Structurally complete Lukasiewicz logics.

Joan Gispert

Facultat de Matemàtiques. Universitat de Barcelona
jgispertb@ub.edu

Workshop on Admissible Rules and Unification II,
Les Diablerets, 2015

Łukasiewicz logics

The Infinite valued Łukasiewicz Calculus, $Ł_\infty$

Axioms:

Ł1. $\varphi \to (\psi \to \varphi)$

Ł2. $(\varphi \to \psi) \to ((\psi \to \nu) \to (\varphi \to \nu))$

Ł3. $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$

Ł4. $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$

Rules:

Modus Ponens. $\{\varphi, \varphi \to \psi\}/\psi$.

Original logic semantics

$$[0,1] = \langle \{a \in \mathbb{R} : 0 \leq a \leq 1\}; \rightarrow, \neg \rangle$$

For all $a, b \in [0,1]$,

$$a \rightarrow b = \begin{cases} 1, & \text{if } a \leq b; \\ 1 - a + b, & \text{otherwise.} \end{cases}, \qquad \neg a = 1 - a.$$

Original logic semantics

$$[0,1] = \langle \{a \in \mathbb{R} : 0 \leq a \leq 1\}; \rightarrow, \neg \rangle$$

For all $a, b \in [0,1]$,

$$a \rightarrow b = \begin{cases} 1, & \text{if } a \leq b; \\ 1 - a + b, & \text{otherwise.} \end{cases}, \qquad \neg a = 1 - a.$$

Let $\Gamma \cup \{\varphi\} \subseteq Prop(X)$, then

$\Gamma \models_{[0,1]} \varphi$ iff
for every $h : Prop(x) \rightarrow [0,1]$, $h(\varphi) = 1$ whenever $h\Gamma = \{1\}$

Completeness Theorems

Weak Completeness Theorem

Theorem (Rose-Rosser 1958, Chang 1959)

$$\vdash_{Ł_\infty} \varphi \textit{ iff } \models_{[0,1]} \varphi$$

Completeness Theorems

Weak Completeness Theorem

Theorem (Rose-Rosser 1958, Chang 1959)

$$\vdash_{Ł_\infty} \varphi \textit{ iff } \models_{[0,1]} \varphi$$

Strong Finite Completeness Theorem

Theorem (Hay 1963)

$$\varphi_1, \ldots, \varphi_n \vdash_{Ł_\infty} \varphi \textit{ iff } \varphi_1, \ldots, \varphi_n \models_{[0,1]} \varphi$$

Algebraic logic

The infinite valued Łukasiewicz calculus $Ł_\infty$ is algebraizable with **MV** *the class of all MV-algebras as its equivalent quasivariety semantics.*

Algebraic logic

Finitary Extensions of $Ł_\infty$ $\longleftrightarrow$ Quasivarieties of **MV**

Algebraic logic

Finitary Extensions of $Ł_\infty$ $\longleftrightarrow$ Quasivarieties of **MV**

Axiomatic Extensions $\longleftrightarrow$ Varieties

Algebraic logic

Finitary Extensions of $Ł_\infty$	$\longleftrightarrow$	Quasivarieties of **MV**
Axiomatic Extensions	$\longleftrightarrow$	Varieties
(Finite) Axiomatization	$\longleftrightarrow$	(Finite) Axiomatization
Deduction Theorem	$\longleftrightarrow$	EDPCR
Local Deduction Theorem	$\longleftrightarrow$	RCEP
Interpolation Theorem	$\longleftrightarrow$	Amalgamation Property

Algebraic logic

Finitary Extensions of $Ł_\infty$	$\longleftrightarrow$	Quasivarieties of **MV**
Axiomatic Extensions	$\longleftrightarrow$	Varieties
(Finite) Axiomatization	$\longleftrightarrow$	(Finite) Axiomatization
Deduction Theorem	$\longleftrightarrow$	EDPCR
Local Deduction Theorem	$\longleftrightarrow$	RCEP
Interpolation Theorem	$\longleftrightarrow$	Amalgamation Property
Structurally Complete Fin. Ext.	$\longleftrightarrow$	Struct. Complete Quas. (Least V-Quasivarieties)

Algebraic logic

Finitary Extensions of $Ł_\infty$	$\longleftrightarrow$	Quasivarieties of $\mathbf{MV}$
$\models^f_{\mathbf{M}}$	$\longleftrightarrow$	$\mathcal{Q}(\mathbf{M})$

Algebraic logic

Finitary Extensions of $Ł_\infty$ $\longleftrightarrow$ Quasivarieties of $\mathbf{MV}$

$$\models^f_{\mathbf{M}} \longleftrightarrow \mathcal{Q}(\mathbf{M})$$

Komori's characterization

Proper Axiomatic Extensions $\longleftrightarrow$ Proper Subvarieties

Algebraic logic

Finitary Extensions of $Ł_\infty$ $\longleftrightarrow$ Quasivarieties of **MV**

$$\models^f_{\mathbf{M}} \longleftrightarrow \mathcal{Q}(\mathbf{M})$$

Komori's characterization

Proper Axiomatic Extensions $\longleftrightarrow$ Proper Subvarieties

$$\vdash_{I,J} \longleftrightarrow \mathcal{V}_{I,J}$$

I, J are two finite subsets of integers ≥ 1 not both empty.

Algebraic logic

Finitary Extensions of $Ł_\infty$ $\longleftrightarrow$ Quasivarieties of **MV**

$$\models^f_{\mathbf{M}} \longleftrightarrow \mathcal{Q}(\mathbf{M})$$

Komori's characterization

Proper Axiomatic Extensions $\longleftrightarrow$ Proper Subvarieties

$$\vdash_{I,J} \longleftrightarrow \mathcal{V}_{I,J}$$

I, J are two finite subsets of integers ≥ 1 not both empty.

$\vdash_{I,J} = \models^f_{\mathbf{M}}$ and $\mathcal{V}_{I,J} = \mathcal{V}(\mathbf{M}) = \mathcal{Q}(\mathbf{M})$

where $\mathbf{M} = \{\mathrm{L}_m \mid m \in I\} \cup \{\mathrm{L}^\omega_n \mid n \in J\}$

Algebraic logic

Structurally Complete Fin. Ext. $\longleftrightarrow$ (Struct. Complete Quas.) Least V-Quasivarieties

Algebraic logic

Structurally Complete Fin. Ext.	$\longleftrightarrow$	(Struct. Complete Quas.) Least V-Quasivarieties
$\models^{f}_{F_{\mathbf{V}}(X)}$	$\longleftrightarrow$	$\mathcal{Q}(F_{\mathbf{V}}(X))$

Algebraic logic

Structurally Complete Fin. Ext.	$\longleftrightarrow$	(Struct. Complete Quas.) Least V-Quasivarieties
$\models^{f}_{F_{\mathbf{V}}(X)}$	$\longleftrightarrow$	$\mathcal{Q}(F_{\mathbf{V}}(X))$

Purpose:
For every variety $\mathbf{V}$ *of MV-algebras to obtain a "nice" class of generators* $\mathbf{M}_{\mathbf{V}}$

$$\models^{f}_{F_{\mathbf{V}}(X)} = \models^{f}_{\mathbf{M}_{\mathbf{V}}} \quad \textit{and} \quad \mathcal{Q}(F_{\mathbf{V}}(X)) = \mathcal{Q}(\mathbf{M}_{\mathbf{V}})$$

Outline

I. MV-preliminaries
II. Varieties and Quasivarieties.
III. Least V-quasivarieties.
IV. Admissibility Theory for Łukasiewicz Logics.
V. Conclusions.

