Stable Canonical Rules and Admissibility I

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Since the 1970's general methods started to develop for classes of non-classical logics.

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- General methods for axiomatizing large classes of logics include Jankov-de Jongh formulas, Fine-Rautenberg formulas, subframe formulas, and canonical formulas (Jankov, de Jongh, Fine, Rautenberg, Zakharyaschev).

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- Fine (1985) introduced subframe formulas and axiomatized large classes of transitive modal logics by these formulas.

- There exist intermediate and transitive modal logics that are not axiomatizable by Jankov or subframe formulas.

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- Jerabek (2009) extended canonical formulas to canonical rules and showed that each intermediate and transitive modal rule system is axiomatizable by canonical rules.

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This gives a positive solution of Zakharyaschev's problem. However, the solution is via rules and not formulas.

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This method relies on locally finite reducts of Heyting and modal algebras.

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Assumption-free single-conclusion modal rules $/\varphi$ can be identified with modal formulas φ.

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3. $\varphi/\Box\varphi \in \mathcal{S}$.
4. $/\varphi \in \mathcal{S}$ for each theorem φ of **K**.
5. If $\Gamma/\Delta \in \mathcal{S}$, then $\Gamma, \Gamma'/\Delta, \Delta' \in \mathcal{S}$.
6. If $\Gamma/\Delta, \varphi \in \mathcal{S}$ and $\Gamma, \varphi/\Delta \in \mathcal{S}$, then $\Gamma/\Delta \in \mathcal{S}$.
7. If $\Gamma/\Delta \in \mathcal{S}$ and s is a substitution, then $s(\Gamma)/s(\Delta) \in \mathcal{S}$.

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We denote the least modal rule system by $\mathbf{S_K}$, and the complete lattice of modal rule systems by $\Sigma(\mathbf{S_K})$.

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If $\rho \in \mathcal{S}$, then we say that the modal rule system $\mathcal{S}$ entails or derives the modal rule ρ, and write $\mathcal{S} \vdash \rho$.

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and for a modal logic L, let $\Sigma(L) = \mathbf{S_K} + \{/\varphi : \varphi \in L\}$ be the corresponding modal rule system.

Modal rule systems and modal logics

Then $\Lambda : \Sigma(\mathbf{S_K}) \to \Lambda(\mathbf{K})$ and $\Sigma : \Lambda(\mathbf{K}) \to \Sigma(\mathbf{S_K})$ are order-preserving maps such that $\Lambda(\Sigma(L)) = L$ for each $L \in \Lambda(\mathbf{K})$ and $\mathcal{S} \supseteq \Sigma(\Lambda(\mathcal{S}))$ for each $\mathcal{S} \in \Sigma(\mathbf{S_K})$.

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Thus, $\Lambda(\mathbf{K})$ embeds isomorphically into $\Sigma(\mathbf{S_K})$. But the embedding is not a lattice embedding.

We say that a modal logic L is axiomatized (over $\mathbf{K}$) by a set Ξ of multiple-conclusion modal rules if $L = \Lambda(\mathbf{S_K} + \Xi)$.

Modal algebras

A modal algebra $\mathfrak{A} = (A, \Diamond)$ is a Boolean algebra A endowed with a unary operator $\Diamond$ satisfying

1. $\Diamond 0 = 0$;
2. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$.

Modal algebras and modal rule systems

A modal algebra $\mathfrak{A} = (A, \Diamond)$ validates a multiple-conclusion modal rule Γ/Δ provided for every valuation V on A, if $V(\gamma) = 1$ for all $\gamma \in \Gamma$, then $V(\delta) = 1$ for some $\delta \in \Delta$.

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If $\mathfrak{A}$ validates Γ/Δ, we write $\mathfrak{A} \models \Gamma/\Delta$, and if $\mathfrak{A}$ refutes Γ/Δ, we write $\mathfrak{A} \not\models \Gamma/\Delta$.

