Hereditary Structural Completeness: the Case of Intermediate Logics

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Outline

Introduction

We will consider the (finitary structural) consequence relations defining intermediate logics and we understand logics as sets of formulas closed under substitutions and modus ponens. We recall that a consequence relation $\vdash$ defining a logic L is *structurally complete* ($\mathcal{SC}$) if every proper extension of $\vdash$ defines a proper extension of L. A logic L is *structurally complete* if a consequence relation defined by the axioms of L and modus ponens is structurally complete.

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Definition

A consequence relation $\vdash$ is said to be *hereditarily structurally complete* ($\mathcal{HSC}$) if $\vdash$ and all its extensions are structurally complete. Logic L is $\mathcal{HSC}$ if L and all its extensions are $\mathcal{SC}$.

Introduction

The notion of hereditary structural completeness ($\mathcal{HSC}$) for intermediate logics was introduced by Citkin [Citkin, 1978], where the following *HSC*-criterion had been proven

Theorem

An intermediate logic L *is* $\mathcal{HSC}$ *if and only if none of the following algebras is a model of* L*:*

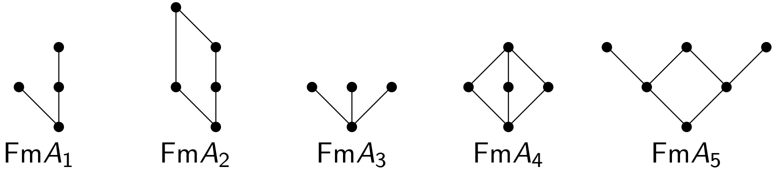

Fig. 1

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In this presentation, we will be concerned with $\mathcal{HSC}$ consequence relations in intermediate logics.

Introduction

Let $\mathscr{L}$ be a class of all intermediate $\mathcal{HSC}$ logics, and $\mathscr{R}$ be a class of all intermediate $\mathcal{HSC}$ consequence relations. The properties of the classes $\mathscr{L}$ and $\mathscr{R}$ are very different:

	Property	$\mathscr{L}$	$\mathscr{R}$
1	Has a smallest element	Yes	No
2	Is countable	Yes	No
3	All members are f. axiomatizable	Yes	No
4	All members are locally tabular	Yes	No (may not have the fmp)

Visser Rules

Definition

Given a logic L, by L° we denote its *structural completion* (or admissible closure [Rybakov, 1997]): the greatest consequence relation having L as a set of theorems.

For instance, $\mathbf{Int}^\circ$ is a consequence relation defined by IPC endowed with Visser rules.

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For instance, **Int**$^\circ$ is a consequence relation defined by IPC endowed with Visser rules.

It was independently observed by both Citkin (1979) and Visser (at the beginning of the 1980s) that the following rules (known as *Visser rules*) are admissible in the intuitionistic propositional logic:

$$V_n := \bigwedge_{j=1}^{n}(A_j \to B_j) \to (A_{n+1} \vee A_{n+2}) \vee D / \bigvee_{i=1}^{n+2}(\bigwedge_{j=1}^{n}(A_j \to B_j) \to A_i) \vee D.$$

Visser Rules

The hereditary structural completeness also has another meaning:

Theorem

Let L *be a logic and* R *be the set of all rules admissible in* L*. The structural completion* L° *is* $\mathcal{HSC}$ *if and only if the rules* R *form a basis of admissible rules for every extension of* L *where all rules* R *are admissible.*

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Corollary

$\mathbf{Int}^\circ$ *is hereditarily structurally complete.*

Properties of $\mathscr{R}$

1. $\mathbf{Int}^\circ$ is minimal in $\mathscr{R}$. Logic L_9 (of single-generated 9-element Heyting algebra) satisfies the $\mathcal{HSC}$-criterion. But Visser rules are not admissible in L_9. Hence $\mathbf{Int}^\circ$ and L_9° are incomparable and $\mathscr{R}$ does not have the smallest element (so, $\mathscr{R}$is not a lattice).

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4. **Int**$^\circ$ does not have the fmp: from [Citkin, 1977] it follows that formula

$$((p \to q) \to (p \lor r)) \to (((p \to q) \to p) \lor ((p \to q) \to r))$$

is not valid in **Int**, but it is valid nevertheless in all finite models of **Int**$^\circ$.