MV-algebra

An *MV-algebra* is an algebra $\langle A, \oplus, \neg, 0\rangle$ satisfying the following equations:

$$
\begin{array}{ll}
\text{MV1} & (x \oplus y) \oplus z \approx x \oplus (y \oplus z) \\
\text{MV2} & x \oplus y \approx y \oplus x \\
\text{MV3} & x \oplus 0 \approx x \\
\text{MV4} & \neg(\neg x) \approx x \\
\text{MV5} & x \oplus \neg 0 \approx \neg 0 \\
\text{MV6} & \neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x.
\end{array}
$$

- $1 =_{def} \neg 0$.
- $x \to y =_{def} \neg x \oplus y$.
- $x \vee y =_{def} (x \to y) \to y$.
- $x \wedge y =_{def} \neg(\neg x \vee \neg y)$.
- $x \odot y =_{def} \neg(\neg x \oplus \neg y)$.

For any MV-algebra A, $a \leq b$ iff $a \to b = 1$ endows A with a distributive lattice-order $\langle A, \vee, \wedge \rangle$, called the *natural order* of A.

An MV-algebra whose natural order is total is said to be an *MV-chain*.

Lattice ordered abelian group

A *lattice-ordered abelian group* (for short, ℓ-group) is an algebra $\langle G, \wedge, \vee, +, -, 0\rangle$ such that $\langle G, \wedge, \vee\rangle$ is a lattice, $\langle G, +, -, 0\rangle$ is an abelian group and satisfies the following equation:

$$(x \vee y) + z \approx (x + z) \vee (y + z)$$

For any ℓ-group G and element $0 < u \in G$, let $\Gamma(G, u) = \langle [0, u], \oplus, \neg, 0\rangle$ be defined by

$$[0, u] = \{a \in G \mid 0 \leq a \leq u\}, \quad a \oplus b = u \wedge (a + b), \quad \neg a = u - a.$$

$\langle [0, u], \oplus, \neg, 0\rangle$ is an MV-algebra.

Examples

- $[0,1] = \Gamma(\mathbb{R}, 1)$,
- $[0,1] \cap \mathbb{Q} = \Gamma(\mathbb{Q}, 1)$,

For every $0 < n < \omega$

- $\mathrm{L}_n = \Gamma(\mathbb{Z}, n) = \langle \{0, 1, \ldots, n\}, \oplus, \neg, 0 \rangle$,
- $\mathrm{L}_n^\omega = \Gamma(\mathbb{Z} \times_{lex} \mathbb{Z}, (n, 0)) =$
  $\langle \{(k, i) : (0, 0) \leq (k, i) \leq (n, 0)\}, \oplus, \neg, 0 \rangle$.
- $\mathrm{L}_n^s = \Gamma(\mathbb{Z} \times_{lex} \mathbb{Z}, (n, s)) =$
  $\langle \{(k, i) : (0, 0) \leq (k, i) \leq (n, s)\}, \oplus, \neg, 0 \rangle$, where $0 \leq s < n$.
- $\mathrm{S_n} = \Gamma(T, n)$ where T is the totally ordered dense subgroup of $\mathbb{R}$ generated by $\sqrt{2} \in \mathbb{R}$ and $1 \in \mathbb{R}$. Notice that $T \cap \mathbb{Q} = \mathbb{Z}$.

The variety **MV**

The class **MV** *of all MV-algebras is a variety.*

The variety **MV**

The class **MV** *of all MV-algebras is a variety.*

Chang's completeness theorem

$$\mathbf{MV} = \mathcal{V}([0,1] \cap \mathbb{Q}) = \mathcal{V}([0,1]).$$

The variety **MV**

The class **MV** *of all MV-algebras is a variety.*

Chang's completeness theorem

$\mathbf{MV} = \mathcal{V}([0,1] \cap \mathbb{Q}) = \mathcal{V}([0,1])$.

Any infinite subalgebra of $[0,1]$ *generates* **MV**

Subvarieties of MV

Subvarieties of **MV**

(Komori)

Two MV-chains generate the same variety iff they have the same order and the same rank.

where the **order of** an MV-chain A is defined by

$$\mathrm{ord}(A) = \begin{cases} n, & \text{if } A \cong \mathrm{L}_n; \\ \infty, & \text{otherwise.} \end{cases}$$

The **rank of** an MV-chain A is defined by

$$\mathrm{rank}(A) = \mathrm{ord}(A/Rad(A)).$$

Subvarieties of **MV**

Theorem (Komori, 1981)

V *is a proper subvariety of* **MV** *iff there exist two finite sets I and J of integers ≥ 1, not both empty, such that*

$$\mathbf{V} = \mathcal{V}(\{\mathrm{L}_m \mid m \in I\} \cup \{\mathrm{L}_n^{\omega} \mid n \in J\}).$$

Subvarieties of **MV**

Theorem (Komori, 1981)

$\mathbf{V}$ is a proper subvariety of $\mathbf{MV}$ iff there exist two finite sets I and J of integers ≥ 1, not both empty, such that

$$\mathbf{V} = \mathcal{V}(\{\mathrm{L}_m \mid m \in I\} \cup \{\mathrm{L}_n^\omega \mid n \in J\}).$$

- *Every proper subvariety of* **MV** *is finitely axiomatizable.*
- *The lattice of all varieties of MV-algebras is a Pseudo-Boolean algebra.*

Let (I, J) be a pair of finite subsets of positive integers, not both empty. (I, J) is a **reduced pair** iff

- For every $n \in I$, there is no $k \in (I \smallsetminus \{n\}) \cup J$ such that $n|k$.
- For every $m \in J$, there is no $k \in J \smallsetminus \{m\}$ such that $m|k$

Let (I, J) be a pair of finite subsets of positive integers, not both empty. (I, J) is a **reduced pair** iff

- For every $n \in I$, there is no $k \in (I \setminus \{n\}) \cup J$ such that $n|k$.
- For every $m \in J$, there is no $k \in J \setminus \{m\}$ such that $m|k$

(Panti)

There is a 1-1 correspondence between proper subvarieties of **MV** *and reduced pairs of finite subsets of positive integers not both empty.*

Let (I, J) be a pair of finite subsets of positive integers, not both empty. (I, J) is a **reduced pair** iff

- For every $n \in I$, there is no $k \in (I \setminus \{n\}) \cup J$ such that $n|k$.
- For every $m \in J$, there is no $k \in J \setminus \{m\}$ such that $m|k$

(Panti)

There is a 1-1 correspondence between proper subvarieties of **MV** *and reduced pairs of finite subsets of positive integers not both empty.*

$\mathcal{V}_{I,J} = \mathcal{V}(\{\mathrm{L}_m \mid m \in I\} \cup \{\mathrm{L}_n^\omega \mid n \in J\})$ is axiomatizable by a single equation in one variable of type $\alpha_{I,J}(x) \approx 1$

Quasivarieties

MV and $\mathcal{V}_{I,J}$ for all reduced (I, J) are quasivarieties.

Quasivarieties

MV and $\mathcal{V}_{I,J}$ for all reduced (I, J) are quasivarieties.

$\mathbf{MV} = \mathcal{Q}([0,1] \cap \mathbb{Q}) = \mathcal{Q}([0,1])$.

Quasivarieties

MV and $\mathcal{V}_{I,J}$ for all reduced (I, J) are quasivarieties.

$$\mathbf{MV} = \mathcal{Q}([0,1] \cap \mathbb{Q}) = \mathcal{Q}([0,1]).$$

However in general, *the quasivariety generated by an infinite subalgebra of* $[0, 1]$ *is* **not** *the class* **MV**.

Quasivarieties

MV and $\mathcal{V}_{I,J}$ for all reduced (I, J) are quasivarieties.

$$\mathbf{MV} = \mathcal{Q}([0,1] \cap \mathbb{Q}) = \mathcal{Q}([0,1]).$$

However in general, *the quasivariety generated by an infinite subalgebra of* $[0,1]$ *is* **not** *the class* **MV**.

