Standard Completeness I: Proof Theoretic Approach

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Standard Completeness

Completeness of axiomatic systems with respect to algebras whose lattice reduct is the real unit interval $[0, 1]$.

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(Hajek 1998) Formalizations of *Fuzzy Logic*

Uninorm (based logics)

Conjunction and implication are interpreted by a particular uninorm/t-norm (or a class of) and its residuum.

- A *uninorm* is a function $* : [0,1]^2 \to [0,1]$ such that for all $x, y, z \in [0,1]$:
  - $x * y = y * x$ (Commutativity)
  - $(x * y) * z = x * (y * z)$ (Associativity)
  - $x \leq y$ implies $x * z \leq y * z$ (Monotonicity)
  - $e \in [0,1] \;\; e * x = x$ (Identity)

  The *residuum* of $*$ is a function $\Rightarrow_*: [0,1]^2 \to [0,1]$ where $x \Rightarrow_* y = max\{z \mid x * z \leq y\}$.
- A *t-norm* is a uninorm in which $e = 1$.

Some standard complete logics

v : Propositions $\rightarrow$ $[0,1]$

- Gödel logic
  $v(A \wedge B) = \min\{v(A), v(B)\}$
  $v(A \vee B) = \max\{v(A), v(B)\}$
  $v(A \rightarrow B) = 1$ if $v(A) \leq v(B)$, and $v(B)$ otherwise
  $v(\bot) = 0$

- UL Uninorm logic (Metcalfe, Montagna 2007)
  $v(A \odot B) = v(A) * v(B)$, $\quad *$ left continous uninorm
  $v(A \vee B) = \max\{v(A), v(B)\}$
  $v(A \rightarrow B) = v(A) \Rightarrow_* v(B)$
  $v(\bot) = 0$
- MTL Monoidal T-norm logic (Godo, Esteva 2001)
  $*$ left continous t-norm

(Uninorm-based) Logics

often described by *adding* axioms to already known logics.

Example

- UL = FLe with $((\alpha \to \beta) \wedge t) \vee ((\beta \to \alpha) \wedge t)$ (linearity)
- MTL = UL with weakening/integrality
- Gödel logic = MTL with contraction $\alpha \to \alpha \odot \alpha$
- SUL = UL with $\alpha \to \alpha \odot \alpha$ and mingle $\alpha \odot \alpha \to \alpha$
- WMTL = MTL with $\neg(\alpha \odot \beta) \vee (\alpha \wedge \beta \to \alpha \odot \beta)$
- ....

(Uninorm-based) Logics

are often described by *adding* axioms to already known logics.

$\boxed{\text{Question}}$ Given a logic L obtained by extending UL with

- $\alpha \odot \alpha \to \alpha$ (mingle)?
- $\alpha^{n-1} \to \alpha^n$ (n-contraction)?
- $\neg(\alpha \odot \beta)^n \vee ((\alpha \wedge \beta)^{n-1} \to (\alpha \odot \beta)^n)$?
- ....

Is L standard complete? (*is it a formalization of Fuzzy Logic?*)

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- ....

Is L standard complete? (*is it a formalization of Fuzzy Logic?*)

Many papers written for individual logics!

Standard Completeness: algebraic approach

Given a logic L:

1. Identify the algebraic semantics of L (L-algebras)
2. Show completeness of L w.r.t. linear, countable L-algebras
3. (Rational completeness): Find an embedding into linear, dense countable L-algebras
4. Dedekind-Mac Neille style completion (embedding into L-algebras with lattice reduct $[0, 1]$)

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- Step 3: problematic (mainly$^{(*)}$ **ad hoc** solutions)

$(*)$ see Paolo's talk!

Standard Completeness: proof theoretic approach

(Metcalfe, Montagna JSL 2007) Given a logic *L*:

- Add Takeuti and Titani's density rule (*p* eigenvariable)

$$\frac{(\alpha \to p) \vee (p \to \beta) \vee \gamma}{(\alpha \to \beta) \vee \gamma} \ (density)$$

  (= *L* + (*density*) is rational complete)
- Show that density produces no new theorems (Rational completeness)
- Dedekind-Mac Neille style completion

Our result

Uniform (and automated) proofs of standard completeness for large classes of axiomatic extensions of UL

How?

