Decidability of the Admissible Rules in Intuitionistic Propositional Logic

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Workshop on Admissibility and Unification 2

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A / $\triangle$ admissible

σA is derivable

$$A \;/\; \Delta \text{ admissible}$$

σC is derivable for some $C \in \Delta$

σA is derivable

$$A \mathrel{|\!\sim} \Delta \text{ admissible}$$

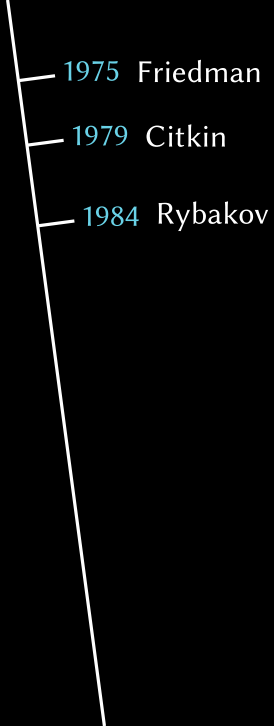
σC is derivable for some $C \in \Delta$

Given a rule A/Δ,
is it admissible?

1975 Friedman

1975 Friedman

1979 Citkin


1975 Friedman
1979 Citkin
1984 Rybakov

1975 Friedman

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1992 Rozière

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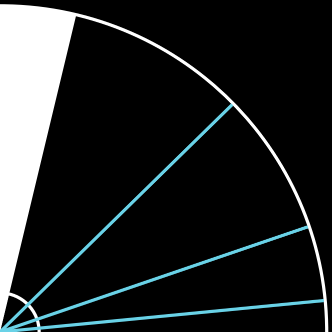
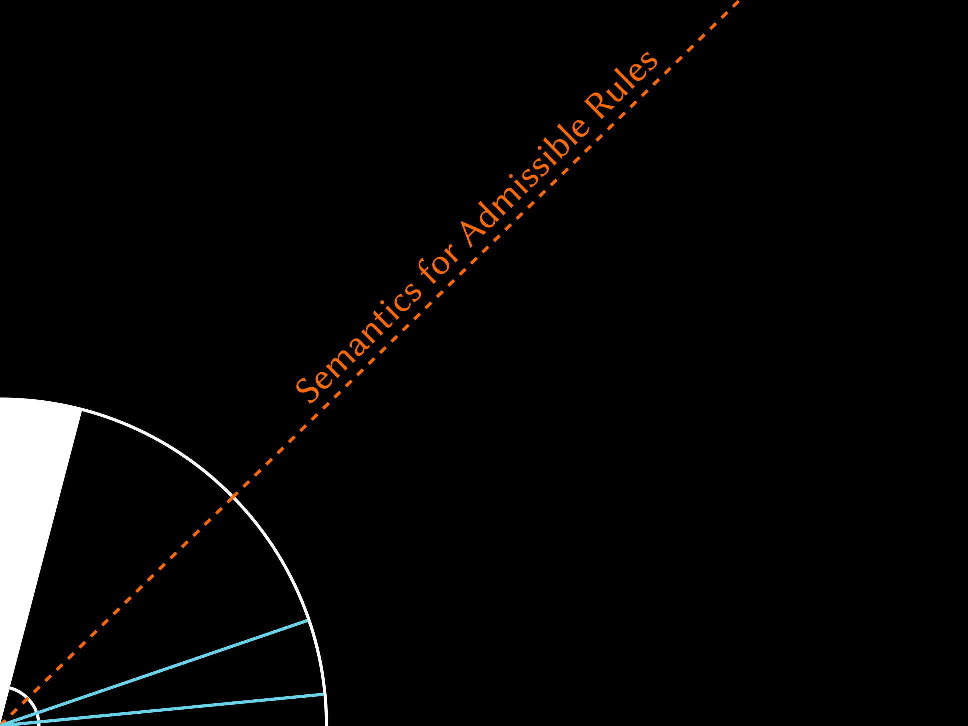
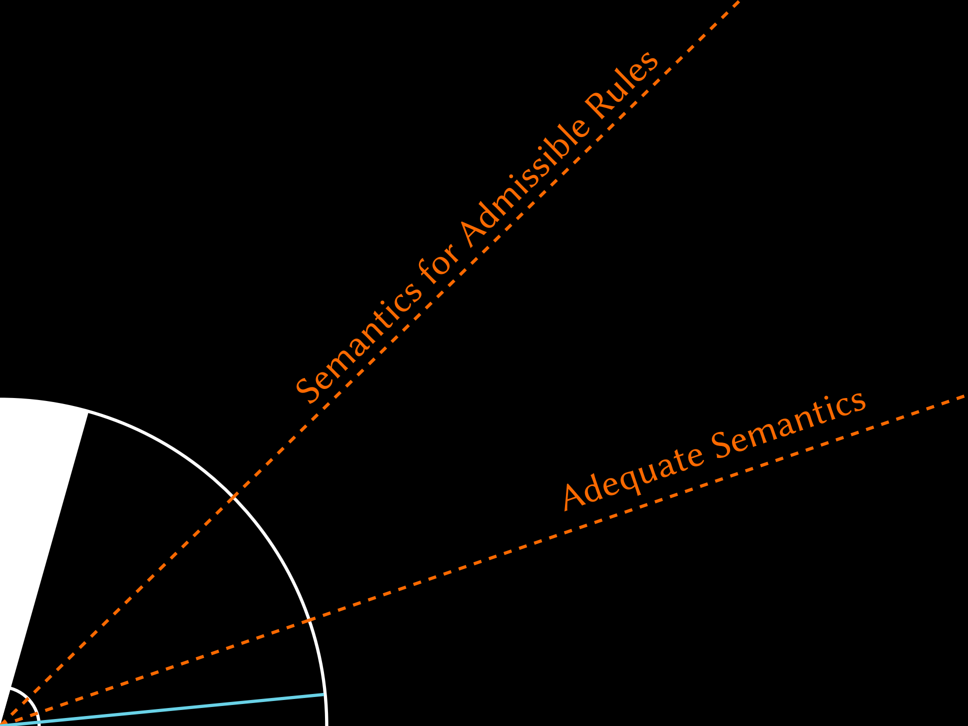
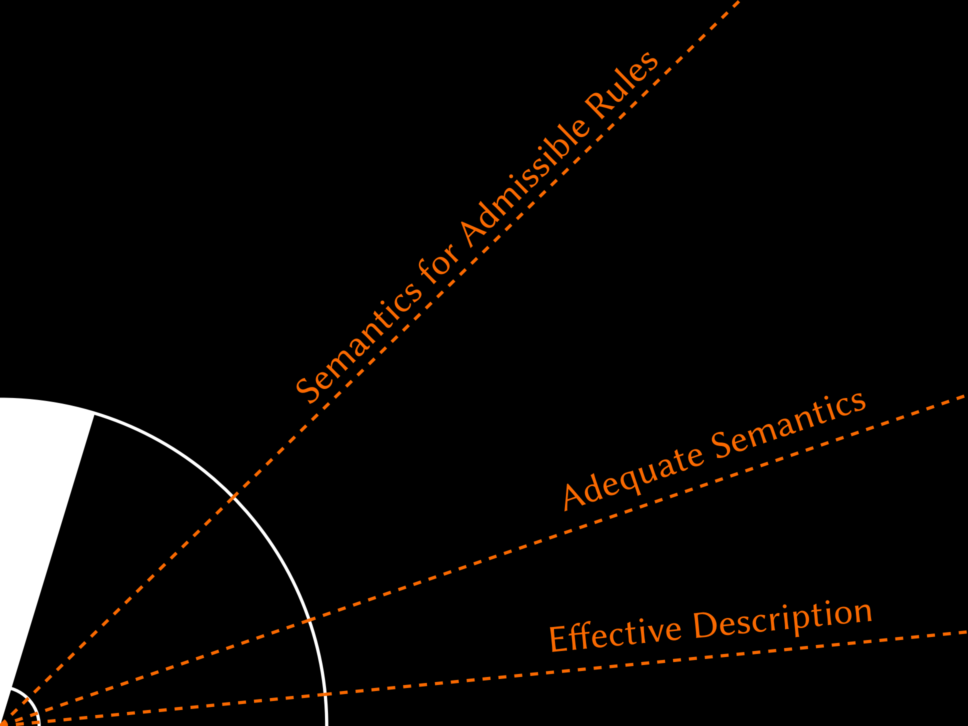
1999 Ghilardi




Semantics for Admissible Rules


Semantics for Admissible Rules
Adequate Semantics



Semantics for Admissible Rules
Adequate Semantics
Effective Description

II

Semantics for Admissible Rules

What is a good notion of semantics for admissibility?

