Non-Transitive Linear Temporal Logic with UNTIL and NEXT, Logical Knowledge Operations, Admissible Rules

V.Rybakov

School of Computing, Mathematics and DT,
Manchester Metropolitan University,
John Dalton Building, Chester Street, Manchester M1 5GD,
U.K.
V.Rybakov@mmu.ac.uk

ABSTRACT:

We study linear temporal logic $\mathcal{LTL}_{NT}$ with non-transitive time (with NEXT and UNTIL) and possible interpretations for logical knowledge operations in this approach. We assume time to be non-transitive, linear and discrete, it is a major innovative part in our paper. Motivation for our approach that time might be non-transitive and comments on possible interpretations of logical knowledge operations are given.

Main results are solutions of decidability problem for $\mathcal{LTL}_{NT}$ and problem of recognizing admissible rules for a version of $\mathcal{LTL}_{NT}$ with an any fixed upper bound for non-transitivity.

We enumerate some open interesting problems within this framework.

LTL and Multi-Agency

Linear temporal logic $\mathcal{LTL}$ (with Until and Next) is very useful instrument in CS (Manna, Pnueli (1992, etc.), Vardi (1995,1998)) ($\mathcal{LTL}$ was used for analyzing protocols of computations, check of consistency, etc.).

The conception of knowledge, and especially the one implemented via multi-agent approach is a popular area in Logic in Computer Science. Various aspects, including interaction and autonomy, effects of cooperation etc were investigated (cf. e.g.. Wooldridge (2003), Lomuscio (2002).

In particular, a multi-agent logic with distances was suggested and studied, satisfiability problem for it was solved (Rybakov et al (2010); conception of Chance Discovery in multi-agent's environment was considered (Rybakov, 2007, etc.); a logic modeling uncertainty via agent's views was investigated (cf. McLean, Rybakov (2013); representation of agent's interaction (as a dual of the common knowledge - an elegant conception suggested and profoundly developed in Fagin et al (2005) was suggested in Rybakov (2009).

The infinite linear Kripke structure is a quadruple

$\mathcal{M} := \langle \mathcal{N}, \leq, \text{Next}, V \rangle$,

where $\mathcal{N}$ is the set of all natural numbers, $\leq$ is the standard order on $\mathcal{N}$, Next is the binary relation, where a Next b means b is the number next to a.

Computational rules for logical operations:

- $\forall p \in Prop \ (\mathcal{M}, a) \Vdash_V p \Leftrightarrow a \in \mathcal{N} \wedge \ a \in V(p)$;

- $(\mathcal{M}, a) \Vdash_V (\varphi \wedge \psi) \Leftrightarrow \ (\mathcal{M}, a) \Vdash_V \varphi \wedge (\mathcal{M}, a) \Vdash_V \psi$;

- $(\mathcal{M}, a) \Vdash_V \neg\varphi \Leftrightarrow not[(\mathcal{M}, a) \Vdash_V \varphi]$;

- $(\mathcal{M}, a) \Vdash_V \mathbf{N}\varphi \Leftrightarrow [[(a$ Next $b) \Rightarrow (\mathcal{M}, b) \Vdash_V \varphi]]$;

- $(\mathcal{M}, a) \Vdash_V \mathbf{Pr}\ \varphi \Leftrightarrow [[(b$ Next $a) \Rightarrow (\mathcal{M}, b) \Vdash_V \varphi]]$;

- $(\mathcal{M}, a) \Vdash_V (\varphi \mathbf{U} \psi) \Leftrightarrow \exists b[(a \leq b) \wedge ((\mathcal{M}, b) \Vdash_V \psi) \wedge \ \forall c[(a \leq c < b) \Rightarrow (\mathcal{M}, c) \Vdash_V \varphi]]$;

- $(\mathcal{M}, a) \Vdash_V (\varphi \mathbf{S} \psi) \Leftrightarrow \exists b[(b \leq a) \wedge ((\mathcal{M}, b) \Vdash_V \psi) \wedge \ \forall c[(a \leq c < b) \Rightarrow (\mathcal{M}, c) \Vdash_V \varphi]]$.

