Uniform Interpolation and the Congruence Lattice

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Joint work with George Metcalfe and Constantine Tsinakis

Workshop on Admissible Rules and Unification 2

31 January – 2 February 2015, Les Diablerets

Uniform interpolation in IPC

Theorem (Pitts, 1992)

For any formula $\phi(\overline{x}, \overline{y})$ *of intuitionistic propositional logic* IPC*, there exist* **left** *and* **right uniform interpolants***,*

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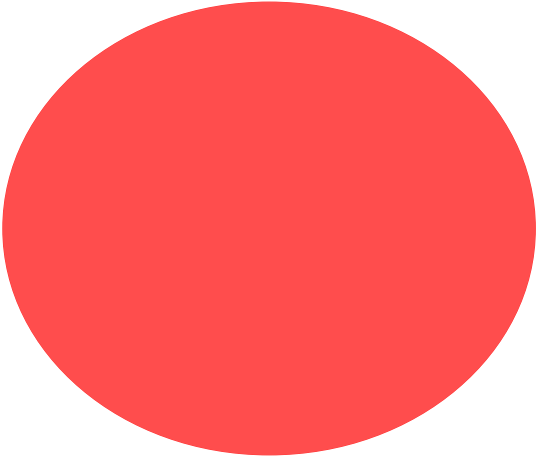
(*) because IPC has interpolation

Right uniform restriction, pictorially

$\mathbf{Fm}(\overline{x}, \overline{y})$



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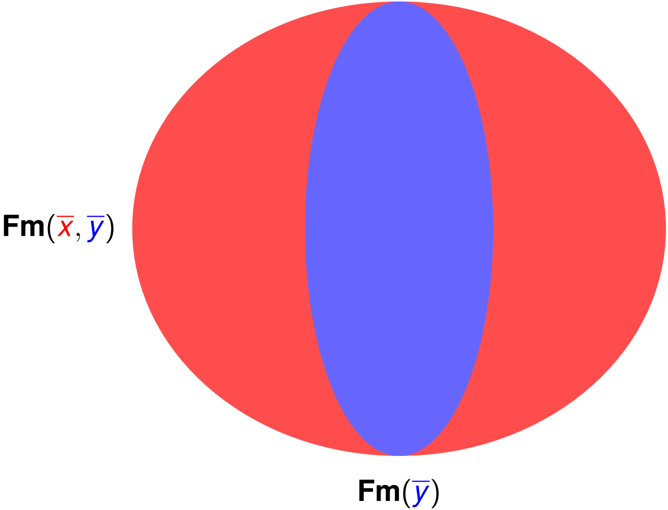

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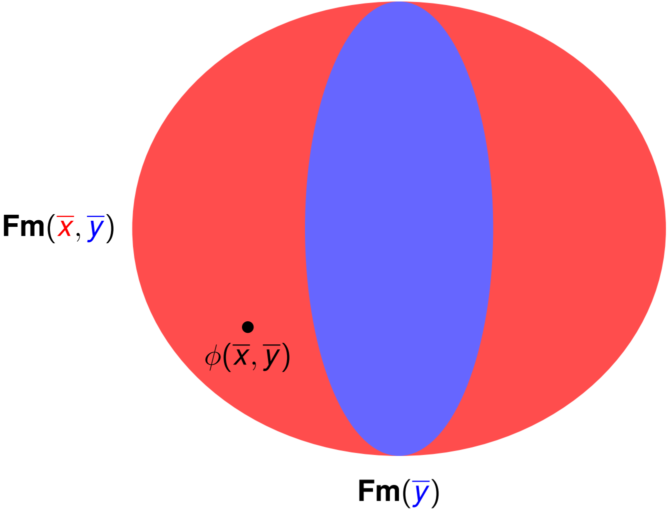

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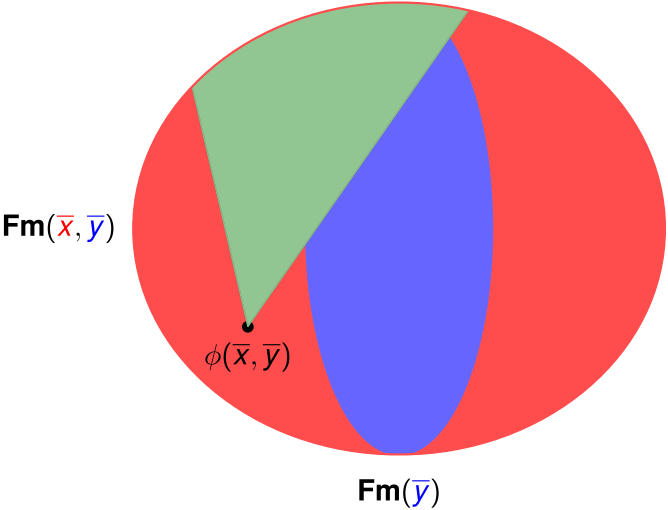