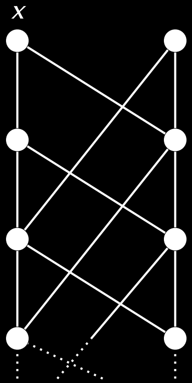
Definition

Say that A/Δ is valid on v, denoted $v \Vdash A/\Delta$, if:

$$v \Vdash A \text{ implies } v \Vdash C \text{ for some } C \in \Delta.$$

Theorem (Rieger, 1949; Nishimura, 1960; Esakia and Grigolia, 1977; Shehtman, 1978; Rybakov, 1984)
For each finite set of variables X, there exists a model $\mathsf{u} : \mathsf{U}(X) \to \mathsf{P}(X)$ *such that:*

$$\mathsf{u} \Vdash A \text{ iff } \vdash A \text{ for all } A \in \mathcal{L}(X).$$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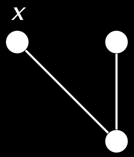
$v \Vdash A/\Delta$ for all $v \in \mathcal{K}$

$A \mathrel{|\!\sim} \Delta$

$$\begin{tikzcd} v \Vdash A/\Delta \text{ for all } v \in \mathcal{K} \arrow[r, bend left, "\text{complete}"] & A \mathrel{|\!\sim} \Delta \end{tikzcd}$$

$$\begin{tikzcd}
\begin{array}{c} v \Vdash A/\Delta \text{ for} \\ \text{all } v \in \mathcal{K} \end{array} \arrow[r, bend left, "\text{complete}"] & A \mathrel{|\!\sim} \Delta \arrow[l, bend left, "\text{sound}"]
\end{tikzcd}$$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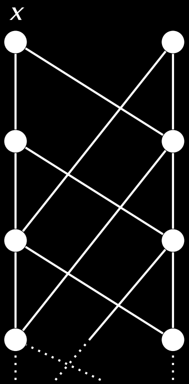
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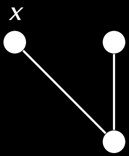
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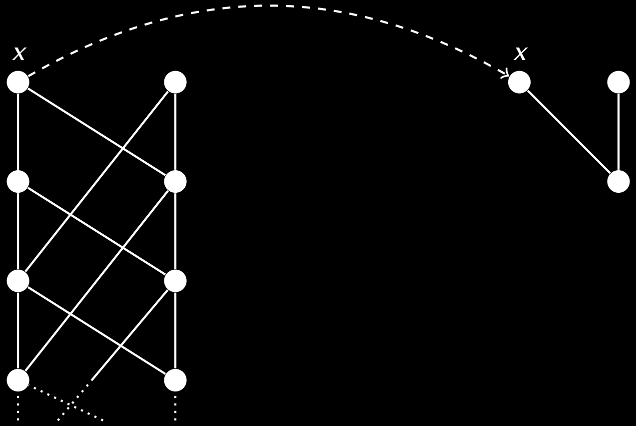

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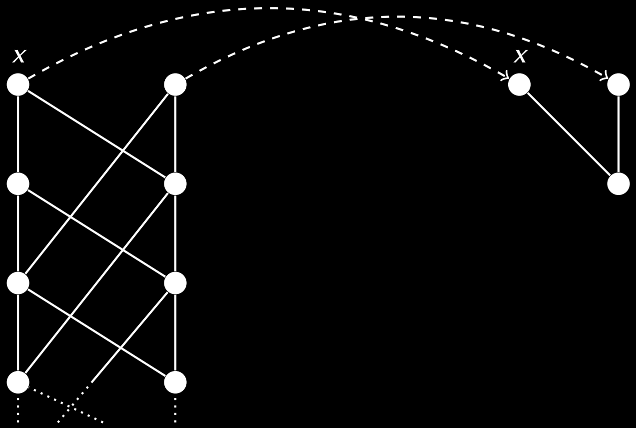
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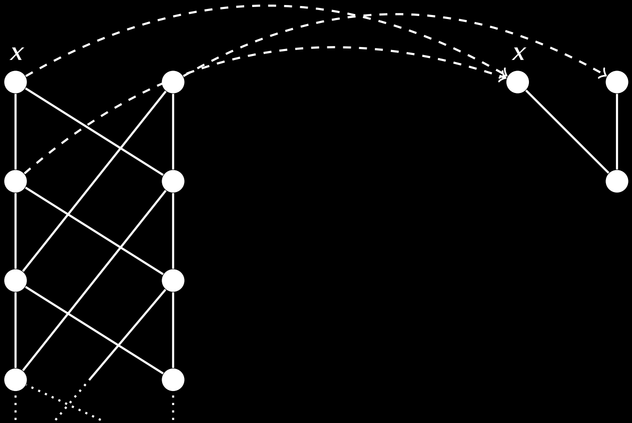

