Ex 1 | Introduction to Speech Processing(67455)

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Question :1

Subsection a:

$$F^{-1}\left[X_1^F(\omega) * X_2^F(\omega)\right] \overset{\text{def of } F^{-1}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} X_1^F(u)\, X_2^F(s-u)\, du\right] e^{2\pi i s\omega} ds$$

Changing the order of integration:

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1^F(u)\left[\int_{-\infty}^{\infty} X_2^F(s-u)\, e^{2\pi i s\omega} ds\right] du$$

By the Shift Theorem (which is given below) we can see that:

$$\int_{-\infty}^{\infty} X_2^F(s-u)\, e^{2\pi i s\omega} ds = e^{2\pi i u\omega} x_2(\omega)$$

Giving us:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1^F(u)\left[\int_{-\infty}^{\infty} X_2^F(s-u)\, e^{2\pi i s\omega} ds\right] du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1^F(u)\, e^{2\pi i u\omega} x_2(\omega)\, du$$

$$= \frac{1}{2\pi} x_2(\omega)\int_{-\infty}^{\infty} X_1^F(u)\, e^{2\pi i u\omega} du$$

Shift Theorem:

$$F\left(x\left(t-t_0\right)\right)(s) = e^{-i2\pi s t_0} X^F(s)$$

Proof:

$$F\left(x\left(t-t_0\right)\right)(s) = \int_{-\infty}^{\infty} x\left(t-t_0\right) e^{-i2\pi s t} dt$$

Notice that:

$$e^{-i2\pi st}e^{i2\pi st} = 1$$

Thus we can multiply the right hand side by this term and get:

$$F\left(x\left(t-t_0\right)\right)(s)$$

$$= \int_{-\infty}^{\infty} x\left(t-t_0\right) e^{-i2\pi st}e^{-i2\pi st}e^{i2\pi st}dt$$

We can now subtitute $u = t - t_0$ and $du = dt$ to finish the proof:

$$F\left(x\left(t-t_0\right)\right)(s)$$

$$= e^{-i2\pi st_0} \int_{-\infty}^{\infty} x(u) e^{-i2\pi su}du$$

$$= e^{-i2\pi st_0} X^F(s)$$

Question :2

Subsection :1

From question 1, we have:

$$\frac{1}{2\pi}\left(X^F(\omega) * S_T^F(\omega)\right) = F\left[x(\omega)\cdot s_T(\omega)\right]$$

$$\overset{\text{prolog property 2}}{=} F\left[x(\omega)\sum_{n=-\infty}^{\infty}\delta(\omega - nT)\right]$$

$$\overset{\text{prolog property 1}}{=} F\left[x_d(\omega)\right] = X_d^F(\omega)$$

Subsection :2

In class we saw that for an integrable function $s(t)$, the following holds:

$$\int_{-\infty}^{\infty} s(t')\,\delta(t-t')\,dt' = s(t)$$

We define $z = \omega - \frac{2\pi n}{T}$ and use the equation above to get:

$$\sum_n \int_{-\infty}^{\infty} X^F(\tilde{\omega})\,\delta\left(\tilde{\omega} - \left(\omega - \frac{2\pi n}{T}\right)\right)$$

$$= \sum_n \int_{-\infty}^{\infty} X^F(\tilde{\omega})\,\delta(\tilde{\omega} - z)$$

Since by definition of δ, we have $\delta\left(\tilde{\omega}-z\right)=0 \iff \tilde{\omega}=z$, we can write:

$$=\sum_n \int\limits_{-\infty}^{\infty} X^F\left(\tilde{\omega}\right)\delta\left(\tilde{\omega}-z\right)$$

$$=\sum_n \int\limits_{-\infty}^{\infty} X^F\left(\tilde{\omega}\right)\delta\left(z-\tilde{\omega}\right)$$

From the equation in class (that was given above) we get:

$$\sum_n \int\limits_{-\infty}^{\infty} X^F\left(\tilde{\omega}\right)\delta\left(z-\tilde{\omega}\right)=\sum_n X^F\left(z\right)$$

$$=\sum_n X^F\left(\omega-\frac{2\pi n}{T}\right)$$

Subsection :3

We start from the left hand side of the equation:

$$X_d^F\left(\omega\right) \overset{1.2.1}{=} \frac{1}{2\pi}\left(X^F\left(\omega\right) * S_T^F\left(\omega\right)\right)$$

$$\overset{\text{definition of } S_T^F}{=} \frac{1}{2\pi}\left(X^F\left(\omega\right) * \frac{2\pi}{T}\sum_n \delta\left(\omega-\frac{2\pi n}{T}\right)\right)$$

$$\overset{\text{linearity of conv}}{=} \frac{1}{T}\sum_n\left(X^F\left(\omega\right) * \delta\left(\omega-\frac{2\pi n}{T}\right)\right)$$