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Uniform (and automated) proofs of standard completeness for large classes of axiomatic extensions of UL

How?

(Step 1) Defining <u>suitable</u> calculi for axiomatic extensions of UL

(Step 2) General conditions for the elimination of the density rule from these calculi

(Step 3) Dedekind-Mac Neille style completion

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(Avron JSL '89)

Hypersequents: $\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n$ where for all $i = 1, \ldots n$, $\Gamma_i \Rightarrow \Pi_i$ is an ordinary sequent

Our base calculus: *FLe*

$$\frac{}{\alpha \Rightarrow \alpha}\ (init) \qquad \frac{}{\Rightarrow t}\ (tr) \qquad \frac{}{f \Rightarrow}\ (fl)$$

$$\frac{}{\Gamma \Rightarrow \top}\ (\top) \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}\ (\bot)$$

$$\frac{\Gamma \Rightarrow \Pi}{t, \Gamma \Rightarrow \Pi}\ (tl) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow f}\ (fr) \qquad \frac{\Gamma \Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi}\ (cut)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\wedge r) \qquad \frac{\alpha_i, \Gamma \Rightarrow \Pi}{\alpha_1 \wedge \alpha_2, \Gamma \Rightarrow \Pi}\ (\wedge l) \qquad \frac{\Gamma \Rightarrow \alpha_i}{\Gamma \Rightarrow \alpha_1 \vee \alpha_2}\ (\vee r)$$

$$\frac{\alpha, \Gamma \Rightarrow \Pi \quad \beta, \Gamma \Rightarrow \Pi}{\alpha \vee \beta, \Gamma \Rightarrow \Pi}\ (\vee l) \qquad \frac{\Gamma \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \Pi}{\Gamma, \alpha \to \beta, \Delta \Rightarrow \Pi}\ (\to l) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\ (\to r)$$

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \odot \beta}\ (\odot r) \qquad \frac{\alpha, \beta, \Gamma \Rightarrow \Pi}{\alpha \odot \beta, \Gamma \Rightarrow \Pi}\ (\odot l)$$

Calculi for axiomatic extensions of *FLe*

E.g. UL = FLe + $((\alpha \to \beta) \wedge t) \vee ((\beta \to \alpha) \wedge t)$ (linearity)

- Cut elimination is not preserved when axioms are added
- (Idea) Axioms are transformed into
  - ‘good’ structural rules
  - in the ‘appropriate’ formalism

Hypersequent Calculus for UL

(UL = FLe + $((\alpha \to \beta) \wedge t) \vee ((\beta \to \alpha) \wedge t)$)

Hypersequent: $\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n$

This calculus is obtained

- embedding sequents into hypersequents in FLe

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta} \ (\to, r)$$

i.e.

$$\frac{G|\alpha, \Gamma \Rightarrow \beta}{G|\Gamma \Rightarrow \alpha \to \beta} \ (\to, r)$$

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$$\text{Hypersequent: } \Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n$$

This calculus is obtained

- embedding sequents into hypersequents in FLe
- adding suitable rules to manipulate the additional layer of structure.

$$\frac{G}{G \mid \Gamma \Rightarrow \alpha}\ (\mathrm{ew}) \qquad\qquad \frac{G \mid \Gamma \Rightarrow \alpha \mid \Gamma \Rightarrow \alpha}{G \mid \Gamma \Rightarrow \alpha}\ (\mathrm{ec})$$

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$$\frac{G \mid \Gamma, \Gamma' \Rightarrow \alpha \quad G \mid \Gamma_1, \Gamma_1' \Rightarrow \alpha'}{G \mid \Gamma, \Gamma_1 \Rightarrow \alpha \mid \Gamma', \Gamma_1' \Rightarrow \alpha'}\ (\mathrm{com})$$

(Avron 1991)