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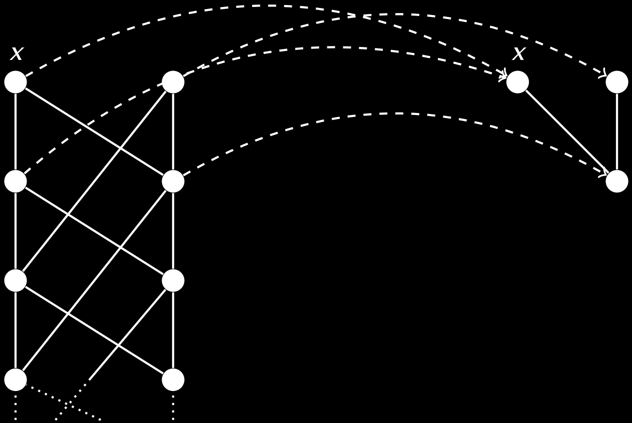
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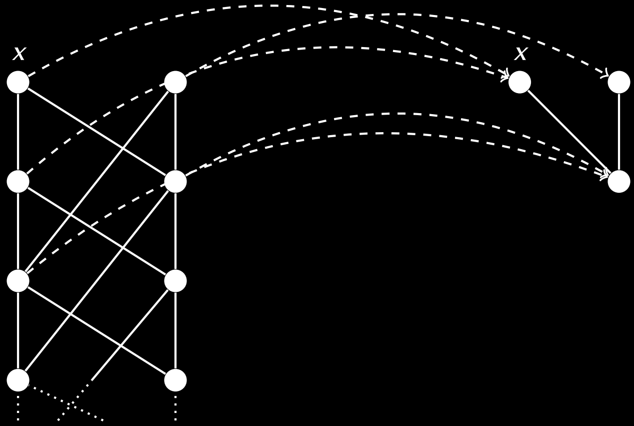
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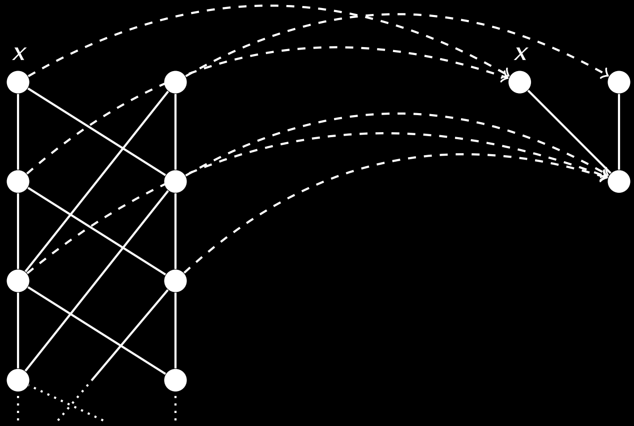
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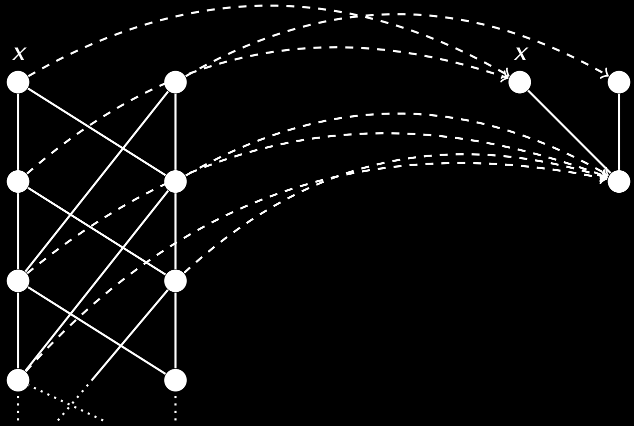
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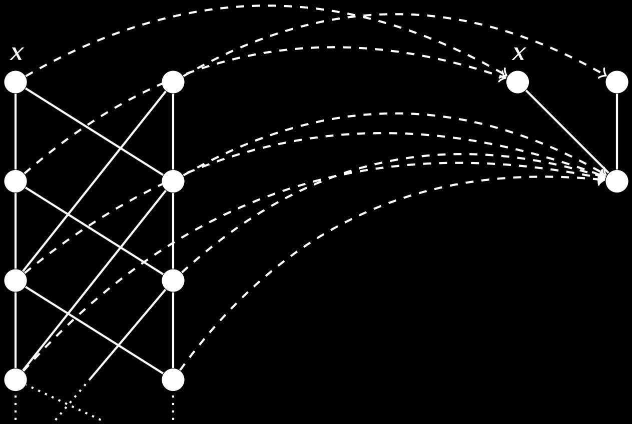
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Definition

