Stable Canonical Rules II

N. Bezhanishvili[1] and **S. Ghilardi**[2]

[1]University of Amsterdam

[2]Università degli Studi di Milano

Les Diablerets,
January 31 - 2015

Multi Conclusion Rules

We review some recent and ongoing work concerning multi-conclusion rules; main topics will be

- rule systems and Hilbert-style calculi for hyper-formulae;
- Jerabek's reformulation of canonical formulae and rule-dichotomy property;
- stable rules as canonical axiomatizations;
- dichotomy property for stable rules (ongoing joint with N. & G. Bezhanishvili, D. Gabelaia, M. Jibladze).

Multi Conclusion Rules

We review some recent and ongoing work concerning multi-conclusion rules; main topics will be

- rule systems and Hilbert-style calculi for hyper-formulae;
- Jerabek's reformulation of canonical formulae and rule-dichotomy property;
- stable rules as canonical axiomatizations;
- dichotomy property for stable rules (ongoing joint with N. & G. Bezhanishvili, D. Gabelaia, M. Jibladze).

Multi Conclusion Rules

We review some recent and ongoing work concerning multi-conclusion rules; main topics will be

- rule systems and Hilbert-style calculi for hyper-formulae;
- Jerabek's reformulation of canonical formulae and rule-dichotomy property;
- stable rules as canonical axiomatizations;
- dichotomy property for stable rules (ongoing joint with N. & G. Bezhanishvili, D. Gabelaia, M. Jibladze).

Multi Conclusion Rules

We review some recent and ongoing work concerning multi-conclusion rules; main topics will be

- rule systems and Hilbert-style calculi for hyper-formulae;
- Jerabek's reformulation of canonical formulae and rule-dichotomy property;
- stable rules as canonical axiomatizations;
- dichotomy property for stable rules (ongoing joint with N. & G. Bezhanishvili, D. Gabelaia, M. Jibladze).

Multi Conclusion Rules

We review some recent and ongoing work concerning multi-conclusion rules; main topics will be

- rule systems and Hilbert-style calculi for hyper-formulae;
- Jerabek's reformulation of canonical formulae and rule-dichotomy property;
- stable rules as canonical axiomatizations;
- dichotomy property for stable rules (ongoing joint with N. & G. Bezhanishvili, D. Gabelaia, M. Jibladze).

Why many conclusions?

A *multiple-conclusion rule* is a pair of finite sets of formulae $\langle \Gamma, S \rangle$.

If $\Gamma = \{\gamma_1, \dots, \gamma_n\}, S = \{\delta_1, \dots, \delta_m\}$, we write the rule $\langle \Gamma, S \rangle$ as Γ / S or as

$$\frac{\gamma_1, \ \dots, \ \gamma_n}{\delta_1 \mid \cdots \mid \delta_m} \ (R)$$

The formulae $\Gamma = \{\gamma_1, \dots . \gamma_n\}$ are said to be the *premises* of the rule (R) and the formulae $S = \{\delta_1, \dots, \delta_m\}$ are said to be the *conclusions* of the rule (R).

The rule (R) is *valid* in a modal algebra $(A, \Box)$ iff for every valuation V

$$V(\gamma_1) = 1 \ \& \ \cdots \ \& \ V(\gamma_n) = 1 \quad \Rightarrow \quad V(\delta_1) = 1 \text{ or } \cdots \text{ or } V(\delta_m) = 1 \ .$$

Thus *rule validity defines a universal class* (not a variety!).

Why many conclusions?

A *multiple-conclusion rule* is a pair of finite sets of formulae $\langle \Gamma, S \rangle$.

If $\Gamma = \{\gamma_1, \ldots, \gamma_n\}, S = \{\delta_1, \ldots, \delta_m\}$, we write the rule $\langle \Gamma, S \rangle$ as Γ / S or as

$$\frac{\gamma_1, \quad \cdots, \quad \gamma_n}{\delta_1 \mid \cdots \mid \delta_m} \; (R)$$

The formulae $\Gamma = \{\gamma_1, \ldots . \gamma_n\}$ are said to be the *premises* of the rule (R) and the formulae $S = \{\delta_1, \ldots, \delta_m\}$ are said to be the *conclusions* of the rule (R).

The rule (R) is *valid* in a modal algebra $(A, \Box)$ iff for every valuation V

$$V(\gamma_1) = 1 \;\&\; \cdots \;\&\; V(\gamma_n) = 1 \quad \Rightarrow \quad V(\delta_1) = 1 \text{ or } \cdots \text{ or } V(\delta_m) = 1 \, .$$

Thus *rule validity defines a universal class* (not a variety!).

