Fair and Efficient Social Decision-Making

CSCI 699

Voting: Committee Selection

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Credit for the slides : Nisarg Shah and Dominik Peters' Tutorial

Voting

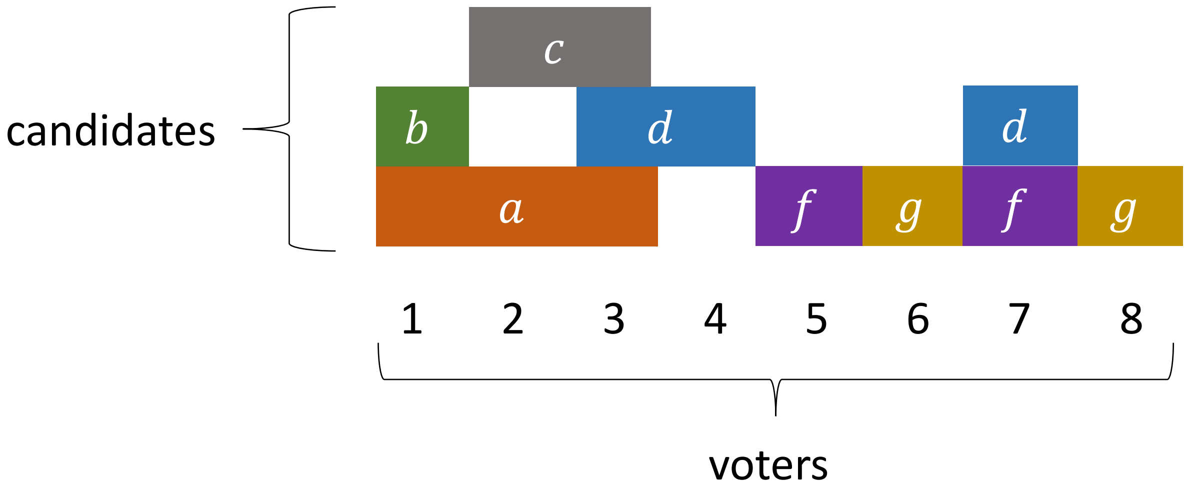
- Set of n agents $N = \{1, \dots, n\}$
- Set of m candidates M
- Votes
  - Ranked ballots $\succ_i$ (e.g., $a \succ_i b \succ_i c$)
  - Cardinal utilities $u_i: M \to \mathbb{R}_{\geq 0}$ (less prominent)
  - Approval ballots $A_i \subseteq M$
    - Equivalent to binary cardinal utilities $c \in A_i \Leftrightarrow u_i(c) = 1$
- Goal
  - Single-winner voting: choose $c^* \in M$
  - Multiwinner voting: choose $S \subseteq M$ with $|S| \leq k$ (for given k)

"ABC" Voting

- Fairness
  - Difficult to define non-trivial fairness notions for single-winner voting
    - Can't give each individual/group "proportionally deserved" utility
  - Much more interesting for multiwinner voting
    - We'll focus on approval ballots, but many of the notions we'll see have been extended to ranked ballots and cardinal utilities

- Approval-Based Multiwinner Voting
  - Each voter i approves a subset of candidates $A_i \subseteq M$
  - A subset of candidates $W \subseteq M, |W| \leq k$ is selected
  - Each voter i gets utility $u_i(W) = |W \cap A_i|$

“ABC” Voting



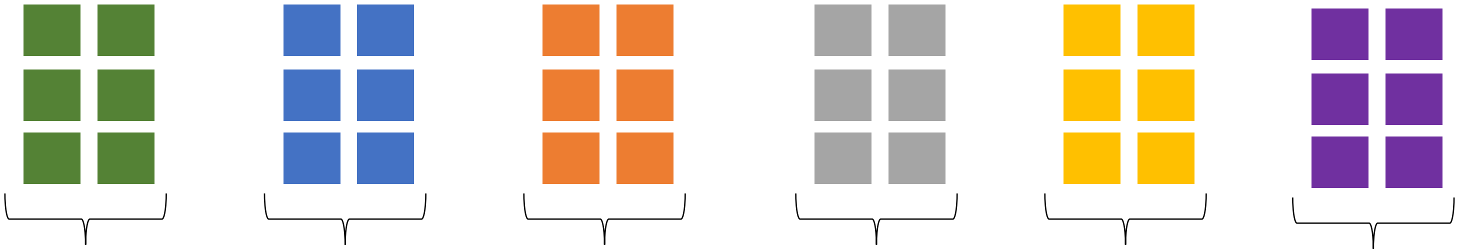
Prominent Rules

- Thiele's Methods [1895]
  - Given a sequence $s = (s_1, s_2, \dots s_k)$, select a committee W that maximizes $\sum_{i \in N} s_1 + s_2 + \cdots + s_{u_i(W)}$

