Standard Completeness II: a novel algebraic approach

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(Joint work with A. Ciabattoni, K.Terui)

Workshop on Admissible Rules II - Les Diablerets

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A novel algebraic approach.

- Reformulation of the proof theoretic methods, with no reference in principle to the proof theory.
- Showing more concretely how to build an embedding into dense algebras
- Extension to the noncommutative cases.

Recall: The usual way to Standard Completeness

Given a logic L:

1. Identify the algebraic semantics of L (L-algebras)
2. Show completeness of L w.r.t. linear, countable L-algebras (L-chains)
3. (Rational completeness): Find an embedding of countable L-chains into dense countable L-chains
4. Dedekind-Mac Neille style completion (embedding into L-algebras with lattice reduct $[0, 1]$)

Algebraic semantics (Step 1.)

- FL-algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, /, \backslash, t, f)$
  - $(A, \wedge, \vee)$ lattice
  - $(A, \cdot, t)$ monoid
  - $f \in A$.
  - $x \cdot y \leq z \Leftrightarrow x \leq z/y \Leftrightarrow y \leq x \backslash z$ for any $x, y, z \in A$

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  - $f \in A$.
  - $x \cdot y \leq z \Leftrightarrow x \leq z/y \Leftrightarrow y \leq x\backslash z$ for any $x, y, z \in A$
- An FL_e-algebra is an FL-algebra where the operation $\cdot$ is commutative (thus $\backslash = /$ and both are denoted with $\rightarrow$)
- An FL_w-algebra is an FL-algebra where f and t are minimum and maximum of the lattice ordering.

Algebraic semantics (Step 1.)

An *FL*-algebra **A** is

- A *Chain* if the lattice ordering is total.
- *Dense* if for any $a, b \in A$ such that $a < b$ there is a $c \in A$ such that $a < c < b$.
- *Bounded* If the lattice ordering has a least element $\perp$ and a maximum element $\top$
- *Complete* if, for any $X \subseteq A$, we have $\vee X, \wedge X \in A$.

Semilinear logics (Step 2.)

Logics complete w.r.t. chains:

- $UL \iff$ bounded FL_e-chains
- $MTL \iff$ bounded FL_{ew}-chains
- $psUL^r \iff$ bounded FL-chains
- $psMTL^r \iff$ bounded FL_w-chains

Semilinear logics (Step 2.)

Logics complete w.r.t. chains:

- $UL + \alpha \iff$ bounded FL_e-chains sat. $t \leq \alpha$
- $MTL + \alpha \iff$ bounded FL_{ew}-chains sat. $t \leq \alpha$
- $psUL^r + \alpha \iff$ bounded FL-chains sat. $t \leq \alpha$
- $psMTL^r + \alpha \iff$ bounded FL_w-chains sat. $t \leq \alpha$

Densifiability (Step 3.)

Definition. A subvariety V of FL algebras is *densifiable*, if for any chain $\mathbf{A}$ in V and $a, b \in A$ such that $a < b$ and for no $c \in A$ we have $a < c < b$ (a, b form a "gap", $a \prec b$)

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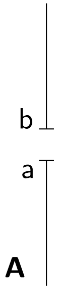

Densifiability (Step 3.)

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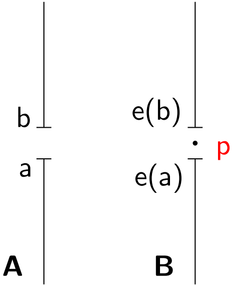

Densifiability

Theorem. (Baldi, Terui 2015) Let V be a densifiable variety. Then every (nontrivial) finite or countable chain in V is embeddable into a countable dense chain in V.

