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Theorem

A modal logic L containing **S4** enjoys projective unification if and only if $S4.3 \subseteq L$.

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- (i) Each finitary consequence operation Cn extending **\$4.3** has a finite basis;
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We provide an uniform basis for all non-passive admissible rules of any $L \in NExt(S4.3)$ consisting of infinitary rules like

$$\frac{\{\Box(\alpha_i \leftrightarrow \alpha_j) \to \alpha_0 : 0 < i < j\}}{\alpha_0}$$

Modal Logic

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Let Var be the set of *propositional variables* and Fm be the set of *modal formulas* in $\{\rightarrow, \perp, \square\}$. For each formula α , let $Var(\alpha)$ denote the (finite) set of variables occurring in α .

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By a *modal logic* we mean any proper subset of *Fm* closed under substitutions, closed under

$$MP: \frac{\alpha \to \beta, \alpha}{\beta}$$
 and $RG: \frac{\alpha}{\Box \alpha}$,

containing all classical tautologies, and

$$\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta).$$

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Each modal logic is an extension of the logic ${\bf K}$ with axiom schemata. ${\bf S4}$ extends ${\bf K}$ with

 $\begin{array}{ll} (\mathcal{T}): \Box \alpha \to \alpha & \text{oraz} & \textbf{(4)}: \Box \Box \alpha \to \Box \alpha. \text{ The logic $\textbf{54.3}$ contains additionally } \Box (\Box \alpha \to \Box \beta) \vee \Box (\Box \beta \to \Box \alpha); \text{ and $\textbf{55}$ is axiomatized with (5)}: \Diamond \Box \alpha \to \Box \alpha. \end{array}$

Given a modal logic L, we define its global entailment relation Cn_L . Thus, $\alpha \in Cn_L(X)$ means that α can be derived from $X \cup L$ using the rules MP and RG.

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By a modal consequence operation we mean any structural consequence operation Cn which extends $Cn_{\mathbf{K}}$. Thus, Cn may be given by extending a modal logic with some inferential rules.

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Modal algebra $\mathcal{A}=(A,\to,\bot,\Box)$ is an extension of a Boolean algebra (A,\to,\bot) with a monadic operator operator \Box (1) $\Box\top=\top;$ (2) $\Box(a\wedge b)=\Box a\wedge\Box b,$ for $a,b\in A$.

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Matrix consequences

Each modal algebra \mathcal{A} generates modal consequence operation $\overrightarrow{\mathcal{A}}$:

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*S*4.3-models

Topological Boolean Algebras (TBA's) are modal algebras which are models for **S4**; i.e., they fulfill

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Any (Kripke) structure $\mathfrak{F} = (V, R)$ consists of a non-empty set V and a binary relation R on V. Each $\mathfrak{F} = (V, R)$ determines a modal algebra \mathfrak{F}^+ , on the power set A = P(V), with $\Box a = \{x \in V : R(x) \subseteq a\}$, where $R(x) = \{y \in V : xRy\}$.

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A subset C of V is called a *cluster* if xRy and yRx, for each $x,y \in C$. Symbols 1, 2 and 3, etc., stand for 1- , 2- and 3-element clusters, which are **S5** models. Then \mathfrak{n}^+ is called a *Henle Algebra*.

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Finite subdirectly irreducible **S4.3**-algebras are *TBA*'s in which all open elements (i.e. such a's that $\Box a = a$) form a chain. They coincide with algebras \mathfrak{F}^+ for finite *quasi chains* \mathfrak{F} which are, in turn, determined by *lists*, i.e. finite sequences of positive integers.

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The following conditions are equivalent for each $L \supseteq S4$: (i) α is L-unifiable; (ii) α has a ground unifier w L; (iii) α is satisfiable in 2; (iv) $\sim \alpha \notin Tr = Log(2)$.

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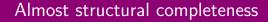
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Passive rules over **S4.3**

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Let $n \ge 0$ and let us consider the sublanguage of Fm spanned on p_1, \ldots, p_n . There are 2^n (Boolean) atoms there and each of them can be represented by the formula

$$p_1^{\sigma(1)} \wedge \cdots \wedge p_n^{\sigma(n)}$$

where $\sigma: \{1, ..., n\} \to \{0, 1\}$, and $p^0 = p$, and $p^1 = p$. Suppose we denote these atoms by: $\theta_1, ..., \theta_{2^n}$

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Theorem

Each finitary consequence operation $Cn \in \mathrm{EXT}(\mathbf{S4.3})$ is an extension of some Cn_L (for $L \supseteq \mathbf{S4.3}$ and some $n \ge 0$) with a finite number of passive rules

$$\frac{\Diamond \theta_1 \wedge \dots \wedge \Diamond \theta_s}{\alpha}$$

where $2 \le s \le 2^n$ and $Var(\alpha) \cap \{p_1, \ldots, p_n\} = \emptyset$.