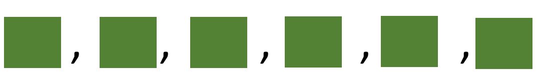
- Examples
  - Approval voting (AV): $s = (1,1,1, \dots 1)$
    - Selects the k candidates with the highest total approvals

Approval Voting

$k = 6$



	24 voters	1 voter	18 voters	12 voters	1 voter	12 voters
1:	+24	+1	+18	+12	+1	+12
2:	+24	+1	+18	+12	+1	+12
3:	+24	+1	+18	+12	+1	+12
4:	+24	+1	+18	+12	+1	+12
5:	+24	+1	+18	+12	+1	+12
6:	+24	+1	+18	+12	+1	+12

W={  }

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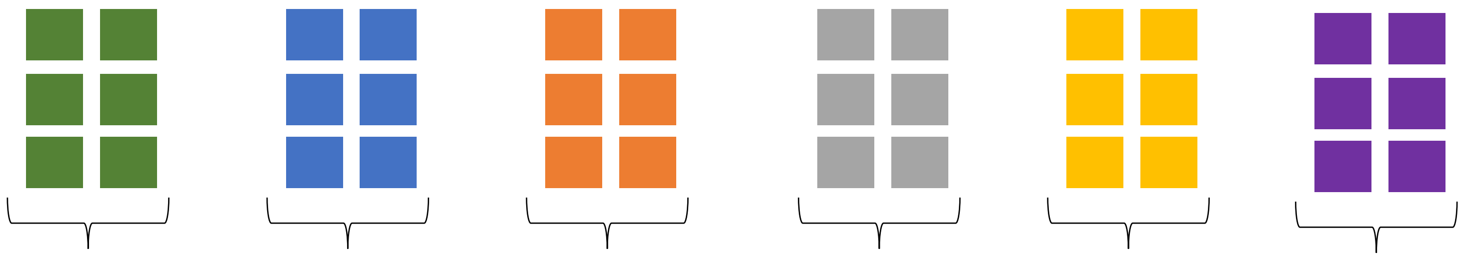
- Examples
  - Approval voting (AV): $s = (1,1,1, \dots 1)$
    - Selects the k candidates with the highest total approvals
  - Chamberlin-Courant (CC): $s = (1,0,0, \dots 0)$
    - Maximizes the number of voters for whom at least one approved candidate is selected

Chamberlin-Courant

$k = 6$



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3:	+0	+0	+0	+0	+0	+0
4:	+0	+0	+0	+0	+0	+0
5:	+0	+0	+0	+0	+0	+0
6:	+0	+0	+0	+0	+0	+0

W={  }

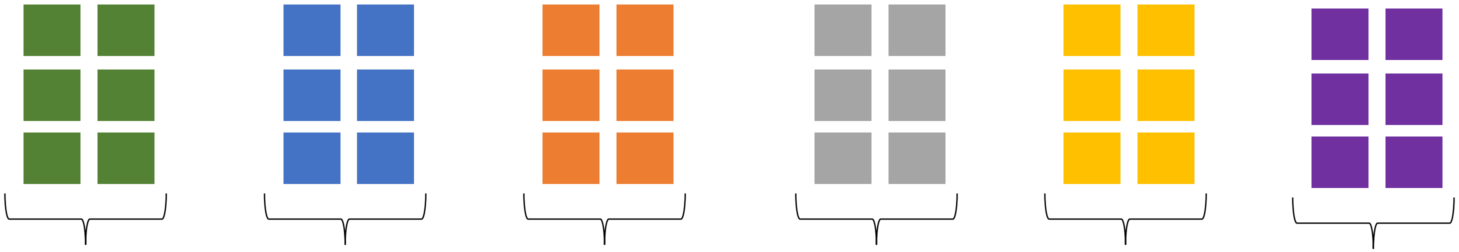
Prominent Rules

- Thiele's Methods [1895]
  - Given a sequence $s = (s_1, s_2, \dots s_k)$, select a committee W that maximizes $\sum_{i \in N} s_1 + s_2 + \cdots + s_{u_i(W)}$

