Admissible Rules and Beyond

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A First Question

What is an **admissible rule**?

Two Informal Answers

(A) *"A rule is* **admissible** *in a system if the set of theorems does not change when the rule is added to the system."*

(B) *"A rule is* **admissible** *in a system if any substitution sending its premises to theorems, sends its conclusion to a theorem."*

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Admissibility in Intuitionistic Logic

The “independence of premises” rule

$$\{\neg p \to (q \vee r)\} \;\Rightarrow\; (\neg p \to q) \vee (\neg p \to r)$$

is not derivable in intuitionistic logic, but it is admissible because. . .

(A) adding it to an axiomatization gives no new theorems

(B) if $\neg\varphi \to (\psi \vee \chi)$ is a theorem, so is $(\neg\varphi \to \psi) \vee (\neg\varphi \to \chi)$.

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The “independence of premises” rule

$$\{\neg p \to (q \lor r)\} \;\Rightarrow\; (\neg p \to q) \lor (\neg p \to r)$$

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Multiple-Conclusion Rules

The "disjunction property"

$$\{p \lor q\} \;\Rightarrow\; \{p,\; q\}$$

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(B) if $\varphi \lor \psi$ is a theorem, either φ or ψ is a theorem.

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A Splitting of the Notions

The "linearity property"

$$\Rightarrow \{p \to q, q \to p\}$$

is admissible in Gödel logic according to. . .

(A) because adding it to an axiomatization gives no new theorems

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A More Exotic Example

The "Takeuti-Titani density rule"

$$\{((\varphi \to p) \lor (p \to \psi)) \lor \chi\} \;\Rightarrow\; (\varphi \to \psi) \lor \chi$$

where p does not occur in φ, ψ, or χ

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More Generally...

What does it mean for a **first-order sentence** such as

$$(\exists x)(\forall y)(x \leq y) \qquad \text{or} \qquad (\forall x)(\exists y)\neg(x \leq y)$$

to be admissible in a logic or class of algebras?

The Main Question

How can these notions of admissibility be **characterized**?

References

We take a “first-order” approach as described in

G. Metcalfe. Admissible Rules: From Characterizations to Applications. *Proceedings of WoLLIC 2012*, LNCS 7456, Springer (2012), 56–69.

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First-Order Logic

Assume the usual terminology of **first-order logic with equality**, using the symbols $\forall$, $\exists$, $\sqcap$, $\sqcup$, $\Rightarrow$, $\sim$, 0, 1, and $\approx$.

Fix an algebraic language $\mathcal{L}$ with terms $\mathrm{Tm}(\mathcal{L})$ and sentences $\mathrm{Sen}(\mathcal{L})$.

For sets of $\mathcal{L}$-equations Γ and Δ, denote by $\Gamma \Rightarrow \Delta$ the **$\mathcal{L}$-clause**

$$(\forall \bar{x})(\sqcap \Gamma \Rightarrow \sqcup \Delta)$$

called an **$\mathcal{L}$-quasiequation** if $|\Delta| = 1$ and a **positive** $\mathcal{L}$-clause if $\Gamma = \emptyset$.

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Admissibility Algebraically

Let $\mathbf{Tm}(\mathcal{L})$ denote the **term algebra of** $\mathcal{L}$, and consider a class of $\mathcal{L}$-algebras K and a set of $\mathcal{L}$-equations Γ.

A K-**unifier** of Γ is a homomorphism $\sigma\colon \mathbf{Tm}(\mathcal{L}) \to \mathbf{Tm}(\mathcal{L})$ such that

$$\mathsf{K} \models \sigma(s) \approx \sigma(t) \quad \text{for all } s \approx t \in \Gamma.$$

We say that an $\mathcal{L}$-**clause** $\Gamma \Rightarrow \Delta$ is K-**admissible** if

$$\sigma \text{ is a K-unifier of } \Gamma \quad \Longrightarrow \quad \sigma \text{ is a K-unifier of some } s \approx t \in \Delta.$$

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An Algebraic Characterization

For any class of $\mathcal{L}$-algebras K and an $\mathcal{L}$-clause $\Gamma \Rightarrow \Delta$,

$$\Gamma \Rightarrow \Delta \text{ is K-admissible} \quad \Leftrightarrow \quad \mathbf{F}_{\mathrm{K}} \models \Gamma \Rightarrow \Delta$$

where $\mathbf{F}_{\mathrm{K}}$ is the **free algebra** of K on countably many generators.

Another Notion of Admissibility

But what about notion (A)

"A rule is **admissible** *in a system if the set of theorems does not change when the rule is added to the system."* ?

Reformulating, consider...

the "system" as a class of $\mathcal{L}$-algebras K

the "rule" as a first-order $\mathcal{L}$-sentence φ

the "theorems" as a set of $\mathcal{L}$-sentences Σ,

when does φ **preserve** Σ **in** K?