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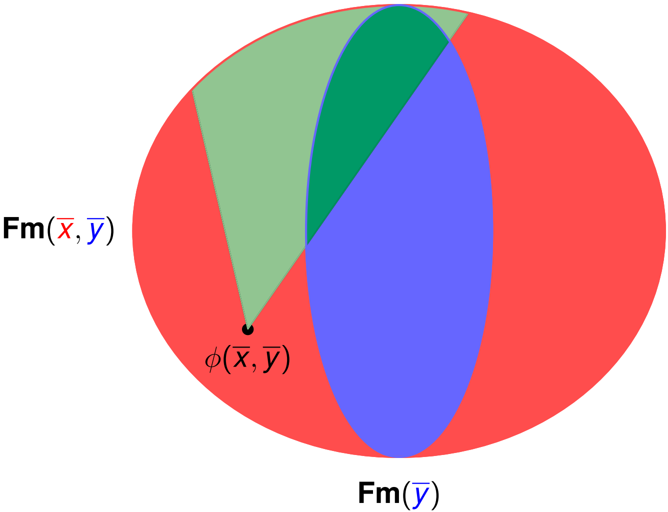

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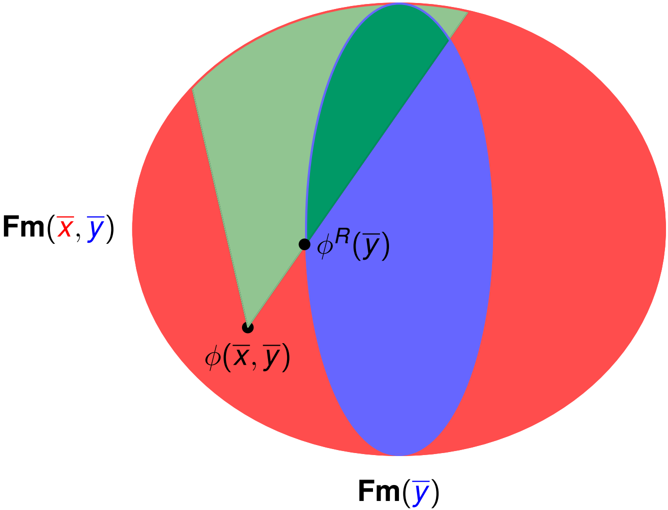

This Talk

Which **varieties of algebras** admit uniform interpolation?

Equational Consequence

The **equational consequence relation** for a variety $\mathcal{V}$ is defined by

$$\Sigma \models_{\mathcal{V}} \alpha \approx \beta \qquad \Leftrightarrow \qquad \begin{array}{c} \text{for all } \mathbf{A} \in \mathcal{V} \text{ and } e : \mathbf{F}_{\mathcal{V}}(\omega) \to \mathbf{A}, \\ \text{if } \Sigma \subseteq \ker(e) \text{ then } \alpha \approx \beta \in \ker(e). \end{array}$$

We will write $\Sigma \models_{\mathcal{V}} \Delta$ to denote that $\Sigma \models_{\mathcal{V}} \varepsilon$ **for all** $\varepsilon \in \Delta$.

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$$\Sigma(\overline{x}, \overline{y}) \models_{\mathcal{V}} \epsilon(\overline{y}) \iff \Sigma^R(\overline{y}) \models_{\mathcal{V}} \epsilon(\overline{y}).$$

Equations and congruences

A set of equations $\Sigma(\overline{x})$ generates a **congruence**, $\Theta(\Sigma)$, on $\mathbf{F}_{\mathcal{V}}(\overline{x})$, the free $\mathcal{V}$-algebra generated by $\overline{x}$.

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A congruence θ on $\mathbf{F}_{\mathcal{V}}(\overline{x})$ is **finitely generated** if $\theta = \Theta(\Sigma)$ for some finite set Σ.

The congruence lattice

For any algebra **A**, the **lattice of congruences** on **A**, Con(**A**), is a complete lattice.

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We denote by $\mathrm{KCon}(\mathbf{A})$ the join-semilattice of **compact congruences.**

Lifting homomorphisms

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The pair (f_*, f^{-1}) is an **adjunction**, i.e., we have

$$f_*(\theta_A) \subseteq \theta_B \iff \theta_A \subseteq f^{-1}(\theta_B),$$

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This adjunction **restricts to compact congruences** iff f^{-1} preserves compact elements.

