CoALa

$$\mathbf{L} = \mathbf{Z}\Gamma\mathbf{Z}^T$$

Joint Shifted Laplacian

$\mathbf{Z}$ consists of the eigenvectors of L

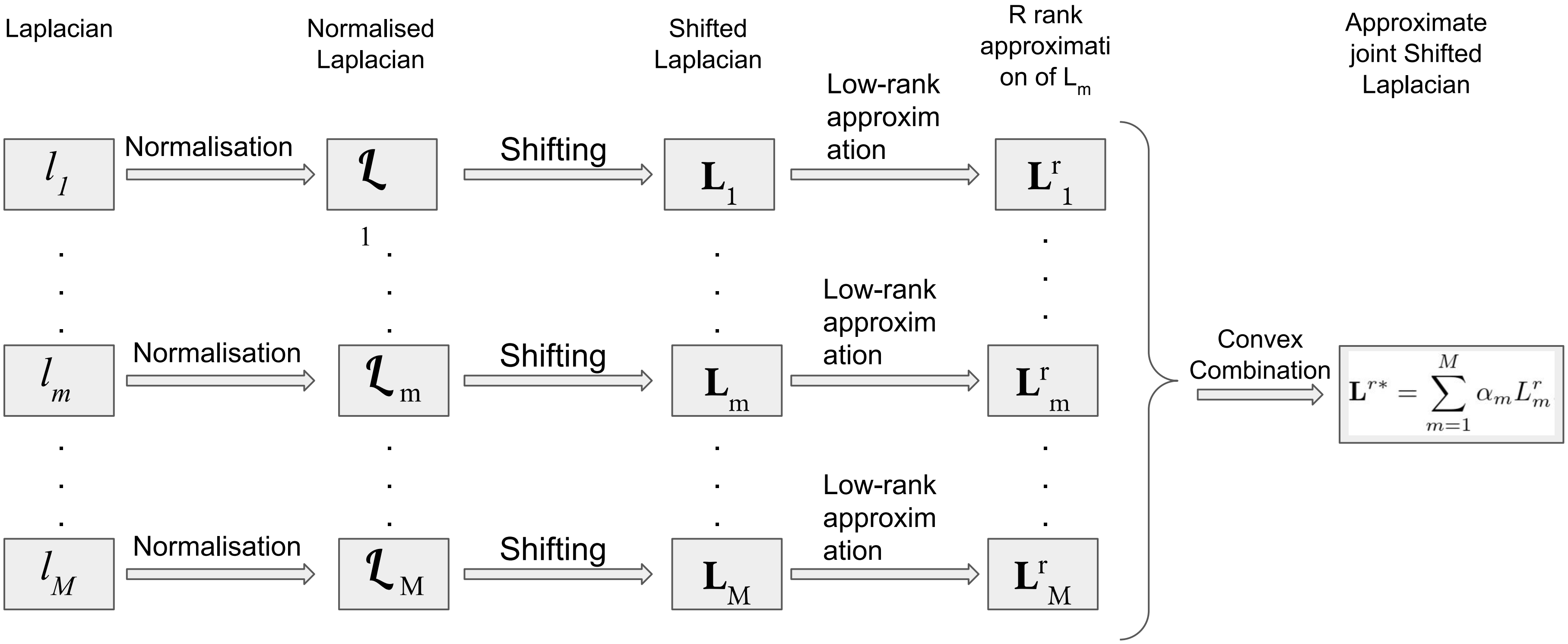
$$\Gamma = diag(\gamma_1, \ldots, \gamma_n)$$

$$\Psi\left(\mathbf{L}^r\right) = \langle \mathbf{Z}^r, \Gamma^r \rangle$$

Represents that eigenspace has rank *r*

$$\mathbf{L}^{r*}$$

Convex combination of best rank *r* approximation of Laplacians L_m of individual modality X_m.



Laplacian
Normalised Laplacian
Shifted Laplacian
R rank approximation of L_m
Approximate joint Shifted Laplacian
l_1
Normalisation
$\mathcal{L}_1$
Shifting
$\mathbf{L}_1$
Low-rank approximation
$\mathbf{L}^r_1$
l_m
Normalisation
$\mathcal{L}_m$
Shifting
$\mathbf{L}_m$
Low-rank approximation
$\mathbf{L}^r_m$
l_M
Normalisation
$\mathcal{L}_M$
Shifting
$\mathbf{L}_M$
Low-rank approximation
$\mathbf{L}^r_M$
Convex Combination
$\mathbf{L}^{r*} = \sum_{m=1}^{M} \alpha_m L^r_m$

If Approximate joint shifted laplacian was constructed the other way around-

$$\boxed{\mathbf{L}_1} \cdots \boxed{\mathbf{L}_m} \cdots \boxed{\mathbf{L}_M}$$

$$\mathbf{L} = \sum_{m=1}^{M} \alpha_m L_m$$

Eigendecomposition

$$\mathbf{L} = \mathbf{Z}\Gamma\mathbf{Z}^T$$

Full rank eigenspace

$$\Psi\left(\mathbf{L}^r\right) = \langle \mathbf{Z}^r, \Gamma^r \rangle$$

Computationally Heavy (comparatively)-

- Adding nxn matrices while constructing joint shifted laplacian.
- Then doing eigendecomposition of nxn matrix.