If S is an infinite subalgebra of $[0,1]$ such that $\frac{1}{2} \notin S$, then

$S \models \neg x \approx x \Rightarrow x \approx 1$ while $[0,1] \not\models \neg x \approx x \Rightarrow x \approx 1$

$$\mathcal{Q}(S) \neq \mathcal{Q}([0,1]) = \mathbf{MV}$$

For every proper subvariety **V** *of* **MV**,

$$\mathbf{V} = \mathcal{V}_{I,J} = \mathcal{Q}(\{\mathrm{L}_m \mid m \in I\} \cup \{\mathrm{L}_m^{\omega} \mid n \in J\}).$$

For every proper subvariety **V** *of* **MV**,

$$\mathbf{V} = \mathcal{V}_{I,J} = \mathcal{Q}(\{\mathrm{L}_m \mid m \in I\} \cup \{\mathrm{L}_m^\omega \mid n \in J\}).$$

However infinite MV-chains of same finite rank n **do not** satisfy the same quasi-equations.

For every proper subvariety **V** *of* **MV**,

$$\mathbf{V} = \mathcal{V}_{I,J} = \mathcal{Q}(\{\mathrm{L}_m \mid m \in I\} \cup \{\mathrm{L}_m^\omega \mid n \in J\}).$$

However infinite MV-chains of same finite rank n **do not** satisfy the same quasi-equations.

$\mathrm{L}_2^1 := \Gamma(\mathbb{Z} \times_{lex} \mathbb{Z}, (2,1)) \models x \approx \neg x \Rightarrow x \approx 1.$

$\mathrm{L}_2^\omega := \Gamma(\mathbb{Z} \times_{lex} \mathbb{Z}, (2,0)) \not\models x \approx \neg x \Rightarrow x \approx 1.$

$$\mathcal{Q}(\mathrm{L}_2^1) \subsetneq \mathcal{Q}(\mathrm{L}_2^\omega) = \mathcal{V}(\mathrm{L}_2^\omega) = \mathcal{V}_{\emptyset,\{2\}}$$

Rational elements

Given $A = \Gamma(G, b)$, the set $\mathrm{Div}(A)$ of **divisors** of A is defined by:

$$\mathrm{Div}(A) = \{n \in \omega \mid \exists c \in G \text{ such that } n\ c = b\}.$$

We say that $a \in A$ is a **rational element** of A if and only if there exist $n \in \mathrm{Div}(A)$ and $0 \leq m \leq n$ such that $a = m\ \frac{b}{n}$.

$$a = \frac{m}{n} \in A \cap \mathbb{Q}.$$

Rational elements

Given $A = \Gamma(G, b)$, the set $\mathrm{Div}(A)$ of **divisors** of A is defined by:

$$\mathrm{Div}(A) = \{n \in \omega \mid \exists c \in G \text{ such that } n\, c = b\}.$$

We say that $a \in A$ is a **rational element** of A if and only if there exist $n \in \mathrm{Div}(A)$ and $0 \leq m \leq n$ such that $a = m\,\frac{b}{n}$.

$$a = \frac{m}{n} \in A \cap \mathbb{Q}.$$

This notion extends the natural definition of a rational element when the MV-algebra is a subalgebra of $[0, 1]$.

Moreover, $\langle A \cap \mathbb{Q}, \oplus, \neg, 0\rangle$ is a subalgebra of A isomorphic to a subalgebra of $[0, 1] \cap \mathbb{Q}$.

Quasivarieties generated by MV-chains

Theorem

Two MV-chains generate the same quasivariety iff they have the same order, the same rank, and they contain the same rational elements.

Quasivarieties generated by MV-chains

Theorem

K *is a quasivariety generated by MV-chains if and only if there are* α, γ, κ *subsets of positive integers, not all of them empty, and for every* $i \in \gamma$*, a nonempty subset* $\gamma(i) \subseteq Div(i)$ *such that*

$$\mathbf{K} = \mathcal{Q}(\{\mathrm{L}_n : n \in \alpha\} \cup \{\mathrm{L}_i^{d_i} : i \in \gamma,\ \ d_i \in \gamma(i)\} \cup \{\mathrm{S}_k : k \in \kappa\}).$$

Quasivarieties generated by MV-chains

Theorem

K *is a quasivariety generated by MV-chains if and only if there are* α, γ, κ *subsets of positive integers, not all of them empty, and for every* $i \in \gamma$, *a nonempty subset* $\gamma(i) \subseteq Div(i)$ *such that*

$$\mathbf{K} = \mathcal{Q}(\{\mathrm{L}_n : n \in \alpha\} \cup \{\mathrm{L}_i^{d_i} : i \in \gamma,\ \ d_i \in \gamma(i)\} \cup \{\mathrm{S}_k : k \in \kappa\}).$$

- *Every quasivariety generated by MV-chains contained in a proper subvariety of* **MV** *is finitely axiomatizable.*
- *For every* $n > 0$, $\mathcal{Q}(\mathrm{S}_n)$ *is not finitely axiomatizable.*
- *The lattice of all quasivarieties generated by MV-chains is a bounded distributive lattice*

Least V-quasivarieties

Let **V** be a variety of algebras of same type . We say that a quasivariety **K** of algebras of same type is a **V-quasivariety**, provided that it generates **V** as a variety. (i.e. $\mathcal{V}(\mathbf{K}) = \mathbf{V}$)

Least V-quasivarieties

Let **V** be a variety of algebras of same type . We say that a quasivariety **K** of algebras of same type is a **V-quasivariety**, provided that it generates **V** as a variety. (i.e. $\mathcal{V}(\mathbf{K}) = \mathbf{V}$)

Let $F_V(X)$ be the free algebra of V with X generators.

Theorem

If X is infinite then, $\mathcal{Q}(F_V(X))$ is the least V-quasivariety.

Least V-quasivarieties

Let **V** be a variety of algebras of same type . We say that a quasivariety **K** of algebras of same type is a **V-quasivariety**, provided that it generates **V** as a variety. (i.e. $\mathcal{V}(\mathbf{K}) = \mathbf{V}$)

Let $F_V(X)$ be the free algebra of V with X generators.

Theorem

If X is infinite then, $\mathcal{Q}(F_V(X))$ is the least V-quasivariety.

Since any variety of MV-algebras can be distinguished by an axiom in just one variable,

Corollary

For every variety V of MV-algebras,
$\mathcal{Q}(F_V(\{x\}))$ is the least V-quasivariety.

From the characterization of quasivarieties generated by MV-chains it can be deduced:

- $\mathcal{Q}(\mathrm{S}_1)$ is the least **MV**-quasivariety generated by chains.

From the characterization of quasivarieties generated by MV-chains it can be deduced:

- $\mathcal{Q}(\mathrm{S}_1)$ is the least **MV**-quasivariety generated by chains.
- $\mathcal{Q}(\mathrm{L}_n^1)$ is the least $\mathcal{V}(\mathrm{L}_n^\omega)$-quasivariety generated by chains.

From the characterization of quasivarieties generated by MV-chains it can be deduced:

- $\mathcal{Q}(\mathrm{S}_1)$ is the least **MV**-quasivariety generated by chains.
- $\mathcal{Q}(\mathrm{L}_n^1)$ is the least $\mathcal{V}(\mathrm{L}_n^\omega)$-quasivariety generated by chains.
- $\mathcal{Q}(\mathrm{L}_n)$ is the least $\mathcal{V}(\mathrm{L}_n)$-quasivariety generated by chains.

From the characterization of quasivarieties generated by MV-chains it can be deduced:

- $\mathcal{Q}(\mathrm{S}_1)$ is the least **MV**-quasivariety generated by chains.
- $\mathcal{Q}(\mathrm{L}_n^1)$ is the least $\mathcal{V}(\mathrm{L}_n^\omega)$-quasivariety generated by chains.
- $\mathcal{Q}(\mathrm{L}_n)$ is the least $\mathcal{V}(\mathrm{L}_n)$-quasivariety generated by chains.
- For every reduced pair (I, J), $\mathcal{Q}(\{\mathrm{L}_n \ : \ n \in I\} \cup \{\mathrm{L}_m^1 \ : \ m \in J\})$ is the least $\mathcal{V}_{I,J}$-quasivariety generated by chains.