- Examples
  - Approval voting (AV): $s = (1,1,1, \dots 1)$
    - Selects the k candidates with the highest total approvals
  - Chamberlin-Courant (CC): $s = (1,0,0, \dots 0)$
    - Maximizes the number of voters for whom at least one approved candidate is selected
  - Proportional Approval Vorting (PAV): $s = (1, {}^{1}/{}_{2}, {}^{1}/{}_{3}, \dots, {}^{1}/{}_{k})$
    - In between AC and CC, but why exactly harmonic scores?

Proportional Approval Voting

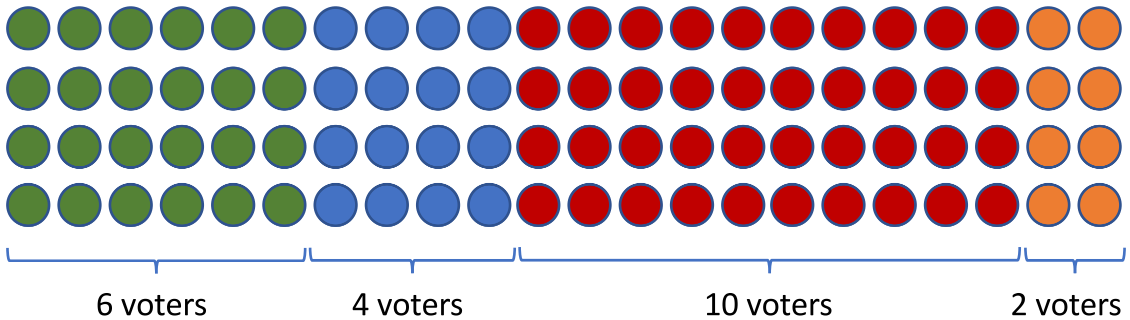
$k = 6$



	24 voters	1 voter	18 voters	12 voters	1 voter	12 voters
1:	+24	+1	+18	+12	+1	+12
2:	+12	+0.5	+9	+6	+0.5	+6
3:	+8	+0.33	+6	+4	+0.33	+4
4:	+6	+0.25	+4.5	+3	+0.25	+3
5:	+4.8	+0.20	+3.6	+2.4	+0.20	+2.4
6:	+4	+0.16	+3	+2	+0.16	+2

W={  }

Why Harmonic Numbers?



$k = 11$

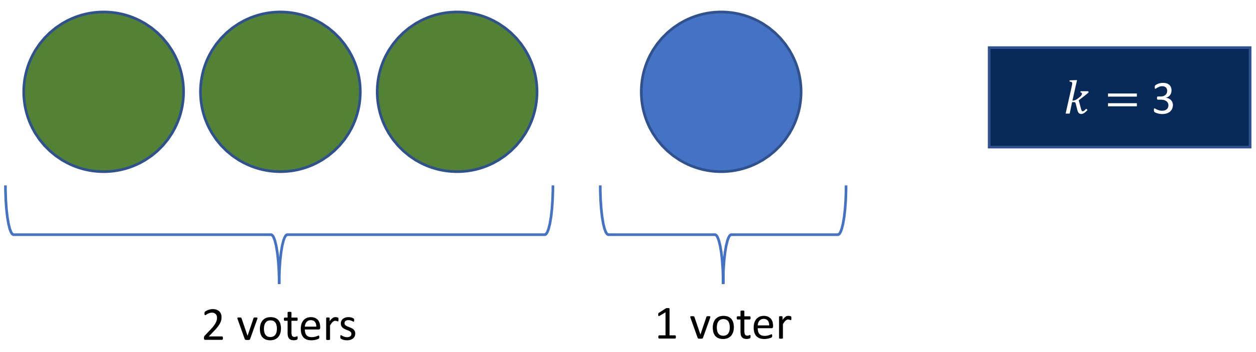
- "Proportionality"
  - ➢ We should select 3 , 2 , 5 , 1