Preframe - Residuated Frames

- A preframe is a structure $(W, W', N, \circ, \varepsilon, \epsilon)$ such that
  - $(W, \circ, \varepsilon)$ is a monoid
  - $N \subseteq W \times W'$
  - $\epsilon \in W'$

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  - $(W, \circ, \varepsilon)$ is a monoid
  - $N \subseteq W \times W'$
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- A *residuated frame* is a preframe with additional operations $\backslash\backslash$ and $//$ satisfying

$$x \circ yNz \Leftrightarrow yNx\backslash\backslash z \Leftrightarrow xNz//y$$

for any $x, y \in W, z \in W'$.

From preframe to residuated frames

We can always extend a preframe $\mathbf{W} = (W, W', N, \circ, \varepsilon, \epsilon)$ to a residuated frame $\tilde{\mathbf{W}} = (W, \tilde{W}', \tilde{N}, \circ, \varepsilon, (\varepsilon, \epsilon, \varepsilon))$, letting

- $\tilde{\mathbf{W}}' := W \times W' \times W$
- $x \; \tilde{N} \; (v_1, z, v_2) \quad \Longleftrightarrow \quad v_1 x v_2 \; N \; z.$

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- $\tilde{\mathbf{W}}' := W \times W' \times W$
- $x \; \tilde{N} \; (v_1, z, v_2) \quad \iff \quad v_1 x v_2 \; N \; z.$
- $(v_1, z, v_2)//y = (v_1, z, y v_2) \quad (v_1, z, v_2)\backslash\backslash x = (v_1 x, z, v_2)$

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- $(v_1, z, v_2)//y = (v_1, z, yv_2) \quad (v_1, z, v_2)\backslash\backslash x = (v_1 x, z, v_2)$

Notice that:

- $xy\ \tilde{N}\ (v_1, z, v_2) \Leftrightarrow x\ \tilde{N}\ (v_1, z, yv_2) \Leftrightarrow y\ \tilde{N}\ (v_1 x, z, v_2)$

Examples of Residuated frames

- Let $\mathbf{A} = (A, \wedge, \vee, \cdot, /, \backslash, t, f)$ be an *FL*-algebra.

  $$W_A = (A, A, N, \circ, f, t)$$

  where N is the lattice ordering $\leq$ of A and $\circ = \cdot$, is *a residuated frame*.
  Letting $\backslash\backslash = \backslash$ and $// = /$ we have that

  $$x \circ yNz \Leftrightarrow yNx\backslash\backslash z \Leftrightarrow xNz//y$$

  is just the residuation property.

Examples of Residuated frames

- Let Fm be the set of formulas in our language, $\vdash_{FL}$ the derivability relation defined by the sequent calculus FL.

  $$W_{FL} = (Fm^*, Fm, N, \circ, \varepsilon, \varepsilon)$$

  where $\circ$ is the comma and N is defined as

  $$\alpha_1 \circ \cdots \circ \alpha_n \; N \; \beta \quad \Leftrightarrow \quad \vdash_{FL} \alpha_1, \ldots, \alpha_n \Rightarrow \beta$$

  is *a preframe*. It can be extended in the canonical way to a residuated frame.

Nuclei on residuated frames

$(W, W', N, \circ, \varepsilon, \epsilon)$ residuated frame, $X \subseteq W$, $Y \subseteq W'$

- $X^{\rhd} = \{y \in W' : XNy\}$
- $Y^{\lhd} = \{w \in W : wNY\}$
- $\gamma_N(X) = X^{\rhd\lhd}$

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$\gamma_N(X)$ is a *closure operator* (1-3), in addition it is a *nucleus* (4).

