Admissible Infinitary Rules in Modal Logic I

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and infinitary consequence operations *Cn*. Why?

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MAIN RESULTS:

- For a consequence operation *Cn* extending m.l. *S*4.3 TFAE:
  (*i*) *Cn* is Almost Structurally Complete (ASCpl),
  (*ii*) *Cn* is finitely approximable, i.e. there is a class $\mathbb{K}$ of *finite* algebras such that $Cn = \overrightarrow{\mathbb{K}}$, where $\overrightarrow{\mathbb{K}}$ is a consequence operation determined by $\mathbb{K}$.

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The key step: all extensions of *S*4.3 enjoy projective unification (D.-W 2009).

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Pre-history

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- has the Finite Frame Prop. (for frames)
- is *finitely axiomatizable* and
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Normal Axiomatic Extensions - the lattice $\mathrm{NExt}\mathbf{S4.3}$

Question: lift the results from theoremhood to derivability

Describe the lattice $\mathrm{EXT}_{\mathit{fin}}\mathbf{S4.3}$

$\star$ Syntactic and semantic descripttion of **finitary** (structural) consequence operations $Cn \in \mathrm{EXT}_{fin}\mathbf{S4.3}$:

• form of *admissible (passive)* rules in consequence oper. Cn,

• Let $\mathcal{K}$ is a class of subdir. irr. S4.3-algebras characterizing $L \in \mathrm{NExt}\mathbf{S4.3}$. Then, for any consequence oper. $Cn \geq Cn_L$:

○ Cn is characterized by a class of algebras of the form of the direct products $\mathcal{A} \times \mathcal{H}_n$, where $\mathcal{A} \in \mathcal{K}$ and $\mathcal{H}_n$ is so called *Henle algebra* with n-atoms, i.e.
Cn has Strong Finite Model Property (*SFMP*).

○ Cn is *finitely based* (can obtained by adding finitely many rules) and decidable

• The lattice $\mathrm{EXT}_{\mathrm{fin}}\mathbf{S4.3}$ of all consequence relations extending S4.3 is countable and distributive (a Heyting algebra).

$Var = \{p_1, p_2, \dots\}$ all propositional variables
Fm all modal formulas built up with $\wedge, \neg, \Box, \top$; Fm_n $\{p_i : i \leq n\}$
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$(Fm, \wedge, \neg, \Box, \top)$ the algebra of modal language, $\varepsilon\colon Var \rightarrow Fm$ substitution; A *modal logic* - any subset L of *Fm* containing all classical tautologies, the axiom
(K) : $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$ and closed under substit. and

$$MP : \frac{\alpha \rightarrow \beta, \alpha}{\beta} \qquad \text{and} \qquad RN : \frac{\alpha}{\Box\alpha}.$$

K the least, **S4** = **K** + $(T) : \Box\alpha \rightarrow \alpha \quad + \quad (4) : \Box\Box\alpha \rightarrow \Box\alpha$.
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$$\mathrm{NExt}\mathbf{S4.3} \ni \quad L \quad \mapsto \quad Cn_L \quad \in \mathrm{EXT}_{\mathrm{fin}}\mathbf{S4.3}$$

its *global consequence relation*; $\alpha \in Cn_L(X)$ means: α can be derived from $X \cup L$ using the rules *MP* and *RN*.

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A *consequence operation*, (A.Tarski) - any mapping $Cn : 2^{Fm} \to 2^{Fm}$ for each $X, Y \subseteq Fm$

$$X \subseteq Cn(X), \qquad Cn(X) \subseteq Cn(X \cup Y), \qquad Cn(Cn(X)) \subseteq Cn(X).$$

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A consequence operation Cn is *finitary* ($Cn \in Fin$),if

$$Cn(X) = \bigcup\{Cn(Y) : \ Y \text{ is finite and } Y \subseteq X\}, \qquad \text{for each } X \subseteq Fm.$$

For any Cn define its 'finitary fragment' Cn_{fin} putting

$$Cn_{fin}(X) = \bigcup\{Cn(Y) : \ Y \text{ is finite and } Y \subseteq X\}, \qquad \text{for each } X \subseteq Fm.$$

Theorem (Deduction Theorem)

If L is a modal logic and $\mathbf{S4} \subseteq L$*, then*
$\alpha \in Cn_L(X, \beta)$ *iff* $\Box\beta \to \alpha \in Cn_L(X)$.