A map $f\colon u \to v$ is definable if there is a substitution σ such that:

$$u, p \Vdash \sigma A \text{ iff } v, f(p) \Vdash A \text{ for all } A.$$

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Definition (de Jongh, 1982)

A model v called exact if there exists a definable map u $\to$ v.

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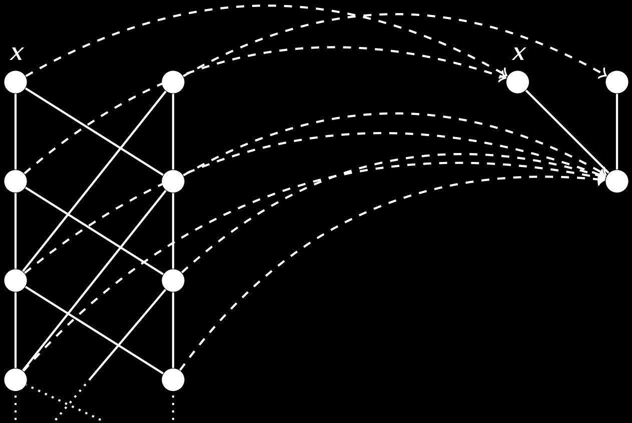
Definition (de Jongh, 1982)

A model v called exact if there exists a definable map u $\to$ v, where u is a universal model.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Theorem

The admissible rules of IPC *are sound and complete with respect to exact models.*

What is a good notion of semantics for admissibility?

What is a good notion of semantics for admissibility?

Exact models!

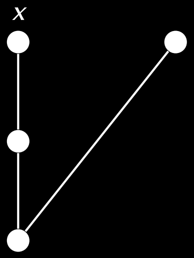
Theorem (Fedorishin and Ivanov, 2003; Goudsmit, 2014b)

The admissible rules of IPC *are not sound and complete with respect to finite exact models.*

