Admissible Rules of (Fragments of) R-Mingle

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R-Mingle

RM	Relevance logic R with Mingle
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RM^t	RM with additional constant t
Language	$\mathcal{L}_t = \{\wedge, \vee, \to, \cdot, \neg, t\}$

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- $\mathcal{R}$ is a basis for the admissible rules of L if $\mathrm{L} + \mathcal{R} = \mathrel{|\!\!\sim}_{\mathrm{L}}$

Corresponding algebraic semantics

$$\mathbf{Z}^\circ = \langle \mathbb{Z} \setminus \{0\}, \min, \max, \rightarrow, \cdot, -, 1 \rangle$$

$$\rightarrow \quad x \rightarrow y := \begin{cases} \max\{-x, y\} & \text{if } x \leq y \\ \min\{-x, y\} & \text{if } x > y \end{cases}$$

$$\cdot \quad x \cdot y := \begin{cases} \min\{x, y\} & \text{if } |x| = |y| \\ y & \text{if } |x| < |y| \\ x & \text{if } |x| > |y| \end{cases}$$

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$$\mathbf{Z}_{2n} = \langle \{-n, \dots, -1, 1, \dots, n\}, \min, \max, \to, \cdot, -, 1 \rangle$$

$$\mathbf{Z}_{2n+1} = \langle \{-n, \dots, -1, 0, 1, \dots, n\}, \min, \max, \to, \cdot, -, 1 \rangle$$

Sugihara Monoids

$\mathcal{SM} =$ $\mathbb{V}(\mathbf{Z}^\circ)$ the variety of Sugihara Monoids generated by $\mathbf{Z}^\circ$.

$\mathcal{SM}$ provides an equivalent algebraic semantics for RM^t

$$\{\psi \approx |\psi| \mid \psi \in \Gamma\} \vDash_{\mathcal{SM}} \varphi \approx |\varphi| \quad \Leftrightarrow: \quad \Gamma \vDash_{\mathcal{SM}} \varphi$$

$$\Leftrightarrow \quad \Gamma \vdash_{\mathrm{RM^t}} \varphi$$

for any rule Γ/φ.

This talk

Bases for admissible rules of the fragments of RM^{t} with the following languages

$$\mathcal{L}_1 = \{\rightarrow, t\}$$

$$\mathcal{L}_2 = \{\rightarrow, \cdot, t\}$$

$\mathcal{L}_m = \{\rightarrow, \neg, t\} = \{\rightarrow, \cdot, \neg, t\}$ multiplicative fragment.

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$\mathcal{SM} \restriction \mathcal{L}_i$	algebraic semantics corresponding to the $\mathcal{L}_i$-fragment of RM^t , $i \in \{1, 2, m\}$
Remark *Raftery, Olson*	$\mathrm{RM}^{\mathrm{t}} \restriction \{\wedge, \to, \mathrm{t}\}$ has empty basis (= it is structurally complete).

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Lemma *S.* $\mathbb{V}(\mathcal{SM} \upharpoonright \mathcal{L}_i) = \mathbb{V}(\mathbf{Z}_4 \upharpoonright \mathcal{L}_i), \quad i \in \{1, 2, m\}$

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- Recall that if for two varieties $\mathbb{V}_1$ and $\mathbb{V}_2$ we have: $\mathbb{V}_1 = \mathbb{V}_2$ iff $(\vdash_{\mathbb{V}_1} \varphi \ \Leftrightarrow \vdash_{\mathbb{V}_2} \varphi$ for all formulas $\varphi)$.

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- A rule is admissible in $\mathrm{RM}^{\mathrm{t}} \upharpoonright \mathcal{L}_{\mathrm{i}}$
  $\Leftrightarrow$ it is admissible in $\mathcal{SM} \upharpoonright \mathcal{L}_i$
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- Interested in algebras s.t. admissibility in $\mathbf{Z}_4 \restriction \mathcal{L}_i$ corresponds to validity in these algebras.
- Then: Axiomatize the quasivarieties generated by these algebras to get an axiomatization of the admissible rules of our fragments.

