Admissible Rules and Unification in the Implication–Negation Fragment of Superintuitionistic Logics

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Terminology

To talk about logic**s**, we need

- **propositional languages** $\mathcal{L}$ consisting of connectives such as $\wedge, \vee, \rightarrow, \neg, \bot, \top$ with specified finite arities;
- sets $\mathrm{Fm}_{\mathcal{L}}$ of $\mathcal{L}$**-formulas** $\psi, \varphi, \chi, \ldots$ built from a countably infinite set of variables $p, q, r, \ldots$;
- endomorphisms on $\boldsymbol{Fm}_{\mathcal{L}}$ called $\mathcal{L}$**-substitutions**.

Logics

Definition

A *logic* L is a *finitary structural consequence relation* on $\mathrm{Fm}_{\mathcal{L}}$, i.e., a set $\mathrm{L} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}) \times \mathrm{Fm}_{\mathcal{L}}$ (writing $\Gamma \vdash_{\mathrm{L}} \varphi$ for $\langle \Gamma, \varphi \rangle \in \mathrm{L}$) satisfying:

- $\{\varphi\} \vdash_{\mathrm{L}} \varphi$ (reflexivity);
- if $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Gamma \cup \Gamma' \vdash_{\mathrm{L}} \varphi$ (monotonicity)
- if $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Gamma \cup \{\varphi\} \vdash_{\mathrm{L}} \psi$, then $\Gamma \vdash_{\mathrm{L}} \psi$ (transitivity)
- if $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Gamma' \vdash_{\mathrm{L}} \varphi$ for some finite $\Gamma' \subseteq \Gamma$ (finitarity)
- if $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma\Gamma \vdash_{\mathrm{L}} \sigma\varphi$ for any $\mathcal{L}$-substitution σ (structurality)

An L-*theorem* is a formula φ such that $\emptyset \vdash_{\mathrm{L}} \varphi$ (abbreviated as $\vdash_{\mathrm{L}} \varphi$).

Derivable and Admissible Rules

Definition

An $\mathcal{L}$-*rule* is an ordered pair Γ/φ where $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ is *finite*.

Definition

For a logic L:

- Γ/φ is L-*derivable*, if $\Gamma \vdash_{\mathrm{L}} \varphi$.
- Γ/φ is L-*admissible*, written $\Gamma \mathrel{|\!\!\sim}_{\mathrm{L}} \varphi$, if for every $\mathcal{L}$-substitution σ:

$$\vdash_{\mathrm{L}} \sigma\psi \quad \text{for all } \psi \in \Gamma \qquad \Rightarrow \qquad \vdash_{\mathrm{L}} \sigma\varphi$$

L is *structurally complete* (SC) if: $\Gamma \vdash_{\mathrm{L}} \varphi$ if and only if $\Gamma \mathrel{|\!\!\sim}_{\mathrm{L}} \varphi$.
L is *hereditarily* SC if all its (axiomatic) extensions are SC.

Note that $\mathrel{|\!\!\sim}_{\mathrm{L}}$ uniquely determines a logic (formally: the minimal logic whose 'finitary fragment' it coincides)

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Example (1): Intuitionistic Logic

The "independence of premises" rule

$$\neg p \to (q \vee r) \;/\; (\neg p \to q) \vee (\neg p \to r)$$

is admissible for intuitionistic logic IPC but

$$\neg p \to (q \vee r) \nvdash_{\mathrm{IPC}} (\neg p \to q) \vee (\neg p \to r).$$

Example (2): Relevant Logics

The "disjunctive syllogism" rule

$$\neg p, p \vee q \ / \ q$$

is admissible but not derivable in the relevant logics R and RM.

Example (3): Modal Logics

The modal rule

$$\Box p \ / \ p$$

is admissible but not derivable in K and $\mathrm{K4}$, while Löb's rule

$$\Box p \to p \ / \ p$$

is admissible and non-derivable for K, but not admissible for $\mathrm{K4}$.

Axiomatizing Admissibility

For a logic L, we are interested in finding a set of rules that "axiomatizes" (over L) the admissible rules of L.

Definition

A *basis* for $\mathrel{|\!\sim}_{\mathrm{L}}$ over L is a set B of rules such that $\mathrel{|\!\sim}_{\mathrm{L}}$ is the smallest logic extending $B \cup \mathrm{L}$.