III

Adequate Semantics

What is a *fair* notion of semantics for admissibility?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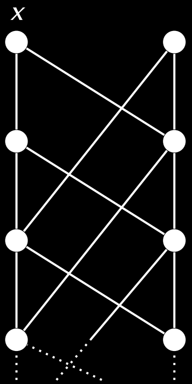
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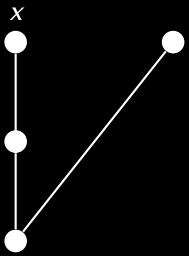
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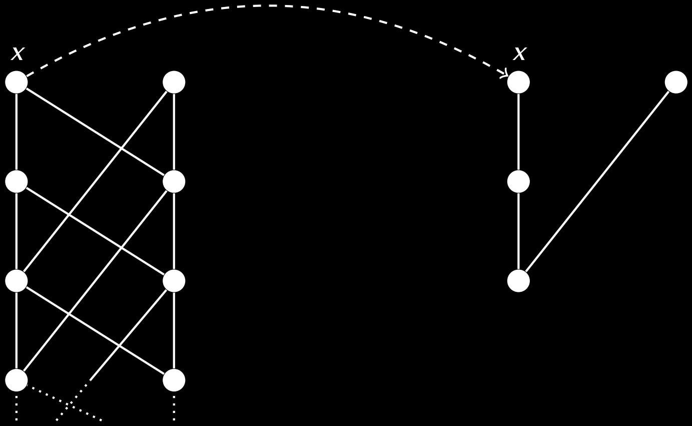
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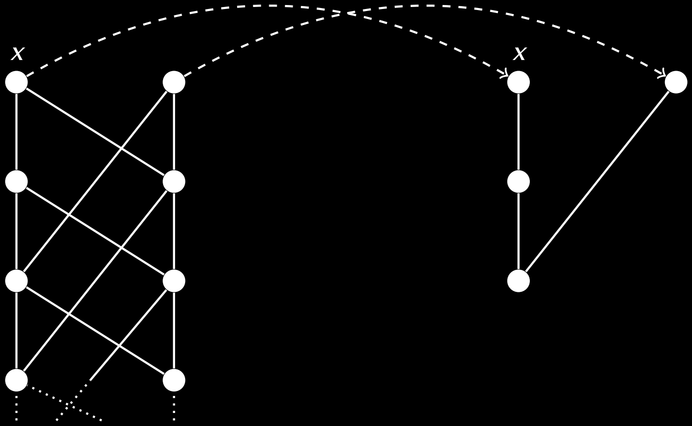
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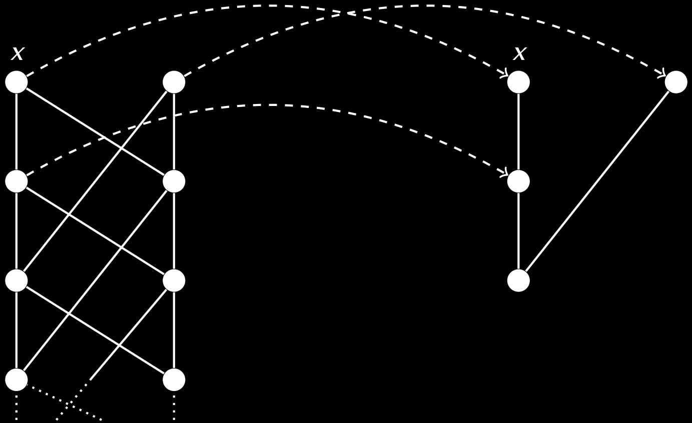
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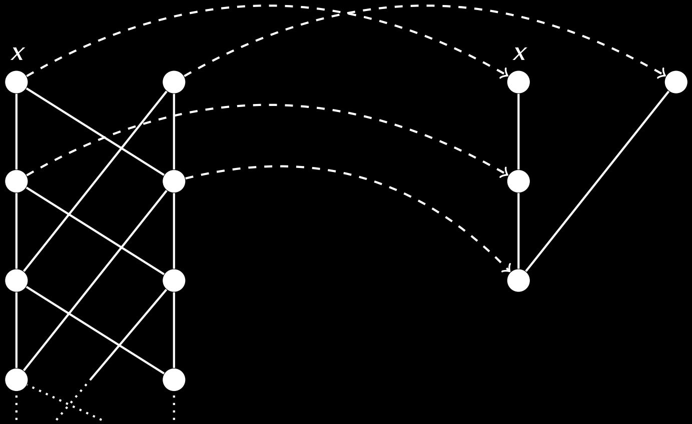
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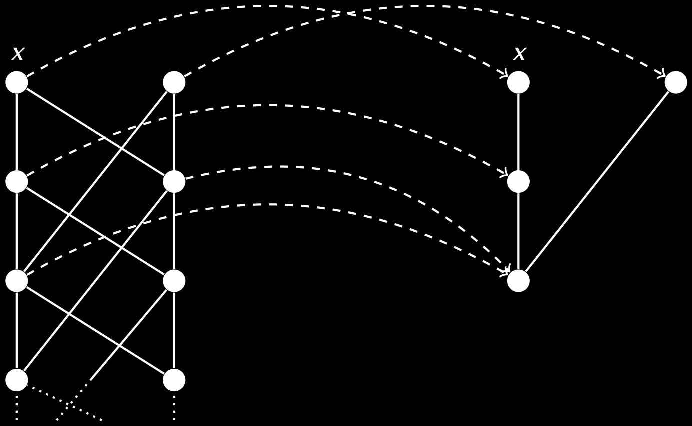
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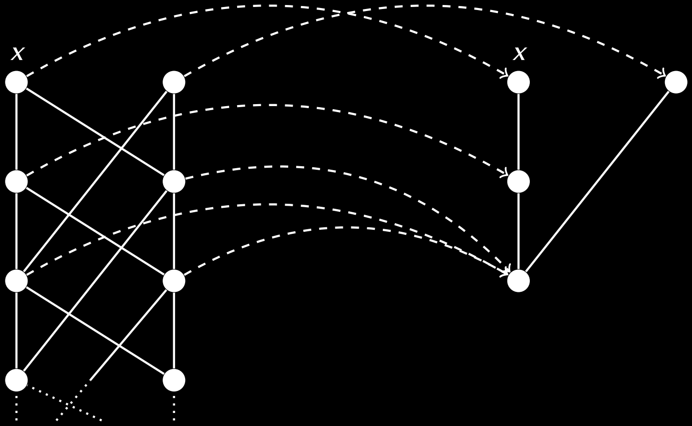
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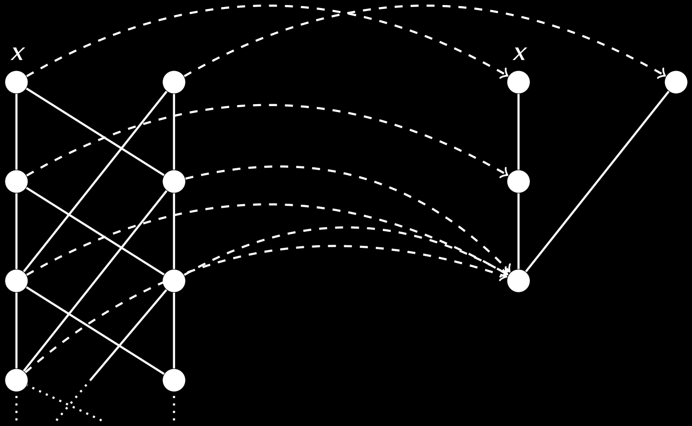
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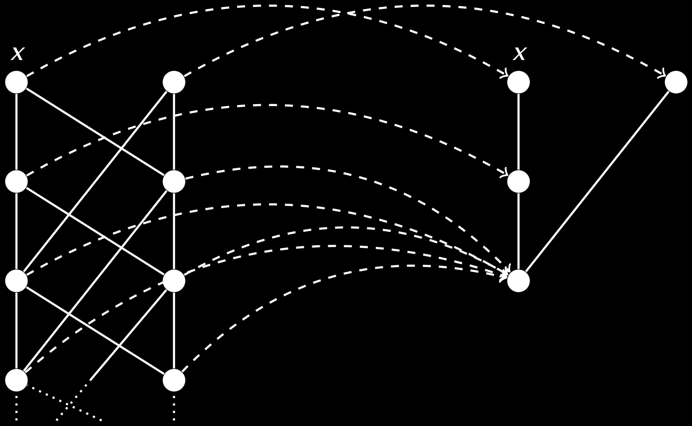
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Definition