1. $X \subseteq \gamma_N(X)$
2. $X \subseteq Y \Rightarrow \gamma_N(X) \subseteq \gamma_N(Y)$
3. $\gamma_N(\gamma_N(X)) = \gamma_N(X)$
4. $\gamma_N(X) \circ \gamma_N(Y) \subseteq \gamma_N(X \circ Y)$

The dual algebra

From a Residuated frame $\mathbf{W} = (W, W', N, \circ, \varepsilon, \epsilon)$ we can build a complete *FL*-algebra, the dual algebra of $\mathbf{W}$.

$$\mathbf{W}^{+} = (\gamma_N[P(W)], \cap, \cup_{\gamma_N}, \circ_{\gamma_N}, \backslash, /, \gamma_N(\varepsilon), \epsilon^{\lhd})$$

Where

- $\gamma_N[P(W)] = \{X \subseteq W \text{ such that } \gamma_N(X) = X\}$
- $X \circ_{\gamma_N} Y := \gamma_N(X \circ Y)$
- $X \cup_{\gamma_N} Y := \gamma_N(X \cup Y)$
- $X \backslash Y := \{y : X \circ \{y\} \subseteq Y\}$
- $Y / X := \{y : \{y\} \circ X \subseteq Y\}$

Densifiability for FL_w chains

Let $\mathbf{A} = (A, \wedge, \vee, \cdot, /, \backslash, t, f)$ be a FL_w-chain which is not dense. Assume $a, b \in A$ form a "gap", $a \prec b$.

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$$xNp \Leftrightarrow x \leq a \quad pNy \Leftrightarrow b \leq y \quad pNp$$

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$$xNp \Leftrightarrow x \leq a \quad pNy \Leftrightarrow b \leq y \quad pNp$$

- Preframe: $((A \cup \{p\})^*, A \cup \{p\}, N, \circ, \varepsilon, f)$, with $\circ$ string concatenation, N defined as:
  - $x[p]Nc \quad \Leftrightarrow \quad x[b] \leq c$.
  - $xNp \quad \Leftrightarrow \quad x \leq a$.
  - $x[p]Np$ always holds.
- We call $\tilde{\mathbf{W}}_{\mathbf{A}}^{p}$ the corresponding residuated frame.

Recall: Density elimination for MTL

Given a density-free derivation, ending in

$$\dfrac{\begin{array}{c}\vdots\, d' \\ G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta\end{array}}{G \mid \Gamma \Rightarrow \Delta}\ (\text{density})$$

- Asymmetric substitution: p is replaced
  - With Δ when occurring on the right
  - With Γ when occurring on the left

Recall: Density elimination for *MTL*

$$\dfrac{\begin{array}{c}\vdots\, d' \\ G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta\end{array}}{G \mid \Gamma \Rightarrow \Delta}\ (\mathrm{EC})$$

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Densifiability for FL_w chains

Let $\mathbf{A} = (A, \wedge, \vee, \cdot, /, \backslash, t, f)$ be an FL_w-chain, $a, b \in A$ such that $a \prec b$. We want to add an element p in between a and b, hence we require:

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Densifiability for FL_w chains

Theorem. (Baldi, Terui 2015) FL_w-chains are densifiable.

Proof Idea. Let A be an FL_w chain with $a, b \in A$ such that $a \prec b$, $\tilde{\mathbf{W}}_{\mathbf{A}}^{p}$ residuated frame. We show the following

1. A is an FL_w chain $\longrightarrow$ $\tilde{\mathbf{W}}_{\mathbf{A}}^{\mathbf{p}+}$ is a complete FL_w-chain.
2. There is an embedding

$$e : x \in A \to \{x\}^{\rhd\lhd} \in \tilde{\mathbf{W}}_{\mathbf{A}}^{p+}$$

3. $\tilde{\mathbf{W}}_{\mathbf{A}}^{\mathbf{p}+}$ "fills the gap" between a and b, i.e.

$$\{a\}^{\rhd\lhd} \subset \{p\}^{\rhd\lhd} \subset \{b\}^{\rhd\lhd}$$

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\end{tikzcd}
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$$\frac{xNz \quad yNw}{xNw \text{ or } yNz}\ (com) \qquad \frac{uv\ N\ z}{uxv\ N\ z}\ (w)$$

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From Densifiability to Standard Completeness

- FL_w-chains are densifiable. We can embedd any FL_w chain $\mathbf{A}$ in the complete FL_w chain $\tilde{\mathbf{W}}_{\mathbf{A}}^{p+}$.

