Uniform Interpolation and Proof Systems

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Workshop on Admissible Rules and Unification II
Les Diablerets, February 2, 2015

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(Ciabattoni, Galatos, Terui) *Fix a sequent calculus and determine which axioms or structural rules, when added, preserve cut-elimination.*

Aim: *Formulate properties that, when violated by a logic, imply that the logic does not have a sequent calculus of a certain form.*

Dfn *A logic L has interpolation if whenever* $\vdash \varphi \to \psi$ *there is a* χ *in the common language* $\mathcal{L}(\varphi) \cap \mathcal{L}(\psi)$ *such that* $\vdash \varphi \to \chi$ *and* $\vdash \chi \to \psi$.

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Dfn *A propositional (modal) logic has uniform interpolation if the interpolant depends only on the premiss or the conclusion: For all* φ *there are formulas* $\exists p\varphi$ *and* $\forall p\varphi$ *not containing p such that for all* ψ *not containing p:*

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Algebraic view (next talk).

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Note *A locally tabular logic that has interpolation, has uniform interpolation.*

$$\exists p\varphi(p,\bar{q}) = \bigwedge\{\psi(\bar{q}) \mid \vdash \varphi(p,\bar{q}) \to \psi(\bar{q})\}$$

$$\forall p\varphi(p,\bar{q}) = \bigvee\{\psi(\bar{q}) \mid \vdash \psi(\bar{q}) \to \varphi(p,\bar{q})\}$$

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Pitts uses Dyckhoff's '92 sequent calculus for IPC.

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Therefore *no modal or intermediate logic without uniform interpolation has such an such a calculus.*

Modularity: *The possibility to determine whether the addition of a new rule will preserve uniform interpolation.*

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where S, S_i *are sequents and* S_0 *contains exactly one formula.*

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Dfn *An axiom is focussed if it is of the form*

$$\Gamma, p \Rightarrow p, \Delta \quad \Gamma, \bot \Rightarrow \Delta \quad \Gamma \Rightarrow \top, \Delta \quad \dots$$

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Cor Classical propositional logic has uniform interpolation.

Cor Intuitionistic propositional logic has uniform interpolation.

Cor Except the seven intermediate logics that have interpolation, no intermediate logic has a terminating sequent calculus that consists of focussed rules and axioms.

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Proof idea:
Define interpolation for rules. For every instance

$$\frac{S_1 \quad \dots \quad S_n}{S_0}\ R$$

of a rule, define the formula $\forall\!\overset{R}{p}\, S_0$ *in terms of* $\forall p\, S_i$ $(i > 0)$. *For example,* $\forall\!\overset{R}{p}\, S_0 \equiv \forall p\, S_1 \wedge \ldots \wedge \forall p\, S_n$.

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Then inductively define

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Prove with induction on the order that for all sequents S *a uniform interpolant* $\forall p S$ *exists.* ⊣

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A logic has uniform interpolation if it satisfies the interpolant properties:

($\forall$l) *for all* p: $\vdash S^a, \forall p S \Rightarrow S^s$;

($\forall$r) *for all* p: $\vdash S^l \cdot (\Rightarrow \forall p S^r)$ *if* $S^l \cdot S^r$ *is derivable and* S^l *does not contain* p.

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From ($\forall$r) *obtain that* $\vdash \psi \to \varphi$ *implies* $\vdash \psi \to \forall p\varphi$, *if* ψ *does not contain* p, *by taking* $S^l = (\psi \Rightarrow)$ *and* $S^r = (\Rightarrow \varphi)$. *Hence* $\vdash \psi \to \varphi \;\Leftrightarrow\; \vdash \psi \to \forall p\varphi$, *if* ψ *does not contain* p.

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A logic satisfies the interpolant properties if it satisfies:

(1) $\{S_j \cdot (\forall p S_j \Rightarrow) \mid 1 \leq j \leq n\} \vdash S_0 \cdot (\forall \overset{R}{p} S_0 \Rightarrow)$.

(2) $\{S_j^l \cdot (\Rightarrow \forall p S_j^r) \mid 1 \leq j \leq n\} \vdash S_0^l \cdot (\Rightarrow \forall \overset{R}{p} S_0^r)$.

(3) *If* S_0^r *is no conclusion of* R *there exists ...*

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A logic satisfies the above properties if it has a terminating calculus that consists of focussed axioms and rules.

$\dashv$

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$$\frac{\Box S_1 \cdot S_0}{S_2 \cdot \Box S_1 \cdot \Box S_0}$$

where S_1 and S_2 consist of multisets and S_0 of multisets and exactly one atom.

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Ex *The following are focussed modal rules.*

$$\frac{\Gamma \Rightarrow p}{\Pi, \Box\Gamma \Rightarrow \Box p, \Sigma}\ \mathrm{R_K} \qquad \frac{\Box\Gamma, p \Rightarrow \Box\Delta}{\Pi, \Box\Gamma, \Box p \Rightarrow \Box\Delta, \Sigma} \qquad \frac{p \Rightarrow \Delta}{\Pi, \Box p \Rightarrow \Box\Delta, \Sigma}\ \mathrm{R_{OK}}$$

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Cor *Any normal modal logic with a balanced terminating calculus that consists of focussed (modal) axioms and rules, has uniform interpolation. (Ex: K)*

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Cor *The logics $K4$ and $S4$ do not have balanced terminating sequent calculi that consist of focussed (modal) axioms and rules.*

- *Extend the method to other modal logics, such as* GL *and* KT.
- *Extend the method to hypersequents.*
- *Use other proof systems than sequent calculi.*

Finis