- $\mathcal{Q}(\mathrm{S}_1)$ *is not the least* **MV***-quasivariety.*

- $\mathcal{Q}(\mathrm{S}_1)$ *is not the least* **MV**-*quasivariety.*

For every $n > 1$,

- $\mathcal{Q}(\mathrm{L}_n^1)$ *is not the least* $\mathcal{V}(\mathrm{L}_n^\omega)$-*quasivariety*

- $\mathcal{Q}(\mathrm{S}_1)$ *is not the least* **MV**-*quasivariety.*

For every $n > 1$,

- $\mathcal{Q}(\mathrm{L}_n^1)$ *is not the least* $\mathcal{V}(\mathrm{L}_n^\omega)$-*quasivariety*

- $\mathcal{Q}(\mathrm{L}_n)$ *is not the least* $\mathcal{V}(\mathrm{L}_n)$-*quasivariety.*

- $\mathcal{Q}(\mathrm{S}_1)$ *is not the least* **MV**-*quasivariety.*

For every $n > 1$,

- $\mathcal{Q}(\mathrm{L}_n^1)$ *is not the least* $\mathcal{V}(\mathrm{L}_n^\omega)$*-quasivariety*
- $\mathcal{Q}(\mathrm{L}_n)$ *is not the least* $\mathcal{V}(\mathrm{L}_n)$*-quasivariety.*
- $\mathcal{Q}(\mathrm{L}_1 \times \mathrm{L}_n)$ *is the least* $\mathcal{V}(\mathrm{L}_n)$*-quasivariety.*

- $\mathcal{Q}(\mathrm{S}_1)$ *is not the least* **MV**-*quasivariety.*

For every $n > 1$,

- $\mathcal{Q}(\mathrm{L}_n^1)$ *is not the least* $\mathcal{V}(\mathrm{L}_n^\omega)$-*quasivariety*
- $\mathcal{Q}(\mathrm{L}_n)$ *is not the least* $\mathcal{V}(\mathrm{L}_n)$-*quasivariety.*
- $\mathcal{Q}(\mathrm{L}_1 \times \mathrm{L}_n)$ *is the least* $\mathcal{V}(\mathrm{L}_n)$-*quasivariety.*
- $\mathcal{Q}(\mathrm{L}_1 \times \mathrm{L}_1) = \mathcal{Q}(\mathrm{L}_1) = \mathbf{B}$ is the least **B**-quasivariety.

Main Results

Theorem

For every $n > 0$, $\mathcal{Q}(\mathrm{L}_1 \times \mathrm{L}_n^1)$ is the least $\mathcal{V}(\mathrm{L}_n^\omega)$-quasivariety.

Main Results

Theorem

For every $n > 0$, $\mathcal{Q}(\mathrm{L}_1 \times \mathrm{L}_n^1)$ is the least $\mathcal{V}(\mathrm{L}_n^\omega)$-quasivariety.

Theorem

For every $n > 0$, $\mathcal{Q}(\mathrm{L}_1 \times \mathrm{L}_n^1)$ is the class of all bipartite algebras in $\mathcal{Q}(\mathrm{L}_n^1)$.

Main Results

If (I, J) is a reduced pair, we write
$\mathcal{Q}_{I,J} := \mathcal{Q}(\{\mathrm{L}_1 \times \mathrm{L}_m \mid m \in I\} \cup \{\mathrm{L}_1 \times \mathrm{L}_n^1 \mid n \in J\})$.

Main Results

If (I, J) is a reduced pair, we write
$\mathcal{Q}_{I,J} := \mathcal{Q}(\{\mathrm{L}_1 \times \mathrm{L}_m \mid m \in I\} \cup \{\mathrm{L}_1 \times \mathrm{L}_n^1 \mid n \in J\})$.

Theorem

$\mathcal{Q}_{I,J}$ is the least $\mathcal{V}_{I,J}$-quasivariety.

Main Results

If (I, J) is a reduced pair, we write
$\mathcal{Q}_{I,J} := \mathcal{Q}(\{\mathrm{L}_1 \times \mathrm{L}_m \mid m \in I\} \cup \{\mathrm{L}_1 \times \mathrm{L}_n^1 \mid n \in J\})$.

Theorem

$\mathcal{Q}_{I,J}$ is the least $\mathcal{V}_{I,J}$-quasivariety.

Theorem

$\mathcal{Q}_{I,J}$ is the class of all bipartite algebras in $\mathcal{Q}(\{\mathrm{L}_m \mid m \in I\} \cup \{\mathrm{L}_n^1 \mid n \in J\})$

Since $\mathrm{L}_1^1 \cong \mathrm{L}_1^\omega$ and L_1^1 is embeddable into $\mathrm{L}_1 \times \mathrm{L}_1^1$

Corollary

$\mathcal{Q}(\mathrm{L}_1 \times \mathrm{L}_1^1) = \mathcal{Q}(\mathrm{L}_1^\omega) = \mathcal{V}(\mathrm{L}_1^\omega)$ *is the least* $\mathcal{V}_{\{1\},\emptyset}$*-quasivariety.*
i.e. $\mathcal{V}_{\{1\},\emptyset}$ *is structurally complete.*

What about the least **MV**-quasivariety, $\mathcal{Q}(\mathcal{M}([0,1]))$?

What about the least **MV**-quasivariety, $\mathcal{Q}(\mathcal{M}([0,1]))$?

Since $\mathrm{L}_1 \times \mathrm{S}_1 \notin \mathcal{Q}(\mathcal{M}([0,1]))$,

$\mathcal{Q}(\mathrm{L}_1 \times \mathrm{S}_1)$ *is not the least* **MV***-quasivariety.*

What about the least **MV**-quasivariety, $\mathcal{Q}(\mathcal{M}([0,1]))$?

Since $\mathrm{L}_1 \times \mathrm{S}_1 \notin \mathcal{Q}(\mathcal{M}([0,1]))$,

$\mathcal{Q}(\mathrm{L}_1 \times \mathrm{S}_1)$ *is not the least* **MV***-quasivariety.*

And moreover since $\mathrm{L}_1 \times \mathrm{L}_n^1 \notin \mathcal{Q}(\mathcal{M}([0,1]))$ for every $n > 1$

There is an infinite numerable chain of quasivarieties, $\mathbf{K}_1, \mathbf{K}_2, \ldots$ *such that*

$$\mathcal{Q}(\mathcal{M}([0,1])) \subsetneq \mathbf{K}_1 \subsetneq \mathbf{K}_2 \subsetneq \cdots \subsetneq \mathcal{Q}(\mathrm{L}_1 \times \mathrm{S}_1)$$

Order structure of least V-quasivarieties

Varieties of MV-algebras are obviously in 1-1 correspondence with all least **V**-quasivarieties.

Order structure of least V-quasivarieties

Varieties of MV-algebras are obviously in 1-1 correspondence with all least **V**-quasivarieties.

However they *do not share same ordered structure (by $\subseteq$)*.

Order structure of least V-quasivarieties

Varieties of MV-algebras are obviously in 1-1 correspondence with all least **V**-quasivarieties.

However they *do not share same ordered structure (by* $\subseteq$*)*.