We now define $\delta'\left(\omega\right)=\delta\left(\omega-\frac{2\pi n}{T}\right)$ and substitute. We then use the definition of the convolution given in the prologue (4):

$$\frac{1}{T}\sum_n\left(X^F\left(\omega\right) * \delta\left(\omega-\frac{2\pi n}{T}\right)\right)$$

$$=\frac{1}{T}\sum_n\left(X^F\left(\omega\right) * \delta'\left(\omega\right)\right)$$

$$=\frac{1}{T}\sum_n \int\limits_{-\infty}^{\infty} X^F\left(\tilde{\omega}\right)\delta'\left(\omega-\tilde{\omega}\right)d\tilde{\omega}$$

Substituting δ back:

$$=\frac{1}{T}\sum_n \int\limits_{-\infty}^{\infty} X^F\left(\tilde{\omega}\right)\delta\left(\omega-\frac{2\pi n}{T}-\tilde{\omega}\right)d\tilde{\omega}$$

And since $\delta\left(\omega-\frac{2\pi n}{T}-\tilde{\omega}\right)=0 \iff \omega-\frac{2\pi n}{T}=\tilde{\omega}$, we can write:

$$\frac{1}{T}\sum_n \int\limits_{-\infty}^{\infty} X^F\left(\tilde{\omega}\right)\delta\left(\omega-\frac{2\pi n}{T}-\tilde{\omega}\right)d\tilde{\omega}$$

$$= \frac{1}{T}\sum_{n}\int_{-\infty}^{\infty} X^F(\tilde{\omega})\,\delta\left(\tilde{\omega}-\left(\omega-\frac{2\pi n}{T}\right)\right)d\tilde{\omega}$$

$$\overset{1.2.2}{=} \frac{1}{T}\sum_{n} X^F\left(\omega-\frac{2\pi n}{T}\right)$$

Subsection :4

Assume towards contradiction that there exists $f_s < 2\cdot f_{max}$ s.t $\forall x_d(t), \forall \mid \omega \mid < \omega_{max} : X_d^F(\omega) = \frac{1}{T}X^F(\omega)$. We define $\overline{\omega} = \omega_{max} + \frac{\pi}{T}$. Since $\overline{\omega} > \omega_{max}$ we have that $X^F(\overline{\omega}) = 0$ (band limited). In addition, according to 1.2.3 we have:

$$X_d^F(\overline{\omega}) = \frac{1}{T}\sum_{n} X^F\left(\overline{\omega}-\frac{2\pi n}{T}\right)$$

We analyze the terms in the summation:

- For $n \leq 0$ we have $\overline{\omega} - \frac{2\pi n}{T} \geq \omega_{max}$ and so $X^F\left(\overline{\omega}-\frac{2\pi n}{T}\right) = 0$
- Hence $\frac{1}{T}\sum_n X^F\left(\overline{\omega}-\frac{2\pi n}{T}\right) = \frac{1}{T}\sum_{n=1} X^F\left(\overline{\omega}-\frac{2\pi n}{T}\right)$

As we've stated, we must have $X^F(\overline{\omega}) = 0$ thus $\frac{1}{T}\sum_{n=1} X^F\left(\overline{\omega}-\frac{2\pi n}{T}\right)$
We now define $\tilde{\omega} = \omega_{max} - \frac{\pi}{T}$. Since we have assumed that $f_s < 2\cdot f_{max}$ we have that $\omega_{max} > \frac{\pi}{T}$ which in turn means that $\tilde{\omega} \in [0, \omega_{max}]$. Hence $X^F(\tilde{\omega}) \neq 0$. Using the result of 1.2.3 again, we get:

$$X_d^F(\tilde{\omega}) = \frac{1}{T}\sum_{n} X^F\left(\tilde{\omega}-\frac{2\pi n}{T}\right)$$

We analyze the terms in the summation:

- For $n \leq -1$ we have $\tilde{\omega} - \frac{2\pi n}{T} \geq \omega_{max}$ and so $X^F\left(\tilde{\omega}-\frac{2\pi n}{T}\right) = 0$
- Hence $\frac{1}{T}\sum_n X^F\left(\tilde{\omega}-\frac{2\pi n}{T}\right) = \frac{1}{T}\sum_{n=0} X^F\left(\tilde{\omega}-\frac{2\pi n}{T}\right)$

