Utrecht University

Admissible rules of propositional dependence logic

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Joint work with Rosalie Iemhoff

Outline

1. propositional dependence logic
2. flat formulas and projective formulas
3. structural completeness

Dependence between first-order variables

First Order Quantifiers:

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 \phi$$

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Theorem (Enderton, Walkoe). Over sentences, all of the above extensions of first-order logic are equivalent to Σ^1_1.

Propositional dependence logic = propositional logic + $=(\vec{p}, q)$

- I will be absent depending on whether he shows up or not.
- Whether it rains depends completely on whether it is summer or not.

	summer	rainy	windy
v_1	0	1	0

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X

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A *team* X = $\{v_1, v_2, v_3, v_4\}$; Y = $\{v_1, v_2, v_3, v_4, v_5\}$

	summer	rainy	windy
v_1	0	1	0
v_2	0	1	1
v_3	1	0	1
v_4	1	0	0
v_5	1	1	1

$$X \models =(p, q), \qquad Y \not\models =(p, q)$$

Propositional dependence logic and its variants

- Well-formed formulas of *propositional dependence logic* (**PD**) are given by the following grammar

$$\phi ::= \; p_i \mid \neg p_i \mid =\!(p_{i_1}, \ldots, p_{i_{n-1}}, p_{i_n}) \mid \phi \wedge \phi \mid \phi \otimes \phi$$

- Well-formed formulas of *propositional intuitionistic dependence logic* (**PID**) are given by the following grammar:

$$\phi ::= p_i \mid \bot \mid =\!(p_{i_1}, \ldots, p_{i_{n-1}}, p_{i_n}) \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi$$

- **PD**$^{\vee}$ is the logic extended from **PD** by adding the connective $\vee$.

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Team semantics: A valuation is a function $v : \mathbb{N} \to \{0, 1\}$.

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Team semantics: A valuation is a function $v : \mathbb{N} \to \{0, 1\}$. A *team* is a set of valuations.

	p_0	p_1	p_2	$\ldots$
v_1	1	0	0	$\ldots$
v_2	1	1	0	$\ldots$
v_4	0	1	0	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Team Semantics

Let X be a team.

- $X \models p_i$ iff for all $v \in X$, $v(i) = 1$;
- $X \models \neg p_i$ iff for all $v \in X$, $v(i) = 0$;
- $X \models \perp$ iff $X = \emptyset$;
- $X \models {=}(p_{i_1}, \ldots, p_{i_n})$ iff for all $v, v' \in X$
  $$\left[\, v(i_1) = v'(i_1), \ldots, v(i_{n-1}) = v'(i_{n-1}) \,\right] \Longrightarrow v(i_n) = v'(i_n);$$
- $X \models \phi \wedge \psi$ iff $X \models \phi$ and $X \models \psi$;
- $X \models \phi \otimes \psi$ iff there exist teams $Y, Z \subseteq X$ with $X = Y \cup Z$ such that
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v_1	1	0	0
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$X = \{v_1, v_2\}$

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$X \models p_0 \qquad X \cup Y \not\models p_0$

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The logics **PD**, **PD**$^\vee$, **PID**

- are downwards closed, that is, $X \models \phi$ and $Y \subseteq X \Longrightarrow Y \models \phi$;
- and have the empty team property, that is, $\emptyset \models \phi$, for all ϕ.

Team Semantics

Let X be a team.