Framing the objective

We want to know the matrix of eigenvectors of matrix $\mathbf{L}^{r^*}$ arranged in descending order of their corresponding eigenvalues.

$$\Psi\left(\mathbf{L}^r\right)=\left\langle\mathbf{Z}^r,\Gamma^r\right\rangle$$

“Full rank” as in: $\mathbf{L}$ has complete information of all eigenpairs of each laplacian as considered during convex combination.

“Full rank” don’t necessarily mean that rank has to be equal to the dimension of the matrix. It just have to satisfy rank plus nullity theorem.

rank(*A)* + nullity(*A)* = the number of columns of *A*

Moreover, because of its ‘full rank’ nature, it contains noise too!

$\Psi\left(\mathbf{L}^{r*}\right)$: That’s why, the approximate eigenspace, constructed from the *r* largest eigenpairs, is expected to preserve the cluster information better.

Reframing the objective

Given these eigenspaces $\Psi(L_m^r)$s, construction of the rank r eigenspace $\Psi\left(\mathbf{L}^{r*}\right)$

Let,

$$V_n = \begin{pmatrix} v_{n1} \\ v_{n2} \\ v_{n3} \end{pmatrix}$$

Assuming all columns of the matrix **A** are linearly independent..

then, for a 3x3 matrix **A** with eigenpairs $\langle \boldsymbol{\lambda_1}, \mathbf{V_1} \rangle$,

$$\mathbf{AV_1} = \boldsymbol{\lambda_1}\mathbf{V_1}$$

$$v_{11}\begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix} + v_{12}\begin{pmatrix} A_{21} \\ A_{22} \\ A_{23} \end{pmatrix} + v_{13}\begin{pmatrix} A_{31} \\ A_{32} \\ A_{33} \end{pmatrix} = \boldsymbol{\lambda_1}\mathbf{V_1}$$

Inference [1]: Hence eigenvectors of a matrix can be represented as a linear combination of its column vectors! That is, eigenvectors of a matrix falls into the column space of that matrix.

Similarly, for rest two eigenpairs

$$v_{12}\begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix} + v_{22}\begin{pmatrix} A_{21} \\ A_{22} \\ A_{23} \end{pmatrix} + v_{23}\begin{pmatrix} A_{31} \\ A_{32} \\ A_{33} \end{pmatrix} = \boldsymbol{\lambda_2 V_2}$$

$$v_{31}\begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix} + v_{32}\begin{pmatrix} A_{21} \\ A_{22} \\ A_{23} \end{pmatrix} + v_{33}\begin{pmatrix} A_{31} \\ A_{32} \\ A_{33} \end{pmatrix} = \boldsymbol{\lambda_3 V_3}$$

Inference [2]: Using definition of the vector space that is vector A and vector B are in a vector space, then A+B would also be in the same vector space, we can argue that linear combination of eigenvectors would also be in the vector space and hence columns of the transformation matrix can also be represented as a linear combination of its eigenvectors.

Inference [3]: From above two inferences, we can say that column space of a matrix is same as the column space of the matrix of eigenvector in columns.

Total number of modalities

$$\mathcal{J}^r = span\left(\bigcup_{m=1}^{M} \underbrace{\mathcal{C}(U_m^r)}_{\text{Column space of matrix of eigenvectors of modality } m}\right)$$

Subspace spanned by the columns of $\mathbf{L}^{r*}$

The catch: We are not going to do the eigendecomposition of the approximate joint laplacian matrix right away.

$\mathbf{U}_m$: The orthonormal basis of the subspace- $\mathcal{J}_m^r$

It is spanning both the subspaces- $\mathcal{J}_m^r$ and $\mathcal{C}(U_m^r)$

With addition of each modality, the new orthonormal basis $\mathbf{U}_{m+1}$ is going to have a new vector such that it is orthogonal to $\mathbf{U}_m$ and should be derived from U^r_{m+1}.

So we begin with computing the residue of each basis vector in U^r_{m+1} wrt the basis $\mathbf{U}_m$.

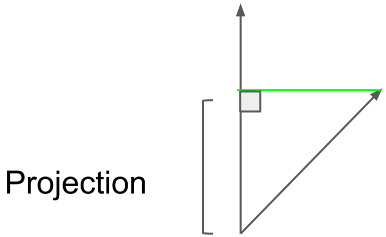
Our aim is to subtract those components of U^r_{m+1} which lies upon $\mathbf{U}_m$.

