j \in [B-1]} \sum_{i \in N} P_{i,j}^{T^*} \cdot u_i + \max \left\{ q_S \cdot \sum_{i \in S} \Pr[\forall t_{j'}^* \in T^*(i, B), t_{j'}^* \setminus S \text{ is positive}] \cdot u_i, \right.$$

$$\left. q_{t_B^* \setminus S} \cdot \sum_{i \in T_B \setminus S} \Pr[\forall t_{j'}^* \in T^*(i; B), t_{j'}^* \setminus (t_B^* \setminus S) \text{ is positive}] \cdot u_i \right\}$$

$$> u(T^*),$$

which contradicts the optimality of T^* and completes the proof of our claim. $\qquad \square$

E Gain of Overlaps for $B \in \{3, 4\}$

Proposition 4. $gain(3) \leq 7/3$ *and* $gain(4) \leq 15/4$.

Proof. We start from the case that $B = 3$. we partition the individuals, that are pooled into at least one test in an optimal testing regime T^*, into seven sets as following: the first three sets, denoted by S_1, S_2 and S_3, consist of individuals that are pooled into only the first, the second and the third test, respectively; the next three sets, denoted by S_4, S_5 and S_6 consist of individuals that are only pooled into the first and the second test, the first and the third test, and the second and the third test, respectively; the last set, denoted by S_7, consists of individuals that are included in all three tests. Then, $u(T^*) \leq \sum_{j \in [7]} q_{S_j} \cdot \sum_{i \in S_j} u_i$, as each $i \in S_j$ is always pooled into a test along with all the individuals in $S_j \setminus \{i\}$ and hence her probability to be included in a test that turns negative is at most q_{S_j} which indicates that probability that all the individuals in S_j are healthy. Now, without loss of generality, assume that $q_{S_j} \sum_{i \in S_j} u_i \geq q_{S_{j+1}} \sum_{i \in S_{j+1}} u_i$ for each $j \in [6]$. Then, we define the non-overlapping testing regime T such that $t_1 = S_1$, $t_2 = S_2$ and $t_3 = S_3$. Notice that $u(T) = \sum_{j \in [3]} q_{S_j} \cdot \sum_{i \in S_j} u_i$, and hence $u(T^*)/u(T) \leq 7/3$.

For the case that $B = 4$, we partition the individuals into 15 sets with a very similar way as above, i.e. the first fours sets consist of individuals that are pooled only into one test, the next six tests consist of individuals that are pooled into exactly two tests, the next 4 tests consist of individuals that are pooled into exactly three tests and the last set consists of individuals that are pooled into all four tests. Then, we use the four available tests to pool individuals from the four sets that have the highest utility and we get an approximation of $15/4$. $\qquad\square$

F Proof of Lemma 3

Proof. Here we show, how the FPTAS, that is introduced in Goldberg and Rudolf [2020] and finds an almost optimal test when there is no any size constraint, can be modified for the case that a test can pool up to G samples.

Using similar notation as in Goldberg and Rudolf [2020], for $i \in [n]$, we denote with $P(i, C, L)$ the maximum probability of a subset of $[i]$ to be negative with sum of utilities exactly C and size exactly L. Then, we modify the dynamic program that was introduced at Equation (6) in Goldberg and Rudolf [2020] as following:

$$P(i, C, L) = \begin{cases} \max\{P(i-1, C, L), q_i \cdot P(i-1, C-u_i, L-1)\} & i \geq 2 \text{ and } u_i < C \\ P(i-1, C, L) & i \geq 2 \text{ and } u_i \geq C \\ q_i & i = 1 \text{ and } u_1 = C \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

With a slight abuse of notation, we denote with $t(P(i, C, L))$ the subset of $[i]$ that satisfies $q_{t(P(i,C,L))} = P(i, C, L)$, $\sum_{\ell \in [t(P(i,C,L))]} u_\ell = C$ and $|t(P(i, C, L))| = L$.

If $\bar{C}$ is an upper bound on the sum of utilities of the optimal test (a straightforward upper bound is $\bar{C} = \sum_{i \in [n]} u_i$), then the optimal test with size at most G that maximizes the utility is given by $t(P(n, C^*, L^*))$ where

$$C^*, L^* = \underset{C \in [\bar{C}], L \in [G]}{\arg\max} \ C \cdot P(n, C, L). \quad (8)$$

Thus, the running time is given by $O(nG\bar{C}) \leq O(n^2\bar{C})$, since $G \leq n$. Then, we see that in order to approximately solve the dynamic programming with a polynomial run-time complexity bound, we should scale down (and round) the utility coefficients whose magnitude determines the running time of the program. We can achieve this by using identical arguments as in Section 3.2 of Goldberg and Rudolf [2020]. We present the whole proof here for completeness.

We scale down the utilities using some factor κ, by setting $\hat{u}_i = \lfloor u_i/\kappa \rfloor$ for each $i \in [n]$. Before, we choose κ, we add some more notation. Let $N_{1/2} = \{i \in [n] : q_i \geq 1/2\}$. Without loss of generality, assume that $N_{1/2} = [h]$ and $[n] \setminus N_{1/2} = \{h+1, \ldots, n\}$. Let $\hat{P}(i, C, L)$ denote the DP

Algorithm 1

1: $\kappa \leftarrow (\epsilon \cdot 1/2 \cdot \max_{i \in N_{1/2}} u_i)/n$
2: $z^* \leftarrow 0; t \leftarrow \emptyset$
3: **for** $j = h+1, \ldots n$ **do**
4:     **if** $\hat{z}(h,j) < q_j \cdot u_j$ **then**
5:         **if** $q_j \cdot u_j > z^*$ **then**
6:             $z^* \leftarrow q_j \cdot u_j$
7:             $t \leftarrow \{j\}$
8:         **end if**
9:     **else**
10:         **if** $\hat{z}(h,j) > z^*$ **then**
11:             $z^* \leftarrow \hat{z}(h,j)$
12:             $t \leftarrow t(\hat{P}(h, \hat{C}_{i,j}, \hat{L}_{i,j}))$
13:         **end if**
14:     **end if**
15: **end for**
16: **return** t

---

in Equation (7) by replacing u_i with $\hat{u}_i$. Moreover, we assume that there exists a dummy individual $n+1$ with $\hat{u}_{n+1} = 0$ and $q_{n+1} = 0$. Then, for $i \in [n]$ and $j > i$ the scaled DP problem is defined as

$$\hat{z}_\kappa(i,j) = \max_{C \in [\bar{C}(i)], L \in [G]} (\kappa \cdot C + u_j) \cdot \hat{P}(i, C, L) \cdot q_j. \tag{9}$$

where $\bar{C}(i) = \sum_{i' \in [i]} u_{i'}$. Let

$$\hat{C}_{i,j}, \hat{L}_{i,j} = \arg\max_{C \in [\bar{C}(i)], L \in [G]} (\kappa \cdot C + u_j) \cdot \hat{P}(i, C, L) \cdot q_j.$$

Note that $t(\hat{P}(i, \hat{C}_{i,j}, \hat{L}_{i,j}) \cup \{j\})$ returns an optimal test by replacing u_i with $\hat{u}_i$ and adding the constraint that for any $\ell \in [i+1, \ldots, n] \setminus \{j\}$, ℓ is not pooled into the test, while j is pooled into it. From Lemma 2, we know that it suffices to evaluate $\hat{z}(i,j)$ for $i \in [h]$ and $j \in \{h+1, \ldots, n\}$ in order to evaluate $\hat{z}_\kappa(n, n+1)$ as at most one individual from $[n] \setminus N_{1/2}$ may be pooled into the test.