- $X \models p_i$ iff for all $v \in X$, $v(i) = 1$;
- $X \models \neg p_i$ iff for all $v \in X$, $v(i) = 0$;
- $X \models \bot$ iff $X = \emptyset$;
- $X \models {=}(p_{i_1}, \dots, p_{i_n})$ iff for all $v, v' \in X$
$$\big[\, v(i_1) = v'(i_1), \dots, v(i_{n-1}) = v'(i_{n-1}) \,\big] \Longrightarrow v(i_n) = v'(i_n);$$
- $X \models \phi \wedge \psi$ iff $X \models \phi$ and $X \models \psi$;
- $X \models \phi \otimes \psi$ iff there exist teams $Y, Z \subseteq X$ with $X = Y \cup Z$ such that
$$Y \models \phi \text{ and } Z \models \psi;$$
- $X \models \phi \vee \psi$ iff $X \models \phi$ or $X \models \psi$;
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Intuitionistic disjunction $\vee$ has the disjunction property:

$$\models \phi \vee \psi \Longrightarrow \models \phi \text{ or } \models \psi.$$

Dependence atoms are definable in the fragment of **PID** and $\mathbf{PD}^{\vee}$ without dependence atoms:

$$\begin{aligned}=\!(p_0, p_1) &\equiv \Big(p_0 \wedge (p_1 \vee \neg p_1)\Big) \otimes \Big(\neg p_0 \wedge (p_1 \vee \neg p_1)\Big)\\ &\equiv (p_0 \vee \neg p_0) \to (p_1 \vee \neg p_1)\end{aligned}$$

	p_0	p_1	p_2
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Observation (de Jongh, Litak)

$\mathbf{PID}^{-}$ is equivalent to inquisitive logic (Ciardelli, Roelofsen 2011).

Fix $N = \{i_1, \ldots, i_n\} \subseteq \mathbb{N}$.

- An *n-valuation* on N is a function $s : N \to \{0, 1\}$.
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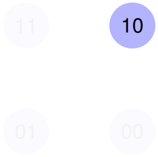

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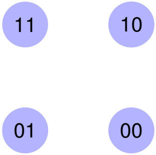

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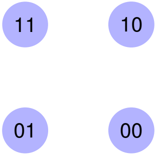

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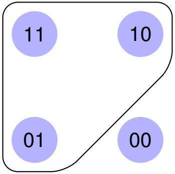

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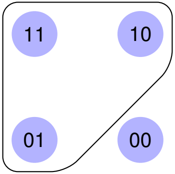

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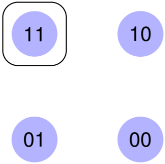

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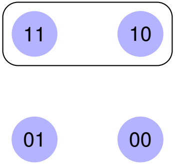

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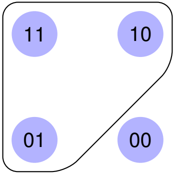

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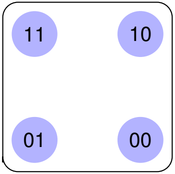

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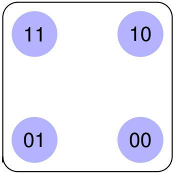

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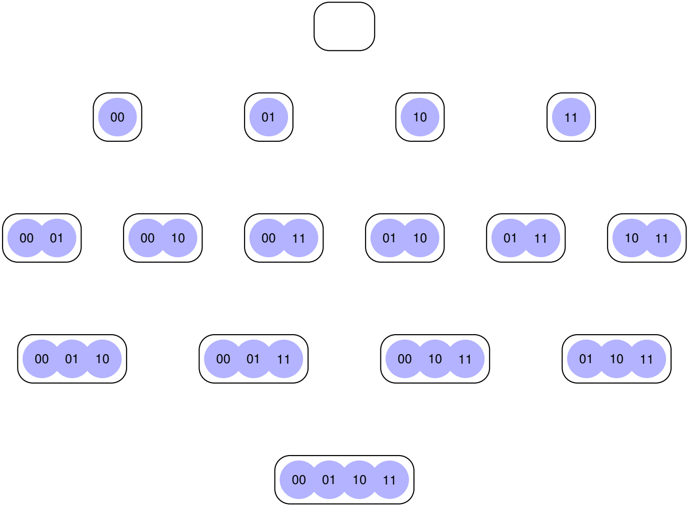
$\wp(\mathbf{2^n})$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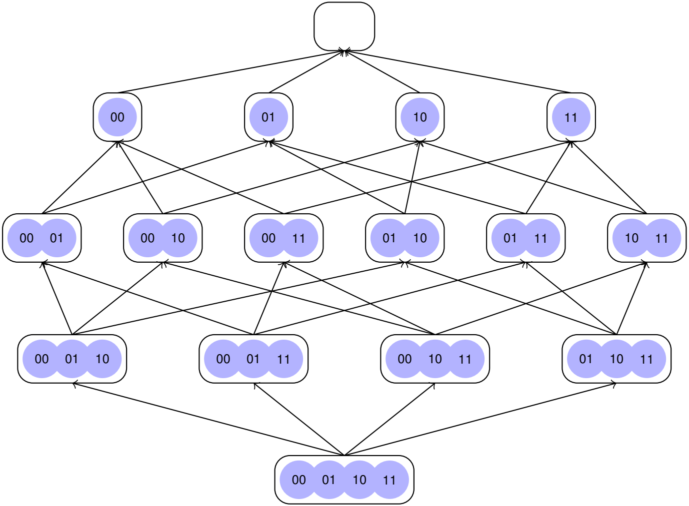
$(\wp(\mathbf{2^n}), \supseteq)$