Intuitionistic Logic

Iemhoff and Rozière established independently that the “Visser rules”:

$$\frac{(\bigwedge_{i=1}^{n}(p_i \to q_i) \to (p_{n+1} \vee p_{n+2})) \vee r}{\bigvee_{j=1}^{n+2}(\bigwedge_{i=1}^{n}(p_i \to q_i) \to p_j) \vee r} \quad \text{n = 1,2,...}$$

provide a basis for the admissible rules of IPC.

Iemhoff has shown that the Visser rules also provide a basis for certain intermediate logics, and Jeřábek has given bases for a wide range of transitive modal logics.

Fragments of IPC

Theorem (Mints)

Implication-less fragments of IPC *are structurally complete*

Theorem (Prucnal)

The implication fragment of IPC *is structurally complete.*

Analogously, the $\{\to, \wedge\}$, $\{\to, \wedge, \neg\}$ fragments of *all* intermediate logics are structurally complete (Minari, Wroński).

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Any fragment of IPC *involving implication and disjunction is not structurally complete.*

What about the implication–negation fragment?

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Admissible rules in the implication–negation fragment of intuitionistic logic

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ABSTRACT

Uniform infinite bases are defined for the single-conclusion and multiple-conclusion admissible rules of the implication–negation fragments of intuitionistic logic IPC and its consistent axiomatic extensions (intermediate logics). A Kripke semantics characterization is given for the (hereditarily) structurally complete implication–negation fragments of intermediate logics, and it is shown that the admissible rules of this fragment of IPC form a PSPACE-complete set and have no finite basis.

Conventions

Convention

Let L be a consistent axiomatic extension of implication–negation fragment IPC with defined constant $\bot =_{\mathrm{def}} \neg(q \to q)$.

For a sequence $\vec{\varphi} = \varphi_1 \dots \varphi_n$ of formulas we write $\vec{\varphi} \to \psi$ instead of $\varphi_1 \to (\varphi_2 \to \cdots (\varphi_{n-1} \to \psi) \cdots)$

Remark

- L enjoys deduction theorem
- L is equal to or weaker than classical logic
- a set for formulas is L-consistent iff it is consistent in classical logic

(My) open problem

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The Wroński Rules

Consider the following “Wroński rules” ($n = 2, 3, \ldots$):

$$(\mathrm{W}'_n) \quad (\vec{p} \to \bot), ((\neg\neg p_1 \to p_1) \to q), \ldots, ((\neg\neg p_n \to p_n) \to q) \;/\; q.$$

Lemma

If $\vdash_{\mathrm{L}} \sigma(\vec{p} \to \bot)$*, then* $\vdash_{\mathrm{L}} \sigma(\neg\neg p_i \to p_i)$ *for some* $p_i \in \vec{p}$

Proof.

Note that $\sigma(p_i) = \vec{\varphi} \to \bot$ for some $i \in \{1, \ldots, n\}$ (otherwise, $\sigma'(v) = \top$ for each variable v gives $\vdash_{\mathrm{L}} \sigma'\sigma(\vec{p} \to \bot)$ and $\vdash_{\mathrm{L}} \top \to \bot$). Hence, $\vdash_{\mathrm{L}} \sigma(\neg\neg p_i \to p_i)$. □

Corollary

(W'_n) *is* L*-admissible for* $n = 2, 3, \ldots$.

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Lemma

(W'_n) *are* not IPC*-admissible (and so* not IPC*-derivable) for* $n = 2, 3, \ldots$.

Proof.

$\sigma p_1 = p \land \neg r \qquad \sigma p_2 = r \qquad \sigma q = \sigma(\neg\neg p_1 \to p_1) \lor \sigma(\neg\neg p_2 \to p_2)$

Then $\vdash_{\mathrm{IPC}} \sigma(p_1 \to (p_2 \to \bot))$ and $\vdash_{\mathrm{IPC}} \sigma((\neg\neg p_i \to p_i) \to q)$

But neither $\nvdash_{\mathrm{IPC}} \sigma(\neg\neg p_1 \to p_1)$ nor $\nvdash_{\mathrm{IPC}} \sigma(\neg\neg p_2 \to p_2)$. Thus by the disjunction property: $\nvdash_{\mathrm{IPC}} \sigma(q)$ $\square$

They are derivable in e.g. Gödel logic ($\mathrm{IPC} + (p \to q) \lor (q \to p)$) and De Morgan logic ($\mathrm{IPC} + \neg p \lor \neg\neg p$).