The following lemma establishes an upper bound on a value of κ that suffices to bound the relative error of solutions of $\hat{z}_\kappa$ in approximating the optimal test within a given $\epsilon > 0$.

Lemma 5. *Let t^* be an optimal test with $\sum_{\ell \in t^*} u_\ell = C^*$ and $|t^*| = L^*$. For a given $\epsilon > 0$, there exist $i \in [h]$, $j \in \{h, \ldots, n+1\}$, with $i < j$, and for $\bar{t} = t^* \setminus \{j\}$ and $\kappa \leq \frac{\epsilon \max_{i \in \bar{t}} q_i \cdot u_i}{n}$ such that*

$$\hat{z}_\kappa(i,j) \geq (1 - \epsilon) \cdot C^* \cdot P(n, C^*, L^*).$$

Proof. First note

$$\sum_{i \in t^*} u_i - \kappa \sum_{i \in t^*} \hat{u}_i = \sum_{i \in t^*} u_i - \kappa \sum_{i \in t^*} \lfloor u_i/\kappa \rfloor \leq \sum_{i \in t^*} u_i - \kappa \sum_{i \in t^*} (u_i/\kappa - 1) \leq \kappa n$$

where the last inequality follows since $|t^*| \leq n$.

Let j be the individual with the smallest probability of being healthy in t^* by breaking ties with respect to individuals that have higher index and let i be the individual with the highest index in $t^* \setminus \{j\}$. We denote with $\hat{t} \subseteq [i]$ the set that maximizes Equation (9). Then, we get that

$$\hat{z}_\kappa(i,j) = q_j \left(\kappa \sum_{\ell \in \hat{t} \setminus \{j\}} \hat{u}_\ell + u_j \right) q_{\hat{t} \setminus \{j\}} = q_j \left(\kappa \sum_{\ell \in t^* \setminus \{j\}} \hat{u}_\ell + u_j \right) q_{t^* \setminus \{j\}}$$

$$\geq \left(1 - \frac{n\kappa}{\sum_{i \in t^*} u_i} \right) \cdot q_{t^*} \sum_{i \in t^*} u_i,$$

19

where the first inequality follows from optimality of $\hat{t}$ under the scaled utilities. Thus, to ensure an ϵ-approximate solution, we need

$$\frac{n\kappa}{\sum_{\ell \in t^*} u_\ell} \leq \epsilon \Leftrightarrow \kappa \leq \frac{\epsilon \cdot \sum_{\ell \in t^*} u_\ell}{n}.$$

Thus, it suffices to choose

$$\kappa \leq \frac{\epsilon \cdot \max_{\ell \in [t^* \setminus \{j\}]} u_\ell \cdot q_\ell}{n} \leq \frac{\epsilon \cdot q_{t^*} \cdot \sum_{i \in t^*} u_i}{n} \leq \frac{\epsilon \cdot \sum_{\ell \in t^*} u_i}{n}$$

$\square$

Since t^* is not known, we should choose a value for κ that satisfies the above lemma. Note that due to Lemma 2, we know that $t^* \cap ([n] \setminus N_{1/2}) \leq 1$.

Algorithm 1, which is an FPTAS for the optimal test, due to Lemma 2, fixes an individual $j \in [n+1] \setminus N_{1/2}$ that is pooled into the optimal test, where $j = n+1$ indicates the case that $t^* \cap ([n] \setminus N_{1/2}) = 0$. Hence, we can apply Lemma 3 by setting $i = h$ and thus $\bar{t} \subseteq N_{1/2}$. Since for each $\ell \in N_{1/2}$, $q_\ell \geq 1/2$, we have that $\max_{\ell \in \bar{t}} q_\ell \cdot u_\ell \geq 1/2 \max_{\ell \in \bar{t}} \cdot u_\ell$. Thus, we can choose κ such that

$$\kappa = \frac{\epsilon \cdot 1/2 \cdot \max_{\ell \in [\bar{t}]} u_\ell}{n} \leq \frac{\epsilon \cdot \max_{\ell \in [\bar{t}]} q_\ell \cdot u_\ell}{n} \leq \frac{\epsilon \cdot q_{t^*} \cdot \sum_{i \in t^*} u_i}{n}$$

where the last inequality follows from optimality of t^*.

Now, we show that Algorithm 1 is an FPTAS for the optimal test.

First note that if $|t^*| = 1$, then Algorithm 1 finds the optimal test in Lines 5-7. Hence, we focus on the case that $|t^*| > 1$. using Lemma 2, we distinguish into two cases:

Case I: $|t^* \setminus N_{1/2}| = 0$. For each given $\epsilon > 0$, κ satisfies the supposition of Lemma 5. So following Lemma 5 with $\bar{C} = \bar{C}(h) = \sum_{\ell \in N_{1/2}} \geq \sum_{\ell \in t^*} \hat{u}_\ell$, for $\hat{C}_{h,n+1}$ and $\hat{L}_{h,n+1}$, we have

$$u(t) = \hat{z}_\kappa(h, n+1) = \kappa \cdot \hat{C}_{h,n+1} \cdot \hat{P}(h, \hat{C}_{h,n+1}, \hat{L}_{h,n+1}) \geq (1-\epsilon) \cdot C^* \cdot P(h, C^*, L^*)$$
$$\geq (1-\epsilon) \cdot C^* \cdot P(n, C^*, L^*)$$

Case II: $|t^* \setminus N_{1/2}| = 1$. Then, for each $\epsilon > 0$, the choice of κ for some $j \in [n] \setminus N_{1/2}$ satisfies

$$\hat{z}_\kappa(h, j) \geq (1-\epsilon) \cdot C^* \cdot P(n, C^*, L^*)$$

where the inequality follows form Lemma 5. The algorithm must determine j since it enumerates all elements of $[n] \setminus N_{1/2}$. in the main loop.