For every $n > 1$,

$$\mathcal{V}_{\{n\},\emptyset} = \mathcal{V}(\mathrm{L}_n) \subseteq \mathcal{V}(\mathrm{L}_n^{\omega}) = \mathcal{V}_{\emptyset,\{n\}}$$

$$\mathcal{Q}_{\{n\},\emptyset} = \mathcal{Q}(\mathrm{L}_1 \times \mathrm{L}_n) \not\subseteq \mathcal{Q}(\mathrm{L}_1 \times \mathrm{L}_n^1) = \mathcal{Q}_{\emptyset,\{n\}}$$

Order structure of least V-quasivarieties

The poset of all least $\mathbf{V}$*-quasivarieties ordered by the inclusion is not a lattice, nor a semilattice.*

Order structure of least V-quasivarieties

The poset of all least **V***-quasivarieties ordered by the inclusion is not a lattice, nor a semilattice.*

$$\langle\{\mathcal{Q}_{I,\emptyset} : (I,\emptyset) \text{ reduced pair}\};\subseteq\rangle \cong \langle\{\mathcal{V}_{I,\emptyset} : (I,\emptyset) \text{ reduced pair}\};\subseteq\rangle$$

$$\langle\{\mathcal{Q}_{\emptyset,J} : (\emptyset,J) \text{ reduced pair}\};\subseteq\rangle \cong \langle\{\mathcal{V}_{\emptyset,J} : (\emptyset,J) \text{ reduced pair}\};\subseteq\rangle$$

The class $\mathbf{B} = \mathcal{V}_{\{1\},\emptyset} = \mathcal{Q}_{\{1\},\emptyset}$ of all Boolean algebras is the smallest among all non trivial quasivarieties.

The class $\mathbf{B} = \mathcal{V}_{\{1\},\emptyset} = \mathcal{Q}_{\{1\},\emptyset}$ of all Boolean algebras is the smallest among all non trivial quasivarieties.

Corollary

$\{\mathcal{V}(\mathrm{L}_1^\omega)\} \cup \{\mathcal{Q}(\mathrm{L}_1 \times \mathrm{L}_p) : p \text{ prime}\}$ is the class of all minimal quasivarieties (in the set of all non trivial MV-quasivarieties different from $\mathbf{B}$*).*

The class $\mathbf{B} = \mathcal{V}_{\{1\},\emptyset} = \mathcal{Q}_{\{1\},\emptyset}$ of all Boolean algebras is the smallest among all non trivial quasivarieties.

Corollary

$\{\mathcal{V}(\mathrm{L}_1^\omega)\} \cup \{\mathcal{Q}(\mathrm{L}_1 \times \mathrm{L}_p) : p \text{ prime}\}$ *is the class of all minimal quasivarieties (in the set of all non trivial MV-quasivarieties different from* $\mathbf{B}$*).*

Corollary

$\mathcal{Q}(\mathcal{M}([0,1]))$ *is the only maximal (in the set of all struct. compl. MV-quasivarieties) and almost minimal (in the set of all non trivial MV-quasivarieties different from* $\mathbf{B}$*), it just contains* $\mathcal{V}(\mathrm{L}_1^\omega)$*.*

Structurally complete axiomatic extensions

Corollary

There are exactly two structurally complete axiomatic extensions of $Ł_\infty$, namely:

- *CPC*
- $\vdash_{\emptyset,\{1\}} = Ł_\infty + 2\varphi^2 \leftrightarrow (2\varphi)^2$.

Axiomatization

Admissible rules for finite valued Łukasiewics logics

- *Let L be an extension of* $Ł_\infty$. *Then L is an extension of a finite valued Łukasiewicz logic iff L is n-contractive for some* $n \in \omega$.
- *Let L be an n-contractive extension of* $Ł_\infty$ *(BL). Then every L-unifiable formula is L-projective.* **(Dzik)**
- *Every finite valued Łukasiewicz logic is almost structurally complete.*
- $\neg(\varphi \vee \neg\varphi)^n / \perp$ *is a basis of passive admissible rules for every n-contractive extension of* $Ł_\infty$. **(Jeřábek)**

Axiomatization

Admissible rules for $Ł_\infty$

(Jeřábek)

- *Infinite basis for $Ł_\infty$-admissible rules.*
- *$Ł_\infty$-admissible rules are not finitely based.*
- *Let L be any extension of $Ł_\infty$ (MTL). Then $\{\neg(\varphi \vee \neg\varphi)^n / \perp \ : n \in \omega\}$ is a basis for passive L-admissible rules.*

More algebraic results

Theorem (G. Metcalfe - C. Röthlisberger)

The following are equivalent for any $B \in \mathbf{S}(F_{\mathbf{K}}(\omega))$

- $\mathbf{K}$ *is almost structurally complete.*
- $\mathcal{Q}(\{A \times B : A \in \mathbf{K}\}) = \mathcal{Q}(F_{\mathbf{K}}(\omega))$.
- $\{A \times B : A \in \mathbf{K}\} \subseteq \mathcal{Q}(F_{\mathbf{K}}(\omega))$.

More algebraic results

Theorem (G. Metcalfe - C. Röthlisberger)

The following are equivalent for any $B \in \mathbf{S}(F_{\mathbf{K}}(\omega))$

- $\mathbf{K}$ *is almost structurally complete.*
- $\mathcal{Q}(\{A \times B : A \in \mathbf{K}\}) = \mathcal{Q}(F_{\mathbf{K}}(\omega))$.
- $\{A \times B : A \in \mathbf{K}\} \subseteq \mathcal{Q}(F_{\mathbf{K}}(\omega))$.

[Our previous algebraic results]

- $\mathcal{Q}_{I,J} = \mathcal{Q}(F_{\mathcal{V}_{I,J}}(\omega))$.
- $\mathcal{Q}_{I,J}$ is the class of all bipartite algebras in $\mathcal{Q}(\{\mathrm{L}_m \mid m \in I\} \cup \{\mathrm{L}_n^1 \mid n \in J\})$.

More algebraic results

Theorem

Let $V_{I,J}$ be a proper subvariety of **MV**.
$\mathcal{Q}(\{\mathrm{L}_n : n \in I\} \cup \{\mathrm{L}_m^1 : m \in J\})$ *is almost structurally complete.*

More algebraic results

Theorem

Let $V_{I,J}$ be a proper subvariety of **MV***.*
$\mathcal{Q}(\{\mathrm{L}_n : n \in I\} \cup \{\mathrm{L}^1_m : m \in J\})$ *is almost structurally complete.*

Corollary

Every quasivariety **K** *such that*
$\mathcal{Q}_{\mathcal{I},\mathcal{J}} \subseteq \mathbf{K} \subseteq \mathcal{Q}(\{\mathrm{L}_n : n \in I\} \cup \{\mathrm{L}^1_m : m \in J\})$ *is almost structurally complete.*

Basis for admissible rules for axiomatic extensions of $Ł_\infty$

- $\mathcal{Q}(\{\mathrm{L}_n : n \in I\} \cup \{\mathrm{L}_m^1 : m \in J\})$ is almost structurally complete.

Basis for admissible rules for axiomatic extensions of Ł$_\infty$

- $\mathcal{Q}(\{\mathrm{L}_n : n \in I\} \cup \{\mathrm{L}_m^1 : m \in J\})$ is almost structurally complete.
- $\mathcal{Q}(\{\mathrm{L}_n : n \in I\} \cup \{\mathrm{L}_m^1 : m \in J\})$ is finitely axiomatizable.

Basis for admissible rules for axiomatic extensions of Ł$_\infty$

- $\mathcal{Q}(\{\mathrm{L}_n : n \in I\} \cup \{\mathrm{L}_m^1 : m \in J\})$ is almost structurally complete.
- $\mathcal{Q}(\{\mathrm{L}_n : n \in I\} \cup \{\mathrm{L}_m^1 : m \in J\})$ is finitely axiomatizable.
- Let $L_{I,J}$ be the axiomatic extension of Ł$_\infty$ associated to $\mathcal{V}_{I,J}$. Then $\neg(\varphi \vee \neg\varphi)^n / \perp$ is a basis for passive $L_{I,J}$-admissible rules where $n = sup(I \cup J) + 1$

Basis for admissible rules for axiomatic extensions of $Ł_\infty$

- $\mathcal{Q}(\{\mathrm{L}_n : n \in I\} \cup \{\mathrm{L}^1_m : m \in J\})$ is almost structurally complete.
- $\mathcal{Q}(\{\mathrm{L}_n : n \in I\} \cup \{\mathrm{L}^1_m : m \in J\})$ is finitely axiomatizable.
- Let $L_{I,J}$ be the axiomatic extension of $Ł_\infty$ associated to $\mathcal{V}_{I,J}$. Then $\neg(\varphi \vee \neg\varphi)^n / \perp$ is a basis for passive $L_{I,J}$-admissible rules where $n = sup(I \cup J) + 1$

Theorem

Admissible rules for proper axiomatic extensions of $Ł_\infty$ are finitely based.