The **linear temporal** logic $\mathcal{LTL}$ is the set of all formulas which are valid in all infinite temporal linear Kripke structures $\mathcal{M}$ based on $\mathcal{N}$ with standard $\leq$ and Next.

Informal Motivation, Discussion, what is Knowledge in Temporal Perspective

It is easy to accept that the knowledge is not absolute and depends on opinions of individuals (agents) who accept a statement as safely true or not, and, yet, on what we actually consider as true knowledge. From, temporal perspective, - some evident trivial observations are:

(i) Human beings remember (at least some) past, but

(ii) they do not know future at all (rather could surmise what will happen in immediate proximity time points);

(iii) individual memory tells to us that the time in past was linear (though there is a chance that it might be only our perception).

Therefore it looks meaningful to look for interpretations of knowledge in linear temporal logic with **accessibility relations and Next directed actually to past.**

Several approaches to define the operation of knowledge: here we will use the unary logical operations K_i with meaning - it is a logical knowledge operation. (Below we consider models for $\mathcal{LTL}$ with interpretation $Next$ as directed to past, and $\leq$ - to be earlier.)

(i) Simple approach: when knowledge was discovered once and since then it always seen to be true:

$$(N, a) \Vdash_V K_1 \varphi \Leftrightarrow \exists b[(N, b+1) \nVdash_V \varphi) \wedge (a \leq b) \wedge (N, b) \Vdash_V \varphi) \wedge$$

$$\forall c[(a \leq c < b) \Rightarrow (N, c) \Vdash_V \varphi]].$$

For first glance, it is a rather plausible interpretation. As bigger b will be, as it would be most reasonable to consider φ as a knowledge. But if $a = b$ this definition actually says to us nothing, this definition then admits one-day knowledge, which is definitely not good. How to avoid it?

(ii) Rigid approach from temporal logic: knowledge if always was true:

$$(N, a) \Vdash_V K_2\varphi \Leftrightarrow (N, a) \Vdash_V \neg(\top \mathbf{U} \neg\varphi).$$

That is perfectly OK, though too rigid, - it assumes that we know all past (and besides it does not admit that knowledge was obtained only since a particular time point).

(iii) Knowledge since parameterizing facts:

$$(N,a) \Vdash_V K_\psi \varphi \Leftrightarrow (N,a) \Vdash_V \varphi \mathbf{U} \psi.$$

This means φ has the stable truth value - true, since some event happened in past (which is modeled now by ψ to be true at a state). Thus, as soon as ψ happened to be true, φ always held true until now. Here we use standard until. The formula ψ may have any desirable value, so, we obtain *knowledge* since ψ.

(iv) Approach: via agents knowledge as voted truth for the valuation:

This is very well established area, cf. the book Fagin (1995) and more contemporary publications.

Here knowledge operations (agents knowledge) were just unary logical operations K_i interpreted as $S5$-modalities, and knowledge operations were introduced via the vote of agents, etc.

We would like to suggest here somewhat **very simple but anyway rather fundamental, and it seems new.**

We assume that all agents have theirs own valuations at the frame N. So, we have n-much agents, and n-much valuations

V_i, and, as earlier, the truth values w.r.t. V_i of any propositional letter p_j at any world $a \in N$.

For applications viewpoint, V_i correspond to agents information about truth of p_j (they may be different). So, V_i is just individual *information* .

How the information can be turned to *local* knowledge? One way is the voted value of truth: we consider a new valuation V, w.r.t. which p_i is true at a if majority (with chosen confidence), biggest part of agents, believes that p_i is true at a. Then we achieve a model with a single (standard) valuation V. Then we can apply any of known approaches. But, we could consider yet individual truth valuations V_i for also all composed formulas φ (in a standard manner), and only then to consider knowledge valuation V for composed formulas φ as voted value via all V_i

(with appointed confidence level). But yet we may use more temporal features, for example:

(v) Approach: via agents knowledge as resolution at evaluation state.