The complexity of the algorithm is determined by at most n evaluations of Equation (9). Hence,

$$O\left(n^3 \bar{C}\right) \subseteq O\left(n^3 \sum_{\ell \in N_{1/2}} \frac{u_\ell}{\kappa}\right) \subseteq O\left(\frac{n^5}{\epsilon}\right).$$

$\square$

G Proof of Theorem 2

Proof. Let T be the testing regime that is returned by ϵ-Greedy and T^* be an optimal non-overlapping testing regime. Without loss of generality, let $N' = \{1, \ldots, n'\}$ be the set of individuals that are pooled into a test in T. Then,

$$u(T^*) = \sum_{j \in [B]} u(t_j^*) = \sum_{j \in [B]} q_{t_j^*} \cdot \left(\sum_{i \in N'} I_{i,j}^{t^*} \cdot u_i\right) + \sum_{j \in [B]} q_{t_j^*} \cdot \left(\sum_{i \in [n] \setminus N'} I_{i,j}^{t^*} \cdot u_i\right),$$

20

and

$$u(T) = \sum_{j \in [B]} u(t_j) = \sum_{j \in [B]} q_{t_j} \cdot \left(\sum_{i \in N'} I_{i,j}^T \cdot u_i \right) = \sum_{j \in [B]} q_{t_j} \cdot \left(\sum_{i \in t_j} u_i \right).$$

Now, let T' be a testing regime such that $t_j' = t_j^* \setminus (t_j^* \cap N')$. In other words, T' is created by removing from T^* any individual in N'. Notice that since every t_j' consists of individuals that are not included in any test in T, it means that all the individuals in t_j' are available at the j-th iteration of the algorithm, and thus we get that for each $j \in [B]$, $u(t_j) \geq (1 - \epsilon)u(t_j')$, as otherwise the algorithm would have chosen t_j' instead of t_j, at the j-th iteration. Thus, we get that

$$u(T) = \sum_{j \in [B]} u(t_j) \geq (1 - \epsilon) \sum_{j \in [B]} u(t_j') = (1 - \epsilon) \cdot u(T'),$$

and also note that

$$u(T') = \sum_{j \in [B]} q_{t_j'} \cdot \left(\sum_{i \in [n] \setminus N'} I_{i,j}^{T'} \cdot u_i \right) = \sum_{j \in [B]} q_{t_j'} \cdot \left(\sum_{i \in [n] \setminus N'} I_{i,j}^{T^*} \cdot u_i \right)$$

$$\geq \sum_{j \in [B]} q_{t_j^*} \cdot \left(\sum_{i \in [n] \setminus N'} I_{i,j}^{T^*} \cdot u_i \right)$$

where the second equality follows from the fact that $I_{i,j}^{T'} = I_{i,j}^{T^*}$ for any $i \in [n] \setminus N'$ and $j \in [B]$, and the last inequality follows from the fact that $q_{t_j'} \geq q_{t_j^*}$ since $t_j' \subseteq t_j^*$ for any $j \in [B]$. Thus,

$$u(T) \geq (1 - \epsilon) \cdot \sum_{j \in [B]} q_{t_j^*} \cdot \left(\sum_{i \in [n] \setminus N'} I_{i,j}^{T^*} \cdot u_i \right).$$

From all the above we have

$$\frac{u(T^*)}{u(T)} = \frac{\sum_{j \in [B]} q_{t_j^*} \cdot \left(\sum_{i \in N'} I_{i,j}^{T^*} \cdot u_i \right) + \sum_{j \in [B]} q_{t_j^*} \cdot \left(\sum_{i \in [n] \setminus N'} I_{i,j}^{T^*} \cdot u_i \right)}{u(T)}$$

$$\leq \frac{\sum_{j \in [B]} q_{t_j^*} \cdot \left(\sum_{i \in N'} I_{i,j}^{T^*} \cdot u_i \right) + \frac{u(T)}{1 - \epsilon}}{u(T)}$$

$$= \frac{\sum_{j \in [B]} q_{t_j^*} \cdot \left(\sum_{i \in t_j^* \cap N'} u_i \right)}{u(T)} + \frac{1}{1 - \epsilon}$$

$$\leq \frac{\sum_{j \in [B]} \left(\sum_{i \in t_j^* \cap N'} q_i \cdot u_i \right)}{u(T)} + \frac{1}{1 - \epsilon}$$

$$= \frac{\sum_{i \in N'} q_i \cdot u_i}{u(T)} + \frac{1}{1 - \epsilon} \tag{10}$$

where the second inequality follows since $q_i \geq q_{t_j}$ when i is included in t_j.

In what follows, we will show that each test, $t_j \in T$ obtains at least a $\frac{(1 - \epsilon)}{4}$ ratio of the maximal possible utility to be gained from individuals in t_j. In other words, we show that the following holds: $u(t_j) = q_{t_j} \cdot \sum_{i \in t_j} u_i > \frac{(1 - \epsilon)}{4} \sum_{i \in t_j} q_i u_i$. To do so, we first consider the case where there exists $i' \in t_j$ such that $q_{i'} < 1/2$. From the definition of the greedy algorithm, we know that

$$(1 - \epsilon) \cdot q_{t_j \setminus \{i'\}} \cdot \sum_{i \in t_j \setminus \{i'\}} u_i \leq q_{t_j} \cdot \sum_{i \in t_j} u_i$$

21

otherwise the algorithm would return $t_j \setminus \{i'\}$ instead of t_j at step j and also,

$$(1 - \epsilon) \cdot q_{i'} \cdot u_{i'} \leq q_{t_j} \cdot \sum_{i \in t_j} u_i$$

as otherwise the algorithm would return $\{i'\}$ instead of t_j at the j-th iteration. Moreover, from Lemma 2, we know that $q_{t_j \setminus \{i'\}} \geq 1/2$ since $q_{i'} < 1/2$. Thus, we get that

$$
\begin{aligned}
q_{t_j} \cdot \sum_{i \in t_j} u_i &\geq \frac{(1-\epsilon)}{2} (q_{t_j \setminus \{i'\}} \cdot \sum_{i \in t_j \setminus \{i\}} u_i + q_{i'} \cdot u_{i'}) \geq \frac{(1-\epsilon)}{2} \left(\frac{1}{2} \cdot \sum_{i \in t_j \setminus \{i\}} u_i + q_{i'} \cdot u_{i'} \right) \\
&= \frac{(1-\epsilon)}{4} \left(\sum_{i \in t_j \setminus \{i'\}} u_i + q_{i'} \cdot u_{i'} \right) \\
&\geq \frac{(1-\epsilon)}{4} \sum_{i \in t_j} q_i u_i.
\end{aligned}
$$

As a second case, assume that for any $i \in t_j$, $q_i \geq 1/2$. We show that for each $i \in t_j$, $q_{t_j \setminus \{i\}} \geq 1/4$. If $q_{t_j} \geq 1/2$, then indeed $q_{t_j \setminus \{i\}} \geq 1/4$. Otherwise, we do the following: In a set S we add individuals that are included in t_j, except for i, one at a time until $q_S \geq 1/2$ and $q_{S \cup \{i\}} < 1/2$ (as $q_{t_j} < 1/2$ notice that such an S should exist). Then, from Lemma 2, we get that $q_{t_j \setminus \{S \cup i\}} \geq 1/2$, and hence $q_{t_j \setminus \{i\}} = q_S \cdot q_{t_j \setminus \{S \cup i\}} \geq 1/4$. Thus, we have that

$$q_{t_j} \cdot \sum_{i \in t_j} u_i = \sum_{i \in t_j} q_{t_j \setminus \{i\}} \cdot q_i \cdot u_i \geq \frac{1}{4} \cdot \sum_{i \in t_j} q_i \cdot u_i \geq \frac{(1-\epsilon)}{4} \sum_{i \in t_j} q_i \cdot u_i.$$

We see that in either case, $q_{t_j} \cdot \sum_{i \in t_j} u_i > \frac{(1-\epsilon)}{4} \cdot \sum_{i \in t_j} q_i \cdot u_i$ hence we get:

$$u(T) = \sum_{j \in [B]} q_{t_j} \cdot \left(\sum_{i \in t_j} u_i \right) > \sum_{j \in [B]} \left(\frac{(1-\epsilon)}{4} \cdot \sum_{i \in t_j} q_i \cdot u_i \right) = \frac{(1-\epsilon)}{4} \sum_{i \in N'} q_i u_i$$

Putting this together with Equation (10), we get that $u(T^*)/u(T) \leq 5/(1-\epsilon)$ and the theorem follows. $\qquad \square$

H Identical Utilities

In this section, we consider the special case where $u_i = u_{i'}$ for each $i, i' \in [n]$. Without loss of generality, assume that $u_i = 1$ for any $i \in [n]$ and $q_i \geq q_{i+1}$ for any $i \in [n-1]$.

