Gödel : the incompleteness theorems

intro

- as much philosophical as it is mathematics
- a deep statement about logic
- a position on what it means for something to be true

why do these results matter to programmers

- halting problem
- debate on if AI is do-able
- your Ambitious Manager
- add your own :)

not included in this talk

- second-order logic
- as little as philo of maths as possible
- focus on the paper at hands, not its far-fetched consequences

some historical context: building mathematics from scratch

- Frege
- Whitehead, Russell,
- Hilbert

more about this in logicomix

Hilbert

- formalism (philo of maths)

  " It calls for a formalization of all of mathematics in axiomatic form, together with a proof that this axiomatization of mathematics is consistent."

Russell & Whitehead

- Principia Mathematica
- apparently takes 300 pages to prove that one plus one equals one
- i haven't read it

Russel's paradox

- 1901
- R = set of of all sets that are not member of themselves
- If R is not a member of itself, then it is a member of itself !
- If R is a member of itself, then it cannot be a member of itself!

Recursion, self-reference

More on this with the lambda calculus talk ?

The liar's paradox

'This statement is false.'

- if it is false, then it is true
- if it is true, then it is false

what is the theorem, actually ?

from the introduction: " these two systems (PM, ZF) are so extensive that all methods of proof used in mathematics today have been formalized int hem, i.e. reduced to a few axioms and rules of inference."

what is the theorem, actually ?

“It may therefore be surmised that these axioms and rules of inference are also sufficient to decide *all* mathematical questions which can in anyway at all be expressed formally in the systems concerned.”

what is the theorem, actually ?

"It is shown below that this is not the case, and that in both the systems mentioned there are in fact relatively simple problems in the theory of ordinary whole numbers which cannot be decided from the axioms."

what is the theorem, actually ?

"This situation is not due in some way to the special nature of the systems set up, but holds for a very extensive class of formal systems, including, in particular, all those arising from the addition of a finite number of axioms to the two systems mentioned."

the first theorem of incompleteness

in short: “if a set of axioms is consistent, then it is incomplete.”

the second theorem of incompleteness

in short: “that no set of axioms can prove its own consistency”

complete

complete: every statement (or its negation) in the system are provable by the same system.

consistent

there are no contradictions.

the core of the proof: Gödel's numbering system

to all symbols in first order logic, we assign a number.

then, we assign a number that is composed in the following way:

we take the numbers assigned previously and use them as powers of prime numbers

this is the Gödel number.

the core of the proof: a statement about prime numbers as factors

since we know that there exists only one prime factorization for any given number, we know that the Gödel number for a given statement is unique.

the core of the proof: building a true statement that cannot be proved

with substitution

- we build a formula
- we then have a Gödel number for this formula
- we can then substitute this number into the formula ifself.

the core of the proof: building a true statement that cannot be proved

example: "The formula with Gödel number sub(y, y, 17) cannot be proved."

- This new formula has a number!
- let's say this number is n
- now let's create a new formula
- we substitude the y in the previous formula by the number n

the core of the proof: building a true statement that cannot be proved

- new formula: “The formula with Gödel number sub(n, n, 17) cannot be proved.”
- this last formula has a number: it must be sub(n,n,17), by definition: “sub(n, n, 17) is the Gödel number of the formula that results from taking the formula with Gödel number n and substituting n anywhere there's a symbol with Gödel number 17.”
- this is exactly this last formula!
- so G is talking about itself. . .

can G be proved?

" If so, this would mean there's some sequence of formulas that proves the formula with Gödel number sub(n, n, 17). But that's the opposite of G, which says no such proof exists. Opposite statements, G and ∼G, can't both be true in a consistent axiomatic system. So the truth of G must be undecidable."

last note

“However, although G is undecidable, it’s clearly true. G says, “The formula with Gödel number sub(n, n, 17) cannot be proved,” and that’s exactly what we’ve found to be the case! Since G is true yet undecidable within the axiomatic system used to construct it, that system is incomplete."

conclusion

note on circularity, from Gödel: “in spite of appearances, there is nothing circular abou tusch a proposition, since it egins by asserting the unprovability of a wholly determinate formula [. . .], and only subsequently (and in some way by accident) does it emerge that this formula is precisely that by which the proposition was itself expressed.”

going further

- second-order logic ?
- is mathematics invented or discovered?
- is the proof circular ?
- obtaining non-trivial examples of undecidability
- proof-checker systems
- AI

resources

the article, translated in English

quanta magazine

logicomix

Nagel and Newman book

article in Nature about this book