Why many conclusions?

Multiple-conclusion rules recently gained attention in the literature from many points of view.

From an algebraic and a semantic point of view, (Kracht 07, Jerabek 09, N. & G. Bezhanishvili & Iemhoff 2014), they constitute an essential tool for investigating classes of algebras beyond varieties and they supply nice canonical formulae axiomatizations.

From a completely different research perspective, *the proof-theoretic oriented community* (since Avron 96) realized that standard sequent formalisms are insufficient to handle complex logics and moved to more expressive hypersequent calculi.

Why many conclusions?

Multiple-conclusion rules recently gained attention in the literature from many points of view.

From an algebraic and a semantic point of view, (Kracht 07, Jerabek 09, N. & G. Bezhanishvili & Iemhoff 2014), they constitute an essential tool for investigating classes of algebras beyond varieties and they supply nice canonical formulae axiomatizations.

From a completely different research perspective, *the proof-theoretic oriented community* (since Avron 96) realized that standard sequent formalisms are insufficient to handle complex logics and moved to more expressive hypersequent calculi.

Why many conclusions?

Multiple-conclusion rules recently gained attention in the literature from many points of view.

From an algebraic and a semantic point of view, (Kracht 07, Jerabek 09, N. & G. Bezhanishvili & Iemhoff 2014), they constitute an essential tool for investigating classes of algebras beyond varieties and they supply nice canonical formulae axiomatizations.

From a completely different research perspective, *the proof-theoretic oriented community* (since Avron 96) realized that standard sequent formalisms are insufficient to handle complex logics and moved to more expressive hypersequent calculi.

Derived Rules

Let K be a set of multiple-conclusion rules; a multiple-conclusion rule Γ/S is *derivable from K* - written $K \vdash \Gamma/S$ iff every modal algebra validating all rules in K validates also Γ/S.

In the terminology of modal rule systems (Jerabek 09, N. & G. Bezhanishvili & Iemhoff 2014), it can be proved that this equivalently means that Γ/S belongs to the smallest modal rule system including K.

What we want to build here is a *Hilbert style* calculus for recognizing $K \vdash \Gamma/S$. This calculus will consequently be complete also for global consequence relation in K.

Derived Rules

Let K be a set of multiple-conclusion rules; a multiple-conclusion rule Γ/S is *derivable from K* - written $K \vdash \Gamma/S$ iff every modal algebra validating all rules in K validates also Γ/S.

In the terminology of modal rule systems (Jerabek 09, N. & G. Bezhanishvili & Iemhoff 2014), it can be proved that this equivalently means that Γ/S belongs to the smallest modal rule system including K.

What we want to build here is a *Hilbert style* calculus for recognizing $K \vdash \Gamma/S$. This calculus will consequently be complete also for global consequence relation in K.

Derived Rules

Let K be a set of multiple-conclusion rules; a multiple-conclusion rule Γ/S is *derivable from K* - written $K \vdash \Gamma/S$ iff every modal algebra validating all rules in K validates also Γ/S.

In the terminology of modal rule systems (Jerabek 09, N. & G. Bezhanishvili & Iemhoff 2014), it can be proved that this equivalently means that Γ/S belongs to the smallest modal rule system including K.

What we want to build here is a *Hilbert style* calculus for recognizing $K \vdash \Gamma/S$. This calculus will consequently be complete also for global consequence relation in K.

Hyperformulae and derivations

Our calculus will manipulate hyperformulae, seen as disjunctions of global assertions (this is the shape of conclusions of our multi-conclusion rules).

A *hyperformula* is a finite set of propositional formulae written in the form

$$\alpha_1 \mid \cdots \mid \alpha_n. \tag{1}$$

We use letters $S, S_1, S', \ldots$ for hyperformulae; the notation $S \mid S'$ means set union and $S \mid \alpha$ and $\alpha \mid S$ stand for $S \mid \{\alpha\}$ and $\{\alpha\} \mid S$, respectively.

Hyperformulae and derivations

Our calculus will manipulate hyperformulae, seen as disjunctions of global assertions (this is the shape of conclusions of our multi-conclusion rules).