H.1 Optimal Testing Regime for constant B

We start by showing that when B is a constant, we can find the optimal non-overlapping testing regime in polynomial time.

Theorem 3. *When the individuals have identical utilities, we can find an optimal non-overlapping testing regime T in time $O(n^{B+1}/B!)$.*

Proof. We start with the following crucial lemma which indicates that there exists an optimal testing regime where the test that has the largest size pools samples of the first k_1 individuals with $k_1 \in [n]$, the test that has the second largest size pools samples of the individuals $k_1 + 1$ to k_2 with $k_2 \in \{k_1 + 1, \ldots, n\}$, the test that has the third largest size pools samples of the individuals $k_2 + 1$ to k_3 with $k_3 \in \{k_2 + 1, \ldots, n\}$ and so on.

Lemma 6. *Let T^* be an optimal testing regime and without loss of generality let $|t_j^*| \geq |t_{j+1}^*|$ for any $j \in [B-1]$. Then, there exists an optimal testing regime T' such that for each $j \in [B]$, $|t_j'| = |t_j^*|$ and*

$$t_j' = \{ \sum_{j' \in [j-1]} |t_{j'}'| + 1, \ldots, \sum_{j' \in [j-1]} |t_{j'}'| + |t_j'| \}.$$

Proof. First, note that for some optimal testing regime, it should hold that for any $i' > i$, if i' is pooled into some test, then i is also pooled into some test, as otherwise the replacement of i' with i cannot worse the expected welfare of the testing regime. Thus, hereinafter, we focus on optimal testing regimes that pool samples of the first k individuals for some $k \in [n]$.

We prove the lemma by induction on the number of tests. Start from the case that $B = 2$. Let T^* be an optimal testing regime with $|t_1^*| \geq |t_2^*|$. Assume that $t_1^* = S_1 \cup S_1'$, where $S_1 \subset \{1, \ldots, |t_1^*|\}$, and $S_1' \subset \{|t_1^*| + 1, \ldots, k\}$ and $t_2^* = S_2 \cup S_2'$, where $S_2 \subset \{1, \ldots, |t_1^*|\}$ and $S_2' \subset \{|t_1^*| + 1, \ldots, k\}$. Since $t_1^* \cup t_2^* = [k]$, we get that $|S_1'| = |S_2|$ and since $|t_1^*| \geq |t_2^*|$, we get that $|S_1| \geq |S_2'|$. Now, consider the testing regime T such that $t_1 = S_1 \cup S_2$ and $t_2 = S_1' \cup S_2'$. Notice that $t_1 \cup t_2 = [k]$, $|t_1| = |t_1^*|$ and $|t_2| = |t_2^*|$. Then, we have

$$u(T^*) = q_{S_1} \cdot q_{S_1'} \cdot |t_1^*| + q_{S_2} \cdot q_{S_2'} \cdot |t_2^*|$$

and

$$u(T) = q_{S_1} \cdot q_{S_2} \cdot |t_1^*| + q_{S_1'} \cdot q_{S_2'} \cdot |t_2^*|.$$

and hence,

$$u(T) - u(T^*) = \left(q_{S_2} - q_{S_1'}\right) \cdot \left(q_{S_1} \cdot |t_1^*| - q_{S_2'} \cdot |t_2^*|\right). \tag{11}$$

Due to optimality of T^*, we have that for any $\hat{S}_1 \subseteq S_1$

$$q_{S_1} \cdot q_{S_1'} \cdot |t_1^*| \geq q_{\hat{S}_1} \cdot q_{S_1'} \cdot (|\hat{S}_1| + |S_1'|)$$

as otherwise if T' is a testing regime with $t_1' = \hat{S}_1 \cup S_1'$ and $t_2' = t_2^*$, then it would hold that $u(T') > u(T^*)$ which is a contradiction. Now, choose arbitrary $\hat{S}_1 \subseteq S_1$ such that $|\hat{S}_1| = |S_2'|$. We know that this is feasible since $|S_1| \geq |S_2'|$. Then, we have

$$q_{S_1} \cdot q_{S_1'} \cdot |t_1^*| \geq q_{\hat{S}_1} \cdot q_{S_1'} \cdot (|S_2'| + |S_2|) \geq q_{S_2'} \cdot q_{S_1'} \cdot |t_2^*|$$

where the second transition follows due to optimally of T^*, and the facts that $|\hat{S}_1| = |S_2'|$ and $|S_1'| = |S_2|$ and the third transition follows since $|S_2'| + |S_2| = |t_2^*|$ and $q_{\hat{S}_1} \geq q_{S_2'}$ as for each $i \in \hat{S}_1$ and each $i' \in S_2'$ it holds that $q_i \geq q_{i'}$. Hence, we have that

$$q_{S_1} \cdot |t_1^*| \geq q_{S_2'} \cdot |t_2^*|.$$

Now from Equation (11), we have that $u(T) \geq u(T^*)$ since $\left(q_{S_2} - q_{S_1'}\right) \geq 0$ as for each $i \in S_2$ and each $i' \in S_1'$ it holds that $q_i \geq q_{i'}$ and $|S_1'| = |S_2|$. Thus, we conclude in a testing regime T, with $|t_1| \geq |t_2|$, $T_1 = \{1 \ldots, |t_1|\}$ and $T_2 = \{|t_1| + 1, \ldots, |t_2|\}$ that is optimal.