Properties of $\mathscr{R}$

Moreover, the following theorem is an immediate consequence of [Citkin, 1977]:

Theorem

If an intermediate logic L *admits* V_1 *and* L° *has the fmp, then* L *is hereditarily structurally complete.*

Recall that there is just countably many $\mathcal{HSC}$ intermediate logics [Citkin, 1978], and there is continuum many extensions of $\mathbf{Int}^\circ$ [Rybakov, 1993].

Corollary

There is continuum many $\mathcal{HSC}$ *consequence relations without fmp.*

Outline

Primitive Quasivarieties

A propositional logic L is algebraizable (in sense of Blok and Pigozzi), if with L we can associate a variety of algebras. Accordingly, with a given algebraizable consequence relation we can associate a quasivariety. And a consequence relation is $\mathcal{HSC}$ if and only if the corresponding quasivariety $\mathbf{Q}$ is *primitive*, that is any proper subquasivariety of $\mathbf{Q}$ can be defined over $\mathbf{Q}$ by identities.

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Theorem ([Gorbunov, 1976])

Any subquasivariety of a primitive quasivariety is primitive. A class of subquasivarieties of a given primitive quasivariety forms a distributive lattice.

Corollary

Every extension of a given $\mathcal{HSC}$ consequence relation is $\mathcal{HSC}$.

Weakly **Q**-Projective Algebras

Definition

Let $\mathbf{Q}$ be a quasivariety and $\mathscr{A} \in \mathbf{Q}$ be an algebra. Algebra $\mathscr{A}$ is $\mathbf{Q}$-*irreducible* if $\mathscr{A}$ is not a subdirect product of algebras of $\mathbf{Q}$ that are not isomorphic to $\mathscr{A}$.

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Proposition

Every weakly $\mathbf{Q}$*-projective algebra is embedded into a free algebra of quasivariety* $\mathbf{Q}$*.*

Weakly **Q**-Projective Algebras

The following theorem gives a simple sufficient condition of primitivity.

Theorem ([Gorbunov, 1976])

If all finitely generated $\mathbf{Q}$*-irreducible algebras of a quasivariety* $\mathbf{Q}$ *are weakly* $\mathbf{Q}$*-projective, then* $\mathbf{Q}$ *is primitive.*

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In case of the locally finite varieties, the sufficient condition is also necessary.

Theorem ([Gorbunov, 1976])

A locally finite quasivariety $\mathbf{Q}$ is primitive if and only if every of its finite $\mathbf{Q}$-irreducible algebra is weakly $\mathbf{Q}$-projective.

Totally Non-Projective Algebras

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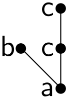
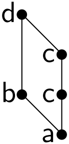
The five algebras from the $\mathcal{HSC}$-criterion are totally non-projective.

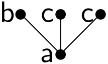
Totally Non-Projective Algebras



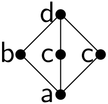
FmA_1



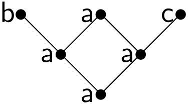
FmA_2



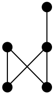
FmA_3

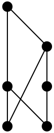


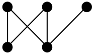
FmA_4

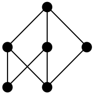


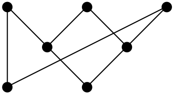
FmA_5

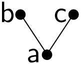

FmA_1'


FmA_2'

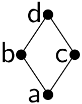

FmA_3'


FmA_4'


FmA_5'



FmC



FmC'

Outline

Structural completions of Extensions of $\mathbf{Int}^\circ$

Since $\mathbf{Int}^\circ$ is $\mathcal{HSC}$, and every extension of an $\mathcal{HSC}$ consequence relation is $\mathcal{HSC}$, we have

Theorem

The structural completions of the following logics are hereditarily structurally complete:

(a) $\mathsf{G_n}$ - *Gödel logics*

(b) KC - *Yankov logic*

(c) LC - *Gödel - Dummett logic*

(d) P - *logic of projective algebras*

(e) RN - *logic of Rieger-Nishimura ladder*

(f) Sm - *Smetanich logic.*

Structural completions of RN_n

In order to demonstrate that a structural completion of a given logic is not $\mathcal{HSC}$ it is enough to show that the consequence relation defined by corresponding cyclic (single-generated) free algebra is not $\mathcal{HSC}$. n-element cyclic algebra we denote by $\mathsf{Z}_n \quad n = 1, 2, \ldots$.