A *hyperformula* is a finite set of propositional formulae written in the form

$$\alpha_1 \mid \cdots \mid \alpha_n. \tag{1}$$

We use letters $S, S_1, S', \ldots$ for hyperformulae; the notation $S \mid S'$ means set union and $S \mid \alpha$ and $\alpha \mid S$ stand for $S \mid \{\alpha\}$ and $\{\alpha\} \mid S$, respectively.

Hyperformulae and derivations

Our calculus will manipulate hyperformulae, seen as disjunctions of global assertions (this is the shape of conclusions of our multi-conclusion rules).

A *hyperformula* is a finite set of propositional formulae written in the form

$$\alpha_1 \mid \cdots \mid \alpha_n. \tag{1}$$

We use letters $S, S_1, S', \ldots$ for hyperformulae; the notation $S \mid S'$ means set union and $S \mid \alpha$ and $\alpha \mid S$ stand for $S \mid \{\alpha\}$ and $\{\alpha\} \mid S$, respectively.

Hyperformulae and derivations

Definition

Let Γ be a set of propositional modal formulae and let K be a set of multiple-conclusion rules. A K-*hyperproof* (or a K-*derivation* or just a *derivation*) under assumptions Γ is a finite list of hyperformulae $S_1, \ldots, S_n$ such that each S_i in it matches one of the following requirements:

(i) S_i is of the kind $\alpha \mid S$, where $\alpha \in \Gamma$ or α is a tautology or α is an instance of the **K** distribution axiom;

(ii) S_i is obtained from hyperformulae preceding it by applying a rule from K or the necessitation rule or the modus ponens rule.

We write $\Gamma \vdash_K S$ to mean that there is a K-derivation ending with S.

Hyperformulae and derivations

An important remark is in order for (ii): when we say that S_i is obtained by applying an inference rule, *we include uniform substitution and weakening in the application of the rule*. Thus, if the rule is

$$\frac{\gamma_1, \quad \dots, \quad \gamma_n}{\delta_1 \mid \cdots \mid \delta_m} \ (R)$$

when we say that S_i is obtained from (R), we mean that there are a hyperformula S and a substitution σ such that S_i is of the kind $S \mid \delta_1\sigma \mid \cdots \mid \delta_m\sigma$ and that there are $j_1, \dots, j_n < i$ such that S_{j_1} is of the kind $S \mid \gamma_1\sigma$, and $\dots$ and S_{j_n} is of the kind $S \mid \gamma_n\sigma$.

Hyperformulae and derivations

In other words, when rule (R) is used, we apply a substitution to its contextual form

$$\frac{\gamma_1 \mid S, \quad \ldots, \; \gamma_n \mid S}{\delta_1 \mid \cdots \mid \delta_m \mid S} \; (R)$$

Proposition

We have $K \vdash \Gamma / S$ iff there is a K-derivation under assumptions Γ ending in S.

Fmp and Bpp

We call a set of multi-conclusion rules K a (modal) inference system.

Definition

An inference system K has the *bounded proof property* (bpp) iff whenever $\Gamma \vdash_K S$ holds, there is a K-hyperproof from Γ of S in which formulae not exceeding the modal degrees of formulae in $\Gamma \cup S$ occur.

Definition

An inference system K has the *finite model property* (fmp) iff whenever $\Gamma \vdash_K S$ does not hold, then there is a finite modal algebra validating all rules from K but not Γ / S.

Fmp and Bpp

We call a set of multi-conclusion rules K a (modal) inference system.

Definition

An inference system K has the *bounded proof property* (bpp) iff whenever $\Gamma \vdash_K S$ holds, there is a K-hyperproof from Γ of S in which formulae not exceeding the modal degrees of formulae in $\Gamma \cup S$ occur.

Definition

An inference system K has the *finite model property* (fmp) iff whenever $\Gamma \vdash_K S$ does not hold, then there is a finite modal algebra validating all rules from K but not Γ/S.

Fmp and Bpp

We call a set of multi-conclusion rules K a (modal) inference system.

Definition

An inference system K has the *bounded proof property* (bpp) iff whenever $\Gamma \vdash_K S$ holds, there is a K-hyperproof from Γ of S in which formulae not exceeding the modal degrees of formulae in $\Gamma \cup S$ occur.

Definition

An inference system K has the *finite model property* (fmp) iff whenever $\Gamma \vdash_K S$ does not hold, then there is a finite modal algebra validating all rules from K but not Γ/S.

Subreductions

E. Jerabek [JSL, 2009] introduced multi-conclusion variants of Zakharyaschev canonical rules. The idea behind such rules is that the rule is refuted precisely whenever there is no subreduction (with conditions) onto a given finite frame.