Finding the bases

Theorem Let $\mathbf{B}$ be an algebra and $\mathbf{F_B}(\omega)$ its free algebra on countably infinite many generators. Then

$$\Gamma/\varphi \text{ is } \mathbf{B}\text{-admissible} \quad \Leftrightarrow \quad \Gamma \models_{\mathbf{F_B}(\omega)} \varphi.$$

Finding the bases

Lemma The following are equivalent:

$$(i) \quad \Gamma/\varphi \text{ is } \mathbf{B}\text{-admissible} \quad \Leftrightarrow \quad \Gamma \vDash_{\mathbf{A}} \varphi$$

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Lemma $\mathbf{A} \subseteq \mathbf{F_B}(\omega), \mathbf{B} \in \mathbb{H}(\mathbf{A}) \quad \Rightarrow \quad \mathbb{Q}(\mathbf{A}) = \mathbb{Q}(\mathbf{F_B}(\omega))$

The algebras in our case

Lemma *S.* Let $\mathbf{Z}_4' \subset (\mathbf{Z}_2 \times \mathbf{Z}_3) \restriction \mathcal{L}_1$, $\mathbf{Z}_4'' \subset (\mathbf{Z}_2 \times \mathbf{Z}_3) \restriction \mathcal{L}_2$, $(\mathbf{Z}_2 \times \mathbf{Z}_3) \restriction \mathcal{L}_m$ be the algebras pictured. Then

$$\begin{array}{rl} (i) & \mathbb{Q}(\mathbf{F}_{\mathbf{Z}_4 \restriction \mathcal{L}_1}(\omega)) = \mathbb{Q}(\mathbf{Z}_4') \\ (ii) & \mathbb{Q}(\mathbf{F}_{\mathbf{Z}_4 \restriction \mathcal{L}_2}(\omega)) = \mathbb{Q}(\mathbf{Z}_4'') \\ (iii) & \mathbb{Q}(\mathbf{F}_{\mathbf{Z}_4 \restriction \mathcal{L}_m}(\omega)) = \mathbb{Q}((\mathbf{Z}_2 \times \mathbf{Z}_3) \restriction \mathcal{L}_m) \end{array}$$

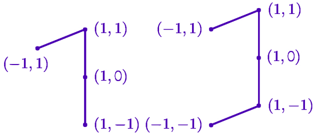

Figure: $\mathbf{Z}_4'$ and $\mathbf{Z}_4''$

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- $\{p, p \Rightarrow q\}/q \qquad (A)$
- $\varphi \leftrightarrow \psi := (\varphi \to \psi) \cdot (\psi \to \varphi)$
- $\{\neg(|p_1| \leftrightarrow \ldots \leftrightarrow |p_n|)\}/q \qquad (R_n), \quad n \in \mathbb{N}.$

The Bases

Lemma *S.* We have the following axiomatizations:

$$
\begin{array}{rl}
(i) & \mathrm{RM^t} \restriction \mathcal{L}_1 + (A) \text{ has equivalent q.v. } \mathbb{Q}(\mathbf{Z}_4') \\
(ii) & \mathrm{RM^t} \restriction \mathcal{L}_2 + (A) \text{ has equivalent q.v. } \mathbb{Q}(\mathbf{Z}_4'') \\
(iii) & \mathrm{RM^t} \restriction \mathcal{L}_\mathrm{m} + (A) + \{(R_n)\}_{n\in\mathbb{N}} \text{ has eq. q.v.} \\
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Theorem S. Then as a Corollary of this lemma

(i) $\{(A)\}$ is a basis for the $\{\to, t\}$- and $\{\to, \cdot, t\}$-fragment of $\mathrm{RM^t}$.

(ii) $\{(A)\} \cup \{(R_n)\}_{n\in\mathbb{N}}$ is a basis for $\mathrm{RM^t} \upharpoonright \{\to, \neg, \mathrm{t}\}$.

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Conjecture We hope to prove the following:

(i) $\mathrm{RM} + (B)$ is almost structurally complete, i.e.,

$$\Gamma \mid\!\sim_{\mathrm{RM}} \varphi \quad \Rightarrow \quad \Gamma \vdash_{\mathrm{RM}} \varphi$$

whenever there is a substitution $\sigma\colon \mathrm{Fm}_{\mathcal{L}} \to \mathrm{Fm}_{\mathcal{L}}$ s.t. for all $\psi \in \Gamma$, $\vdash_{\mathrm{RM}} \sigma(\psi)$.

(ii) The admissible rules of RM have basis $\{(B)\} \cup \{(R_n)\}_{n\in\mathbb{N}}$.

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