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A *modal algebra* $\mathcal{A} = (A, \wedge, \neg, \Box, \top)$, $\Box(a \wedge b) = \Box a \wedge \Box b$, $\Box\top = \top$; $\quad \mathrm{Log}(\mathcal{A}) = \{\alpha : v(\alpha) = \top,$ for all $v : Var \to A\}$, for a class $\mathbb{K}$, $\quad \mathrm{Log}(\mathbb{K}) = \bigcap\{\mathrm{Log}(\mathcal{A}) : \mathcal{A} \in \mathbb{K}\}$,

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$\alpha \in \overrightarrow{\mathcal{A}}(\emptyset)$ iff $\alpha \in \mathrm{Log}(\mathcal{A})$. For a class $\mathbb{K}$ of modal algebras, put

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a consequence operation *Cn* is *finitely approximable* ($Cn \in$ FA) if there is a strongly adequate family of finite algebras for *Cn*, i.e. $Cn = \bigwedge_i \overrightarrow{\mathcal{A}_i}$ where $\mathcal{A}_i$ is finite for each i.If

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(i) If $X \in Sat(\mathcal{A}_t)$ *for each* $t \in T$, *then* $\overrightarrow{\mathbf{P}_{t\in T}\mathcal{A}_t}(X) = \bigcap_{t\in T} \overrightarrow{\mathcal{A}_t}(X)$.

(ii) If $X \notin Sat(\mathcal{A}_t)$ *for some* $t \in T$, *then* $\overrightarrow{\mathbf{P}_{t\in T}\mathcal{A}_t}(X) = Fm$.

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it easily follows that $\overrightarrow{\mathbb{K}} \leq \overrightarrow{\mathcal{A}}$ if $\mathcal{A} \in SP(\mathbb{K})$. We also have

Theorem

Let $\mathbb{K}$ *be a class of modal algebras and Cn be a modal consequence operation such that* $\overrightarrow{\mathbb{K}} \leq Cn$. *Then there is a class* $\mathbb{L} \subseteq SP(\mathbb{K})$ *such that* $Cn = \overrightarrow{\mathbb{L}}$.

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$\mathcal{A}$ is a finite *TBA*, extend the set *Var* of prop. var. with fresh variables p_a, for each $a \in A$. The *diagram of* $\mathcal{A}$ is

$$\Delta(\mathcal{A}) = \bigwedge\{(p_a \to p_b) \leftrightarrow p_{a\to b} : a, b \in A\} \wedge (p_\bot \leftrightarrow \bot) \wedge \bigwedge\{\Box p_a \leftrightarrow p_{\Box a} :$$

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Theorem

(i) Let $\mathcal{A}$ be a finite TBA and $v : Var \to A$. Then, for $\alpha \in Fm$:

$$\alpha \leftrightarrow p_{v(\alpha)} \in Cn_{S4}(\{\Delta(\mathcal{A})\} \cup \{p \leftrightarrow p_{v(p)} : p \in Var\}).$$

(ii) Let $\mathcal{A}$ be a finite s.i. TBA and $\mathcal{B}$ be any TBA. Then $\chi_{\mathcal{A}} \notin Log(\mathcal{B}) \iff \mathcal{A}$ is embeddable in some homomorphic image of $\mathcal{B}$.

A *frame* $\mathfrak{F} = (V, R)$: a set V (worlds), $R \subseteq V \times V$
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$\mathfrak{1}$, $\mathfrak{2}$, $\mathfrak{3}$,..., $\mathfrak{n}$ denote 1- , 2- , 3- ,... n-element clusters, respectiv.
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A modal algebra $\mathcal{A}$ is a *Henle algebra* if $\Box a = \bot$ for each $a \neq \top$.
Henle algebras are s.i. (simples) for **S5**.

$\mathfrak{n}^+$ is the Henle algebra with n generators.
Note: $\mathfrak{1}^+ = \mathbf{2} =^{def} (\{\bot, \top\}, min, \neg, \Box)$, with $\Box a = a$.

Frames for $L \in NExt(\mathbf{S4.3})$ - chains of clusters, lists, covering r. (K.Fine)

A *unifier for a formula α in a logic L* - $\varepsilon(\alpha) \in L$.

A *projective unifier for α in L* is a unifier such that
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The rule X/β is *admissible* for a modal consequence oper. *Cn*,
if $\varepsilon[X] \subseteq Cn(\emptyset) \Rightarrow \varepsilon(\beta) \in Cn(\emptyset)$,
for every substitution ε.
The rule is *derivable* for *Cn* (is *Cn*–derivable), if $\beta \in Cn(X)$.