Basis of admissible rules

Theorem (1)

The set $\{(\mathrm{W}'_n) \mid n = 2, 3, \dots\}$ *is a basis for the admissible rules of* L.

Theorem (2)

The set of admissible rules of the implication–negation fragment of IPC *is PSPACE-complete.*

Note that: admissibility in full intuitionistic logic is co-NEXP-complete

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Note that: admissibility in full intuitionistic logic is co-NEXP-complete

A hint of the proof of Theorem (1)

Let us by L^{W} denote the logic $\mathrm{L} + \{(\mathrm{W}'_n) \mid n = 2, 3, \dots\}$.

Our goal is to show $\mathrel{|\!\sim}_{\mathrm{L}} = \vdash_{\mathrm{L}^{\mathrm{W}}}$ clearly: $\mathrel{|\!\sim}_{\mathrm{L}} \supseteq \vdash_{\mathrm{L}^{\mathrm{W}}}$

A rule Γ/φ is *simple*, if each $\psi \in \Gamma$ is *simple*, i.e., of the form:

$$\psi_1 \to (\psi_2 \to (\cdots(\psi_n \to \chi)\cdots)) \qquad \text{where}$$

(i) *either* $\chi = \bot$ and all ψ_i's are atoms *or*

(ii) χ is atom and all ψ_i's are either atoms or have the form $p \to q$

Lemma (1)

There is a polynomial-time algorithm producing for a given finite set Γ of formulas a set Π of simple formulas such that:

$$\Gamma \mathrel{|\!\sim}_{\mathrm{L}} \varphi \textit{ iff } \Pi \mathrel{|\!\sim}_{\mathrm{L}} \varphi \qquad \Pi \vdash_{\mathrm{L}^{\mathrm{W}}} \varphi \textit{ iff } \Gamma \vdash_{\mathrm{L}^{\mathrm{W}}} \varphi.$$

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For each Γ we define a set Ψ_Γ of supersets of Γ such that:

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Let Γ/φ be a simple rule. If $\Gamma \mathrel{|\!\!\!\sim}_{\mathrm{L}} \varphi$, then $\Delta \vdash_{\mathrm{L}} \varphi$ for each $\Delta \in \Psi_\Gamma$.

Lemma (3)

Let Γ/φ be a simple rule. If $\Delta \vdash_{\mathrm{L}} \varphi$ for each $\Delta \in \Psi_\Gamma$, then $\Gamma \vdash_{\mathrm{Lw}} \varphi$

How do we do that?

$$\Psi_\Gamma = \{\Gamma \cup \{\neg\neg p \to p \mid p \in Y\} \mid Y \subseteq \mathrm{Var}(\Gamma) \text{ and } \Gamma, \mathrm{Var}(\Gamma) \setminus Y \nvdash_{\mathrm{L}} \bot\}.$$

For a given Γ: the task whether $\Gamma \cup \{\neg\neg p \to p \mid p \in Y\} \in \Psi_\Gamma$ is in NP.

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Let us by $\mathrel{|\!\sim}^s_{\mathrm{L}}$ denote the admissibility problem for simple rules.

Proof.

We know that theorems of L are PSPACE-hard, thus so is $\mathrel{|\!\sim}_{\mathrm{L}}$.

We present an NPSPACE algorithm for $\not\mathrel{|\!\sim}^s_{\mathrm{L}}$, thus $\mathrel{|\!\sim}^s_{\mathrm{L}}$ is in PSPACE and so (by Lemma (1)) is $\mathrel{|\!\sim}_{\mathrm{L}}$. Take a simple rule Γ/φ and

- nondeterministically guess some $X \subseteq \mathrm{Var}(\Gamma)$
- check whether $\Gamma' = \Gamma \cup \{\neg\neg p \to p \mid p \in X\} \in \Psi_\Gamma$ NP
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If it is the case, then by Lemma (2): $\Gamma \not\mathrel{|\!\sim}^s_{\mathrm{L}} \varphi$. $\square$

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Corollaries

Corollary (1)

A logic L is (hereditarily) structurally complete iff $\{(\mathrm{W}'_n) \mid n \geq 2\} \subseteq \mathrm{L}$.

Corollary (2)

If L is the fragment of some intermediate logic L' with the disjunction property, then L is not structurally complete.

Proof.