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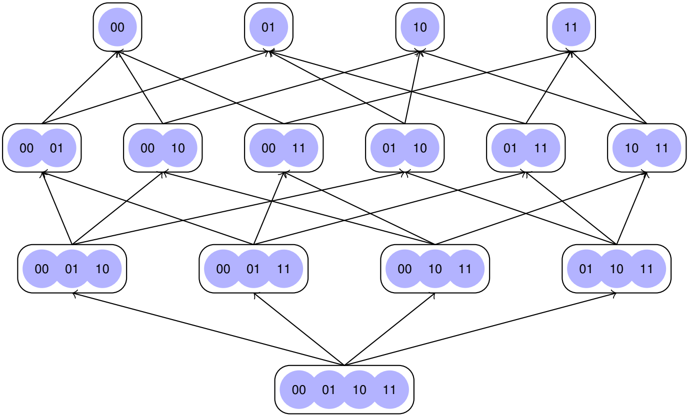
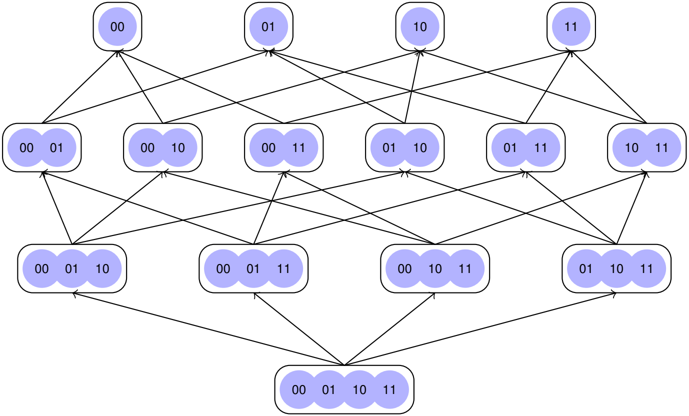


A Medvedev frame: $(\wp(\mathbf{2^n}) \setminus \{\emptyset\}, \supseteq)$



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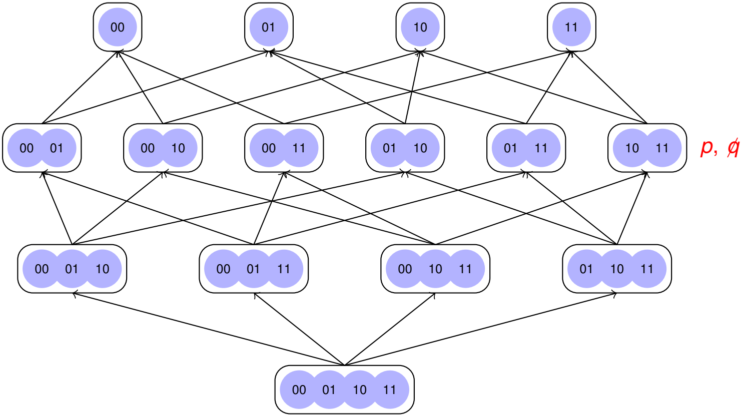