Modal rule systems and universal classes

If $\Gamma = \{\phi_1, \ldots, \phi_n\}$, $\Delta = \{\psi_1, \ldots, \psi_m\}$, and $\phi_i(\underline{x})$ and $\psi_j(\underline{x})$ are the terms in the first-order language of modal algebras corresponding to the ϕ_i and ψ_j, then $\mathfrak{A} \models \Gamma/\Delta$ iff $\mathfrak{A}$ is a model of the universal sentence $\forall \underline{x}\ (\bigwedge_{i=1}^{n} \phi_i(\underline{x}) = 1 \to \bigvee_{j=1}^{m} \psi_j(\underline{x}) = 1)$.

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A class of modal algebras is a universal class iff it is closed under isomorphisms, subalgebras, and ultraproducts.

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Single-conclusion rule systems and quasi-varieties

Modal algebra $\mathfrak{A}$ validates a single-conclusion modal rule Γ/ψ iff $\mathfrak{A}$ is a model of the sentence $\forall \underline{x}\ (\bigwedge_{i=1}^{n} \phi_i(\underline{x}) = 1 \to \psi(\underline{x}) = 1)$, where $\Gamma = \{\phi_1, \ldots, \phi_n\}$ and $\phi_i(\underline{x})$ and $\psi(\underline{x})$ are the terms in the first-order language of modal algebras corresponding to the ϕ_i and ψ.

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A class of modal algebras is a universal Horn class iff it is a quasi-variety (closed under isomorphisms, subalgebras, products, and ultraproducts).

Correspondence

Let $\mathcal{S}$ be a modal rule system and $\mathcal{U}$ be the universal class corresponding to $\mathcal{S}$.

Then the variety corresponding to the modal logic $\Lambda(\mathcal{S})$ is the variety generated by $\mathcal{U}$.

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The varieties of all modal algebras, **K4**-algebras and **S4**-algebras are not locally finite.

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Theorem.

- (Diego, 1966). The variety of implicative semilattices is locally finite.
- (Folklore). The variety of distributive lattices is locally finite.

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Theorem.

- (G. B and N. B., 2009). Every intermediate logic is axiomatizable by $(\wedge, \to, 0)$-canonical formulas.
- (G. B and N. B., 2013). Every intermediate logic is axiomatizable by $(\wedge, \vee, 0, 1)$-canonical formulas.

Connection with filtrations

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The modern account is discussed in Ghilardi (2010) and van Alten et al. (2013).

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$$x \sim_\Sigma y \text{ iff } (\forall \varphi \in \Sigma)(x \models \varphi \Leftrightarrow y \models \varphi).$$

Let $X' = X/{\sim_\Sigma}$ and let $V'(p) = \{[x] : x \in V(p)\}$, where $[x]$ is the equivalence class of x with respect to $\sim_\Sigma$.

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Definition. For a binary relation R' on X', we say that the triple $\mathfrak{M}' = (X', R', V')$ is a filtration of $\mathfrak{M}$ through Σ if the following two conditions are satisfied:

(F1) $xRy \Rightarrow [x]R'[y]$.

(F2) $[x]R'[y] \Rightarrow (\forall \Diamond\varphi \in \Sigma)(y \models \varphi \Rightarrow x \models \Diamond\varphi)$.

Stable homomorphisms and CDC

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It is easy to see that $h : A \to B$ is stable iff $h(\Box a) \leq \Box h(a)$ for each $a \in A$.

Stable homomorphisms and CDC

The key concepts for developing an algebraic approach to filtrations are stable homomorphisms and the closed domain condition (CDC).

Definition. Let $\mathfrak{A} = (A, \Diamond)$ and $\mathfrak{B} = (B, \Diamond)$ be modal algebras and let $h : A \to B$ be a Boolean homomorphism. We call h a stable homomorphism provided $\Diamond h(a) \leqslant h(\Diamond a)$ for each $a \in A$.

It is easy to see that $h : A \to B$ is stable iff $h(\Box a) \leq \Box h(a)$ for each $a \in A$.

Stable homomorphisms were studied by G. B., Mines, Morandi (2008), Ghilardi (2010), and Coumans, van Gool (2012).

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Definition. Let $\mathfrak{A} = (A, \Diamond)$ and $\mathfrak{B} = (B, \Diamond)$ be modal algebras and let $h : A \to B$ be a stable homomorphism. We say that h satisfies the closed domain condition (CDC) for $D \subseteq A$ if $h(\Diamond a) = \Diamond h(a)$ for $a \in D$.