(Almost) Structural Completeness

Cn is *structurally complete* (*Cn* $\in$ SCpl, see Pogorzelski [**?**])
each admissible rule for *Cn* is *Cn*-derivable
Cn $\in$ SCpl_{fin} means that each finitary rule which is admissible for *Cn* is *Cn*–derivable.

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S5, and many others **S4.3**,... contrary to **S4**, is not structurally complete only because the following rule is passive, hence, admissible, but not derivable:

$$P_2 : \frac{\Diamond\alpha \wedge \Diamond{\sim}\alpha}{\beta}$$

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Theorem

If Cn is modal consequence operation extending $S4.3M$, where (M) : $\Box\Diamond\alpha \to \Diamond\Box\alpha$ (McKinsey's axiom), then

$$Cn \in ASCpl \iff Cn \in SCpl$$

.

SCpl$\neq$ SCpl$_{fin}$ and ASCpl$\neq$ ASCpl$_{fin}$, more exactly:.

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$$\varrho : \ \frac{\{\sim \Box(p_i \leftrightarrow p_j) : 0 < i < j\}}{\bot} \qquad \varrho' : \ \frac{\{\Box(p_i \leftrightarrow p_j) \rightarrow p_0 : 0 < i < j\}}{p_0}$$

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are valid in every finite TBA $\mathcal{A}$ i.e. derivable for the cons.op. $\overrightarrow{\mathcal{A}}$
Hence, by FMP ϱ and ϱ' admissible in $S4.3$ and all extensions.
But,
ϱ' and ϱ are not derivable for $S4.3$ (nor $S4.3M$)
p_0 cannot be deduced, in $S4.3$, from any finite subset of
$\{\Box(p_i \leftrightarrow p_j) \to p_0 : 0 < i < j\}$.

OBSERVATION: attempts to extend the concept of projective unifier to infinite sets of formulas fails

Theorem

(i) Each finitary consequence operation $Cn \geq Cn_{S4.3}$ *has a finite basis (of finitary rules) and* $Cn = C_{fin}$ *for some* $C \in FA$.
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$$p_1^{\sigma(1)} \wedge \cdots \wedge p_n^{\sigma(n)}$$

where $\sigma\colon \{1, \ldots, n\} \to \{0, 1\}$, and $p^0 = p$, and $p^1 = \sim p$.
Suppose that all these atoms are listed as: $\theta_1, \ldots, \theta_{2^n}$.

Corollary

Each Cn $\in \mathrm{EXT}_{fin}(\mathbf{S4.3})$ *is and extension of* Cn_L*, for some* $L \in NExt(\mathbf{S4.3})$*,* $n \geq 0$*, with a finite number of passive rules having the form*

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$$\frac{\Diamond\theta_1 \wedge \cdots \wedge \Diamond\theta_s}{\alpha}$$

where $2 \leq s \leq 2^n$ *and* $Var(\alpha) \cap \{p_1, \ldots, p_n\} = \emptyset$*.*

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The rule $\Diamond\theta_1 \wedge \cdots \wedge \Diamond\theta_s/\alpha$ is valid in the complex algebra of a cluster of $\leq s - 1$ elements, it is not valid in the cluster $\mathfrak{s}$ of s elements.

Theorem

If $Cn \in \mathrm{EXT}(\mathbf{S4.3})$ *and* $Log(\mathbb{K}) = Cn(\emptyset)$ *for some class* $\mathbb{K} \subseteq \mathrm{Alg}(\mathbf{S4.3})_{si\,fin}$*, then one can find* $\mathbb{K}_1 \supseteq \cdots \supseteq \mathbb{K}_m$ *such that*

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$$\mathbb{L} = \mathbb{K}_m \cup \Big((\mathbb{K}_{m-1} \setminus \mathbb{K}_m) \times (\mathfrak{m} - \mathfrak{1})^+) \Big) \cup \cdots \cup \Big((\mathbb{K}_1 \setminus \mathbb{K}_2) \times \mathfrak{1}^+ \Big).$$

Each quasivariety of $S4.3$-algebras is generated by a class of finite $S4.3$-algebras of the form:

- s.i. algebra, or
- the direct product of a s.i. algebra and a Henle algebra.

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With all assumptions/ notations of the above Theorem we have

Theorem

If $\overrightarrow{\mathbb{K}} \leq Cn \leq \overrightarrow{\mathbb{L}}$ and $Cn =_{fin} \overrightarrow{\mathbb{L}}$, then $Cn = \overrightarrow{\mathbb{L}}$.