Now, suppose that the claim holds for $B - 1$. We will show that it holds for B. Let T^* be an optimal testing regime with $|t_j^*| \geq |t_{j+1}^*|$ for any $j \in [B]$. Using the induction hypothesis, we can construct an optimal testing regime T' such that $|t_j'| = |t_j^*|$ for any $j \in [B - 1]$, $t_B' = t_B^*$ and there are no $i \in t_j'$ and $i' \in t_{j'}'$ with $i' < i$ and $j' > j$. Then, in round j, for any t_j' and t_B^*, from induction base we construct t_j'' and t_B^j such that $|t_j''| = |t_j'| = |t_j^*|$ and $|t_B^j| = |t_B^*|$ and there are no $i \in t_j''$ and $i' \in t_B^j$ with $i' < i$. Thus, after $n - 1$ rounds, we have $T'' = (t_1'', \ldots, t_{B-1}'', t_B^{n-1})$ which is optimal and satisfies the property of the statement. $\square$

Using Lemma 6, we can find an optimal testing as following. For any $k \in [n]$ and any $k_1 \geq k_2 \ldots \geq k_B$ with $\sum_{\ell \in [B]} k_\ell = k$, calculate the welfare of T such that

$$t_j = \{ \sum_{\ell \in [j-1]} k_\ell + 1, \ldots \sum_{\ell \in [j-1]} k_\ell + k_j \},$$

and return the testing regime that has the highest welfare. Hence, we need time at most $n \cdot n^B / B!$ to find the optimal testing regime as for each k, each k_ℓ takes up to n values and for each of the cases, we order the k_ℓ's in a decreasing order, meaning that among the $B!$ different ways of ordering them we are interested only for the case that $k_1 \geq k_2 \ldots \geq k_B$. $\square$

H.2 Greedy Algorithm

Here, we show that when the utilities are identical, we can find an e-approximate testing regime with respect to the optimal non-overlapping testing regime, for any value B. Specifically, we consider a variation of the greedy algorithm that we introduced in Section 3.3 which we denote as *var-Greedy* and is defined as following: *var-Greedy* runs B rounds, and in each round j, includes in test t_j individuals that have not been pooled into any other test yet in an decreasing order with respect to their probability of being healthy until the utility of the test is not worsen. Note that *var-Greedy* always returns a testing regime T where samples of the first $n' \in [n]$ individuals are pooled into some test, i.e. $\cup_{j \in [B]} t_j = [n']$.

We start with the following lemma.

Lemma 7. *If var-Greedy returns a testing regime that pools samples of the first n' individuals, then there exists an optimal testing regime that pools samples of the first n'' individuals with $n'' \leq n'$.*

Proof. Let T^* be an optimal non-overlapping testing regime that satisfies the properties of Lemma 6, i.e. for each $j \in B$
$$t_j^* = \{i_{j-1}^* + 1, \ldots, i_j^*\}$$
with $i_0^* = 0$ and $i_{j-1}^* < i_j^*$. We denote with T the testing regime that is returned by *var-Greedy*, where for each $j \in [B]$, $t_j = \{i_{j-1} + 1, \ldots, i_j\}$, with $i_0 = 0$ and $i_{j-1} < i_j$. We show that for each $j \in [B]$, $i_j^* \leq i_j$. Suppose for contradiction that t_j^* is the first test such that $i_j^* > i_j$. Due to the structure of T^* and T, this means that $\cup_{j' \in [j-1]} t_{j'}^* \subseteq \cup_{j' \in [j-1]} t_{j'}$, and hence $i_{j-1}^* \leq i_{j-1}$. Given that *var-Greedy* did not pool $i_j + 1$ in t_j, we have that

$$q_{i_{j-1}+1} \cdot \ldots \cdot q_{i_j} \cdot |t_j| > q_{i_{j-1}+1} \cdot \ldots \cdot q_{i_j} \cdot q_{i_j+1} \cdot (|t_j| + 1)$$

$$\Rightarrow \frac{|t_j|}{|t_j| + 1} > q_{i_j+1}.$$

as otherwise, from the definition of *var-Greedy*, $i_j + 1$ would have been included in t_j.

Note that,

$$q_{i_j^*} < \frac{|t_j|}{|t_j| + 1} \leq \frac{|t_j^*| - 1}{|t_j^*|}$$

where the first transition follows since for any $i' > i$ it holds $q_i \geq q_{i'}$ and $|t_j|/(|t_j| + 1) > q_{i_j+1}$, and the second transition holds since $|t_j^*| - 1 \geq |t_j|$.

Thus, we have that,

$$q_{t_j^* \setminus \{i_j^*\}} \cdot (|t_j^*| - 1) > q_{t_j^* \setminus \{i_j^*\}} \cdot q_{i_j^*} \cdot |t_j^*|.$$

This means that $u(t_j^* \setminus \{i_j^*\}) > u(t_j^*)$ and hence, we have that if T' is the testing regime with $t_{j'}' = t_{j'}^*$ for each $j' \neq j$ and $t_j' = t_j^* \setminus \{i_j^*\}$, then $u(T') > u(T^*)$ which is a contradiction.

We conclude that for each $j \in [B]$, $i_j^* \leq i_j$, and the statement follows. $\qquad\square$

Now, we are ready to show that for each instance *var-Greedy* returns an e-approximate testing regime.

Theorem 4. *var-Greedy returns an e-approximate testing regime.*

Proof. Let T be the testing that is returned by *var-Greedy* which pools the first $n' \leq n$ individuals. We start by showing that for each i, $P_i^T \geq q_i \cdot \frac{1}{e}$.

Consider an individual i that is included in test t_j of size k. Note for each $i' \in t_j$, we have that $q_{i'} \geq (k-1)/k$, as otherwise we would have that

$$q_{i'} \prod_{i'' \in t_j \setminus \{i'\}} q_{i''} \cdot k < \prod_{i'' \in t_j \setminus \{i'\}} q_{i''} \cdot (k-1)$$

which is a contradiction. Thus, we get that

$$P_i^T = q_{t_j} = q_i \cdot \prod_{i' \in t_j \setminus \{i\}} q_{i'} \geq q_i \cdot \left(\frac{k-1}{k}\right)^{k-1} \geq q_i \cdot \frac{1}{e}. \tag{12}$$

From Lemma 7, we know that it exists an optimal non-overlapping testing regime T^* that pools the first n'' individuals with $n'' \leq n'$. Then, we have

$$\frac{u(T^*)}{u(T)} = \frac{\sum_{i \in [n'']} P_i^T \cdot u_i}{\sum_{i \in [n']} P_i^T \cdot u_i} \leq \frac{\sum_{i \in [n']} q_i \cdot u_i}{\sum_{i \in [n']} q_i \cdot \frac{1}{e} \cdot u_i} \leq e,$$

where the third transition follows from Equation (12). $\qquad\square$

I Proof of Proposition 3

We begin by providing a simple upper bound on the attainable welfare for any testing regime $T \in \mathcal{T}^B$.