Basis for admissible rules for axiomatic extensions of $Ł_\infty$

Theorem

Let (I, J) be a reduced pair, then a base of admissible rules for $\vdash_{I,J}$ is given by

- $Ł1,\ Ł2,\ Ł3,\ Ł4 + M.P.$
- $\alpha_{I,J}(\gamma)$.
- $[(\neg\varphi)^{p-1} \leftrightarrow \varphi] \vee [\psi \leftrightarrow \chi] \ / \ \psi \leftrightarrow \chi$
  for every prime number $p \in Div(J) \smallsetminus Div(I)$
- $[(\neg\varphi)^{q-1} \leftrightarrow \varphi] \vee [\psi \leftrightarrow \chi] \ / \ \alpha_{Iq,\emptyset}(\gamma) \vee (\psi \leftrightarrow \chi)$
  for every prime number $q \in Div(I)$, where $I_q = \{n \in I : q|n\}$
- $\neg(\varphi \vee \neg\varphi)^n / \perp$
  where $n = sup(I \cup J) + 1$

Conclusions

- Komori's type characterization of $\mathcal{Q}_{I,J}$ the least $\mathcal{V}_{I,J}$-quasivariety for every proper subvariety $\mathcal{V}_{I,J}$ of **MV**.

Conclusions

- Komori's type characterization of $\mathcal{Q}_{I,J}$ the least $\mathcal{V}_{I,J}$-quasivariety for every proper subvariety $\mathcal{V}_{I,J}$ of **MV**.
- $\mathcal{Q}_{I,J}$ is the class of bipartite algebras in the least $\mathcal{V}_{I,J}$-quasivariety generated by chains.

Conclusions

- Komori's type characterization of $\mathcal{Q}_{I,J}$ the least $\mathcal{V}_{I,J}$-quasivariety for every proper subvariety $\mathcal{V}_{I,J}$ of **MV**.
- $\mathcal{Q}_{I,J}$ is the class of bipartite algebras in the least $\mathcal{V}_{I,J}$-quasivariety generated by chains.
- This second statement is not true for the case of **MV**.

Conclusions

- Komori's type characterization of $\mathcal{Q}_{I,J}$ the least $\mathcal{V}_{I,J}$-quasivariety for every proper subvariety $\mathcal{V}_{I,J}$ of **MV**.
- $\mathcal{Q}_{I,J}$ is the class of bipartite algebras in the least $\mathcal{V}_{I,J}$-quasivariety generated by chains.
- This second statement is not true for the case of **MV**.
- The poset of least V-quasivarieties of **MV** is not a lattice nor a semilattice. Restricted to suitable classes it behaves like the lattice of subvarieties of **MV**.

Conclusions

- Komori's type characterization of $\mathcal{Q}_{I,J}$ the least $\mathcal{V}_{I,J}$-quasivariety for every proper subvariety $\mathcal{V}_{I,J}$ of **MV**.
- $\mathcal{Q}_{I,J}$ is the class of bipartite algebras in the least $\mathcal{V}_{I,J}$-quasivariety generated by chains.
- This second statement is not true for the case of **MV**.
- The poset of least V-quasivarieties of **MV** is not a lattice nor a semilattice. Restricted to suitable classes it behaves like the lattice of subvarieties of **MV**.
- Description of all minimal quasivarieties.

Conclusions

Using the algebraic results

- Description of all structurally complete axiomatic extensions of $Ł_\infty$.

Conclusions

Using the algebraic results

- Description of all structurally complete axiomatic extensions of $Ł_\infty$.
- Admissible rules for proper axiomatic extensions of $Ł_\infty$ are finitely based.

Conclusions

Using the algebraic results

- Description of all structurally complete axiomatic extensions of $Ł_\infty$.
- Admissible rules for proper axiomatic extensions of $Ł_\infty$ are finitely based.
- Basis for admissible rules for proper axiomatic extensions of $Ł_\infty$

Thank you for your attention

- Gispert, J (2014) Least V-quasivarieties of MV-algebras, *Fuzzy Sets and systems*, http://dx.doi.org/10.1016/j.fss.2014.07.011.
- Gispert, J., Mundici, D. (2005) MV-algebras: a variety for magnitudes with archimedean units. *Algebra Universalis*, **53**, p. 7-43.
- Gispert, J (2002) Universal classes of MV-algebras with applications to many-valued logics, *Mathematical Logic Quarterly*, **48** , p. 581-601.
- Gispert, J., Mundici, D., Torrens, A. (1999) Ultraproducts of Z with an application to Many-Valued Logics. *Journal of Algebra*, **219**, p. 214-233.
- Gispert, J., Torrens, A. (1998) Quasivarieties generated by simple MV-algebras. *Studia Logica*, **61**, p. 79-99.

Sketch of the proof:

Theorem

For every $n > 0$, $\mathcal{Q}(\mathrm{L}_1 \times \mathrm{L}_n^1)$ is the least $\mathcal{V}(\mathrm{L}_n^\omega)$-quasivariety.

To prove $\mathcal{Q}(\mathrm{L}_1 \times \mathrm{L}_n^1) \subseteq \mathcal{Q}(F_V(\{g\}))$ we prove that

$\mathrm{L}_1 \times \mathrm{L}_n^1$ *is embeddable into* $F_V(\{g\})$

Characterization of $F_V(\{g\})$

DiNola, Grigolia, Panti

Let $V = \mathcal{V}(\mathrm{L}_n^\omega)$ *then* $F_V(\{g\})$ *is the subalgebra of*

$$\prod_{\substack{k|n \\ h<k \\ (k,h)=1}} \left(\mathrm{L}_k^h\right)^2$$

generated by g *defined as* $g(k,h) = (a, \neg a)$ *where* a *is the only one generator of* L_k^h *such that* $a \leq \neg a$.

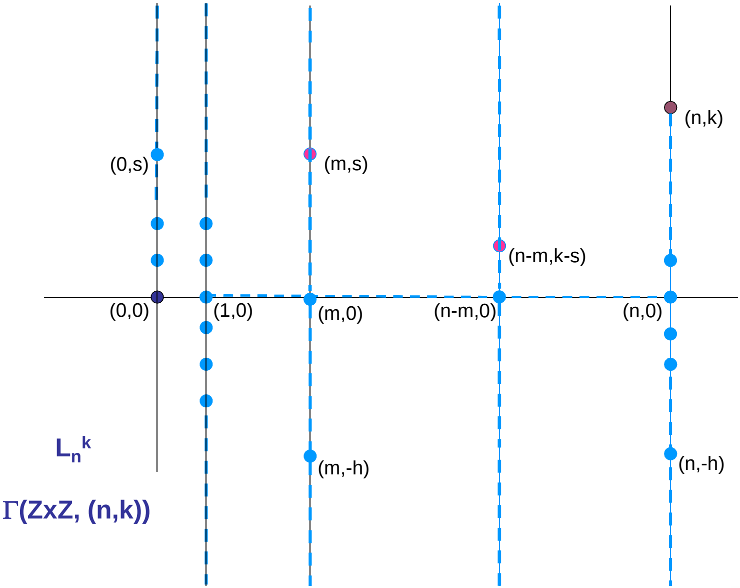
Let B be the subalgebra of $\prod_{\substack{k|n \\ h<k \\ (k,h)=1}} \mathrm{L}_k^h$ generated by v defined as $v(k,h) = a$ where a is the only one generator of L_k^h such that $a \leq \neg a$.

Let B be the subalgebra of $\prod_{\substack{k|n \\ h<k \\ (k,h)=1}} \mathrm{L}_k^h$ generated by v defined as $v(k,h) = a$ where a is the only one generator of L_k^h such that $a \leq \neg a$.