An example

$$
\begin{prooftree}
\AxiomC{$\beta \Rightarrow \beta$}
\AxiomC{$\alpha \Rightarrow \alpha$}
\RightLabel{(com)}
\BinaryInfC{$\alpha \Rightarrow \beta \mid \beta \Rightarrow \alpha$}
\RightLabel{$(\rightarrow,\mathrm{r})$}
\UnaryInfC{$\alpha \Rightarrow \beta \mid \;\Rightarrow \beta \rightarrow \alpha$}
\RightLabel{$(\rightarrow,\mathrm{r})$}
\UnaryInfC{$\Rightarrow t \quad \Rightarrow \alpha \rightarrow \beta \mid \;\Rightarrow \beta \rightarrow \alpha$}
\AxiomC{$\Rightarrow t$}
\RightLabel{$2\mathrm{x}(\wedge,\mathrm{r})$}
\BinaryInfC{$\Rightarrow (\alpha \rightarrow \beta) \wedge t \mid \;\Rightarrow (\beta \rightarrow \alpha) \wedge t$}
\RightLabel{$(\vee_{\mathrm{i}},\mathrm{r})$}
\UnaryInfC{$\Rightarrow (\alpha \rightarrow \beta) \wedge t \mid \;\Rightarrow ((\alpha \rightarrow \beta) \wedge t) \vee ((\beta \rightarrow \alpha) \wedge t)$}
\RightLabel{$(\vee_{\mathrm{i}},\mathrm{r})$}
\UnaryInfC{$\Rightarrow ((\alpha \rightarrow \beta) \wedge t) \vee ((\beta \rightarrow \alpha) \wedge t) \mid \;\Rightarrow ((\alpha \rightarrow \beta) \wedge t) \vee ((\beta \rightarrow \alpha) \wedge t)$}
\RightLabel{(EC)}
\UnaryInfC{$\Rightarrow ((\alpha \rightarrow \beta) \wedge t) \vee ((\beta \rightarrow \alpha) \wedge t)$}
\end{prooftree}
$$

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Algorithmic introduction of analytic calculi I

Definition (Classification; -, Galatos and Terui, LICS 2008)

The classes $\mathcal{P}_n, \mathcal{N}_n$ of positive and negative $\boxed{\text{axioms/equations}}$ are:

- $\mathcal{P}_0 ::= \mathcal{N}_0 ::=$ atomic formulas
- $\mathcal{P}_{n+1} ::= \mathcal{N}_n \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \odot \mathcal{P}_{n+1} \mid t \mid \bot$
- $\mathcal{N}_{n+1} ::= \mathcal{P}_n \mid \mathcal{P}_{n+1} \to \mathcal{N}_{n+1} \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid f \mid \top$

$$\begin{tikzcd}
\mathcal{P}_0 \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & \mathcal{P}_1 \arrow[r] \arrow[dr] & \mathcal{P}_2 \arrow[r] \arrow[dr] & \mathcal{P}_3 \arrow[r] \arrow[dr] & \mathcal{P}_4 \arrow[r] \arrow[dr] & \cdots \\
\mathcal{N}_0 \arrow[r] \arrow[ur] & \mathcal{N}_1 \arrow[r] \arrow[ur] & \mathcal{N}_2 \arrow[r] \arrow[ur] & \mathcal{N}_3 \arrow[r] \arrow[ur] & \mathcal{N}_4 \arrow[r] \arrow[ur] & \cdots
\end{tikzcd}$$

Examples

Class	Axiom	Name
$\mathcal{N}_2$	$\alpha \to t$, $\bot \to \alpha$	weakening
	$\alpha \to \alpha \odot \alpha$	contraction
	$\alpha \odot \alpha \to \alpha$	expansion
	$\alpha^n \to \alpha^m$	knotted axioms
	$\neg(\alpha \wedge \neg\alpha)$	weak contraction
$\mathcal{P}_2$	$\alpha \vee \neg\alpha$	excluded middle
	$(\alpha \to \beta) \vee (\beta \to \alpha)$	prelinearity
$\mathcal{P}_3$	$\neg\alpha \vee \neg\neg\alpha$	weak excluded middle
	$\neg(\alpha \odot \beta) \vee (\alpha \wedge \beta \to \alpha \odot \beta)$	(wnm)
$\mathcal{N}_3$	$((\alpha \to \beta) \to \beta) \to ((\beta \to \alpha) \to \alpha)$	Lukasiewicz axiom
	$(\alpha \wedge \beta) \to \alpha \odot (\alpha \to \beta)$	divisibility