Lemma 8. *Suppose that $(t^*) \in \mathcal{T}^1$ is optimal. For any $B \leq 1$. If $T \in \mathcal{T}^B$, then it follows that $u(T) \leq B \cdot u(t^*)$.*

Proof. Suppose that $T \in \mathcal{T}^B$. As before, we write its welfare as follows:

$$
\begin{aligned}
u(T^*) &= \sum_{i \in [n]} u_i \cdot P_i^T \\
&= \sum_{i \in [n]} \sum_{j \in [B]} u_i \cdot P_{i,j}^T \\
&= \sum_{j \in [B]} \sum_{i \in [n]} u_i \cdot P_{i,j}^T \\
&= \sum_{j \in [B]} \sum_{i \in t_j} u_i \cdot P_{i,j}^T \\
&\leq \sum_{j \in [B]} q_{t_j} \sum_{i \in [n]} u_i \\
&= \sum_{j \in [B]} u(t_j) \\
&\leq \sum_{j \in [B]} u(t^*) \\
&= B \cdot u(t^*)
\end{aligned}
$$

The initial equalities in the above equations arise from re-arranging the sum and from the fact that $P_{i,j}^T = 0$ when $i \notin t_j$. As for the first inequality, We can express and bound the probability that t_j is pivotal for i in T:

$$P_{i,j}^T = q_{t_j} \prod_{t_\ell \in T(i:j)} (1 - q_{t_\ell \setminus t_j}) \leq q_{t_j}.$$

Finally, due to optimality of t^*, it follows that $u(t) \leq u(t^*)$ for any feasible pooled test $t \subseteq [n]$. This finishes the proof of the claim. $\qquad\square$

Suppose that we consider a population instance where the population to be tested, $[n]$ can be partitioned into C clusters, such that all individuals in the i-th cluster have utility given by u_i and probability of infection given by p_i. In addition, suppose that the i-th cluster contains $n_i > 0$ individuals. Using the above lemma, we can show that if the population sizes of each cluster permit repeating identical copies of the optimal test in $\mathcal{T}^1$, then this is optimal. Notice that this is what result of executing the greedy algorithm with $\epsilon = 0$ (which can be done efficiently by brute force if the number of clusters is not too large).

Proposition 5. *Suppose that $\{t^*\} \in \mathcal{T}^1$ is optimal and that in addition, $B \cdot |t^* \cap C_i| \leq n_i$ for each cluster. Let $T^* \in \mathcal{T}^B$ be a testing regime that simply repeats t^* in disjoint copies B times. It follows that T^* is optimal.*

J   Practical implementations for non-overlapping testing

We state the convex program for determining optimal non-overlapping testing and describe a conic formulation for the single test case as well as a mixed-integer formulation for approximating optimal non-overlapping with one or more tests. Without loss of generality, we can assume that the testing budget B is at most the population size n, and so pool sizes lies between 1 and G.

For each test $j \in [B]$, we introduce an indicator vector $x^j \in \{0,1\}^n$ with $x_i^j = 1$ if individual i is included in j and $x_i^j = 0$ otherwise, and let variable w^j denote its expected utility $w^j = u \cdot x^j \prod_{i \in [n]} q_i^{x_i^j}$. We impose pool sizes between 1 and G with constraints $1 \le \sum_{i \in [n]} x_i^j \le G$ for all $j \in [B]$, and non-overlapping testing with constraints $\sum_{j \in [B]} x_i^j \le G$ for all $i \in [n]$. Our objective is to maximise welfare $\sum_{j \in [B]} w^j$. In order to isolate the non-linear elements of the optimization problem, we reformulate the convex program with additional variables below; variables l^j denote the log of w^j, and variables y^j and z^j allow us to isolate the non-linear elements of the expressions into constraints (13b) and (13d).

$$\max \quad \sum_{j \in [B]} w^j \tag{13a}$$

$$\text{s.t.} \quad w^j = \exp l^j, \qquad\qquad \forall j \in [B], \tag{13b}$$

$$l^j = y^j + \sum_{i \in [n]} x_i^j \log q_i, \quad \forall j \in [B], \tag{13c}$$

$$y^j = \log z^j, \qquad\qquad \forall j \in [B], \tag{13d}$$

$$z^j = u \cdot x^j, \qquad\qquad \forall j \in [B], \tag{13e}$$

$$\sum_{j \in [B]} x_i^j \le 1, \qquad\qquad \forall i \in [n], \tag{13f}$$

$$\sum_{i \in [n]} x_i^j \ge 1, \qquad\qquad \forall j \in [B], \tag{13g}$$

$$\sum_{i \in [n]} x_i^j \le G, \qquad\qquad \forall j \in [B], \tag{13h}$$

$$x_i^j \in \{0,1\}, \qquad\qquad i \in [n], \forall j \in [B] \tag{13i}$$

J.1   A conic program for a single test

Suppose we wish to allocate a single test. In this setting, we can eliminate the exponential constraint (13b) by changing the objective to $\max l^1$. Note that the remaining non-linear constraints (13d) can be relaxed to $y^j \le \log z^j$ without affecting the outcome, and formulated as conic constraints $(z^j, 1, y^j) \in K_{\exp}$, where $K_{\exp}$ is the exponential cone defined as

$$K_{\exp} = \{(x_1, x_2, x_3) \mid x_1 \ge x_2 e^{x_3/x_2}, x_2 > 0\} \cup \{(x_1, 0, x_3) \mid x_1 \ge 0, x_3 \le 0\}.$$

The resulting mixed-integer conic optimization program can be solved efficiently[9] with conic solvers such as MOSEK (https://mosek.com). In our simulations, our Greedy algorithm repeatedly solves a conic program to allocate a single test.

J.2   A mixed-integer linear programming approximation

If we wish to allocate more than one test, it is not longer possible to express the problem as a conic program. Instead, we formulate a mixed-integer linear program (MILP) that approximates an optimal non-overlapping solution. In order to make the problem tractable, we assume that the utility vector u is integral and non-negative. Note that integrality is without loss of generality, as the problem is invariant to scaling of utilities. We describe how the non-linear constraints (13b) and (13d) can respectively be captured exactly and approximately by a collection of integer linear constraints.

---

[9]Example running times are shown in the experiments section of this paper.

Handling the logarithmic constraints. We can replace (13d) with integer linear constraints as follows. Fix some test $j \in [B]$. Note that z^j only takes integral values in the range $[L, U]$, where $L = \min_i u_i$ and $U = G \max_i u_i$. We introduce an indicator vector $\gamma^j \in \{0,1\}^{[L,U]}$ indexed by $k \in [L, U]$ with constraints $\sum_{k \in [L,U]} \gamma_k^j = 1$ and $\sum_{k \in [L,U]} k \cdot \gamma_k^j = z^j$ to encode which value z holds, and ensure $y^j = \log(z^j)$ with the constraint $y^j = \sum_{k \in [L,U]} \log(k) \cdot \gamma_k^j$.

Approximating the exponential constraints. We now describe how to approximate (13b) from above by a piecewise-linear function f using integer linear constraints. Fix some test $j \in [B]$. Note first that we can relax the equality in (13b) to $w^j \le \exp(l^j)$ without affecting the outcome. The variable l^j only takes values between $A = \min_i(\log u_i) + G \min_i(\log q_i)$ and $B = \log(G \max_i u_i) + \max_i(\log q_i)$ (and these values will be generically non-integral). We approximate $\exp$ from above by a piecewise-linear function $f : [A, B] \to \mathbb{R}$ with K linear segments. (Here the parameter K is given exogenously.) Partitioning $[A, B]$ into K parts $[c_k, c_{k+1}], k \in [K]$, we define the k-th line segment as the linear function $f_k(x) = a_k x + b_k$ on domain $[c_k, c_{k+1}]$ with slope $a_k = \frac{\exp c_{k+1} - \exp c_k}{c_{k+1} - c_k}$ and residual $b_k = \exp c_{k+1} - a_k c_{k+1}$. Note that the number of integer variables in the MILP increases with K, so this parameter must be chosen judiciously. Moreover, given a fixed number of segments K, we wish to determine a partitioning of $[A, B]$ that minimises the approximation error $\varepsilon = \max_{x \in [A,B]}(f(x) - \exp(x))$. In our implementation, we apply binary search techniques to numerically determine the partition of $[A, B]$ such that the error $\max_{x \in [c_k - c_{k+1}]}(f_k(x) - \exp(x))$ is the same for all parts $[c_k, c_{k+1}]$, which minimises ε.