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(ii)The structurally complete extension of Cn_L is determined by

$$\mathbb{K} \times \mathbf{1}^+ = \{ \mathcal{B} \times \mathbf{1}^+ : \mathcal{B} \in \mathbb{K} \}.$$

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Since ϱ is **S4.3**-admissible and ε is a unifier for any $\Box(p_i \leftrightarrow p_j) \rightarrow p_0$ with 0 < i < j, we get $\varepsilon(p_0) \in$ **S4.3**. Thus, we obtain (for $\beta = p_0$)

$$p_0 \in Cn_{S4.3}(\{\Box(p_i \leftrightarrow p_j) \rightarrow p_0 : 0 < i < j\})$$

which is impossible. Consequently, the set $\{\Box(p_i \leftrightarrow p_j) \rightarrow p_0 : 0 < i < j\}$ cannot have a projective unifier in **S4.3** (and any weaker logic).

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Infinitary Admissible Rules

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Let Σ be the family of all strictly increasing number-theoretic functions. For each function $f \in \Sigma$, let r_f be

$$\frac{\{[\bigwedge_{n < j \le f(n)} \bigvee_{0 < i \le n} \Box(p_i \leftrightarrow p_j)] \to p_0 : n > 0\}}{p_0}$$

Example

Let f(n) = n + 1 for each n. Then r_f is equivalent to the rule ϱ :

$$\frac{\{\Box(p_i \leftrightarrow p_j) \to p_0 : 0 < i < j\}}{p_0}$$

Example

Let f(n) = n + 2 for each n. Then r_f is equivalent to

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Thus, similarly as ϱ each rule r_f is admissible for any extension of **S4.3**. For any consequence operation Cn, let Cn^{Σ} be the extension of Cn with the rules $\{r_f\}_{f\in\Sigma}$. We prove that $\{r_f\}_{f\in\Sigma}$ is a rule basis for all admissible non-passive (infinitary) rules of any $Cn\in \mathrm{EXT}(\mathbf{S4.3})$:

Note that the rules r_f are unifiable, i.e. they are non-passive as p_0/\top is a unifier for each of the premises.

Lemma

The rule r_f , for any $f \in \Sigma$, is valid in any finite algebra A, i.e.

$$p_0 \in \overrightarrow{\mathcal{A}}(\{[\bigwedge_{n < j < f(n)} \bigvee_{0 < i \le n} \Box(p_i \leftrightarrow p_j)] \rightarrow p_0 : n > 0\})$$

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Theorem

$$Cn_{fin}^{\Sigma} = Cn^{\Sigma} = Cn^{ASCpl} =_{fin} Cn$$
, for any $Cn \in EXT(\mathbf{S4.3})$.

Corollaries

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It also follows from the above result that almost structurally complete consequence operations are hereditary almost structurally complete, i.e.

Corollary

If
$$Cn, Cn' \in \mathrm{EXT}(\mathbf{S4.3})$$
, then

$$Cn \in ASCpl \land Cn \leq Cn' \Rightarrow Cn' \in ASCpl.$$

$$Cn \in FA \land Cn \leq Cn' \Rightarrow Cn' \in FA$$
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Corollary

If
$$Cn$$
, $Cn' \in \mathrm{EXT}(\mathbf{S4.3})$, then

$$\mathit{Cn} \in \mathit{ASCpI} \land \mathit{Cn} \leq \mathit{Cn'} \Rightarrow \mathit{Cn'} \in \mathit{ASCpI}.$$

$$Cn \in FA \land Cn \leq Cn' \Rightarrow Cn' \in FA$$
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The lattice ASCpl(**S4.3**)

The lattice ASCpI(S4.3)

Corollary

ASCpl(\$4.3) is a sublattice of $\mathrm{EXT}(\$4.3)$. The lattices ASCpl(\$4.3) and $\mathrm{EXT}_{fin}(\$4.3)$ are isomorphic and as the lattice isomorphisms one can take the mappings $Cn\mapsto Cn^\Sigma$ and $Cn\mapsto Cn_{fin}$.

The lattice $ASCpl(\mathbf{S4.3})$

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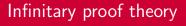
The consequence operation $\overrightarrow{\mathbb{K}}$, where $\mathbb{K}=\mathrm{Alg}(\mathbf{S4.3})_{si\,fin}$, is the least element of the lattice $ASCpl(\mathbf{S4.3})$. This consequence operation can also be given as the ASCpl extension of $\mathbf{S4.3}$, that is as the extension of $Cn_{\mathbf{S4.3}}$ with the rules $\{r_f\}_{f\in\Sigma}$.

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Corollary

For each formula $\alpha \in Cn^{\Sigma}(X)$ we have:

- **1.** either a finitary proof (of α) in which we apply MP, RG and some finitary passive rules valid for Cn (with respect to some formulas in $L \cup X$);
- **2.** or α is given by a single application of one of the rules r_f with respect to formulas which have finitary proofs as above.

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In other words each formula $\alpha \in \mathit{Cn}^\Sigma(X)$ has a syntactic proof of the type $\leq \omega + 1$. Although the rules r_f , for $f \in \Sigma$, look artificial they generate a natural (and simple) proof system. The only problem is that the rule basis $\{r_f \colon f \in \Sigma\}$ has the cardinality \mathfrak{c} . We suspect that $\mathit{Cn}^\Sigma_{\mathbf{54}.3}$ has no countable rule basis though we failed to prove it and left this problem open.