Party-List PR

- Party-list instances
  - For all $i, j \in N$: either $A_i = A_j$ or $A_i \cap A_j = \emptyset$
  - For all $i \in N$: $|A_i| \geq k$
- Lower quota for party-list instances
  - For every party-list instance, $u_i(W) \geq \left\lfloor k \cdot {}^{n_i}/_{n} \right\rfloor$ for all $i \in N$, where $n_i = \left|\left\{j \in N : A_j = A_i\right\}\right|$

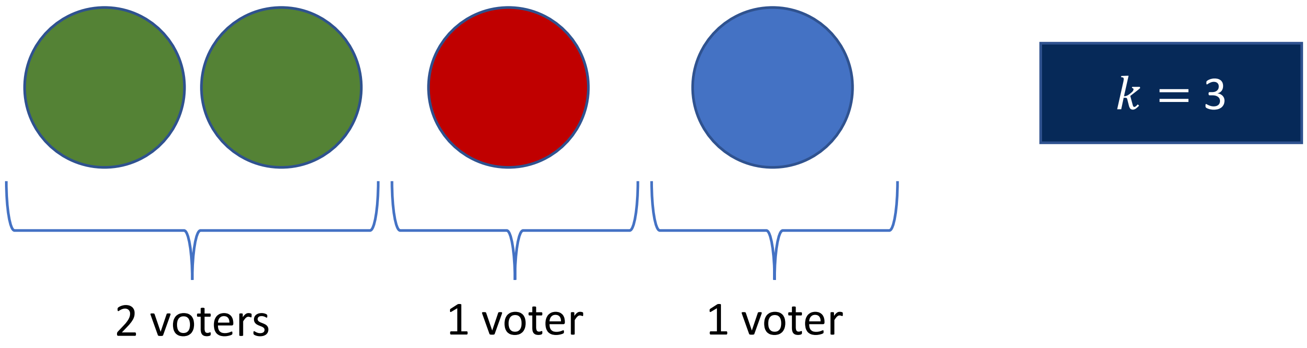
Party-List PR

- AV violates lower quota for party-list instances
  - 4 candidates $\{a, b, c, d\}$, $k = 3$
  - 2 voters approve $\{a, b, c\}$ and 1 voter approves d

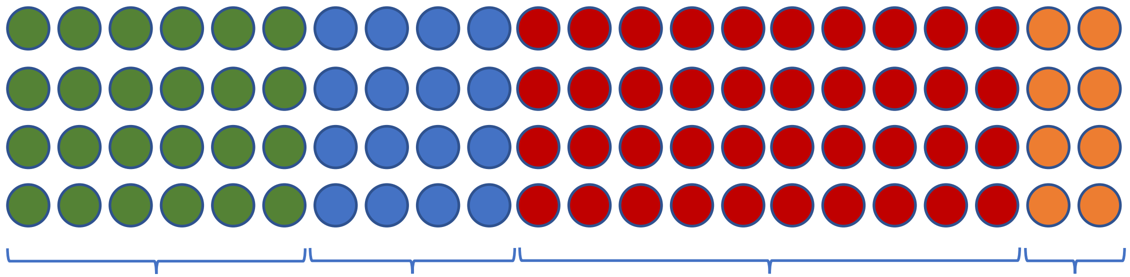


Party-List PR

- CC violates lower quota for party-list instances
  - 6 candidates $\{a, b, c, d\}, k = 3$
  - 2 voters approve $\{a, b\}$, 1 voter approves $\{c\}$, 1 voter approves $\{d\}$



Intuition Behind PAV



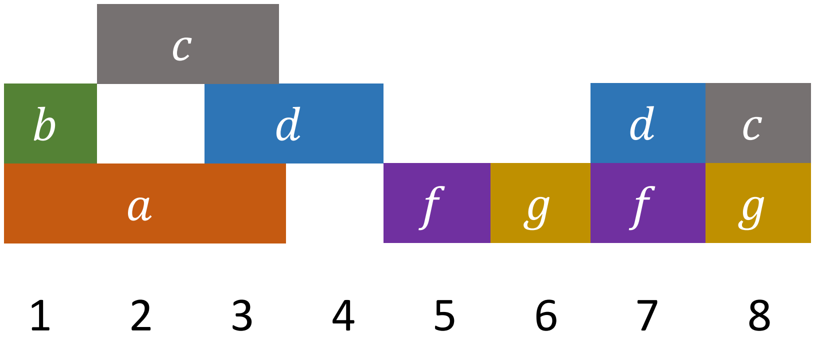
$k = 11$

- Party-list PR
  - We should select 3 green, 2 blue, 5 red, 1 orange
  - PAV would have the desired result because:
    - 3rd green, 2nd blue, 5th red, 1st orange have the same marginal contribution = 2
    - We'll see a formal proof of PAV satisfying something stronger later
    - PAV known to be the only Thiele's method (and subject to additional axioms the only ABC rule) achieving this