We introduce the indicator vector $\delta^j \in \{0,1\}^K$ to encode in which part $[c_k, c_{k+1}]$ the value of l^j lies, as well as the vector $v^j \in \mathbb{R}^K$ whose k-th entry agrees with l^t if l^j lies in the k-th part, and is 0 otherwise. This is guaranteed by constraints $\sum_{k \in [K]} \delta_k^j = 1$, $l^j = \sum_{k \in [K]} v_k^j$ and $c_k \cdot \delta_k^j \le v_k^j \le c_{k+1} \cdot \delta_k^j, \forall k \in [K]$. Finally, we require that $w^j \le f_k(l^j)$ for the k-th part $[c_k, c_{k+1}]$ that l^t lies in. This is expressed by constraint $w^j \le \sum_{k \in [K]} a_k v_k^j + b_k \cdot \delta_k^j$.

Bounding the approximation error. Recall that the piecewise-function f with K segments approximates $\exp$ on domain $[A, B]$ from above with error ε. Let $\sigma(x) = \sum_{j \in [B]} \exp(l^t)$ and $\sigma'(x) = \sum_{j \in [B]} f(l^t)$ respectively denote the corresponding objective values of the convex program (13) and the MILP described above for testing x. Let x^* denote an optimal non-overlapping testing, so x^* maximises σ, and x' be an optimal solution for the MILP. Clearly, x^* and x' are both feasible for both programs and satisfy $\sigma(x') \le \sigma(x^*)$ as well as $\sigma'(x^*) \le \sigma'(x')$. By construction of f, we have $\sigma(x) \le \sigma'(x)$ and $\sigma(x) \ge \sigma'(x) - \varepsilon B$, which implies $\sigma(x^*) \le \sigma'(x^*) \le \sigma'(x') \le \sigma(x') + T\varepsilon$. Here ε is the additive approximation error of f with regard to $\exp$. Hence, $0 \le \sigma(x^*) - \sigma(x') \le \varepsilon B$. This allows us to compute a bound on the additive gap between the welfare achieved by the optimal solution of our MILP and the optimal non-overlapping testing. In our experimental section, we list the values of K used, together with the additive welfare gap bound.

Clustering individuals. In order to speed up the computation, we can consider groups of individuals with the same utilities and health probabilities as *clusters*. Clusters are particularly pertinent when utilities are integral and health probabilities are discretized, as is the case in our pilot study. Suppose we have C clusters. We introduce a population vector $n \in \mathbb{N}_0^C$ so that n_i denotes the number of individuals in cluster $i \in [C]$. In order to incorporate clustering into the MILP, we now let the index i refer to a cluster (instead of an individual), and allow variables x_i^j to take arbitrary non-negative integral values (instead of binary values in (13i)); these values represent the number of individuals from cluster i that are included in test j. Additionally, we relax the non-overlapping test constraint (13f) to $\sum_{j \in [B]} x_i^j \le n_i$. As an aside, it is not difficult to show that if cluster populations are much larger than the testing budget at hand, then non-overlapping tests are optimal.

The full MILP with clustering. For completeness, we state the full MILP with clustering below. Note that constraints (14b)–(14f) capture the exponential constraint (13b), while (14h)–(14j) capture the logarithmic constraint (13d).

$$\max \quad \sum_{j \in [B]} w^j \tag{14a}$$

$$\text{s.t.} \quad w^j \leq \sum_{k \in [K]} a_k v_k^j + b_k \cdot \delta_k^j \quad \forall j \in [B], \tag{14b}$$

$$\sum_{k \in [K]} \delta_k^j = 1 \quad \forall j \in [B], \tag{14c}$$

$$\sum_{k \in [K]} v_k^j = l^j \quad \forall j \in [B], \tag{14d}$$

$$c_k \cdot \delta_k^j \leq v_k^j \quad \forall j \in [B], k \in [K], \tag{14e}$$

$$c_{k+1} \cdot \delta_k^j \geq v_k^j \quad \forall j \in [B], k \in [K], \tag{14f}$$

$$l^j = y^j + \sum_{i \in [C]} x_i^j \log q_i \quad \forall j \in [B], \tag{14g}$$

$$1 = \sum_{k \in [L,U]} \gamma_k^j \quad \forall j \in [B], \tag{14h}$$

$$z^j = \sum_{k \in [L,U]} k \cdot \gamma_k^j \quad \forall j \in [B], \tag{14i}$$

$$y^j = \sum_{k \in [L,U]} \log(k) \cdot \gamma_k^j \quad \forall j \in [B], \tag{14j}$$

$$z^j = u \cdot x^j, \quad \forall j \in [B], \tag{14k}$$

$$\sum_{j \in [B]} x_i^j \leq n_i, \quad \forall i \in [C], \tag{14l}$$

$$\sum_{i \in [C]} x_i^j \geq 1, \quad \forall j \in [B], \tag{14m}$$

$$\sum_{i \in [C]} x_i^j \leq G, \quad \forall j \in [B], \tag{14n}$$

$$x_i^j \in \mathbb{N}_0, \quad \forall j \in [B], i \in [C], \tag{14o}$$

$$v_k^j \in \mathbb{R}, \quad \forall i \in [C], k \in [K], \tag{14p}$$

$$\delta_k^j \in \{0,1\}, \quad \forall i \in [C], k \in [K], \tag{14q}$$

$$\gamma_k^j \in \{0,1\}, \quad \forall i \in [C], k \in [L,U] \tag{14r}$$

K   Additional figures and tables from experiments

Here we show figures and summary tables for our experiments comparing Approx and Greedy with population size $n = 250$ and pool size constraints $G \in \{5, 10\}$. Also shown are our comparisons between 1-Overlap and 2-Overlap.
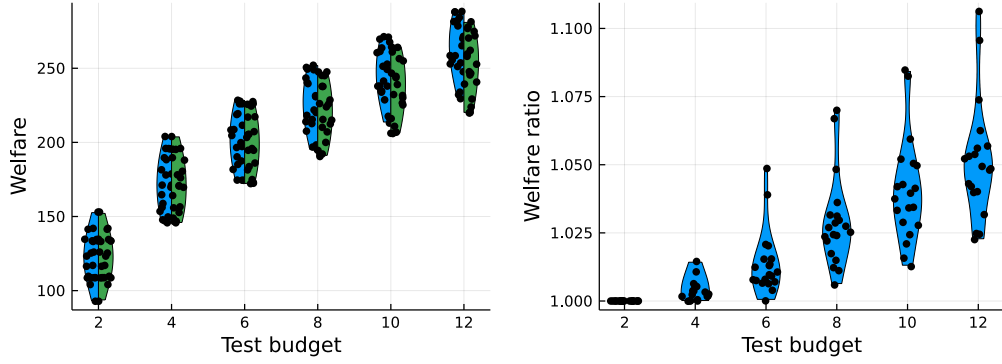
Figure 2: Outcomes of Greedy and Approx with population size $n = 250$, pool size constraint $G = 10$ and testing budgets $B \in \{2, 4, \ldots, 12\}$. Left: Welfares achieved by Approx (left regions, blue) and Greedy (right regions, red). Right: Ratios between the welfares of Approx and Greedy. In both figures, each black dot corresponds to one of the 20 randomly generated populations.