Frames in this section are assumed to be *transitive*.

Definition

Let (W, R, P) be a descriptive frame and (F, R_F) be a finite frame. A partial surjective map $f : W \supseteq dom(f) \longrightarrow F$ is a *subreduction* iff the following hold:

- $x, y \in dom(f)$ and xRy imply $f(x)R_F f(y)$;
- if $f(x)Ru$ then there is y such that xRy and $f(y) = u$;
- $f^{-1}(u) \in P$, for all $u \in F$.

Subreductions

E. Jerabek [JSL, 2009] introduced multi-conclusion variants of Zakharyaschev canonical rules. The idea behind such rules is that the rule is refuted precisely whenever there is no subreduction (with conditions) onto a given finite frame.

Frames in this section are assumed to be *transitive*.

Definition

Let (W, R, P) be a descriptive frame and (F, R_F) be a finite frame. A partial surjective map $f : W \supseteq dom(f) \longrightarrow F$ is a *subreduction* iff the following hold:

- $x, y \in dom(f)$ and xRy imply $f(x)R_F f(y)$;
- if $f(x)Ru$ then there is y such that xRy and $f(y) = u$;
- $f^{-1}(u) \in P$, for all $u \in F$.

Subreductions

A *domain* d in F is an upward closed subset of F; a subreduction f satisfies the (global) closed domain condition for d iff there is no $x \in W \setminus dom(f)$ such that $f(\uparrow x) = d$.

Definition

Let (F, R) be a finite frame and D be a set of domains in F; the canonical rule $\gamma(F, D)$ is the multi-conclusion rule:

$$\frac{\{\delta_{ij} \mid i \neq j\},\ \{\rho_{ij} \mid a_i R_F a_j\},\ \{\overline{\rho}_{ij} \mid \text{not } a_i R_F a_j\},\ \{\phi_d \mid d \in D\}}{\neg x_{a_1} \mid \cdots \mid \neg x_{a_n}}\ \gamma(F, D)$$

where we suppose that $F = \{a_1, \dots, a_n\}$ and

- $\delta_{ij} \ := \ \neg(x_{a_i} \wedge x_{a_j})$;
- $\rho_{ij} \ := \ x_{a_i} \to \Diamond x_{a_j}$;
- $\overline{\rho}_{ij} \ := \ \neg(x_{a_i} \wedge \Diamond x_{a_j})$;
- $\phi_d \ := \ \bigwedge_{a_i \in d} \Diamond x_{a_i} \to \bigvee_i x_{a_i} \vee \bigvee_{a_i \notin d} \Diamond x_{a_i}$.

Subreductions

A *domain* d in F is an upward closed subset of F; a subreduction f satisfies the (global) closed domain condition for d iff there is no $x \in W \setminus dom(f)$ such that $f(\uparrow x) = d$.

Definition

Let (F, R) be a finite frame and D be a set of domains in F; the canonical rule $\gamma(F, D)$ is the multi-conclusion rule:

$$\frac{\{\delta_{ij} \mid i \neq j\},\ \{\rho_{ij} \mid a_i R_F a_j\},\ \{\overline{\rho}_{ij} \mid \text{not } a_i R_F a_j\},\ \{\phi_d \mid d \in D\}}{\neg x_{a_1} \mid \cdots \mid \neg x_{a_n}}\ \gamma(F, D)$$

where we suppose that $F = \{a_1, \ldots, a_n\}$ and

- $\delta_{ij} \ := \ \neg(x_{a_i} \wedge x_{a_j})$;
- $\rho_{ij} \ := \ x_{a_i} \to \Diamond x_{a_j}$;
- $\overline{\rho}_{ij} \ := \ \neg(x_{a_i} \wedge \Diamond x_{a_j})$;
- $\phi_d \ := \ \bigwedge_{a_i \in d} \Diamond x_{a_i} \to \bigvee_i x_{a_i} \vee \bigvee_{a_i \notin d} \Diamond x_{a_i}$.

Subreductions

Proposition

A general frame (W, R, P) refutes $\gamma(F, D)$ iff there is a subreduction from (W, R, P) to (F, R_F) satisfying the (global) closed domain condition for all $d \in D$.

The following completeness result guarantees that one can replace any axiomatization whatsoever with an axiomatization via canonical rules:

Theorem

Given a rule Γ/S one can always find a finite set of canonical rules equivalent to it over K4.

