Admissible Infinitary Rules in Modal Logic. Part II

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Les Diablerets, 1 February, 2015

Abstract

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• *Almost structurally complete consequence operations extending* **S4**.**3**, (in preparation).

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Each unifiable formula has a projective unifier in **S4**.3*.*
Consequently, each modal consequence operation extending **S4**.3 *is (structurally)-complete with respect finitary non-passive rules.*

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A modal logic L containing **S4** *enjoys projective unification if and only if* $\mathbf{S4.3} \subseteq L$.

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(i) Each finitary consequence operation Cn extending **S4.3** *has a finite basis;*
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Theorem

A modal consequence operation Cn extending **S4.3** *is structurally complete with respect to infinitary non-passive rules iff Cn is FA.*

We provide an uniform basis for all non-passive admissible rules of any $L \in \mathrm{NExt}(\mathbf{S4.3})$ consisting of infinitary rules like

$$\frac{\{\Box(\alpha_i \leftrightarrow \alpha_j) \to \alpha_0 : 0 < i < j\}}{\alpha_0}$$

Let *Var* be the set of *propositional variables* and *Fm* be the set of *modal formulas* in $\{\rightarrow, \bot, \Box\}$. For each formula α, let *Var*(α) denote the (finite) set of variables occurring in α.

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By a *modal logic* we mean any proper subset of *Fm* closed under substitutions, closed under

$$MP : \frac{\alpha \to \beta, \alpha}{\beta} \qquad \text{and} \qquad RG : \frac{\alpha}{\Box\alpha},$$

containing all classical tautologies, and

$$\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta).$$

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Each modal logic is an extension of the logic **K** with axiom schemata. **S4** extends **K** with
$(T) : \Box\alpha \to \alpha$ oraz $(4) : \Box\Box\alpha \to \Box\alpha$. The logic **S4.3** contains additionally $\Box(\Box\alpha \to \Box\beta) \vee \Box(\Box\beta \to \Box\alpha)$; and **S5** is axiomatized with $(5) : \Diamond\Box\alpha \to \Box\alpha$.

Given a modal logic L, we define its *global entailment relation* Cn_L. Thus, $\alpha \in Cn_L(X)$ means that α can be derived from $X \cup L$ using the rules MP and RG.

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If **S4** $\subseteq L$, *then* $\quad \alpha \in Cn_L(X, \beta) \quad$ *iff* $\ \Box\beta \to \alpha \in Cn_L(X)$.

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The lattice ordering in each case is determined by inclusion.

Modal algebra $\mathcal{A} = (A, \to, \bot, \Box)$ is an extension of a Boolean algebra $(A, \to, \bot)$ with a monadic operator operator $\Box$

(1) $\quad \Box\top = \top;$ $\qquad$ (2) $\quad \Box(a \wedge b) = \Box a \wedge \Box b, \quad$ for $a, b \in A$.

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It means each modal logic has na adequate class of modal algebras.

Each modal algebra $\mathcal{A}$ generates modal consequence operation $\overrightarrow{\mathcal{A}}$:

$$\alpha \in \overrightarrow{\mathcal{A}}(X) \Leftrightarrow \big(v[X] \subseteq \{\top\} \Rightarrow v(\alpha) = \top,\ \text{dla } v\colon Var \to A\big).$$

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For consequence operations over (**S4.3**)*: $FA \cap Fin = SF$.*

Topological Boolean Algebras (*TBA*'s) are modal algebras which are models for **S4**; i.e., they fulfill

(3) $\Box a \leq a$; (4) $\Box\Box a = \Box a$, for each $a \in A$.

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Any *(Kripke) structure* $\mathfrak{F} = (V, R)$ consists of a non-empty set V and a binary relation R on V. Each $\mathfrak{F} = (V, R)$ determines a modal algebra $\mathfrak{F}^+$, on the power set $A = P(V)$, with $\Box a = \{x \in V : R(x) \subseteq a\}$, where $R(x) = \{y \in V : xRy\}$.

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A subset C of V is called a *cluster* if xRy and yRx, for each $x, y \in C$. Symbols $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$, etc., stand for 1- , 2- and 3-element clusters, which are **S5** models. Then $\mathfrak{n}^+$ is called a *Henle Algebra*.

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Finite subdirectly irreducible **S4**.**3**-algebras are *TBA*'s in which all open elements (i.e. such a's that $\Box a = a$) form a chain. They coincide with algebras $\mathfrak{F}^+$ for finite *quasi chains* $\mathfrak{F}$ which are, in turn, determined by *lists*, i.e. finite sequences of positive integers.

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*The following conditions are equivalent for each $L \supseteq$ **S4**:*
(i) α is L-unifiable; *(ii) α has a ground unifier w L;*
(iii) α is satisfiable in $\mathbf{2}$; *(iv) $\sim\alpha \notin \mathbf{Tr} = Log(\mathbf{2})$.*

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A unifier ε is said to be *projective* for X if $\varepsilon(\beta) \leftrightarrow \beta \in Cn_L(X)$, for each formula β; we say that L has *projective unification*, if each L-unifiable formula has a projective unifier is L.

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Passive rules over **S4.3**

Let $n \geq 0$ and let us consider the sublanguage of Fm spanned on $p_1, \ldots, p_n$. There are 2^n (Boolean) atoms there and each of them can be represented by the formula

$$p_1^{\sigma(1)} \wedge \cdots \wedge p_n^{\sigma(n)}$$

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Theorem

Each finitary consequence operation $Cn \in \mathrm{EXT}(\mathbf{S4.3})$ is an extension of some Cn_L (for $L \supseteq \mathbf{S4.3}$ and some $n \geq 0$) with a finite number of passive rules

$$\frac{\Diamond\theta_1 \wedge \cdots \wedge \Diamond\theta_s}{\alpha}$$

where $2 \leq s \leq 2^n$ and $Var(\alpha) \cap \{p_1, \ldots, p_n\} = \emptyset$.