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- The logic $PsMTL^r$ is Standard Complete.

From Densifiability to Standard Completeness

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- We can embedd any FL_w-chain into a dense, complete FL_w-chain.
- The logic $PsMTL^r$ is Standard Complete.

What about subvarieties / Axiomatic extensions of $PsMTL^r$?

Densifiability for FL_w chains

Problem. For which equations $t \leq \alpha$ the FL_w-chains satisfying $t \leq \alpha$ are densifiable?

Proof Idea. Let A be an FL_w chain satisfying $t \leq \alpha$ with $a, b \in A$ such that $a \prec b$, $\tilde{\mathbf{W}}_{\mathbf{A}}^{p}$ residuated frame. We need to show

1. A is an FL_w chain satisfying $t \leq \alpha \longrightarrow \tilde{\mathbf{W}}_{\mathbf{A}}^{\mathbf{p}+}$ is a complete FL_w-chain satisfying $t \leq \alpha$
2. There is an embedding

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3. $\tilde{\mathbf{W}}_{\mathbf{A}}^{\mathbf{p}+}$ "fills the gap" between a and b, i.e.

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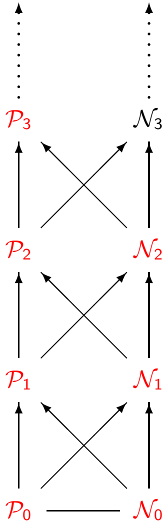
The substructural hierarchy for FL



(Ciabattoni, Galatos, Terui 2012).
Sets $\mathcal{P}_n, \mathcal{N}_n$ of terms defined by:

$$
\begin{aligned}
\mathcal{P}_0, \mathcal{N}_0 &:= \text{Variables} \\
\mathcal{P}_{n+1} &:= \mathcal{N}_n \mid \mathcal{P}_{n+1} \cdot \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid t \mid \bot \\
\mathcal{N}_{n+1} &:= \mathcal{P}_n \mid \mathcal{P}_{n+1} \backslash \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} / \mathcal{P}_{n+1} \\
&:= \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid f \mid \top
\end{aligned}
$$

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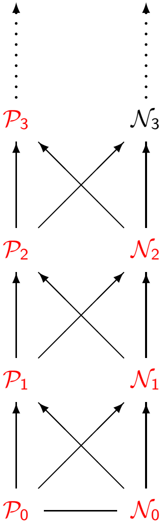

- If a residuated frame $\tilde{W}$ satisfies an *analytic quasi equation*

$$\frac{t_1 N u_1 \quad \text{and} \quad \ldots \quad \text{and} \quad t_m N u_m}{t_0 N u_0}\ (q)$$

  W^+ satisfies the corresponding $\mathcal{N}_2$ equation $t \leq \alpha$.

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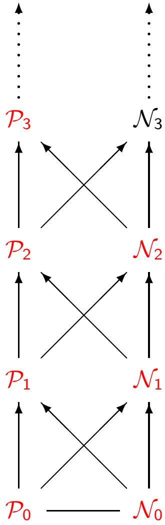

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- If a residuated frame $\tilde{W}$ satisfies an *analytic clause*

$$\frac{t_1 N u_1 \quad \text{and} \quad \ldots \quad \text{and} \quad t_m N u_m}{t_{m+1} N u_{m+1} \ \text{or} \ \ldots \ \text{or} \ t_n N u_n} \ (q)$$

  W^+ satisfies the corresponding $\mathcal{P}_3$ equation $t \leq \alpha$

Examples

$\mathcal{N}_2$ includes:

$t \leq xy\backslash yx$	*(e)*
$t \leq x\backslash xx$	*(c)*
$t \leq x^k\backslash x^n$	*(knotted axioms, $n, k \geq 0$)*
$t \leq \sim(x \wedge \sim x)$	*(no-contradiction)*