Algorithmic introduction of analytic calculi II

Theorem (AC, Galatos, Terui 2008)

Algorithm to transform (almost all)

- *axioms α up to the class $\mathcal{N}_2$ into good structural rules in sequent calculus*
- *axioms α up to the class $\mathcal{P}_3$ into good structural rules in hypersequent calculus*

$$
\begin{tikzcd}
\mathcal{P}_0 \arrow[r] \arrow[d, leftrightarrow] \arrow[dr] & \mathcal{P}_1 \arrow[r] \arrow[dr] & \mathcal{P}_2 \arrow[r] \arrow[dr] & \mathcal{P}_3 \arrow[r] \arrow[dr] & \mathcal{P}_4 \arrow[r] \arrow[dr] & \cdots \\
\mathcal{N}_0 \arrow[r] \arrow[ur] & \mathcal{N}_1 \arrow[r] \arrow[ur] & \mathcal{N}_2 \arrow[r] \arrow[ur] & \mathcal{N}_3 \arrow[r] \arrow[ur] & \mathcal{N}_4 \arrow[r] \arrow[ur] & \cdots
\end{tikzcd}
$$

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(AC, Galatos, Terui 2011,2012,Submitted) and (almost all)

- algebraic equations $1 \leq \alpha$ up to the class $\mathcal{N}_2$ are preserved under DM-completion
- algebraic equations $1 \leq \alpha$ up to the class $\mathcal{P}_3$ are preserved under DM-completion when applied to s.i. algebras

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Density vs Cut

Takeuti and Titani's rule (p eigenvariable)

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$$\boxed{\frac{(\alpha \to p) \vee (p \to \beta) \vee \gamma}{(\alpha \to \beta) \vee \gamma}}$$

- $$\frac{G \,|\, \Gamma \Rightarrow p \,|\, \Sigma, p \Rightarrow \Delta}{G \,|\, \Gamma, \Sigma \Rightarrow \Delta}\ (density)$$

  where p is does not occur in the conclusion.

- $$\frac{G \,|\, \Gamma \Rightarrow A \quad G \,|\, \Sigma, A \Rightarrow \Delta}{G \,|\, \Gamma, \Sigma \Rightarrow \Delta}\ (cut)$$

Density elimination

- Similar to cut-elimination
- Proof by induction on the length of derivations

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(AC, Metcalfe 2008) Given a density-free derivation, ending in

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- Asymmetric substitution: p is replaced
  - With Δ when occuring on the right
  - With Γ when occuring on the left

$$\dfrac{\begin{array}{c}\vdots\, d' \\ G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta\end{array}}{G \mid \Gamma \Rightarrow \Delta}\ \text{(EC)}$$

Problem with (com)

$$
\dfrac{\begin{array}{c}
\dfrac{p \Rightarrow p \qquad \begin{array}{c}\vdots \\ \Pi \Rightarrow \Psi\end{array}}{\Pi \Rightarrow p \mid p \Rightarrow \Psi}\,(com) \\
\vdots\, d \\
G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta
\end{array}}{G \mid \Gamma \Rightarrow \Delta}\,(\mathrm{D})
\qquad\qquad
\dfrac{\begin{array}{c}
\dfrac{\Gamma \Rightarrow \Delta \qquad \begin{array}{c}\vdots \\ \Pi \Rightarrow \Psi\end{array}}{\Pi \Rightarrow \Delta \mid \Gamma \Rightarrow \Psi}\,(com) \\
\vdots\, d^* \\
G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta
\end{array}}{G \mid \Gamma \Rightarrow \Delta}\,(\mathrm{EC})
$$

- $p \Rightarrow p$ axiom
- $\Gamma \Rightarrow \Delta$ not an axiom

Solution (with weakening)

(AC, Metcalfe 2008)

$$\dfrac{\dfrac{p \Rightarrow p \qquad \begin{matrix}\vdots \\ \Pi \Rightarrow \Psi\end{matrix}}{\begin{matrix}\Pi \Rightarrow p \mid p \Rightarrow \Psi \\ \vdots\, d \\ G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta\end{matrix}}\,(com)}{G \mid \Gamma \Rightarrow \Delta}\,\text{(D)}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta\end{matrix} \qquad \begin{matrix}\vdots \\ \Pi \Rightarrow \Psi\end{matrix}}{\begin{matrix}\Pi \Rightarrow \Delta \mid \Gamma \Rightarrow \Psi \\ \vdots\, d^* \\ G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta\end{matrix}}\,(cut)}{G \mid \Gamma \Rightarrow \Delta}\,\text{(EC)}$$