Subreductions

Proposition

A general frame (W, R, P) refutes $\gamma(F, D)$ iff there is a subreduction from (W, R, P) to (F, R_F) satisfying the (global) closed domain condition for all $d \in D$.

The following completeness result guarantees that one can replace any axiomatization whatsoever with an axiomatization via canonical rules:

Theorem

Given a rule Γ/S one can always find a finite set of canonical rules equivalent to it over $K4$.

Subreductions

Proposition

A general frame (W, R, P) refutes $\gamma(F, D)$ iff there is a subreduction from (W, R, P) to (F, R_F) satisfying the (global) closed domain condition for all $d \in D$.

The following completeness result guarantees that one can replace any axiomatization whatsoever with an axiomatization via canonical rules:

Theorem

Given a rule Γ/S one can always find a finite set of canonical rules equivalent to it over $K4$.

Subframe case

A subframe rule is a canonical rule of the kind $(F, \emptyset)$ and a cofinal subframe rule is a rule of the kind $(F, \{\emptyset\})$.

Theorem

Any rule system over K4 axiomatized by subframe and cofinal-subframe canonical rules has the finite model property.

It is not clear however how to get bpp:

Open Problem

Find a way to modify a subframe rule in such a way that the resulting system has the bpp.

Subframe case

A subframe rule is a canonical rule of the kind $(F, \emptyset)$ and a cofinal subframe rule is a rule of the kind $(F, \{\emptyset\})$.

Theorem

Any rule system over $K4$ axiomatized by subframe and cofinal-subframe canonical rules has the finite model property.

It is not clear however how to get bpp:

Open Problem

Find a way to modify a subframe rule in such a way that the resulting system has the bpp.

Subframe case

A subframe rule is a canonical rule of the kind $(F, \emptyset)$ and a cofinal subframe rule is a rule of the kind $(F, \{\emptyset\})$.

Theorem

Any rule system over $K4$ axiomatized by subframe and cofinal-subframe canonical rules has the finite model property.

It is not clear however how to get bpp:

Open Problem

Find a way to modify a subframe rule in such a way that the resulting system has the bpp.

Dichotomy property

The following result is very interesting and gives new insight into rule admissibility problems:

Theorem (Jerabek 2009)

Over various common logics (including K4, S4, GL, ...), a canonical rule is admissible or equivalent to an assumption-free rule.

The proof of the above theorem gives a new way of building bases for admissible rules and a new proof of decidability of the admissibility problem for the above logics.

Dichotomy property

The following result is very interesting and gives new insight into rule admissibility problems:

Theorem (Jerabek 2009)

Over various common logics (including $K4$, $S4$, GL, $\ldots$), a canonical rule is admissible or equivalent to an assumption-free rule.

The proof of the above theorem gives a new way of building bases for admissible rules and a new proof of decidability of the admissibility problem for the above logics.

Dichotomy property

The following result is very interesting and gives new insight into rule admissibility problems:

Theorem (Jerabek 2009)

Over various common logics (including $K4$, $S4$, GL, $\ldots$), a canonical rule is admissible or equivalent to an assumption-free rule.

The proof of the above theorem gives a new way of building bases for admissible rules and a new proof of decidability of the admissibility problem for the above logics.

Homomorphic Images and Stability

A *stable embedding* of a modal algebra $\mathfrak{A} = (A, \Diamond)$ into a modal algebra $\mathfrak{B} = (B, \Diamond)$ is an injective Boolean morphism $\mu : A \to B$ such that we have $\Diamond\mu(x) \leq \mu(\Diamond x)$ for all $x \in A$.

A class $\mathcal{C}$ of modal algebras is said to be *stable* iff whenever $\mathfrak{B} \in \mathcal{C}$ and $\mathfrak{A}$ has a stable embedding into $\mathfrak{B}$, then $\mathfrak{A} \in \mathcal{C}$ too.

We have dual notions for general frames. $\mathfrak{F} = (W, R, P)$ is a *homomorphic image* of $\mathfrak{F}' = (W', R', P')$ iff there is a continuous (i.e. $S \in P \Rightarrow f^{-1}(S) \in P'$) surjective map $f : W' \to W$ such that xRy implies $f(x)R'f(y)$ for all $x, y \in W'$.

A class of (ordinary, general or descriptive) frames is said to be *stable* iff it is closed under homomorphic images.