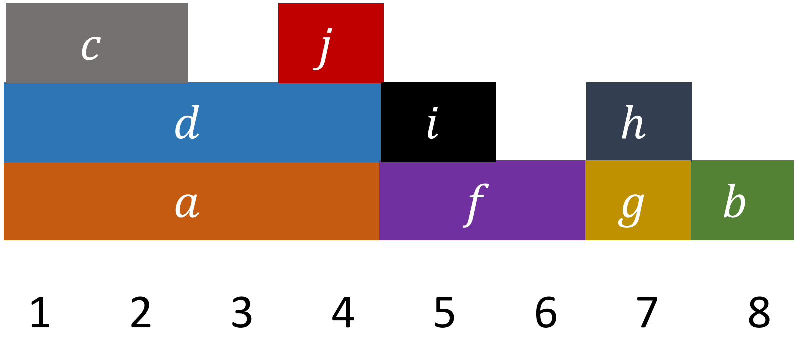
Fairness for General Instances

- Issues
  - No well-separated "groups" of voters
  - A subset of voters may not be "fully cohesive" (having identical approval sets)

- We want to provide a utility guarantee to
  - ...every possible subset (group) of voters that is...
  - ...sufficiently large and cohesive and...
  - ...their guarantee scales with their size and cohesiveness



Fairness for General Instances

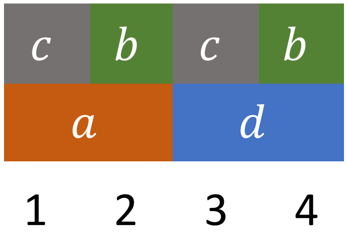


- For all $S \subseteq N$
- If $|S| \geq \ell \cdot {}^{n}/_{k}$ (large) and $|\cap_{i \in S} A_i| \geq \ell$ (cohesive)
- Then $|W \cap_{i \in S} A_i| \geq \ell$
- Question: Is this property always satisfiable?

First Attempt

- For all $S \subseteq N$
- If $|S| \geq \ell \cdot {}^{n}/_{k}$ (large) and $|\bigcap_{i \in S} A_i| \geq \ell$ (cohesive)
- Then $|W \cap_{i \in S} A_i| \geq \ell$

- k=2

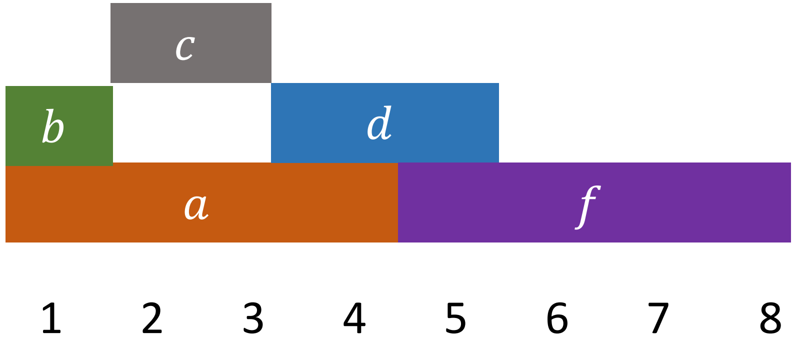


- $A_1 \cap A_2 = a$
- $A_2 \cap A_4 = b$
- $A_1 \cap A_3 = c$
- $A_3 \cap A_4 = d$

Justified Representation (JR)

- Definition: W satisfies JR if
  - For all $S \subseteq N$
  - If $|S| \geq {}^{n}/_{k}$ (large) and $|\bigcap_{i \in S} A_i| \geq 1$ (cohesive)
  - Then $u_i(W) \geq 1$ for some $i \in S$
  - "If a group deserves one candidate and has a commonly approved candidate, then not every member should get 0 utility"