If L is SC, then: $\vdash_{\mathrm{L}'} \neg(p_1 \wedge p_2) \to ((\neg\neg p_1 \to p_1) \vee (\neg\neg p_2 \to p_2))$.
The independence of premises rule is admissible for any intermediate logic with the disjunction property. Hence

$$\vdash_{\mathrm{L}'} \neg(p_1 \wedge p_2) \to (\neg\neg p_1 \to p_1) \qquad \text{or} \qquad \vdash_{\mathrm{L}'} \neg(p_1 \wedge p_2) \to (\neg\neg p_2 \to p_2)$$

Thus L' is classical logic, a contradiction. $\square$

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If L is SC, then: $\vdash_{\mathrm{L}'} \neg(p_1 \wedge p_2) \to ((\neg\neg p_1 \to p_1) \vee (\neg\neg p_2 \to p_2))$.
The independence of premises rule is admissible for any intermediate logic with the disjunction property. Hence

$$\vdash_{\mathrm{L}'} \neg(p_1 \wedge p_2) \to (\neg\neg p_1 \to p_1) \quad \text{or} \quad \vdash_{\mathrm{L}'} \neg(p_1 \wedge p_2) \to (\neg\neg p_2 \to p_2)$$

Thus L' is classical logic, a contradiction. $\square$

Kripke frames: characterization of (W'_n)

Recall: a frame is Church–Rosser if every finite set of elements with a lower bound also has an upper bound.

Definition

A frame is *n-almost-Church–Rosser (n-aCR)* if each set of at most n *non-maximal* elements which has a lower bound has an upper bound.

A frame F is *almost-Church–Rosser (aCR)* if it is n-aCR for all $n \in \mathbb{N}$.

Lemma

(W'_n) *is valid in a frame F iff F is n-aCR* $(n = 2, 3, \ldots)$.

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Kripke frames: corollaries of the characterization 1

Theorem

Let L be the implication–negation fragment of an intermediate logic L'. Then L is (hereditarily) structurally complete iff all L'-frames are aCR.

Proof.

We can assume that L' is axiomatized over IPC by formulas involving implication and negation only.

By McKay's theorem, L' is Kripke complete and so is L.

Recall that L is (hereditarily) structurally complete iff $W' \subseteq \mathrm{L}$.

But since L is Kripke complete, $W' \subseteq \mathrm{L}$ iff all L-frames validate W'
iff all L-frames are aCR. □

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Kripke frames: corollaries of the characterization 2

Theorem

The set of admissible rules of the implication–negation fragment of IPC *has no finite basis.*

Proof.

We show that for each $n \geq 2$, the rules $\{(W_i') \mid 0 \leq i \leq n\}$ do not form a basis

From McKay's theorem we know that the logic

$$\mathrm{L} = \mathrm{IPC} + \{(W_i') \mid 0 \leq i \leq n\}$$

is Kripke complete w.r.t. the class of all n-aCR Kripke frames.

Clearly there is an n-aCR frame which is not $n+1$-aCR.

Hence (W_{n+1}') is not derivable in L. □

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Unification type

An L-unifier of Γ is a substitution σ s.t. $\vdash_{\mathrm{L}} \sigma[\Gamma]$. L-unifiers of Γ can be ordered by the 'generality':

$$\sigma_1 \leq_{\mathrm{L}} \sigma_2 \quad \text{iff} \quad \text{there is } \sigma \text{ s.t. } \sigma_1 = \sigma\sigma_2$$

$\mathcal{C}$ is a *minimal complete set of* L*-unifiers* (MCSU) for Γ if

- for any L-unifier σ for Γ, there exists $\sigma' \in \mathcal{C}$ such that $\sigma \leq_{\mathrm{L}} \sigma'$.
- for any $\sigma_1, \sigma_2 \in \mathcal{C}$, if $\sigma_1 \leq_{\mathrm{L}} \sigma_2$, then $\sigma_1 = \sigma_2$.

L has *unitary unification type* if each Γ has a singleton MCSU

L has *finitary unification type* if each Γ has a finite MCSU and it has not the unitary UT.

Theorem

Classical logic has unitary unification type. All others have finitary unification type.

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Summary and open problems

For any axiomatic extension of implication–negation fragment of Intuitionistic logic we have

- characterized when it is (hereditarily) structurally complete
- described a basis of its admissible rules
- show that it has finitary unification type (unless it is the classical)

For the fragment of Intuitionistic logic we showed that admissible rules have no-finite basis and form a PSPACE-complete set.

Open (?) problem: solve these issues of implication–disjunction and implication–disjunction–negation fragments