Key idea

Let $(A, \Diamond)$ and $(B, \Diamond)$ be modal algebras, (X, R) and (Y, R) be their duals, $h : A \to B$ be a Boolean homomorphism and $f : Y \to X$ be the dual of h.

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Let $(A, \Diamond)$ and $(B, \Diamond)$ be modal algebras, (X, R) and (Y, R) be their duals, $h : A \to B$ be a Boolean homomorphism and $f : Y \to X$ be the dual of h.

Then

1. h is one-to-one iff f is onto.
2. h is stable iff f is stable (that is, xRy implies $f(x)Rf(y)$).
3. h is a modal homomorphism iff f is a p-morphism.
4. If h is stable but not a modal homomorphism it may still be the case that $h(\Diamond a) = \Diamond h(a)$ for some $a \in D \subseteq A$.

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1. Being a stable homomorphism dually corresponds to satisfying condition (F1) in the definition of filtration.
2. Satisfying (CDC) dually corresponds to satisfying condition (F2) in the definition of filtration.

Filtrations and finite refutation patterns

Refutation Pattern Theorem.

1. If $\mathbf{S_K} \nvdash \Gamma/\Delta$, then there exist $(\mathfrak{A}_1, D_1), \ldots, (\mathfrak{A}_n, D_n)$ such that each $\mathfrak{A}_i = (A_i, \Diamond_i)$ is a finite modal algebra, $D_i \subseteq A_i$, and for each modal algebra $\mathfrak{B} = (B, \Diamond)$, we have $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is $i \leq n$ and a stable embedding $h : A_i \rightarrowtail B$ satisfying (CDC) for D_i.

2. If $\mathbf{K} \nvdash \varphi$, then there exist $(\mathfrak{A}_1, D_1), \ldots, (\mathfrak{A}_n, D_n)$ such that each $\mathfrak{A}_i = (A_i, \Diamond_i)$ is a finite modal algebra, $D_i \subseteq A_i$, and for each modal algebra $\mathfrak{B} = (B, \Diamond)$, we have $\mathfrak{B} \not\models \varphi$ iff there is $i \leq n$ and a stable embedding $h : A_i \rightarrowtail B$ satisfying (CDC) for D_i.

Proof sketch

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Let Σ be the set of subformulas of $\Gamma \cup \Delta$, A' be the Boolean subalgebra of A generated by $V(\Sigma)$, and $\mathfrak{A}' = (A', \Diamond')$ be a filtration of $\mathfrak{A}$ through Σ.

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Then $\mathfrak{A}'$ is a finite modal algebra refuting Γ/Δ. In fact, $|A'| \leqslant m$, where $m = 2^{2^{|\Sigma|}}$ is the size of the free Boolean algebra on $|\Sigma|$-generators.

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Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ be the list of all finite modal algebras $\mathfrak{A}_i = (A_i, \Diamond_i)$ of size $\leqslant m$ refuting Γ/Δ.

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Let V_i be a valuation on A_i refuting Γ/Δ; that is, $V_i(\gamma) = 1_{A_i}$ for each $\gamma \in \Gamma$ and $V_i(\delta) \neq 1_{A_i}$ for each $\delta \in \Delta$. Set $D_i = \{V_i(\psi) : \Diamond\psi \in \Sigma\}$.

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Key step: Given a modal algebra $\mathfrak{B} = (B, \Diamond)$, we show that $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is $i \leq n$ and a stable embedding $h : A_i \rightarrowtail B$ satisfying (CDC) for D_i.