Axiomatic extensions of MTL

MTL = UL + weakening/integrality

Theorem (Baldi, A.C. ,Spendier 2013, 2014)

MTL + all $\mathcal{P}_3$ axioms leading to **semi-anchored** *rules admits density elimination*

$$\text{Ex.}\quad \frac{\begin{array}{cc} G \mid \Gamma_2, \Gamma_1, \Delta_1 \Rightarrow \Pi_1 & G \mid \Gamma_1, \Gamma_3, \Delta_1 \Rightarrow \Pi_1 \\ G \mid \Gamma_1, \Gamma_1, \Delta_1 \Rightarrow \Pi_1 & G \mid \Gamma_2, \Gamma_3, \Delta_1 \Rightarrow \Pi_1 \end{array}}{G \mid \Gamma_2, \Gamma_3 \Rightarrow \ \mid \Gamma_1, \Delta_1 \Rightarrow \Pi_1}\ (wnm)$$

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Automated transformation of axioms into rules and check whether the latter are semi-anchored:

http://www.logic.at/people/lara/axiomcalc.html

Solution (without weakening)

(AC, Metcalfe 2008)

$$\dfrac{\begin{array}{c}\Pi, p \Rightarrow p \\ \vdots\, d \\ G \mid \Gamma \Rightarrow p \mid p \Rightarrow \Delta\end{array}}{G \mid \Gamma \Rightarrow \Delta}\,(\mathrm{D}) \qquad \dfrac{\begin{array}{c}\Pi \Rightarrow t \\ \vdots\, d^* \\ G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta\end{array}}{G \mid \Gamma \Rightarrow \Delta}\,(\mathrm{EC})$$

- We substitute:
  - $p \Rightarrow p$ with $\Rightarrow t$.
  - p with Δ when occurring on the right.
  - p with Γ when occurring on the left.
- We introduce suitable cuts

Works for *UL*. How about extensions?

Axiomatic extensions of UL?

Standard completeness has been shown for:

- UL + contraction and mingle (Metcalfe and Montagna, JSL 2007)
- UL with n-contraction $\alpha^{n-1} \to \alpha^n$ and n-mingle $\alpha^n \to \alpha^{n-1}$ ($n > 2$) (Wang, FSS 2012)
- UL + with knotted axioms $\alpha^k \to \alpha^j$ ($j, k > 1$) (Baldi, Soft Computing 2014)

... many open problems and no uniform method ...

A case study: UL with $A \to A \odot A$

(P. Baldi, AC 2014)

$$\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{\vdots}}{\Pi, p, p \Rightarrow p}\,(c)}{\Pi, p \Rightarrow p}}{\vdots} \qquad \dfrac{\dfrac{\dfrac{\Rightarrow t}{\vdots}}{\Pi, \Gamma \Rightarrow t}\,(?)}{\Pi \Rightarrow t}$$

$$\dfrac{G|\Gamma \Rightarrow p|p \Rightarrow \Delta}{G|\Gamma \Rightarrow \Delta}\,(D) \qquad \dfrac{G|\Gamma \Rightarrow \Delta|\Gamma \Rightarrow \Delta}{G|\Gamma \Rightarrow \Delta}\,(ec)$$

- We substitute:
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A case study: UL with $A \to A \odot A$

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$$\dfrac{\dfrac{\begin{array}{c} p \Rightarrow p \\ \vdots \\ \Pi, p, p \Rightarrow p \end{array}}{\begin{array}{c} \Pi, p \Rightarrow p \\ \vdots \\ G|\Gamma \Rightarrow p|p \Rightarrow \Delta \end{array}}\,(c)}{G|\Gamma \Rightarrow \Delta}\,(D)
\qquad
\dfrac{\dfrac{\begin{array}{c} \Rightarrow t \\ \vdots \\ \Pi, \Gamma \Rightarrow t \end{array}}{\begin{array}{c} \Pi \Rightarrow t \\ \vdots \\ G|\Gamma \Rightarrow \Delta|\Gamma \Rightarrow \Delta \end{array}}\,(?)}{G|\Gamma \Rightarrow \Delta}\,(ec)$$