But notice that $\frac{1}{T}\sum_{n=0} X^F\left(\tilde{\omega}-\frac{2\pi n}{T}\right) = \frac{1}{T}\sum_{n=1} X^F\left(\overline{\omega}-\frac{2\pi n}{T}\right)$. This is a contradiction since $0 \neq X_d^F(\tilde{\omega}) = X_d^F(\overline{\omega}) = 0$.
For the other direction, we assume that $f_s > 2f_{max}$. Let $x_d(t)$ be a discrete signal such that $F(x_d(t)) = X_d^F(\omega)$ and $\mid \omega \mid \leq \omega_{max}$. According to 1.2.3 we have:

$$X_d^F(\omega) = \frac{1}{T}\sum_{n} X^F\left(\omega-\frac{2\pi n}{T}\right)$$

Notice that for the terms where $n \neq 0$, we have $\mid \omega - \frac{2\pi n}{T} \mid > \mid \omega - 2n\omega_{max} \mid > \omega_{max}$ thus $X^F\left(\omega - \frac{2\pi n}{T}\right) = 0$. This means that:

$$\frac{1}{T}\sum_{n} X^F\left(\omega-\frac{2\pi n}{T}\right) = \frac{1}{T}X^F(\omega)$$

Question :3

Using the result of 1.2.3, we can calculate $X_d^F(\omega)$ for the different frequencies (considering also the negative frequencies as the hint suggests). The following shows the summation, and then explicitly shows the terms that we get for $n = -1, 0, 1$, as the other terms obviously results in irrelevant values:

- $\omega = 2\pi \cdot 5000$,

$$\frac{1}{T}\sum_n X^F\left(\omega - \frac{2\pi n}{T}\right) =$$

$$\frac{1}{T}\left[... + X^F\left(\underbrace{2\pi \cdot 5000 + 2\pi \cdot 8000}_{=2\pi\cdot 13000}\right) + X^F\left(2\pi \cdot 5000\right) + X^F\left(\underbrace{2\pi \cdot 5000 - 2\pi \cdot 8000}_{=2\pi\cdot(-3000)}\right)...\right]$$

  Since we are ignoring negative values in our representation, the above summation has no terms which give non-zero frequency.

- $\omega = 2\pi \cdot 1000$,

$$\frac{1}{T}\sum_n X^F\left(\omega - \frac{2\pi n}{T}\right) =$$

$$\frac{1}{T}\left[... + X^F\left(\underbrace{2\pi \cdot 1000 + 2\pi \cdot 8000}_{=2\pi\cdot 9000}\right) + X^F\left(2\pi \cdot 1000\right) + X^F\left(\underbrace{2\pi \cdot 1000 - 2\pi \cdot 8000}_{=2\pi\cdot(-7000)}\right)...\right]$$

  In the above summation , we get the original frequency $2\pi \cdot 1000$

- $\omega = 2\pi \cdot (-1000)$,

$$\frac{1}{T}\sum_n X^F\left(\omega - \frac{2\pi n}{T}\right) =$$

$$\frac{1}{T}\left[... + X^F\left(\underbrace{2\pi \cdot (-1000) + 2\pi \cdot 8000}_{=2\pi\cdot 7000}\right) + X^F\left(2\pi \cdot (-1000)\right) + X^F\left(\underbrace{2\pi \cdot (-1000) - 2\pi \cdot 8000}_{=2\pi\cdot(-9000)}\right)...\right]$$

  This summation does not add any positive frequency either.

- $\omega = 2\pi \cdot (-5000)$,

$$\frac{1}{T}\sum_n X^F\left(\omega - \frac{2\pi n}{T}\right) =$$

$$\frac{1}{T}\left[... + X^F\left(\underbrace{2\pi \cdot (-5000) + 2\pi \cdot 8000}_{=2\pi\cdot 3000}\right) + X^F\left(2\pi \cdot (-5000)\right) + X^F\left(\underbrace{2\pi \cdot (-5000) - 2\pi \cdot 8000}_{=2\pi\cdot(-13000)}\right)...\right]$$

  We can see that we have the positive frequency $2\pi \cdot 3000$.

We conclude that the frequencies that will appear in the Fourier transform of the discrete measured signal are $2\pi \cdot 1000$ and $2\pi \cdot 3000$.