$\mathcal{P}_3$ includes:

$t \leq x \vee \sim x$	(excluded middle)
$t \leq \sim x \vee \sim\sim x$	(weak excluded middle)
$t \leq \sim(x \cdot y) \vee (x \wedge y\backslash x \cdot y)$	(wnm)
$t \leq \sim(x \cdot y)^n \vee ((x \wedge y)^{n-1}\backslash(x \cdot y)^n)$	(wnm^n)
$t \leq p_0 \vee (p_0\backslash p_1) \vee \cdots \vee (p_0 \wedge \cdots \wedge p_{k-1}\backslash p_k)$	(bounded size k)
$t \leq (x^{n-1}\backslash x \cdot y) \vee (y\backslash x \cdot y)$	(Ω_n)

Densifiability for FL_w chains

1.

$$\begin{tikzcd}
\mathbf{A} \text{ is an } FL_w \text{ chain sat. } t \le \alpha \arrow[rr, dashed] \arrow[dr] & & \tilde{\mathbf{W}}_{\mathbf{A}}^{p+} \text{ is a } FL_w \text{ chain sat. } t \le \alpha \\
& \tilde{\mathbf{W}}_{\mathbf{A}}^{p} \text{ satisfies } (com),(w),\ (q) \arrow[ur, "\mathcal{N}_2,\ \mathcal{P}_3"'] &
\end{tikzcd}$$

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\begin{tikzcd}
\mathbf{A} \text{ is an } FL_w \text{ chain sat. } t \leq \alpha \arrow[rr, dashed] \arrow[dr, "\text{semi-anchored}"'] & & \tilde{\mathbf{W}}_{\mathbf{A}}^{p+} \text{ is a } FL_w \text{ chain sat. } t \leq \alpha \\
& \tilde{\mathbf{W}}_{\mathbf{A}}^{p} \text{ satisfies } (com),(w),\ (q) \arrow[ur, "\mathcal{N}_2,\ \mathcal{P}_3"'] &
\end{tikzcd}
$$

Example

The equation $t \leq \sim(x \cdot y) \vee (x \wedge y \backslash x \cdot y)$ is equivalent to the clause

$$\frac{xu\ N\ z \quad \text{and} \quad xy\ N\ z \quad \text{and} \quad uy\ N\ z \quad \text{and} \quad uu\ N\ z}{xy\ N\ \epsilon \text{ or } u\ N\ z}\ (wnm)$$

- (x, z) unanchored

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- (x, z) unanchored
- (u, z) anchored

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- $xu\ N\ z$ contains (x, z) unanchored $\Longrightarrow$ $uu\ N\ z$ contains (u, z) anchored .

Example

The equation $t \leq (x^2 \backslash x \cdot y) \vee (y \backslash x \cdot y)$ is equivalent to the analytic clause

$$\frac{yx \; N \; z_1 \quad \text{and} \quad wx \; N \; z_1 \quad \text{and} \quad yx \; N \; z_2 \quad \text{and} \quad wx \; N \; z_2}{wy \; N \; z_2 \text{ or } x \; N \; z_1} \; (\Omega_3)$$

- (y, z_1) unanchored

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- (y, z_1) unanchored
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- $yx\ N\ z_1$ contains (y, z_1) unanchored $\Longrightarrow$ $yx\ N\ z_2$ contains (y, z_2) anchored

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\begin{tikzcd}
\mathbf{A} \text{ is an } FL_w \text{ chain sat. } t \leq \alpha \arrow[rr, dashed] \arrow[dr, "\text{semi-anchored}"'] & & \tilde{\mathbf{W}}_{\mathbf{A}}^{p+} \text{ is a } FL_w \text{ chain sat. } t \leq \alpha \\
& \tilde{\mathbf{W}}_{\mathbf{A}}^{p} \text{ satisfies } (com),(w),\ (q) \arrow[ur, "\mathcal{N}_2,\ \mathcal{P}_3"'] &
\end{tikzcd}
$$

Densifiability for FL_w chains

Theorem. (Baldi, Terui 2015) FL_w chains satisfying a semi-anchored equation $t \leq \alpha$ are densifiable.