  - Question: Find all the committees that satisfy JR for $k = 2$

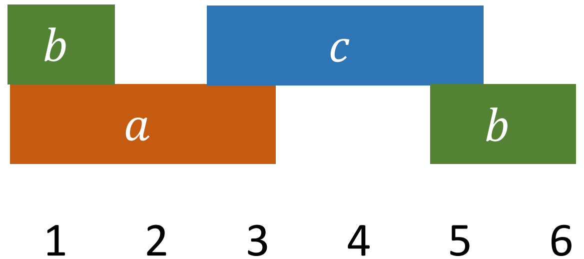


Justified Representation (JR)

- **Definition:** W satisfies JR if
  - For all $S \subseteq N$
  - If $|S| \geq n/k$ (large) and $|\cap_{i \in S} A_i| \geq 1$ (cohesive)
  - Then $u_i(W) \geq 1$ for some $i \in S$
  - "If a group deserves one candidate and has a commonly approved candidate, then not every member should get 0 utility"

  - **Question:** Can we ask $u_i(W) \geq 1$ for all $i \in S$?

  - $k = 2$



Justified Representation (JR)

- Approval Voting violates JR

$k = 3$

	24 voters	12 voters
1:	+24	+12
2:	+24	+12
3:	+24	+12

Justified Representation

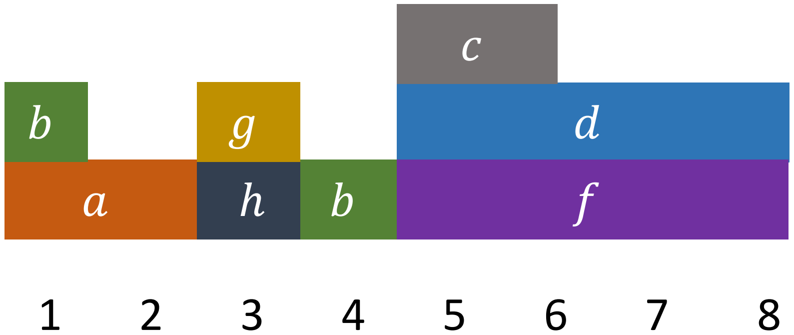
- Theorem: Chamberlin-Courant satisfies JR
- Proof:
- Suppose CC selects W, which violates JR
- Then, there is a group $S \subseteq N$ such that
  - $|S| \geq n/_k$
  - No $i \in S$ is "covered" ($u_i(W) = 0 \; \forall i \in S$)
  - There is a candidate $c^* \in \cap_i A_i$
- Since W covers less than n voters in total, some $c \in W$ covers (is approved by) less than $n/_k$ voters
- Replacing c with c^* gives a new committee that covers strictly more voters, a contradiction to W already maximizing this metric!

Extended Justified Representation (EJR)

- Definition: W satisfies EJR if
  - For all $S \subseteq N$ and $\ell \in \{1, \dots, k\}$
  - If $|S| \geq \ell \cdot {}^{n}/_{k}$ (large) and $|\bigcap_{i \in S} A_i| \geq \ell$ (cohesive)
  - Then $u_i(W) \geq \ell$ for some $i \in S$
  - "If a group deserves ℓ candidates and has ℓ commonly approved candidates, then not every member should get less than ℓ utility"
  - JR imposes this but only for $\ell = 1$, so EJR $\Rightarrow$ JR

Extended Justified Representation (EJR)

- Question: What is a committee that satisfies EJR? Is there a committee that satisfies EJR but not JR?



Extended Justified Representation (EJR)

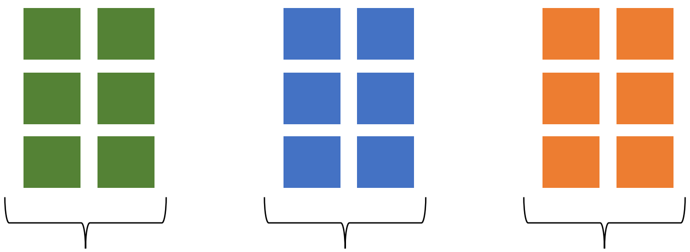
- Question: What is the relationship between JR, EJR and proportionality in the case of party lists?