Lifting homomorphisms, pictorially

For any $\overline{x}$ and $\overline{y}$ finite, let $i : \mathbf{F}_{\mathcal{V}}(\overline{y}) \hookrightarrow \mathbf{F}_{\mathcal{V}}(\overline{x}, \overline{y})$ denote the natural inclusion map.

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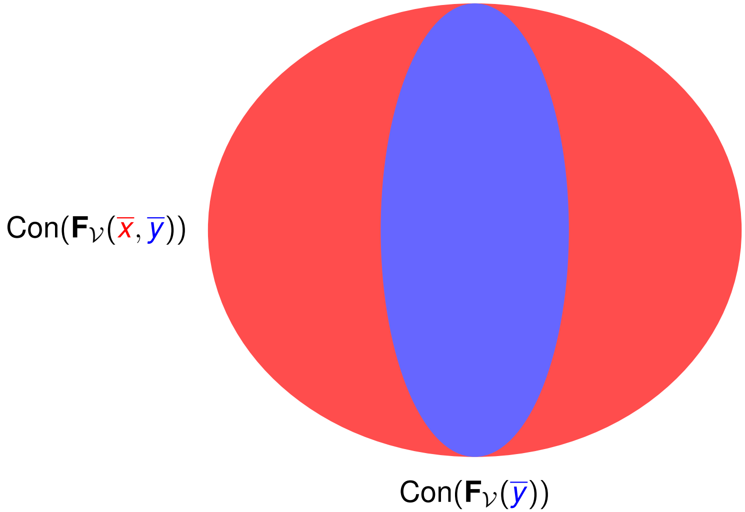

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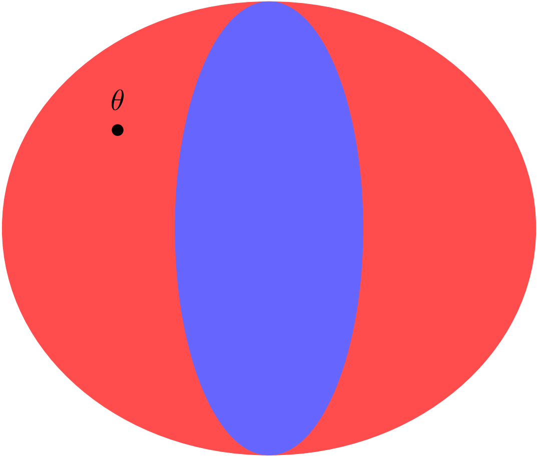

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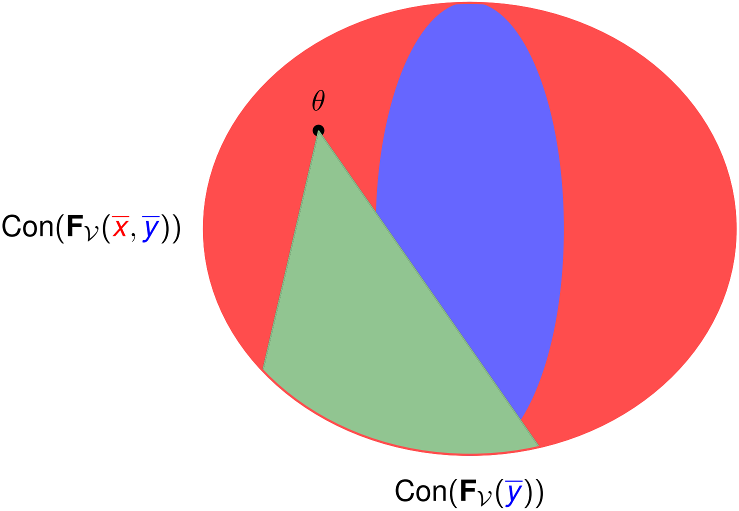

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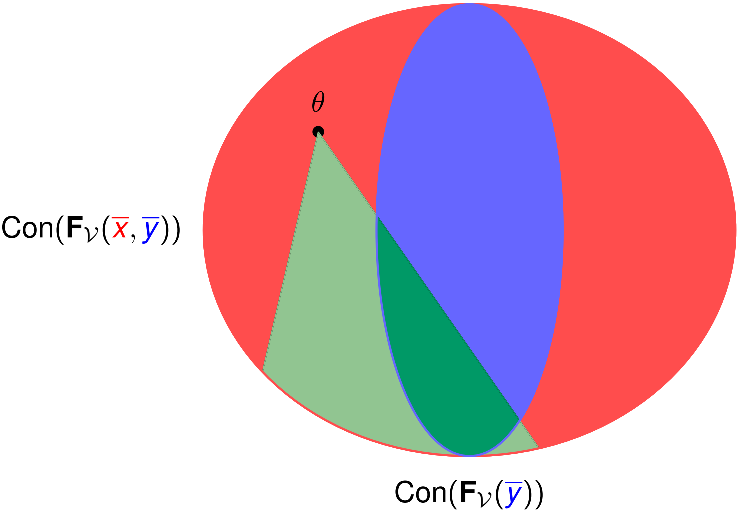