- (?) is replaced by a subderivation obtained by substituting:
  - $p \Rightarrow p$ (axiom) with $\Pi, \Gamma \Rightarrow t$ (derivable).
  - p with Δ when occurring on the right.
  - p with Γ when occurring on the left.
- We introduce suitable cuts

A case study: UL with $A \to A \odot A$

$$
\cfrac{\cfrac{\begin{array}{c} p \Rightarrow p \\ \vdots \\ \Pi, p, p \Rightarrow p \end{array}}{\begin{array}{c} \Pi, p \Rightarrow p \\ \vdots \\ G|\Gamma \Rightarrow p|p \Rightarrow \Delta \end{array}}\,(c)}{G|\Gamma \Rightarrow \Delta}\,(D)
\qquad
\cfrac{\cfrac{\begin{array}{c} \Rightarrow t \\ \vdots \\ \Pi, \Gamma \Rightarrow t \end{array}}{\begin{array}{c} \Pi \Rightarrow t \\ \vdots \\ G|\Gamma \Rightarrow \Delta|\Gamma \Rightarrow \Delta \end{array}}\,(?)}{G|\Gamma \Rightarrow \Delta}\,(ec)
$$

The same idea works for (mingle) and for all sequent structural rules (= $\mathcal{N}_2$ axioms)

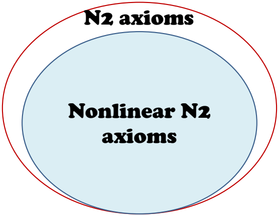
$$
\frac{G|S_1 \quad \dots G|S_m}{G|\Pi, \Gamma_1, \dots, \Gamma_n \Rightarrow \Psi}\,(r)
$$

s.t. if $R(S_i) = \Psi$ then none of Γ_i appears only once in $L(S_i)$.

Axiomatic extensions of UL

Theorem (AC, Baldi Submitted 2014)

UL + nonlinear $\mathcal{N}_2$ axioms (and/or mingle) admits density elimination



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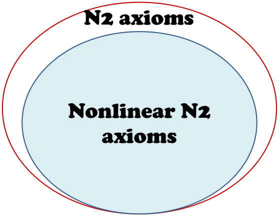

Conjecture: $UL + \mathcal{N}_2$ axioms admits density elimination

Closing the cycle

Uniform (and automated) proofs of standard completeness for large classes of axiomatic extensions of UL

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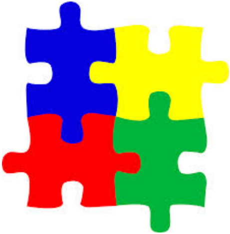

Our methods applies to

Known Logics

- UL + contraction *and* mingle (Metcalfe, Montagna 2007)
- $MTL + \alpha^{n-1} \to \alpha^n$ ($n \geq 3$) (Baldi, 2014)
- $MTL + \neg(\alpha \odot \beta) \vee ((\alpha \wedge \beta) \to (\alpha \odot \beta))$ (Noguera et al.08)
- $MTL + \alpha^{n-1} \to \alpha^n$ (AC, Esteva, Godo 2002)
- ...

New Fuzzy Logics

- UL + contraction *or* mingle
- $UL + f \odot \alpha^k \to \alpha^n$ ($n > 1$)
- $MTL + \neg(\alpha \odot \beta)^n \vee ((\alpha \wedge \beta)^{n-1} \to (\alpha \odot \beta)^n)$, for all $n > 1$
- ...

The big picture

Theory and tools for the investigation of non-classical logics

- Analytic calculi (sequent, hypersequent, nested, display calculi ...)
- Exploitation:
  - standard completeness
  - new semantic foundations (e.g. paraconsistent logics)
  - interpolation
  - properties of algebraic structures
  - ....

”Non-classical Proofs: Theory, Applications and Tools”, research project 2012-2017 (START prize – Austrian Research Fund)