1. JR $\Longrightarrow$ party-list PR
2. EJR $\Longrightarrow$ party-list PR
3. None
4. Both

Extended Justified Representation (EJR)

- Chamberlin-Courant violates EJR

$k = 3$



	24 voters	1 voter	1 voter
1:	+24	+1	+1
2:	+0	+0	+0
3:	+0	+0	+0

Extended Justified Representation (EJR)

- Theorem [Aziz et al. (2016)]: PAV satisfies EJR
- Proof:
- Suppose PAV selects W, which violates EJR
  - $PAV(W) = \sum_{i \in N} 1 + \frac{1}{2} + \cdots + \frac{1}{u_i(W)}$
- Then, there is a group $S \subseteq N$ and $\ell \in \{1, \ldots, k\}$ such that
  - $|S| \geq \ell \cdot {}^{n}/_{k}$
  - $u_i(W) < \ell, \ \forall i \in S$
  - $|\cap_{i \in S} A_i| \geq \ell \Rightarrow$ there exists $c^* \in \cap_{i \in S} \ A_i \setminus W$ (Why?)
- Consider $\widetilde{W} = W \cup \{c^*\}$
  - $PAV(\widetilde{W}) \geq PAV(W) + |S| \cdot \frac{1}{\ell} \geq PAV(W) + \frac{n}{k}$
- Claim: Can remove some $c \in \widetilde{W}$ and lower score by $< \frac{n}{k}$

Extended Justified Representation (EJR)

- Claim: Can remove some $c \in \widetilde{W}$ and lower score by $< \frac{n}{k}$
- Proof:
  - Suffices to prove that average reduction across $c \in \widetilde{W}$ is less than $\frac{n}{k}$
  - Reduction when removing $c \in \widetilde{W} = \sum_{i:c \in A_i} \frac{1}{u_i(\widetilde{W})}$
  - Average reduction:

$$\frac{1}{k+1} \cdot \sum_{c \in \widetilde{W}} \sum_{i:c \in A_i} \frac{1}{u_i(\widetilde{W})} = \frac{1}{k+1} \cdot \sum_{i \in N} \sum_{c \in A_i \cap \widetilde{W}} \frac{1}{u_i(\widetilde{W})}$$

$$= \frac{1}{k+1} \cdot \sum_{i \in N} 1$$

$$= \frac{n}{k+1} < \frac{n}{k}$$

Computation of PAV

- Computing PAV is NP-complete

- What about a greedy approximation?
  - Sequential PAV
    - $W \leftarrow \emptyset$
    - **while** $|W| < k$ **do**
      - Find c which maximizes $PAV(W \cup \{c\})$
      - $W \leftarrow W \cup \{c\}$
  - Achieves at least $\left(1 - \frac{1}{e}\right)$ fraction of optimal PAV score
    - PAV score is a submodular function
  - But fails to satisfy EJR

Computation of PAV

- In practice, exact PAV solution can be computed via a BILP
- Binary variables:
  - $y_c \rightarrow$ Is candidate c selected?
  - $x_{i,\ell} \rightarrow$ Is $u_i(\{c : y_c = 1\}) \geq \ell$?
- Maximize $\sum_{i \in N} \sum_{\ell=1}^{k} \frac{1}{\ell} \cdot x_{i,\ell}$

  subject to $\sum_{\ell=1}^{k} x_{i,\ell} = \sum_{c \in A_i} y_c$ for all i

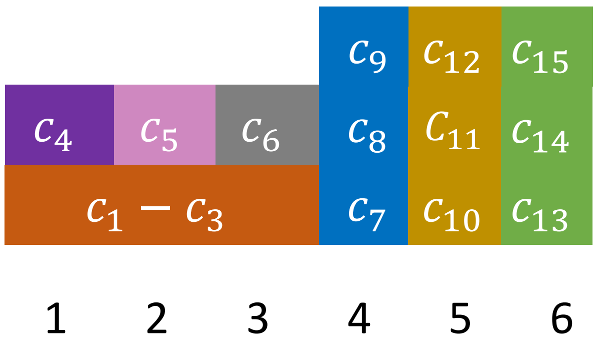
  $\sum_c y_c = k$

  $y_c, x_{i,\ell} \in \{0,1\}$ for all i, ℓ, c

← Why does this work?

Is EJR enough?