Proof Idea. Let A be an FL_w chain satisfying $t \leq \alpha$ with $a, b \in A$ such that $a \prec b$, $\tilde{\mathbf{W}}^p_{\mathbf{A}}$ residuated frame. We show that the following hold

1. A is an FL_w chain satisfying $t \leq \alpha \longrightarrow \tilde{\mathbf{W}}^{\mathbf{p}+}_{\mathbf{A}}$ is a complete FL_w-chain satisfying $t \leq \alpha$
2. There is an embedding

$$e : x \in A \to \{x\}^{\rhd\lhd} \in \tilde{\mathbf{W}}^{p+}_{\mathbf{A}}$$

3. $\tilde{\mathbf{W}}^{\mathbf{p}+}_{\mathbf{A}}$ "fills the gap" between a and b, i.e.

$$\{a\}^{\rhd\lhd} \subset \{p\}^{\rhd\lhd} \subset \{b\}^{\rhd\lhd}$$

From Densifiability to Standard Completeness

- FL_w-chains satisfying a semi-anchored equation $t \leq \alpha$ are densifiable.
- We can embedd any FL_w-chain satisfying $t \leq \alpha$ into a dense, complete FL_w-chain satisfying $t \leq \alpha$.
- The logic $PsMTL^r + \alpha$ is standard complete.

Densifiability for FL_e chains

Let $\mathbf{A} = (A, \wedge, \vee, \cdot, /, \backslash, t, f)$ be an FL_e-chain and $a, b \in A$ such that $a \prec b$.

- Preframe $((A \cup \{p\})^*, A \cup \{p\}, N, t, f)$, with N defined as follows:
  - $xp^n Nc \quad \Leftrightarrow \quad xb^n \leq c$.
  - $xNp \quad \Leftrightarrow \quad x \leq a$.
  - $xp^{n-1}pNp \quad \Leftrightarrow \quad xb^{n-1} \leq t$.
- Residuated frame $\tilde{\mathbf{W}}_{\mathbf{A}}^{p}$ defined in the standard way.

Recall: Density Elimination for *UL*

(Ciabattoni, Metcalfe 2008)

$$\dfrac{\begin{array}{c}\Pi, p \Rightarrow p \\ \vdots\, d \\ G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta\end{array}}{G \mid \Gamma \Rightarrow \Delta}\ (\mathrm{D})$$

- We substitute:
  - $p \Rightarrow p$ with $\Rightarrow t$ (axiom)
  - p with Δ when occurring on the right.
  - p with Γ when occurring on the left.

Recall: Density Elimination

(Ciabattoni, Metcalfe 2008)

$$\dfrac{\begin{array}{c}\Pi \Rightarrow t \\ \vdots\, d^* \\ G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta\end{array}}{G \mid \Gamma \Rightarrow \Delta}\ (\mathrm{EC})$$

- We substitute:
  - $p \Rightarrow p$ with $\Rightarrow t$.
  - p with Δ when occurring on the right.
  - p with Γ when occurring on the left.

Conclusions

- Standard completeness for axiomatic extensions of *UL* with nonlinear $\mathcal{N}_2$ axioms.
- Standard completeness for extensions of *MTL*, $psMTL^r$ with semi-anchored axioms/equations
- An algebraic version of the proof theoretical approach via residuated frames.

Some open problems

- Find a *necessary* condition for standard completeness for MTL extended with axioms within the class $\mathcal{P}_3$ in the substructural hierarchy.
- Prove that any axiomatic extension of UL with axioms within the class $\mathcal{N}_2$ in the substructural hierarchy is standard complete.
- Logics with involutive negation. Long standing open problem: standard completeness of IUL.