Homomorphic Images and Stability

A *stable embedding* of a modal algebra $\mathfrak{A} = (A, \Diamond)$ into a modal algebra $\mathfrak{B} = (B, \Diamond)$ is an injective Boolean morphism $\mu : A \to B$ such that we have $\Diamond\mu(x) \leq \mu(\Diamond x)$ for all $x \in A$.
A class $\mathcal{C}$ of modal algebras is said to be *stable* iff whenever $\mathfrak{B} \in \mathcal{C}$ and $\mathfrak{A}$ has a stable embedding into $\mathfrak{B}$, then $\mathfrak{A} \in \mathcal{C}$ too.

We have dual notions for general frames. $\mathfrak{F} = (W, R, P)$ is a *homomorphic image* of $\mathfrak{F}' = (W', R', P')$ iff there is a continuous (i.e. $S \in P \Rightarrow f^{-1}(S) \in P'$) surjective map $f : W' \to W$ such that xRy implies $f(x)R'f(y)$ for all $x, y \in W'$.
A class of (ordinary, general or descriptive) frames is said to be *stable* iff it is closed under homomorphic images.

Homomorphic Images and Stability

A *stable embedding* of a modal algebra $\mathfrak{A} = (A, \Diamond)$ into a modal algebra $\mathfrak{B} = (B, \Diamond)$ is an injective Boolean morphism $\mu : A \to B$ such that we have $\Diamond\mu(x) \leq \mu(\Diamond x)$ for all $x \in A$.
A class $\mathcal{C}$ of modal algebras is said to be *stable* iff whenever $\mathfrak{B} \in \mathcal{C}$ and $\mathfrak{A}$ has a stable embedding into $\mathfrak{B}$, then $\mathfrak{A} \in \mathcal{C}$ too.

We have dual notions for general frames. $\mathfrak{F} = (W, R, P)$ is a *homomorphic image* of $\mathfrak{F}' = (W', R', P')$ iff there is a continuous (i.e. $S \in P \Rightarrow f^{-1}(S) \in P'$) surjective map $f : W' \to W$ such that xRy implies $f(x)R'f(y)$ for all $x, y \in W'$.
A class of (ordinary, general or descriptive) frames is said to be *stable* iff it is closed under homomorphic images.

Homomorphic Images and Stability

A *stable embedding* of a modal algebra $\mathfrak{A} = (A, \Diamond)$ into a modal algebra $\mathfrak{B} = (B, \Diamond)$ is an injective Boolean morphism $\mu : A \to B$ such that we have $\Diamond\mu(x) \leq \mu(\Diamond x)$ for all $x \in A$.
A class $\mathcal{C}$ of modal algebras is said to be *stable* iff whenever $\mathfrak{B} \in \mathcal{C}$ and $\mathfrak{A}$ has a stable embedding into $\mathfrak{B}$, then $\mathfrak{A} \in \mathcal{C}$ too.

We have dual notions for general frames. $\mathfrak{F} = (W, R, P)$ is a *homomorphic image* of $\mathfrak{F}' = (W', R', P')$ iff there is a continuous (i.e. $S \in P \Rightarrow f^{-1}(S) \in P'$) surjective map $f : W' \to W$ such that xRy implies $f(x)R'f(y)$ for all $x, y \in W'$.
A class of (ordinary, general or descriptive) frames is said to be *stable* iff it is closed under homomorphic images.

Canonical Stable Rules

Given a domain a clopen $d \in P'$, we say that a stable map f from $\mathfrak{F} = (W, R, P)$ into $\mathfrak{F}' = (W', R', P')$ satisfies the closed domain condition for d iff $f^{-1}(\Diamond d) = \Diamond f^{-1}(d)$ i.e. iff for all x

$$d \cap {\uparrow} f(x) \neq \varnothing \Rightarrow d \cap f({\uparrow} x) \neq \varnothing.$$

We introduce now another class of rules, called ‘canonical stable rules’ (see N. & G. Bezhanishvili, R. Iemhoff (2014). No transitivity is assumed now.

Canonical Stable Rules

Definition

Let (F, R) be a finite frame and D be a set of domains in F; the canonical stable rule $\sigma(F, D)$ is the multi-conclusion rule:

$$\frac{\bigvee_{i=1}^{n} x_{a_i},\ \{\delta_{ij} \mid i \neq j\},\ \{x_{a_i} \to \Box \bigvee_{a_i R b} x_b\}_i,\ \{\phi_d \mid d \in D\}}{\neg x_{a_1} \mid \cdots \mid \neg x_{a_n}}\ \sigma(F, D)$$

where we suppose that $F = \{a_1, \ldots, a_n\}$ and

- $\delta_{ij} \ := \ \neg(x_{a_i} \wedge x_{a_j})$;
- $\phi_d \ := \ \bigwedge_i \bigwedge_{b \in d, a_i R b} (x_b \to \Diamond x_b)$.