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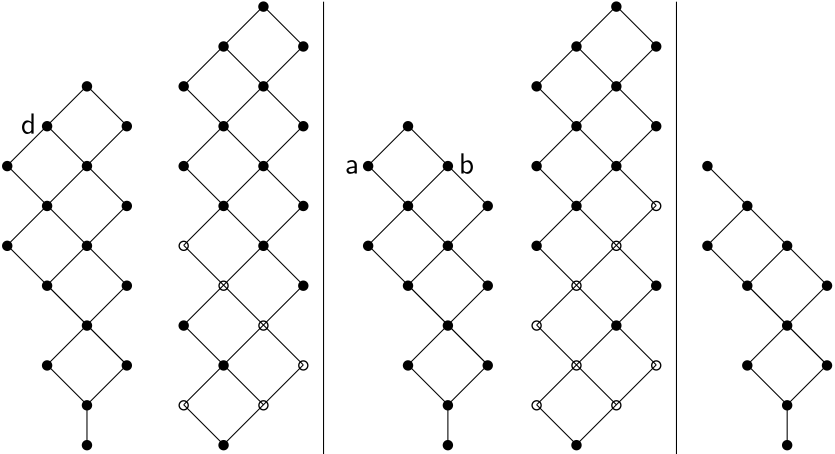
Cardinalities of cyclic free algebras of RN_n for $n \leq 10$ are presented in the following table. For all $n \geq 8$, the cyclic free algebra of RN_n is isomorphic to Z_n.

n	2	3	4	5	6	7	8	9	10
Card.	4	6	4	8	6	10	8	9	10

Theorem

For every $n < 11$, the structural completions RN_n° is $\mathcal{HSC}$. For every $k >= 5$, the structural completion of RN_{2k+1} is not $\mathcal{HSC}$. For every $k > 10$, the structural completion of RN_{2k} is not $\mathcal{HSC}$.

Structural completions of RN_n



$Z_2 \oplus Z_4 \oplus Z_{12}$ $\quad Z_{22} \quad$ $Z_2 \oplus Z_4 \oplus Z_{10}$ $\quad Z_{22} \quad$ $Z_2 \oplus Z_4 \oplus Z_7$

Structural completions of BD_n

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Let L *be a locally tabular intermediate logic. If* Z_{22} *is a model of* L*, then* L° *is not* $\mathcal{HSC}$*.*

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By $\mathsf{BD}_n, n = 1, 2, 3, \ldots$ we denote the logic of frames of depth at most $n + 1$ and by $\mathbf{BD}_n$ – a corresponding variety of Heyting algebras.

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By $\mathrm{BD}_n, n = 1, 2, 3, \dots$ we denote the logic of frames of depth at most $n + 1$ and by $\mathbf{BD}_n$ – a corresponding variety of Heyting algebras.

Since Z_{4n} is a free cyclic algebra of $\mathbf{BD}_n$, the following holds.

Theorem

Structural completions of BD_n *for all* $n > 5$ *are not* $\mathcal{HSC}$*.*

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Theorem

Structural completions of BD_n for all $n > 5$ are not $\mathcal{HSC}$.

Note that the structural completion of every logic admitting all Visser rules is $\mathcal{HSC}$. On the other hand, for all $n > 5$ the logics BD_n admit restricted Visser rules, and their structural completions are not $\mathcal{HSC}$.

Structural completions in intermediate logics

L	Description	L is $\mathcal{HSC}$	L° is $\mathcal{HSC}$
Int	(intuitionistic logic)	No	Yes
$\mathsf{BD_n}$	(depth at most n)	No for $n > 1$	No for $n > 5$
$\mathsf{D_n}$	(Gabbay - de Jongh)	No for $n > 2$	?
$\mathsf{G_n}$	(Gödel logics)	Yes	Yes
KC	(Yankov logic)	No	Yes
KP	(Kreisel-Putnam logic)	No	?
LC	(Gödel - Dummett logic)	Yes	Yes
$\mathsf{M_n}$	(at most n maximal nodes)	No for $n > 2$	?
ML	(Medvedev logic)	No	No
P	(logic of projective algebras)	Yes	Yes
RN	(logic of Z)	No	Yes
RN_n	(logic of Z_n)	No for $n = 7$ and $n > 9$	No for $n > 21$
Sm	(Smetanich logic)	Yes	Yes

Open Problems

1. Are there distinct from **ML** structurally complete logics which are not *HSC*?
2. Can a not finitely axiomatizable logic have a finitely axiomatizable structural completion?
3. Is there a weakly **Q**-projective algebra (in a quasivariety **Q**) that is not **Q**-projective?

Thanks

Thank you for your patience and kind attention.

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