Figure 3: Outcomes of Greedy and Approx with population size $n = 250$, pool size constraint $G = 250$ and testing budgets $B \in \{2, 4, \ldots, 12\}$. Left: Welfares achieved by Approx (left regions, blue) and Greedy (right regions, red). Right: Ratios between the welfares of Approx and Greedy. In both figures, each black dot corresponds to one of the 20 randomly generated populations.
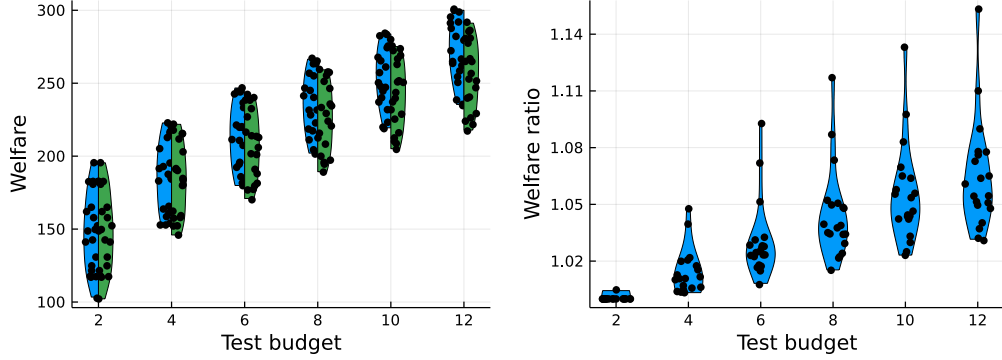
$n = 250, G = 10$	**Approx**			**Greedy**	
Budget	Welfare*	Guarantee (add.)	Time*	Welfare*	Time*
2	122.76	0.231791	149 ms	122.76	29 ms
4	172.159	0.463581	589 ms	171.571	71 ms
6	202.063	0.695372	3491 ms	199.419	110 ms
8	225.892	0.927162	14857 ms	219.713	159 ms
10	245.886	1.15895	64137 ms	236.491	193 ms
12	263.537	1.39074	289254 ms	250.941	239 ms

Table 2: Summary showing welfare and computation time for Approx and Greedy on populations of size $n = 250$ and pool size constraint $G = 10$, with testing budgets $B \in \{2, 4, \ldots, 12\}$. Starred columns show mean values computed over the 20 random populations. We also provide additive approximation guarantee of Approx (compared to optimal non-overlapping welfare).

$n = 250, G = 250$	**Approx**			**Greedy**	
Budget	Welfare*	Guarantee (add.)	Time*	Welfare*	Time*
2	148.772	0.597331	994 ms	148.747	29 ms
4	186.096	1.19466	3459 ms	183.692	57 ms
6	213.514	1.79199	8847 ms	207.428	85 ms
8	235.677	2.38932	22724 ms	225.906	120 ms
10	254.626	2.98666	47394 ms	241.59	149 ms
12	271.445	3.58399	176396 ms	255.292	175 ms

Table 3: Summary showing welfare and computation time for Approx and Greedy on populations of size $n = 250$ and pool size constraint $G = 250$, with testing budgets $B \in \{2, 4, \ldots, 12\}$. Starred columns show mean values computed over the 20 random populations. We also provide additive approximation guarantee of Approx (compared to optimal non-overlapping welfare).

L Towards overlapping testing.

In Section 3.2, we show that the gain of allowing overlapping testing, compared to non-overlapping testing, is at most 4. Despite extensive computational searching, no example with a gain of more than $7/6$ has been found. In order to better understand the average-case gain, we conduct computational experiments in which we generate 20 populations of size 10, with utilities and probabilities respectively drawn from $\{1, 2, 3\}$, and $\{0, 0.1, \ldots, 1\}$. The pool size is unbounded and testing budgets are $B \in \{2, 3, 4\}$. We note that our choice of population size is constrained by the fact that computing optimal overlapping tests is significantly more computationally intensive. For the same reason, we restrict ourselves to comparing non-overlapping testing with 2-overlapping testing; in the latter case, individuals are permitted to lie in at most two tests. In order to compute optimal non-overlapping and 2-overlapping regimes, we formulate the optimization problems as an (exact) MILP parametrised by the overlap k, and refer to the resulting approach as k-Overlap. The results of running 1-Overlap and 2-Overlap on our 20 populations are shown in Fig. 4 and Table 4; they indicate that the gain of 2-Overlap is non-negligible but limited.
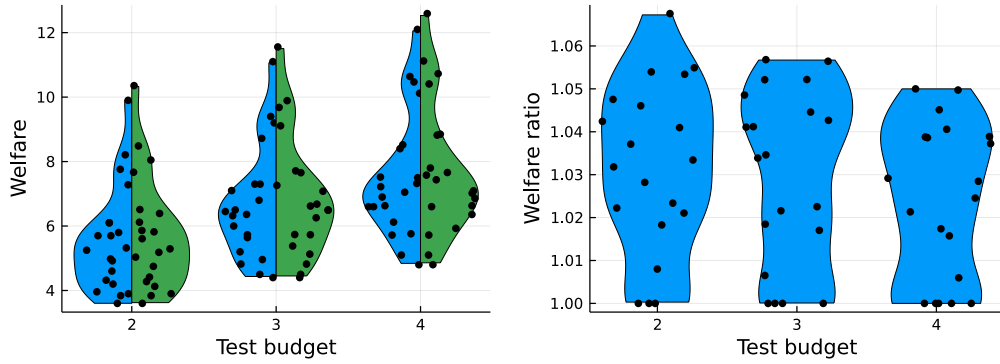

Figure 4: Outcomes of 1-Overlap and 2-Overlap with population size $n = 10$, pool size constraint $G = 10$ and testing budgets $B \in \{2, 3, 4\}$. Left: Welfares achieved by 1-Overlap (left regions, blue) and 2-Overlap (right regions, red). Right: Ratios between the welfares of 1-Overlap and 2-Overlap. In both figures, each black dot corresponds to one of the 20 randomly generated populations.

$n = 10, G = 10$	**1-Overlap**		**2-Overlap**	
Budget	Welfare*	Time*	Welfare*	Time*
2	5.5721	58 milliseconds	5.76428	104 ms
3	6.69	102 milliseconds	6.90898	2107 ms
4	7.5545	136 milliseconds	7.75466	94627 ms

Table 4: Summary showing welfare and computation time for 1-Overlap and 2-Overlap on populations of size $n = 10$ and pool size constraint $G = 10$, with testing budgets $B \in \{2, 3, 4\}$. Starred columns show mean values computed over the 20 random populations.