A map $f\colon u \to v$ is adequate if there exists a substitution such that:

$$u, p \Vdash \sigma A \text{ iff } v, f(p) \Vdash A \text{ for all } A.$$

Definition

A map $f\colon u \to v$ is Σ-adequate if there exists a substitution such that:

$$u, p \Vdash \sigma A \text{ iff } v, f(p) \Vdash A \text{ for all } A \in \Sigma.$$

Definition

A map $f\colon u \to v$ is Σ-adequate if there exists a substitution such that:

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Definition

A model v is Σ-adequately exact if there exists a Σ-adequate map $\mathrm{u} \to v$.

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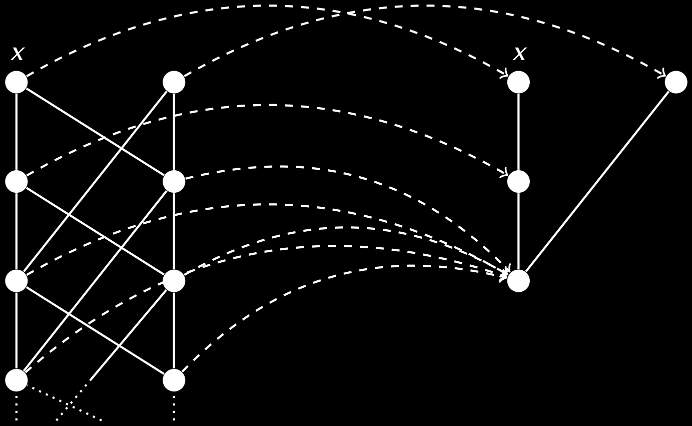
Definition

A model v is Σ-adequately exact if there exists a Σ-adequate map $\mathrm{u} \to v$, where u is a universal model.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Theorem

The admissible rules of IPC *are sound and complete with respect to exact models.*

Theorem

The admissible rules of IPC *from an adequate set* Σ *are sound and complete with respect to* Σ*-adequately exact models.*

Theorem

Let $A \in \Sigma$ and $\Delta \subseteq \Sigma$. The following are equivalent:

1. $A \mathrel{|\!\sim} \Delta$;
2. $v \Vdash A/\Delta$ *for all* Σ*-adequately exact models* v *of size at most* $2^{|\Sigma|}$.