$k = 12$



Fully Justified Representation (FJR)

- Definition: W satisfies FJR if
  - For all $S \subseteq N, T \subseteq M$ and $\beta \in \{1, \dots, k\}$
  - If $|S| \geq |T| \cdot {}^{n}/_{k}$ (large) and $u_i(T) \geq \beta, \forall i \in S$ (cohesive)
  - Then $u_i(W) \geq \beta$ for some $i \in S$
  - "If a group deserves ℓ candidates and can propose a set of ℓ candidates from which each member gets at least β utility, then not every member should get less than β utility"
  - EJR imposes this but only for $\beta = |T|$, which would imply $T \subseteq \cap_{i \in S} A_i$, so we just wrote $|\cap_{i \in S} A_i| \geq \ell$
  - FJR $\Rightarrow$ EJR

- Bad news: PAV (and every other known "natural" rule) violates FJR

Fully Justified Representation (FJR)

- FJR is satisfiable via a simple polynomial-time greedy rule

- Greedy Cohesive Rule (GCR):
  - $W \leftarrow \emptyset$
  - $N^a \leftarrow N$ ("active voters")
  - **while** $\exists \beta > 0,\ S \subseteq N^a,\ T \subseteq M \setminus W$
    s.t. $|S| \geq |T| \cdot \frac{n}{k}$ and $\min_{i \in S} u_i(T) \geq \beta$ **do**
    - Pick such (β, S, T) with the highest β (break ties arbitrarily)
    - $W \leftarrow W \cup T, N^a \leftarrow N^a \setminus S$
  - **return** W

- Greedily find the most cohesive group of voters and add their suggested group of candidates

Fully Justified Representation (FJR)

- Theorem[Peters et al. (2022)]: Greedy Cohesive Rule satisfies FJR

- Proof:

- Suppose for contradiction that Greedy Cohesive Rule does not satisfy FJR
  - Then, there is a group $S \subseteq N, T \subseteq M$ and $\beta \in \{1, \ldots, k\}$ such that
  - $|S| \geq |T| \cdot {}^{n}/_{k}$ and $u_i(T) \geq \beta, \forall i \in S$
  - $u_i(W) < \beta, \ \forall i \in S$

- Let i^* be the first agent in S that was removed from N^a as part of the group S' and subcommittee T', with $\min_{i \in S'} u_i(T') \geq \beta'$

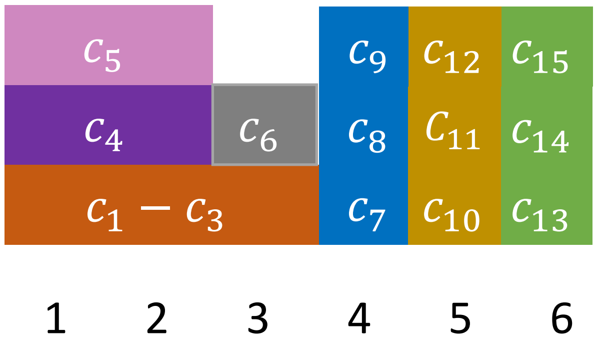
- Just before S' is chosen, S was available since i^* is the first agent in S that that was removed from N^a

- From the definition of the algorithm, this means $\min_{i \in S'} u_i(T') \geq \min_{i \in S} u_i(T) \Longrightarrow \beta' \geq \beta$

- But since $T' \subseteq W$, we get $\beta > u_{i^*}(W) \geq u_{i^*}(T') \geq \beta'$ which is a contradiction

Is FJR enough?

$k = 12$



Core

- Definition: W satisfies core if
  - For all $S \subseteq N$ and $T \subseteq M$
  - If $|S| \geq |T| \cdot {}^{n}/_{k}$ (large)
  - Then $u_i(W) \geq u_i(T)$ for some $i \in S$
  - "If a group can afford T, then T should not be a strict Pareto improvement for the group"
  - FJR only imposes $\max_{i \in S} u_i(W) \geq \min_{i \in S} u_i(T)$, so core ⇒ FJR

- **Major open question**
  - For ABC voting, does there always exist a committee in the core?

Notes

- Other fairness definitions
  - EJR+, SJR, AJR, PJR, PRJ+, UJR, CS, proportionality degree, …
  - See [Justified Representation wiki]() for more details