Stable canonical rules

Definition. Let $\mathfrak{A} = (A, \Diamond)$ be a finite modal algebra and let D be a subset of A. For each $a \in A$ we introduce a new propositional letter p_a and define the stable canonical rule $\rho(\mathfrak{A}, D)$ associated with $\mathfrak{A}$ and D as Γ/Δ, where:

$$
\begin{aligned}
\Gamma \quad = \quad & \{p_{a \vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \cup \\
& \{p_{\neg a} \leftrightarrow \neg p_a : a \in A\} \cup \\
& \{\Diamond p_a \to p_{\Diamond a} : a \in A\} \cup \\
& \{p_{\Diamond a} \to \Diamond p_a : a \in D\},
\end{aligned}
$$

and

$$
\Delta \quad = \quad \{p_a \leftrightarrow p_b : a, b \in A, a \neq b\}.
$$

Stable canonical rules

Stable Canonical Rule Theorem. Let $\mathfrak{A} = (A, \Diamond)$ be a finite modal algebra, $D \subseteq A$, and $\mathfrak{B} = (B, \Diamond)$ be a modal algebra. Then $\mathfrak{B} \not\models \rho(\mathfrak{A}, D)$ iff there is a stable embedding $h : A \rightarrowtail B$ satisfying (CDC) for D.

Stable canonical rules

Corollary.

1. If $\mathbf{S_K} \not\vdash \Gamma/\Delta$, then there exist $(\mathfrak{A}_1, D_1), \ldots, (\mathfrak{A}_n, D_n)$ such that each $\mathfrak{A}_i = (A_i, \Diamond_i)$ is a finite modal algebra, $D_i \subseteq A_i$, and for each modal algebra $\mathfrak{B} = (B, \Diamond)$, we have:

$$\mathfrak{B} \models \Gamma/\Delta \text{ iff } \mathfrak{B} \models \rho(\mathfrak{A}_1, D_1), \ldots, \rho(\mathfrak{A}_n, D_n).$$

2. If $\mathbf{K} \not\vdash \varphi$, then there exist $(\mathfrak{A}_1, D_1), \ldots, (\mathfrak{A}_n, D_n)$ such that each $\mathfrak{A}_i = (A_i, \Diamond_i)$ is a finite modal algebra, $D_i \subseteq A_i$, and for each modal algebra $\mathfrak{B} = (B, \Diamond)$, we have:

$$\mathfrak{B} \models \varphi \text{ iff } \mathfrak{B} \models \rho(\mathfrak{A}_1, D_1), \ldots, \rho(\mathfrak{A}_n, D_n).$$

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By the Refutation Pattern Theorem, there exist $(\mathfrak{A}_1, D_1), \ldots, (\mathfrak{A}_n, D_n)$ such that each $\mathfrak{A}_i = (A_i, \Diamond_i)$ is a finite modal algebra, $D_i \subseteq A_i$, and for each modal algebra $\mathfrak{B} = (B, \Diamond)$, we have $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is $i \leq n$ and a stable embedding $h : A_i \rightarrowtail B$ satisfying (CDC) for D_i.

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By the Stable Canonical Rule Theorem, this is equivalent to the existence of $i \leq n$ such that $\mathfrak{B} \not\models \rho(\mathfrak{A}_i, D_i)$.

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By the Refutation Pattern Theorem, there exist $(\mathfrak{A}_1, D_1), \ldots, (\mathfrak{A}_n, D_n)$ such that each $\mathfrak{A}_i = (A_i, \Diamond_i)$ is a finite modal algebra, $D_i \subseteq A_i$, and for each modal algebra $\mathfrak{B} = (B, \Diamond)$, we have $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is $i \leq n$ and a stable embedding $h : A_i \rightarrowtail B$ satisfying (CDC) for D_i.

By the Stable Canonical Rule Theorem, this is equivalent to the existence of $i \leq n$ such that $\mathfrak{B} \not\models \rho(\mathfrak{A}_i, D_i)$.

Thus, $\mathfrak{B} \models \Gamma/\Delta$ iff $\mathfrak{B} \models \rho(\mathfrak{A}_1, D_1), \ldots, \rho(\mathfrak{A}_n, D_n)$.

Main Theorem

1. Each modal rule system $\mathcal{S}$ over $\mathbf{S_K}$ is axiomatizable by stable canonical rules.
2. Each modal logic L is axiomatizable by stable canonical rules.

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Also our axiomatization requires to work with all finite modal algebras. It is not sufficient to work with only finite s.i. modal algebras.

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Part 2 yields a solution of an open problem of Zakharyaschev. However, our solution is by means of multiple-conclusion rules rather than formulas.

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Various applications of this method will be discussed in Part II of the talk by Silvio Ghilardi.

Thank you!