Magnitude of components of U^r_{m+1} on $\mathbf{U}_m$

$$S_{m+1} = \mathbf{U}_m^T U_{m+1}^r$$

Projection of components of U^r_{m+1} on $\mathbf{U}_m$

$$P_{m+1} = \mathbf{U}_m S_{m+1}$$



This is the residue component of U^r_{m+1}

$$Q_{m+1} = U_{m+1}^r - P_{m+1}$$

Then Gram- Schmidt process is applied on Q matrix

For a linearly independent set $Q = \{v_1, ..., v_k\}$, we need to construct an orthogonal set $Q' = \{u_1, ..., u_k\}$

$$
\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1, & \mathbf{e}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \\
\mathbf{u}_2 &= \mathbf{v}_2 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_2), & \mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \\
\mathbf{u}_3 &= \mathbf{v}_3 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_3), & \mathbf{e}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \\
\mathbf{u}_4 &= \mathbf{v}_4 - \mathrm{proj}_{\mathbf{u}_1}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_2}(\mathbf{v}_4) - \mathrm{proj}_{\mathbf{u}_3}(\mathbf{v}_4), & \mathbf{e}_4 &= \frac{\mathbf{u}_4}{\|\mathbf{u}_4\|} \\
&\vdots & &\vdots \\
\mathbf{u}_k &= \mathbf{v}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\mathbf{u}_j}(\mathbf{v}_k), & \mathbf{e}_k &= \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}.
\end{aligned}
$$

In this way we will get basis Υ_{m+1} spanned by the columns of Q_{m+1}

The required ‘sufficient basis’ would become-

$$\mathbf{U}_{m+1} = \begin{bmatrix} \mathbf{U}_m & \Upsilon_{m+1} \end{bmatrix}$$

And hence, finally-

$$\mathbf{U}_M = \begin{bmatrix} \Upsilon_1 & \Upsilon_2 & \dots & \Upsilon_M \end{bmatrix}$$

Let the eigendecomposition of approximate shifted graph laplacian be

$$\mathbf{L}^{r*} = \mathbf{V}\Pi\mathbf{V}^T$$

As proved previously, eigenvector of the laplacian would span its column space also. Which in turn is nothing but-$\mathcal{J}^r$

We know the basis of $\mathcal{J}^r$ is $\mathbf{U}_M$

Hence, $\mathbf{V}$ must be just a transformation of $\mathbf{U}_M$ about a rotation matrix $\mathbf{R}$

$$\mathbf{V} = \mathbf{U}_M \mathbf{R}$$

$$\mathbf{L}^{r*} = \sum_{m=1}^{M} \alpha_m U_m^r \Sigma_m^r (U_m^r)^T,$$

$$\Rightarrow \mathbf{V}\Pi\mathbf{V}^T = \sum_{m=1}^{M} \alpha_m U_m^r \Sigma_m^r (U_m^r)^T,$$

$$\Rightarrow (\mathbf{U}_M\mathbf{R})\Pi(\mathbf{U}_M\mathbf{R})^T = \sum_{m=1}^{M} \alpha_m U_m^r \Sigma_m^r (U_m^r)^T$$

$$\Rightarrow \mathbf{R}\Pi\mathbf{R}^T = \mathbf{U}_M^T \left(\sum_{m=1}^{M} \alpha_m U_m^r \Sigma_m^r (U_m^r)^T \right) \mathbf{U}_M$$

$$\Rightarrow \mathbf{R}\Pi\mathbf{R}^T = \sum_{m=1}^{M} \alpha_m \mathbf{U}_M^T U_m^r \Sigma_m^r (U_m^r)^T \mathbf{U}_M,$$

$$\Rightarrow \mathbf{R}\Pi\mathbf{R}^T = \sum_{m=1}^{M} \alpha_m \begin{bmatrix} \Upsilon_1^T \\ \vdots \\ \Upsilon_M^T \end{bmatrix} U_m^r \Sigma_m^r (U_m^r)^T \begin{bmatrix} \Upsilon_1 & \dots & \Upsilon_M \end{bmatrix}$$

$$\Rightarrow \mathbf{R}\Pi\mathbf{R}^T = \sum_{m=1}^{M} \alpha_m H_m$$

$$H_m = [\Upsilon_1 \ldots \Upsilon_M]^T U_m^r \Sigma_m^r (U_m^r)^T [\Upsilon_1 \ldots \Upsilon_M]$$

$H_m \in \mathbb{R}^{(Mr \times Mr)}$

$$H_m(i,j) = \begin{cases} \Upsilon_i^T U_m^r \Sigma_m^r (U_m^r)^T \Upsilon_j & \text{if } \ i \leq m \text{ and } \ j \leq m, \\ 0 & \text{if } \ i > m \text{ or } \ j > m. \end{cases}$$

$$\mathbf{H} = \sum_{m=1}^{M} \alpha_m H_m;$$

$$\mathbf{H} = \mathbf{R}\Pi\mathbf{R}^T$$

Problem boiled down to perform eigendecomposition of a MrxMr matrix!