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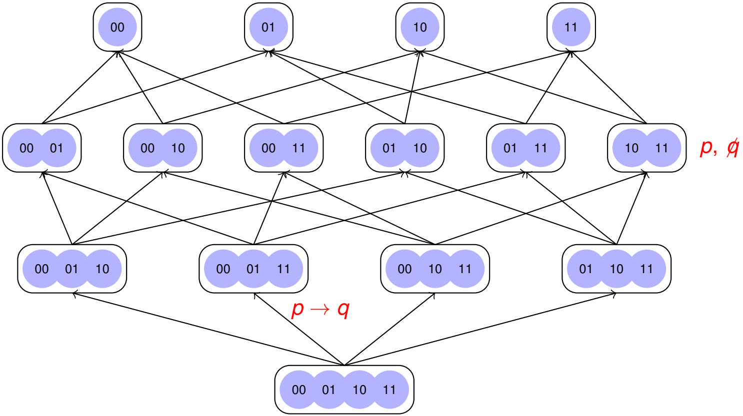

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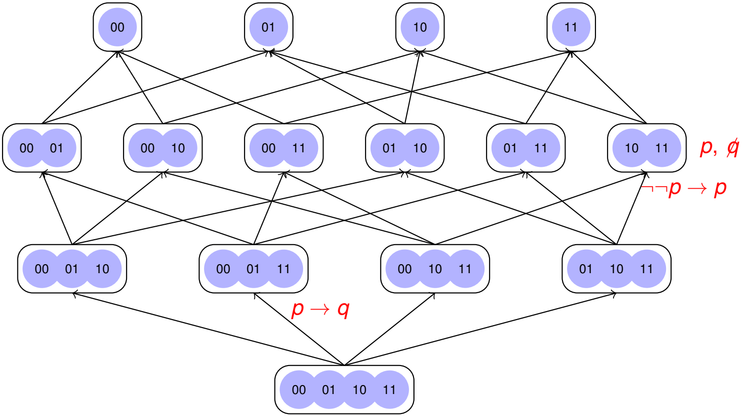

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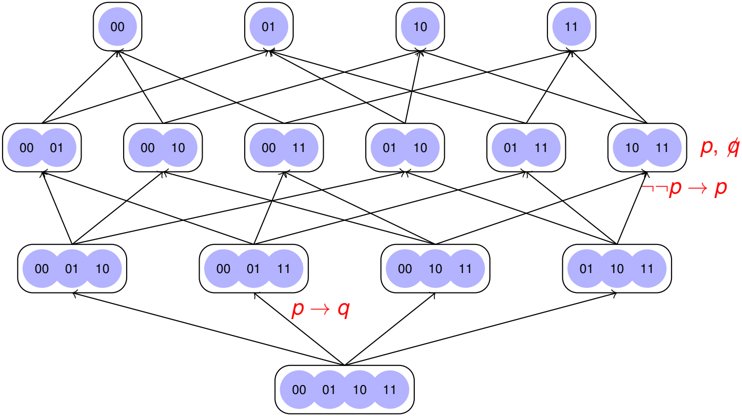

[Ciardelli, Roelofsen 2011]:

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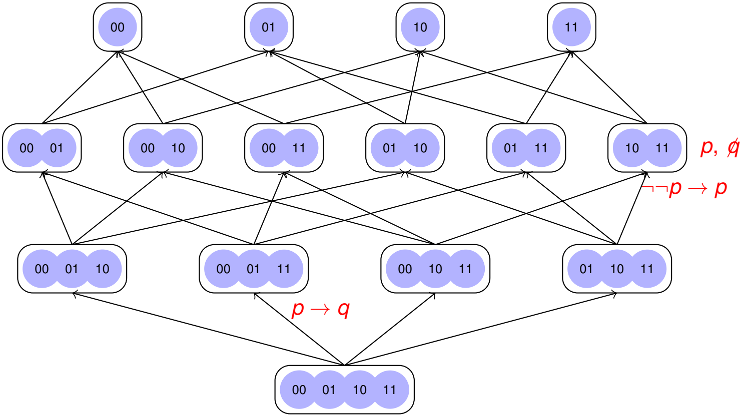