Completeness

Proposition

A general frame (W, R, P) refutes $\sigma(F, D)$ iff there is a stable surjectyive map from (W, R, P) onto (F, R_F) satisfying the (global) closed domain condition for all $d \in D$.

We have a completeness result here too (without transitivity hypothesis):

Theorem

Given a rule Γ / S one can always find a finite set of canonical rules equivalent to it over **K**, **K4**, **S4**.

Completeness

Proposition

A general frame (W, R, P) refutes $\sigma(F, D)$ iff there is a stable surjectyive map from (W, R, P) onto (F, R_F) satisfying the (global) closed domain condition for all $d \in D$.

We have a completeness result here too (without transitivity hypothesis):

Theorem

Given a rule Γ / S one can always find a finite set of canonical rules equivalent to it over $\mathbf{K}, \mathbf{K4}, \mathbf{S4}$.

Completeness

Proposition

A general frame (W, R, P) refutes $\sigma(F, D)$ iff there is a stable surjectyive map from (W, R, P) onto (F, R_F) satisfying the (global) closed domain condition for all $d \in D$.

We have a completeness result here too (without transitivity hypothesis):

Theorem

Given a rule Γ/S one can always find a finite set of canonical rules equivalent to it over $\mathbf{K}, \mathbf{K4}, \mathbf{S4}$.

Fmp and Bpp

A stable rule is a canonical stable rule of the kind $(F, \emptyset)$. A modal calculus K is *stable* iff so is the class of modal algebras validating it (equivalently: the class of descriptive frames validating it).

Theorem (N. & G. Bezhanishvili & Iemhoff 2014)

(i) *A modal calculus K is stable iff it is axiomatizable via stable rules.*

(ii) *A stable modal calculus enjoys fmp.*

Bpp can be obtained for stable rules over **K** as follows (it is still an open problem over **K4**). We need to modify the stable rule $\sigma(F, \emptyset)$.

Fmp and Bpp

A stable rule is a canonical stable rule of the kind $(F, \emptyset)$. A modal calculus K is *stable* iff so is the class of modal algebras validating it (equivalently: the class of descriptive frames validating it).

Theorem (N. & G. Bezhanishvili & Iemhoff 2014)

(i) *A modal calculus K is stable iff it is axiomatizable via stable rules.*

(ii) *A stable modal calculus enjoys fmp.*

Bpp can be obtained for stable rules over **K** as follows (it is still an open problem over **K4**). We need to modify the stable rule $\sigma(F, \emptyset)$.

Fmp and Bpp

A stable rule is a canonical stable rule of the kind $(F, \emptyset)$. A modal calculus K is *stable* iff so is the class of modal algebras validating it (equivalently: the class of descriptive frames validating it).

Theorem (N. & G. Bezhanishvili & Iemhoff 2014)

(i) *A modal calculus K is stable iff it is axiomatizable via stable rules.*

(ii) *A stable modal calculus enjoys fmp.*

Bpp can be obtained for stable rules over **K** as follows (it is still an open problem over **K4**). We need to modify the stable rule $\sigma(F, \emptyset)$.

Fmp and Bpp

A stable rule is a canonical stable rule of the kind $(F, \emptyset)$. A modal calculus K is *stable* iff so is the class of modal algebras validating it (equivalently: the class of descriptive frames validating it).

Theorem (N. & G. Bezhanishvili & Iemhoff 2014)

(i) *A modal calculus K is stable iff it is axiomatizable via stable rules.*

(ii) *A stable modal calculus enjoys fmp.*

Bpp can be obtained for stable rules over **K** as follows (it is still an open problem over **K4**). We need to modify the stable rule $\sigma(F, \emptyset)$.

Fmp and Bpp

Rule $\sigma(F, D)$ is modified into the rule $\sigma^+(F, D)$ below:

$$\frac{\bigvee_{i=1}^{n} x_{a_i}, \quad \bigwedge_{i\neq j} \neg(x_{a_i} \wedge x_{a_j}), \quad \bigwedge_{i=1}^{n}(x_{a_i} \to \Box r_{a_i}), \quad \bigwedge_{i=1}^{n}(r_{a_i} \to \bigvee_{b\in R_F(a_i)} x_b)}{\neg x_{a_1} \mid \cdots \mid \neg x_{a_n}}$$

Lemma

Rules $\sigma(F, D)$ and $\sigma^+(F, D)$ are inter-derivable.