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Preserving Sentences

Definition

For a class of $\mathcal{L}$-algebras K and $\Sigma \subseteq \text{Sen}(\mathcal{L})$, we set

$$\mathrm{Th}_\Sigma(\mathsf{K}) = \{\psi \in \Sigma : \mathsf{K} \models \psi\}$$

and say that $\varphi \in \text{Sen}(\mathcal{L})$ **preserves** Σ **in** K if

$$\mathrm{Th}_\Sigma(\mathsf{K}) = \mathrm{Th}_\Sigma(\{\mathbf{A} \in \mathsf{K} : \mathbf{A} \models \varphi\}).$$

If $\Theta \subseteq \text{Sen}(\mathcal{L})$ axiomatizes K, then φ preserves Σ in K if for all $\psi \in \Sigma$:

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Preserving Equations

Theorem

The following are equivalent for any $\mathcal{L}$-quasiequation φ:

(i) φ *is* K*-admissible*

(ii) $\mathbf{F}_{\mathsf{K}} \models \varphi$

(iii) φ *preserves* $\mathcal{L}$*-equations in* K

(iv) $\mathsf{K} \subseteq \mathbb{V}(\{\mathbf{A} \in \mathsf{K} : \mathbf{A} \models \varphi\})$,

and if K *is a quasivariety,*

(v) *each* $\mathbf{B} \in \mathsf{K}$ *is a homomorphic image of an* $\mathbf{A} \in \mathsf{K}$ *such that* $\mathbf{A} \models \varphi$.

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Preserving Positive Clauses

Theorem

The following are equivalent for any $\mathcal{L}$-clause φ:

(i) φ *is* K*-admissible*

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(iv) $\mathsf{K} \subseteq \mathbb{U}^{+}(\{\mathbf{A} \in \mathsf{K} : \mathbf{A} \models \varphi\})$,

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Example

For the variety BA of **Boolean algebras** in a language $\mathcal{L}_{\mathrm{Bool}}$,

$$\varphi = (\forall x)((x \approx \bot) \sqcup (x \approx \top))$$

preserves $\mathcal{L}_{\mathrm{Bool}}$-equations in BA, but $\mathbf{F}_{\mathrm{BA}} \not\models \varphi$.

Note that $\neg\varphi$, equivalent to

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Beyond Clauses

The **Skolem form** $\mathrm{sk}(\varphi)$ of a prenex $\varphi \in \mathrm{Sen}(\mathcal{L})$ results by repeating

$$(\forall \bar{x})(\exists y)\varphi(\bar{x}, y) \qquad \Longrightarrow \qquad (\forall \bar{x})\varphi(\bar{x}, f(\bar{x})) \qquad f \text{ new.}$$

Then for any $\Theta \cup \{\psi\} \subseteq \mathrm{Sen}(\mathcal{L})$:

$$\Theta \cup \{\varphi\} \models \psi \quad \Leftrightarrow \quad \Theta \cup \{\mathrm{sk}(\varphi)\} \models \psi.$$

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Preservation under Skolemization

Let K be a class of $\mathcal{L}$-algebras, $\mathcal{L}'$ an extension of $\mathcal{L}$, and K$'$ the class of $\mathcal{L}'$-algebras whose $\mathcal{L}$-reducts are in K.

Theorem

The following are equivalent for any $\Sigma \cup \{\varphi\} \subseteq \mathrm{Sen}(\mathcal{L})$:

(1) φ *preserves* Σ *in* K

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Applications?

For a class of algebras K, we often seek a "distinguished subclass" $\mathsf{K}' \subseteq \mathsf{K}$ such that for all equations (quasiequations, etc.) φ,

$$\mathsf{K}' \models \varphi \qquad \Leftrightarrow \qquad \mathsf{K} \models \varphi.$$

For example:

- Boolean algebras and the two-element Boolean algebra
- modal algebras and perfect modal algebras
- Gödel algebras and dense Gödel chains
- lattice-ordered groups and automorphisms of $\mathbb{R}$.

Algebraically, we want to establish $\mathbb{V}(\mathsf{K}) = \mathbb{V}(\mathsf{K}')$ $(\mathbb{Q}(\mathsf{K}) = \mathbb{Q}(\mathsf{K}'), \text{etc.})$.

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For a class of algebras K, we often seek a “distinguished subclass” $\mathsf{K}' \subseteq \mathsf{K}$ such that for all equations (quasiequations, etc.) φ,

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A Proof System GLat for Lattices

Axioms

$$\frac{}{s \leq s}\ (\text{ID})$$

Cut rule

$$\frac{s \leq u \quad u \leq t}{s \leq t}\ (\text{CUT})$$

Left rules

$$\frac{s_i \leq t}{s_1 \wedge s_2 \leq t}\ (\wedge\Rightarrow)_i \quad (i = 1,2)$$

$$\frac{s_1 \leq t \quad s_2 \leq t}{s_1 \vee s_2 \leq t}\ (\vee\Rightarrow)$$

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Example: Boundedness in Lattices

The following $\mathcal{L}_{\text{Lat}}$-sentence expresses **boundedness**:

$$\varphi_{\text{BD}} = (\exists x)(\exists y)(\forall z)((x \leq z) \sqcap (z \leq y)).$$

Skolemizing, we obtain

$$(\forall z)((\bot \leq z) \sqcap (z \leq \top)).$$

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Linearity and Density in Gödel Algebras

The following $\mathcal{L}_{\mathrm{Lat}}$-sentence expresses **linearity** and **density**:

$$\varphi_{\mathrm{DC}} = (\forall x)(\forall y)(\exists z)(((x \le y) \sqcup (y \le x)) \sqcap (((x \le z) \sqcup (z \le y)) \Rightarrow (x \le y))).$$

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