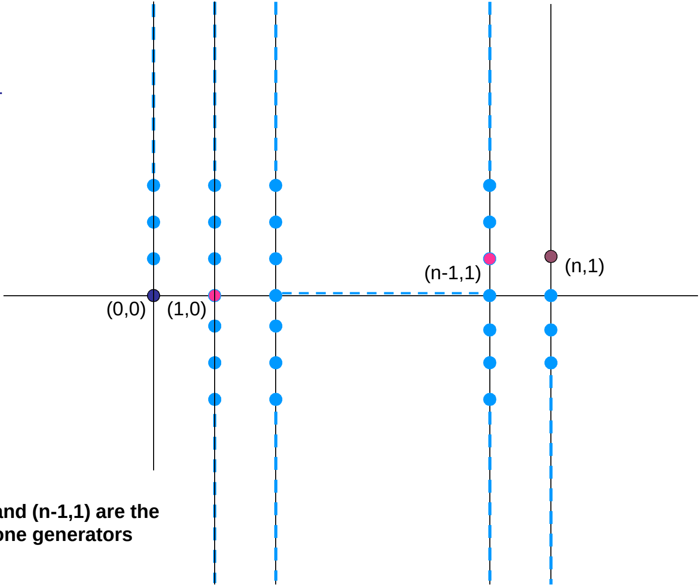
- B *is a subalgebra of* $F_V(\{g\})$ *generated by* $g \wedge \neg g$.

Let B be the subalgebra of $\prod_{\substack{k|n \\ h<k \\ (k,h)=1}} \mathrm{L}_k^h$ generated by v defined as $v(k,h) = a$ where a is the only one generator of L_k^h such that $a \leq \neg a$.

- B *is a subalgebra of* $F_V(\{g\})$ *generated by* $g \wedge \neg g$.
- B *can be decomposed as* $\mathrm{C} \times \mathrm{D}_n$ *where* D_n *is the subalgebra of* $\mathrm{L}_n^{n-1} \times \mathrm{L}_n^1$ *generated by* (a_{n-1}, a_1), a_{n-1} *the generator of* L_n^{n-1} *and* a_1 *the generator of* L_n^1.

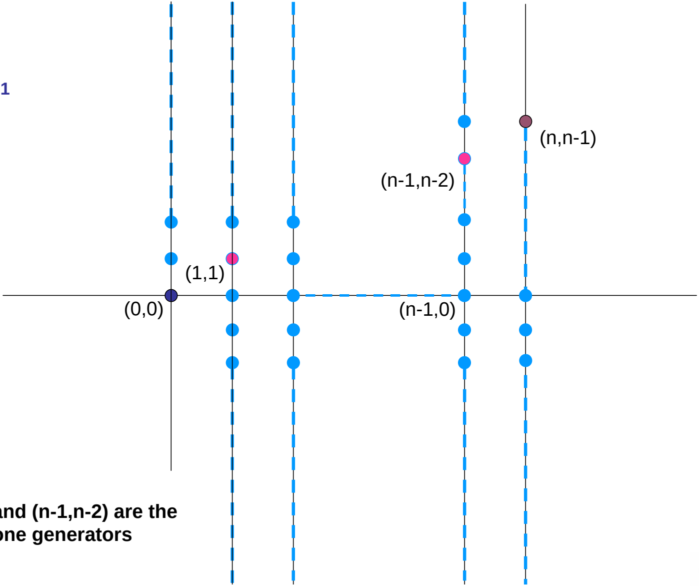

(n,k)
(0,s)
(m,s)
(n-m,k-s)
(0,0)
(1,0)
(m,0)
(n-m,0)
(n,0)
L_n^k
(m,-h)
(n,-h)
Γ(ZxZ, (n,k))

L_n^1

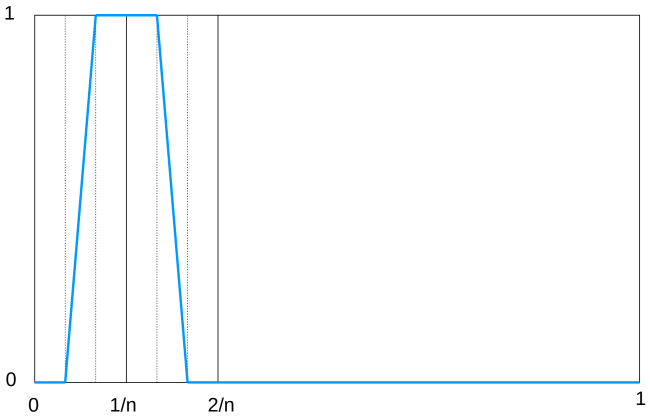


(1,0) and (n-1,1) are the only one generators

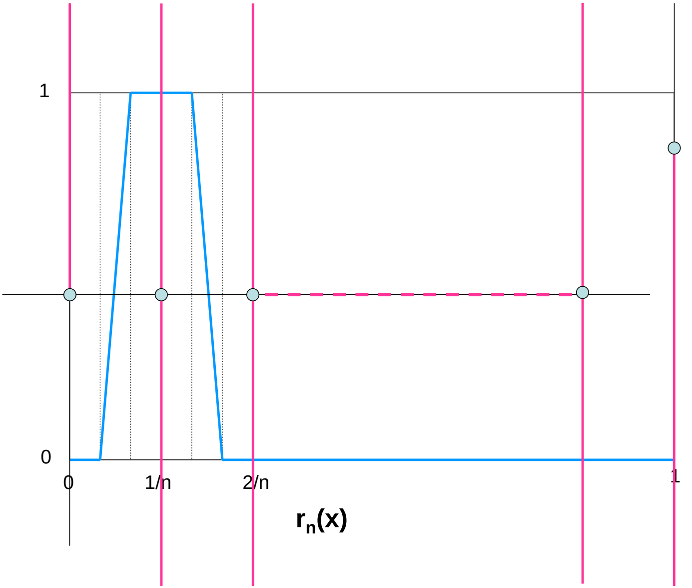
L_n^{n-1}



(1,1) and (n-1,n-2) are the only one generators



$r_n(x)$

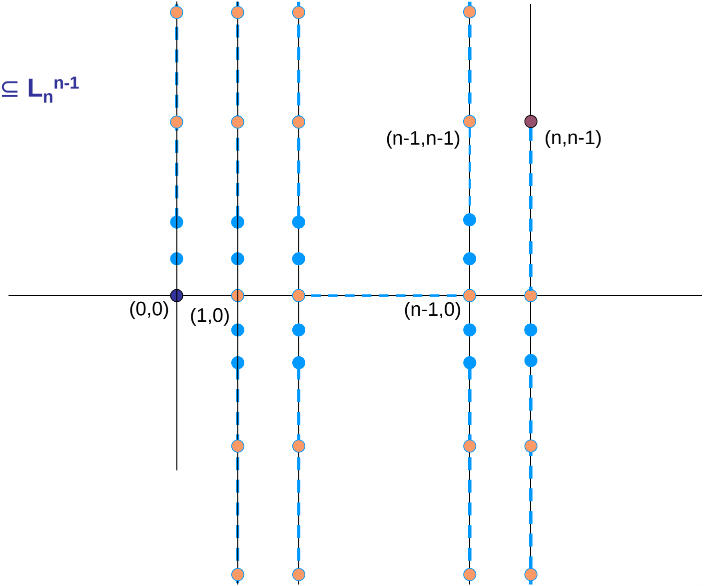

1
0
0
1/n
2/n
1
$r_n(x)$

- B *can be decomposed as* $\mathrm{C} \times \mathrm{D}_n$

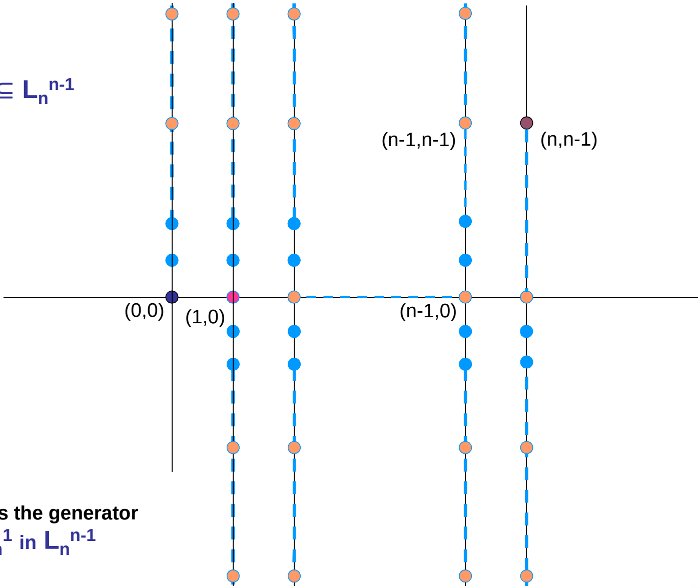
- B *can be decomposed as* $\mathrm{C} \times \mathrm{D}_n$