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Since ϱ is **S4.3**-admissible and ε is a unifier for any $\Box(p_i \leftrightarrow p_j) \to p_0$ with $0 < i < j$, we get $\varepsilon(p_0) \in \mathbf{S4.3}$. Thus, we obtain (for $\beta = p_0$)

$$p_0 \in Cn_{\mathbf{S4.3}}(\{\Box(p_i \leftrightarrow p_j) \to p_0 : 0 < i < j\})$$

which is impossible. Consequently, the set $\{\Box(p_i \leftrightarrow p_j) \to p_0 : 0 < i < j\}$ cannot have a projective unifier in **S4.3** (and any weaker logic).

FA extensions

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Corollary

$$\overrightarrow{\mathbb{K} \times \mathfrak{1}^+} = Cn_L^{SCpl} \quad \textit{and} \quad \overrightarrow{\mathbb{K}} = Cn_L^{ASCpl}.$$

Let Σ be the family of all strictly increasing number-theoretic functions. For each function $f \in \Sigma$, let r_f be

$$\frac{\{[\bigwedge_{n<j\leq f(n)} \bigvee_{0<i\leq n} \Box(p_i \leftrightarrow p_j)] \rightarrow p_0 : n > 0\}}{p_0}$$

Example

Let $f(n) = n + 1$ for each n. Then r_f is equivalent to the rule ϱ:

$$\frac{\{\Box(p_i \leftrightarrow p_j) \rightarrow p_0 : 0 < i < j\}}{p_0}$$

□

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Lemma

The rule r_f, for any $f \in \Sigma$, is valid in any finite algebra $\mathcal{A}$, i.e.

$$p_0 \in \overrightarrow{\mathcal{A}}(\{[\bigwedge_{n<j\leq f(n)} \bigvee_{0<i\leq n} \Box(p_i \leftrightarrow p_j)] \to p_0 : n > 0\})$$

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Thus, similarly as ϱ each rule r_f is admissible for any extension of **S4**.**3**. For any consequence operation Cn, let Cn^{Σ} be the extension of Cn with the rules $\{r_f\}_{f\in\Sigma}$. We prove that $\{r_f\}_{f\in\Sigma}$ is a rule basis for all admissible non-passive (infinitary) rules of any $Cn \in \mathrm{EXT}(\mathbf{S4.3})$:

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Theorem

$Cn_{fin}{}^{\Sigma} = Cn^{\Sigma} = Cn^{ASCpl} =_{fin} Cn$, for any $Cn \in \mathrm{EXT}(\mathbf{S4.3})$.

Corollaries

Corollary

Let $Cn \in \mathrm{EXT}(\mathbf{S4.3})$. Then Cn is almost structurally complete ($Cn \in ASCpl$) iff the rules $\{r_f\}_{f \in \Sigma}$ are derivable for Cn (i.e. $Cn = Cn^{\Sigma}$).

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Let $Cn \in \mathrm{EXT}(\mathbf{S4.3})$*. Then Cn is almost structurally complete (*$Cn \in ASCpl$*) iff the rules* $\{r_f\}_{f \in \Sigma}$ *are derivable for Cn (i.e.* $Cn = Cn^{\Sigma}$*).*

It also follows from the above result that almost structurally complete consequence operations are hereditary almost structurally complete, i.e.

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If $Cn, Cn' \in \mathrm{EXT}(\mathbf{S4.3})$*, then*

$$Cn \in ASCpl \wedge Cn \leq Cn' \Rightarrow Cn' \in ASCpl.$$

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Corollary

$ASCpl(\mathbf{S4.3})$ is a sublattice of $\mathrm{EXT}(\mathbf{S4.3})$. The lattices $ASCpl(\mathbf{S4.3})$ and $\mathrm{EXT}_{fin}(\mathbf{S4.3})$ are isomorphic and as the lattice isomorphisms one can take the mappings $Cn \mapsto Cn^{\Sigma}$ and $Cn \mapsto Cn_{fin}$.

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The consequence operation $\overrightarrow{\mathbb{K}}$, where $\mathbb{K} = \mathrm{Alg}(\mathbf{S4.3})_{si\,fin}$, is the least element of the lattice $ASCpl(\mathbf{S4.3})$. This consequence operation can also be given as the *ASCpl* extension of **S4**.**3**, that is as the extension of $Cn_{\mathbf{S4.3}}$ with the rules $\{r_f\}_{f \in \Sigma}$.

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ASCpl(**S4**.3) *is a sublattice of* EXT(**S4**.3). *The lattices* *ASCpl*(**S4**.3) *and* EXT_{fin}(**S4**.3) *are isomorphic and as the lattice isomorphisms one can take the mappings* $Cn \mapsto Cn^{\Sigma}$ *and* $Cn \mapsto Cn_{fin}$.

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Corollary

For each formula $\alpha \in Cn^{\Sigma}(X)$ we have:
1. *either a finitary proof (of α) in which we apply MP, RG and some finitary passive rules valid for Cn (with respect to some formulas in $L \cup X$);*
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In other words each formula $\alpha \in Cn^{\Sigma}(X)$ has a syntactic proof of the type $\leq \omega + 1$. Although the rules r_f, for $f \in \Sigma$, look artificial they generate a natural (and simple) proof system. The only problem is that the rule basis $\{r_f : f \in \Sigma\}$ has the cardinality $\mathfrak{c}$. We suspect that $Cn^{\Sigma}_{\mathbf{S4.3}}$ has no countable rule basis though we failed to prove it and left this problem open.