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If a logic $L \in \mathrm{NExt}(\mathbf{S4.3})$ *is tabular, then* $\mathrm{EXT}(L) \subseteq SF$ *and consequently* $\mathrm{EXT}(L) = \mathrm{EXT}_{fin}(L)$.

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A finitary modal consequence operation $Cn \in \mathrm{EXT}(\mathbf{S4.3})$ *is finitely approximable,* $Cn \in FA$ *iff* Cn *is strongly finite,* $Cn \in SF$*; i.e.* $FA \cap Fin = SF$.

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Theorem

A modal consequence operation $Cn \in \mathrm{EXT}(\mathbf{S4.3})$ is finitely approximable iff Cn is almost structurally complete; i.e. $FA = ASCpl$.

Corollary

If Cn is a finitary consequence operation extending **S4.3**, *then Cn is almost structurally complete iff Cn is strongly finite.*

Corollary

Let $L \in \mathrm{NExt}(\mathbf{S4.3})$ *and* $\mathbb{L}$ *be an adequate class of finite* **S4.3***-algebras for L. Then* $\overrightarrow{\mathbb{L} \times \mathbf{2}}$ *is the structurally complete extension of* Cn_L, *i.e.* $Cn_L^{SCpl} = \overrightarrow{\mathbb{L} \times \mathbf{2}}$.

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The lattices $ASCpl(\mathbf{S4.3})$ and $\mathrm{EXT}_{fin}(\mathbf{S4.3})$ are isomorphic,
Similarly $ASCpl(\mathbf{L})$ and $\mathrm{EXT}_{fin}(\mathbf{L})$.
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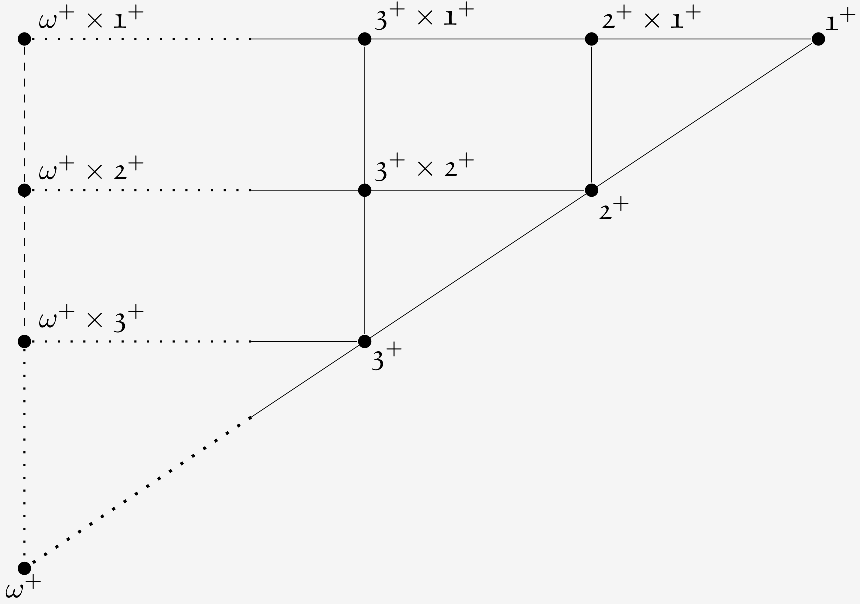
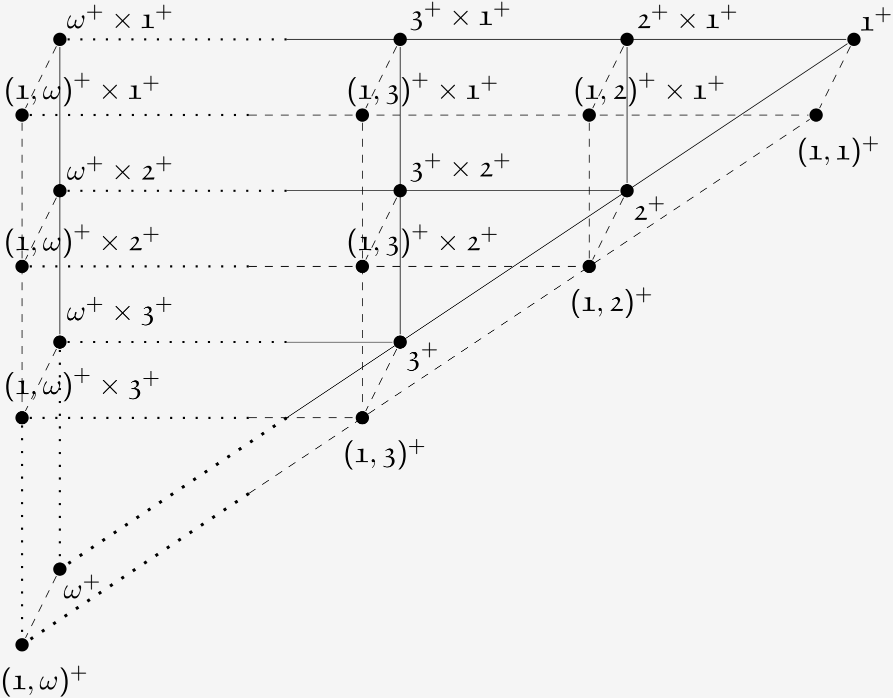
Σ an infinite basis for infinitary admissible non-passive rules - in Part II.