- L_n^1 *is embeddable into* D_n.
  D_n is the subalgebra of $\mathrm{L}_n^{n-1} \times \mathrm{L}_n^1$ *generated by* $((1,1),(1,0))$
  $(1,1)$ *is the generator of* L_n^{n-1} *and* $(1,0)$ *is the generator of* L_n^1.

$L_n^1 \subseteq L_n^{n-1}$

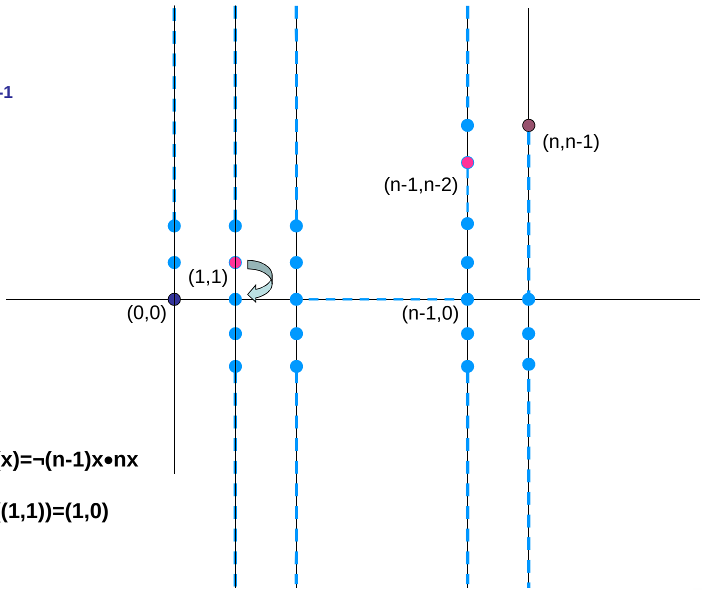


$L_n^1 \subseteq L_n^{n-1}$



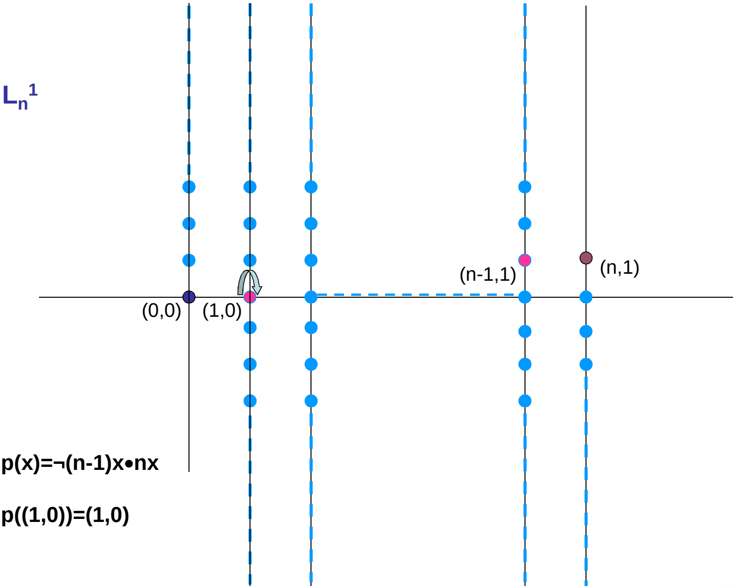
(1,0) is the generator of L_n^1 in L_n^{n-1}

L_n^{n-1}



p(x)=¬(n-1)x•nx

p((1,1))=(1,0)



p(x)=¬(n-1)x•nx

p((1,0))=(1,0)

- B *is a subalgebra of* $F_V(\{g\})$ *generated by* $g \wedge \neg g$.
- B *can be decomposed as* $\mathrm{C} \times \mathrm{D}_n$ *where* D_n *is the subalgebra of* $\mathrm{L}_n^{n-1} \times \mathrm{L}_n^1$ *generated by* (a_{n-1}, a_1), a_{n-1} *the generator of* L_n^{n-1} *and* a_1 *the generator of* L_n^1.
- L_n^1 *is embeddable into* D_n

- B *is a subalgebra of* $F_V(\{g\})$ *generated by* $g \wedge \neg g$.
- B *can be decomposed as* $\mathrm{C} \times \mathrm{D}_n$ *where* D_n *is the subalgebra of* $\mathrm{L}_n^{n-1} \times \mathrm{L}_n^1$ *generated by* (a_{n-1}, a_1), a_{n-1} *the generator of* L_n^{n-1} *and* a_1 *the generator of* L_n^1.
- L_n^1 *is embeddable into* D_n

$$\mathrm{L}_1 \times \mathrm{L}_n^1 \subseteq \mathrm{L}_1 \times \mathrm{D}_n$$

- B *is a subalgebra of* $F_V(\{g\})$ *generated by* $g \wedge \neg g$.
- B *can be decomposed as* $\mathrm{C} \times \mathrm{D}_n$ *where* D_n *is the subalgebra of* $\mathrm{L}_n^{n-1} \times \mathrm{L}_n^1$ *generated by* (a_{n-1}, a_1), a_{n-1} *the generator of* L_n^{n-1} *and* a_1 *the generator of* L_n^1.
- L_n^1 *is embeddable into* D_n

$$\mathrm{L}_1 \times \mathrm{L}_n^1 \subseteq \mathrm{L}_1 \times \mathrm{D}_n \subseteq \mathrm{C} \times \mathrm{D}_n$$

- B *is a subalgebra of* $F_V(\{g\})$ *generated by* $g \wedge \neg g$.
- B *can be decomposed as* $\mathrm{C} \times \mathrm{D}_n$ *where* D_n *is the subalgebra of* $\mathrm{L}_n^{n-1} \times \mathrm{L}_n^1$ *generated by* (a_{n-1}, a_1), a_{n-1} *the generator of* L_n^{n-1} *and* a_1 *the generator of* L_n^1.
- L_n^1 *is embeddable into* D_n

$$\mathrm{L}_1 \times \mathrm{L}_n^1 \subseteq \mathrm{L}_1 \times \mathrm{D}_n \subseteq \mathrm{C} \times \mathrm{D}_n \subseteq \mathrm{B}$$

- B *is a subalgebra of* $F_V(\{g\})$ *generated by* $g \wedge \neg g$.
- B *can be decomposed as* $\mathrm{C} \times \mathrm{D}_n$ *where* D_n *is the subalgebra of* $\mathrm{L}_n^{n-1} \times \mathrm{L}_n^1$ *generated by* (a_{n-1}, a_1), a_{n-1} *the generator of* L_n^{n-1} *and* a_1 *the generator of* L_n^1.
- L_n^1 *is embeddable into* D_n

$$\mathrm{L}_1 \times \mathrm{L}_n^1 \subseteq \mathrm{L}_1 \times \mathrm{D}_n \subseteq \mathrm{C} \times \mathrm{D}_n \subseteq \mathrm{B} \subseteq F_V(\{g\})$$