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PID, $\mathbf{PD}^{\vee}$, **PD** *are maximal downwards closed logics, i.e., for* $\mathsf{L} \in \{\mathbf{PID}, \mathbf{PD}^{\vee}, \mathbf{PD}\}$,

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Theorem (Y.)

Every instance of $\vee$ *and* $\to$ *is definable in* **PD**, *but* $\vee$ *and* $\to$ *are not uniformly definable in* **PD**.

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$$\Theta_X := \begin{cases} & \text{for } \mathbf{PD}; \\ & \text{for } \mathbf{PID}. \end{cases}$$

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$$\Theta_X := \begin{cases} \bigotimes_{v \in X} (p_{i_1}^{v(i_1)} \wedge \dots \wedge p_{i_n}^{v(i_n)}), & \text{for } \mathbf{PD}; \\ & \text{for } \mathbf{PID}. \end{cases}$$

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- $[\![\phi(p_{i_1}, \dots, p_{i_n})]\!] := \{X \subseteq \mathbf{2^n} \mid X \models \phi\}$,
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Theorem (Ciardelli, Huuskonen, Y.)

PID, $\mathbf{PD}^{\vee}$, **PD** *are maximal downwards closed logics, i.e., for* $\mathsf{L} \in \{\mathbf{PID}, \mathbf{PD}^{\vee}, \mathbf{PD}\}$,

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In particular, $\mathbf{PID} \equiv \mathbf{PD}^{\vee} \equiv \mathbf{PD}$.

Proof. We only treat **PID** and $\mathbf{PD}^{\vee}$. First, consider an n-team:

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$$\Theta_X := \begin{cases} \bigotimes_{v \in X} (p_{i_1}^{v(i_1)} \wedge \dots \wedge p_{i_n}^{v(i_n)}), & \text{for } \mathbf{PD}; \\ \neg\neg \bigvee_{v \in X} (p_{i_1}^{v(i_1)} \wedge \dots \wedge p_{i_n}^{v(i_n)}), & \text{for } \mathbf{PID}. \end{cases}$$

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Hence $[\![\bigvee_{X \in \mathcal{K}} \Theta_X]\!] = \mathcal{K}$. □

Theorem. **PID**, **PD**$^{\vee}$, **PD** are sound and complete w.r.t. their deductive systems.

Corrolary. For every formula ϕ of **PID** and **PD**$^{\vee}$, $\phi \dashv\vdash \bigvee_{i \in I} \Theta_{X_i}$.

A Hilbert Style deductive system for PID^{-} (Ciardelli, Roelofsen 2011)

Axioms:

- all substitution instances of **IPC** axioms
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$$\text{(KP)} \qquad (\neg p_i \to (p_j \vee p_k)) \to ((\neg p_i \to p_j) \vee (\neg p_i \to p_k)).$$

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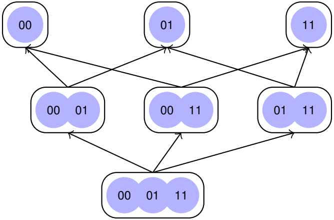

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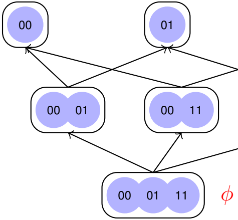

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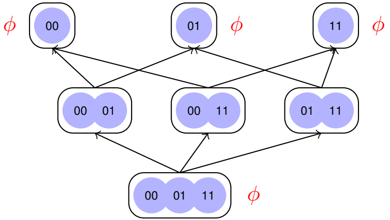