The lattice $ASCpl(\mathbf{S5})$, Not all $Cn \in \mathrm{EXT}_{fin}(\mathbf{S5})$ have a single str. ad. matrix, e.g. $(1^+ \times 3^+) \cap 2^+$.:




$\omega^+ \times 1^+$
$3^+ \times 1^+$
$2^+ \times 1^+$
1^+
$(1, \omega)^+ \times 1^+$
$(1, 3)^+ \times 1^+$
$(1, 2)^+ \times 1^+$
$(1, 1)^+$
$\omega^+ \times 2^+$
$3^+ \times 2^+$
2^+
$(1, \omega)^+ \times 2^+$
$(1, 3)^+ \times 2^+$
$(1, 2)^+$
$\omega^+ \times 3^+$
3^+
$(1, \omega)^+ \times 3^+$
$(1, 3)^+$
ω^+
$(1, \omega)^+$

ASCpl(**S4.5**)

a Logic - a variety $\mathbb{K} = \mathrm{HSP}\mathbb{K}$, a finitary (modal) consequence operation - a quasivariety of (modal) algebras, $\mathbb{K} = \mathrm{ISPP_U}\mathbb{K}$, an arbitrary, (modal) consequence operation - a prevariety of (modal) algebras, $\mathbb{K} = \mathrm{ISP}\mathbb{K}$.

a set of *quasiequations*, that is, by *equational implications* of the form

$$(ei) \quad \Big(\bigwedge_{i \in I} p_i(\overline{x}) = q_i(\overline{x})\Big) \quad \Rightarrow \quad p(\overline{x}) = q(\overline{x})$$

a prevariety, $\mathbb{K} = \mathrm{ISP}\mathbb{K}$ is defined by a set of *generalized equational implications*, (ei), where the set I of indices can be infinite;

Let $\mathrm{F}_{\mathbb{K}}$ be the ω-generated free algebra in $\mathbb{K}$. Admissibility-validity of the (generalized) equational implication (ei) in the free algebra $\mathrm{F}_{\mathbb{K}}$,

a generalized equational implication (ei) is *passive* if $\Big(\bigwedge_{i \in I} p_i(\overline{x}) = q_i(\overline{x})\Big)$ is not satisfiable in $\mathrm{F}_{\mathbb{K}}$.

A prevariety $\mathbb{K}$ is *(almost) structurally complete* iff every generalized equational implication (ei) which is valid in $\mathrm{F}_{\mathbb{K}}$ (and is not passive), is valid in the whole prevariety $\mathbb{K}$.

Theorem

For a prevariety $\mathbb{K}$ of S4.3-algebras TFCE:

(i) *$\mathbb{K}$ is almost structurally complete*

(ii) *$\mathbb{K}$ is generated by its finite members.*

(iii) *every generalized equational implication (ei) which is valid in $\mathrm{F}_{\mathbb{K}}$ and its premises are satisfiable in $\mathrm{F}_{\mathbb{K}}$, is valid in the whole prevariety $\mathbb{K}$.*

Let Σ be a set of all increasing functions $f : \mathbb{N} \to \mathbb{N}$. The following equational implications $(ei)_f$, for $f \in \Sigma$:

$$\bigwedge_{n>0} \{[\bigwedge_{n<j\leq f(n)} \bigvee_{0<i\leq n} \Box(p_i \leftrightarrow p_j)] \to p_0\} = 1 \;\Rightarrow\; p_0 = 1$$

form a basis for admissible generalized equational implications in $\mathbb{K}$.