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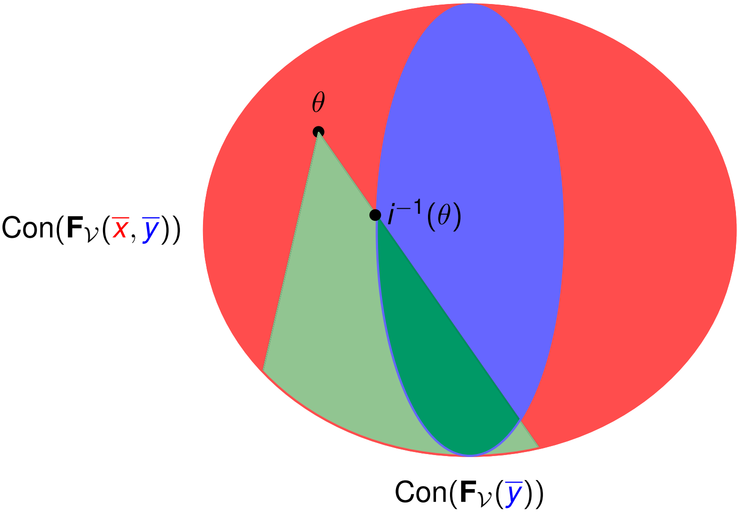

Right uniform restriction and existence of adjoints

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For any variety $\mathcal{V}$, the following are equivalent:

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(3) *For finitely presented $\mathbf{A}, \mathbf{B} \in \mathcal{V}$ and any homomorphism $f \colon \mathbf{A} \to \mathbf{B}$, the map $f_* \colon \mathrm{KCon}(\mathbf{A}) \to \mathrm{KCon}(\mathbf{B})$ has a right adjoint.*

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However, the varieties of **groups** and of **S4-algebras** do not have right uniform restriction.

Left uniform restriction

Definition

$\mathcal{V}$ has **left uniform restriction** iff for any finite set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a finite set of equations $\Sigma^L(\overline{y})$ such that for any set of equations $\Pi(\overline{y})$:

$$\Pi(\overline{y}) \models_{\mathcal{V}} \Sigma(\overline{x}, \overline{y}) \iff \Pi(\overline{y}) \models_{\mathcal{V}} \Sigma^L(\overline{y}).$$

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(2) *$\mathcal{V}$ has left uniform restriction* **and** *for finite $\overline{x}$, the join-semilattice $\mathrm{KCon}(\mathbf{F}_{\mathcal{V}}(\overline{x}))$ is residuated.*

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For example,

$$\Sigma := \{\top \approx ((x \rightarrow z) \wedge (y \rightarrow z)) \rightarrow z\}$$

is a consequence of both $\top \approx x$ and $\top \approx y$, i.e.,

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but there is no $\Delta(x, y)$ satisfying

$$\Delta \models_{\mathcal{ISL}} \Sigma, \quad \{\top \approx x\} \models_{\mathcal{ISL}} \Delta, \quad \text{and} \quad \{\top \approx y\} \models_{\mathcal{ISL}} \Delta.$$

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$\iff$

$\mathcal{V}$ is congruence-distributive **and** *for finite $\overline{x}$, $\overline{y}$,*
i_* : $\mathrm{Con}(\mathbf{F}_{\mathcal{V}}(\overline{x})) \to \mathrm{Con}(\mathbf{F}_{\mathcal{V}}(\overline{x}, \overline{y}))$ *preserves intersections.*

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Uniform interpolation is equivalent to uniform restriction + deductive interpolation, and also to the existence of adjoints for **countably generated** free algebras:

Theorem

The following are equivalent for any variety $\mathcal{V}$:

(1) *$\mathcal{V}$ has right uniform interpolation.*

(2) *For any countable X and $Y \subseteq X$, the natural embedding of* $\mathrm{KCon}(\mathbf{F}_{\mathcal{V}}(Y))$ *into* $\mathrm{KCon}(\mathbf{F}_{\mathcal{V}}(X))$ *has a right adjoint.*

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In particular, under certain conditions (e.g., for varieties of Heyting and modal algebras), uniform interpolation for $\mathcal{V}$ implies the existence of a **model completion** for the first-order theory of $\mathcal{V}$.

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Can we weaken these conditions to cover other classes of algebras, e.g., quasi-varieties, universal classes?

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