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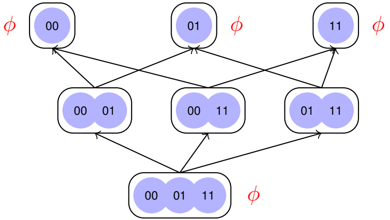

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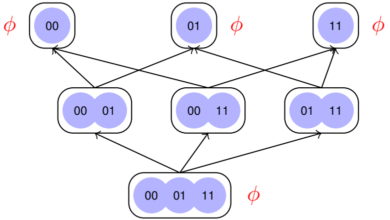

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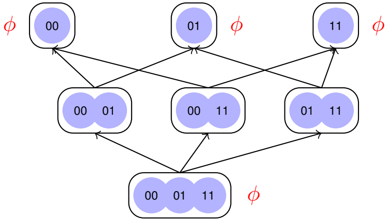

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Lemma. In **PID**, a formula ϕ is flat if $\vdash \phi \leftrightarrow \neg\neg\phi$.

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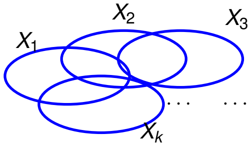

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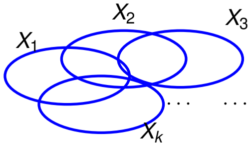

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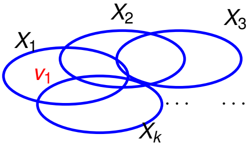

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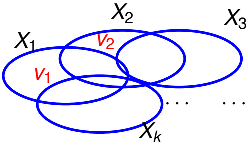

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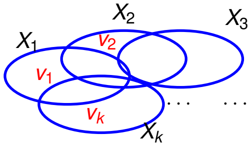

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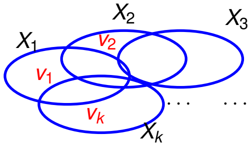

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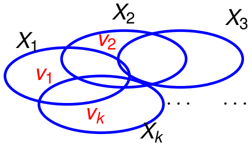

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We say that a substitution σ of a logic L is well-behaved, if

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Recall: Well-formed formulas of **PD** are built from the following grammar:

$$\phi ::= \; p_i \mid \neg p_i \mid {=}(p_{i_1}, \ldots, p_{i_k}) \mid \phi \wedge \phi \mid \phi \otimes \phi$$

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Definition

A substitution σ is called a flat substitution if $\sigma(p)$ is flat for all p.

Lemma

Flat substitutions are well-behaved in **PID** *and* **PD**.

Proof. For **PID**, it follows from (Ciardelli). For **PD**, nontrivial. □

Definition (Projective formula)

Let S be a set of well-behaved substitutions of a logic L. An L-formula ϕ is said to be *S-projective* if there exists $\sigma \in \mathsf{S}$ such that

(1) $\vdash_{\mathsf{L}} \sigma(\phi)$

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For formulas ϕ *of the logics* **PD** *and* **PID***, TFAE:*

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Definition

Let S be a set of well-behaved substitutions of a logic L.

- A rule ϕ/ψ of L is said to be a *S-admissible rule*, in symbols $\phi \mathrel{|\!\!\sim}^{\mathsf{S}}_{\mathsf{L}} \psi$ or simply $\phi \mathrel{|\!\!\sim}^{\mathsf{S}} \psi$, if for all $\sigma \in \mathsf{S}$,

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- L is said to be *S-structurally complete* if every S-admissible rule of L is derivable, i.e., $\phi \mathrel{|\!\!\sim}^{\mathsf{S}}_{\mathsf{L}} \psi \implies \phi \vdash_{\mathsf{L}} \psi$.

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Theorem

PID *and* **PD** *are* F*-structurally complete, where* F *is the set of all flat substitutions.*

Proof. Since $\phi \equiv \bigvee_{i \in I} \Theta_{X_i}$, where each Θ_{X_i} is S-projective, for every formula ϕ of the logics. $\square$