What is a *fair* notion of semantics for admissibility?

What is a *fair* notion of semantics for admissibility?

Adequately exact models!

IV

Effective Description

When is a model
Σ-adequately exact?

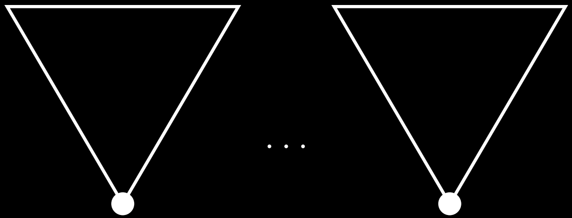
Theorem (Citkin, 1977)

A finite submodel of a universal model is exact iff it is extendible.

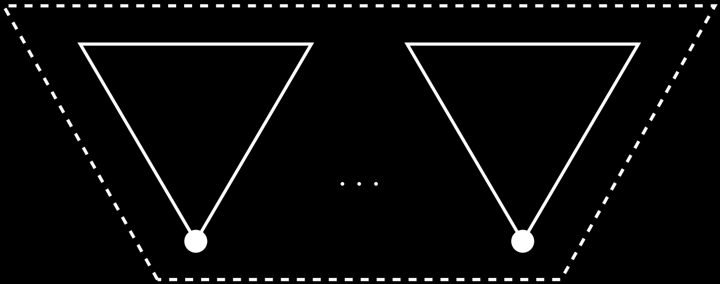
Theorem (Ghilardi, 1999)
A definable submodel of a universal model is exact iff it is extendible.

Extendible

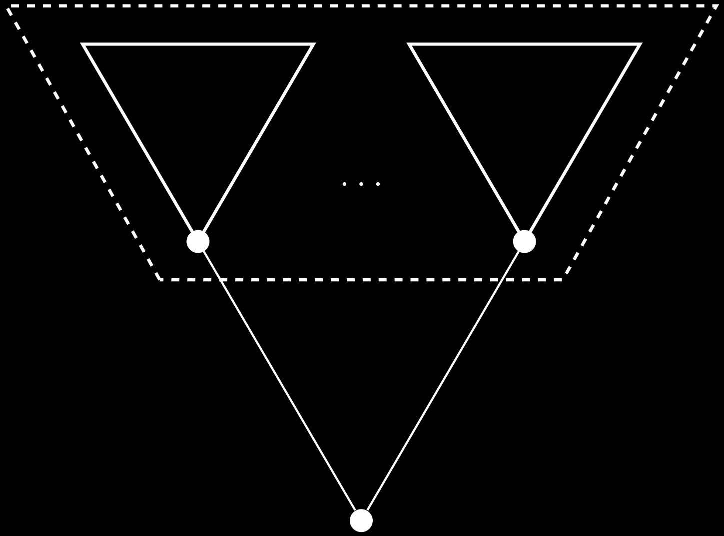
Extendible



Extendible



Extendible



Theorem
Let $U \subseteq \mathsf{U}(X)$ be an upset. Now, U is extendible iff for each finite $W \subseteq U$ there exists a $p \in U$ such that $W \subseteq {\uparrow}p$ and for all $A \to B$:

$$p \Vdash A \to B \text{ iff } (W \Vdash A \to B \text{ and } p \Vdash A \text{ implies } p \Vdash B).$$

Definition
Let $U \subseteq \mathsf{U}(X)$ be an upset. Now, U is Σ-extendible iff for each finite $W \subseteq U$ there exists a $p \in U$ such that $W \subseteq {\uparrow}p$ and for all $A \to B \in \Sigma$:

$$p \Vdash A \to B \text{ iff } (W \Vdash A \to B \text{ and } p \Vdash A \text{ implies } p \Vdash B).$$

Theorem
A finite submodel of a universal model is Σ-adequately exact iff it is Σ-extendible.

When is a model

Σ-adequately exact?

If it's Σ-extendible.

Given a rule A/Δ,
is it admissible?

Given a rule A/Δ, is it admissible?

Take Σ as all subformulae of A and Δ. Compute whether $v \Vdash A/\Delta$ for all Σ-extendible models of size at most $2^{|\Sigma|}$. If it is, then yes. Otherwise, no.

The admissible rules of IPC are decidable.

The admissible rules of IPC are decidable.

Rybakov (1984)
see Goudsmit (2014a) for more details.

Intuitionistic Rules

Admissible Rules of Intermediate Logics

Jeroen P. Goudsmit

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