Fmp and Bpp

Rule $\sigma(F, D)$ is modified into the rule $\sigma^+(F, D)$ below:

$$\frac{\bigvee_{i=1}^{n} x_{a_i}, \quad \bigwedge_{i \neq j} \neg(x_{a_i} \wedge x_{a_j}), \quad \bigwedge_{i=1}^{n}(x_{a_i} \to \Box r_{a_i}), \quad \bigwedge_{i=1}^{n}(r_{a_i} \to \bigvee_{b \in R_F(a_i)} x_b)}{\neg x_{a_1} \mid \cdots \mid \neg x_{a_n}}$$

Lemma

Rules $\sigma(F, D)$ and $\sigma^+(F, D)$ are inter-derivable.

Bpp for Stable Calculi

Theorem (N. B. & S. G. 2014)

Any modal calculus axiomatized by rules of the kind $\sigma^+(F, D)$ enjoys bpp and fmp.

Corollary

Let $\mathcal{C}$ be a stable class of (ordinary) Kripke frames such that membership of a finite frame to $\mathcal{C}$ is decidable. Then validity of a formula (more generally, of a rule) in $\mathcal{C}$ is decidable as well.

Bpp for Stable Calculi

Theorem (N. B. & S. G. 2014)

Any modal calculus axiomatized by rules of the kind $\sigma^+(F, D)$ enjoys bpp and fmp.

Corollary

Let $\mathcal{C}$ be a stable class of (ordinary) Kripke frames such that membership of a finite frame to $\mathcal{C}$ is decidable. Then validity of a formula (more generally, of a rule) in $\mathcal{C}$ is decidable as well.

Dichotomy property

The following result is ongloing work (N. B. & S. G. with G. Bezhanishvili, M. Jibladze and D. Gabelaia):

Theorem

Over **K4** *and* **S4***, a stable canonical rule is admissible or equivalent to an assumption-free rule.*

The above result extends to **IPC** (it is an open problem over **K**). Again, the proof builds bases for admissible rules and shows decidability of admissibility.

Dichotomy property

The following result is ongloing work (N. B. & S. G. with G. Bezhanishvili, M. Jibladze and D. Gabelaia):

Theorem

Over **K4** *and* **S4***, a stable canonical rule is admissible or equivalent to an assumption-free rule.*

The above result extends to **IPC** (it is an open problem over **K**). Again, the proof builds bases for admissible rules and shows decidability of admissibility.

Dichotomy property

The following result is ongloing work (N. B. & S. G. with G. Bezhanishvili, M. Jibladze and D. Gabelaia):

Theorem

Over **K4** *and* **S4***, a stable canonical rule is admissible or equivalent to an assumption-free rule.*

The above result extends to **IPC** (it is an open problem over **K**). Again, the proof builds bases for admissible rules and shows decidability of admissibility.

Conclusions

- canonical stable rules are a powerful method for analyzing modal logics and also rule admissibility;
- nevertheless, the behavior of canonical stable rules might be different from that of canonical ryules;
- there are still challenging basic open problems in the area.

THANK FOR ATTENTION !

Conclusions

- canonical stable rules are a powerful method for analyzing modal logics and also rule admissibility;
- nevertheless, the behavior of canonical stable rules might be different from that of canonical ryules;
- there are still challenging basic open problems in the area.

THANK FOR ATTENTION !

Conclusions

- canonical stable rules are a powerful method for analyzing modal logics and also rule admissibility;
- nevertheless, the behavior of canonical stable rules might be different from that of canonical ryules;
- there are still challenging basic open problems in the area.

THANK FOR ATTENTION !

Conclusions

- canonical stable rules are a powerful method for analyzing modal logics and also rule admissibility;
- nevertheless, the behavior of canonical stable rules might be different from that of canonical ryules;
- there are still challenging basic open problems in the area.

THANK FOR ATTENTION !

Conclusions

- canonical stable rules are a powerful method for analyzing modal logics and also rule admissibility;
- nevertheless, the behavior of canonical stable rules might be different from that of canonical ryules;
- there are still challenging basic open problems in the area.

THANK FOR ATTENTION !