Here we suggest a way starting similar as in the case (iv) above until introduction of different valuations V_i of agent's opinion. But now we suggest

$$(N,a) \Vdash_V K\varphi \Leftrightarrow \forall i[(N,a) \Vdash_{V_i} \Diamond\varphi \wedge \Box[\neg\varphi \rightarrow \mathbf{N}\neg\varphi].$$

In this case, if we will allow then usage of nested knowledge operations for K in formulas (together with several valuations V_i for agent's information) and the derivative valuation V (for all cases when we evaluate $K\varphi$ (regardless for which agent (i.e. V_i)), no decision procedure (for the logic based at this approach) is known. We think that to study it is an interesting open question.

Summarizing these observations, we think that the linear temporal logic is very promising tool for subtle definitions what could be logical knowledge operations.

Why Time Might Be Non-Transitive ?

View (i). *Computational view.* Inspections of protocols for computations are limited by time resources and have non-uniform length (yet, in any point of inspection, verification may refer to stored old protocols).

Therefore, if we interpret our models as the ones reflecting verification of computations, the amount of check points is finite, but not all of them might be in disposal to the given time point.

View (ii). *Agent's-admin's view.* We may consider states (worlds of our model) as checkpoints of admin's (agents) for inspections of states of network in past. Any admin has allowed amount of inspections for previous states, but only within the areas of its(his/her) responsibility (by security or another reasons). So, the accessibility is not transitive again.

View (iii). *Agent's-users's view.* If we consider the sates of the models as the content of web pages available for users, any surf step is accessibility relation, and starting from any web page user may achieve, using links in hypertext(s) some foremost available web sites. The latter one may have web links which are available only for individuals possessing passwords for accessibility. And users having password may continue web surf, etc. Clearly that in this approach, web surfing looks as non-transitive relation. Here, if we interpret web surf as time-steps, the accessibility is intransitive.

View (iv). *View on time in past for collecting knowledge.* In human perception, only some finite intervals of time in past (not in future) are available to individuals to inspect evens and to record knowledge collected to current time state. The time is past in our feelings looks as linear and has only a finite amount of memory to remember information and events.

There, in past, at foremost available (memorable) time point, individuals again had a memorable interval of time with collected information, and so forth ... So, the time in past, generally speaking, looks as not transitive.

View (v). *View in past for individuals as agents with opposition.* Here the picture is similar to the case (iv) above, but we may consider the knowledge as the collection of facts which about only majority (not compulsory all) experts (agents) have affirmative positive opinion. And again ....

Linear Temporal Logic Based at Non-transitive Time, Preparation

Definition. *A* **linear non-transitive possible-worlds frame** *is*

$$\mathcal{F} := \langle N, \leq, \text{Next}, \bigcup_{i \in N} R_i \rangle,$$

where each R_i is the standard linear order ($\leq$) on the interval $[i, m_i]$, where $m_i \in N, m_i > i$ and $m_{i+1} > m_i$. We fix notation $t(i) := m_i$; $a\ Next\ b \Leftrightarrow b = a + 1$.

We now may define a model $\mathcal{M}$ on $\mathcal{F}$ by introducing a valuation V on $\mathcal{F}$ and extend it on all formulas as earlier, but for formulas of sort $\varphi \mathbf{U} \psi$ we define the truth value as follows:

Definition. *For any $a \in N$:*

$$(\mathcal{M}, a) \Vdash_V (\varphi \ \mathbf{U} \ \psi) \quad \Leftrightarrow$$

$$\exists b[(aR_a b) \wedge ((\mathcal{M}, b) \Vdash_V \psi) \wedge \forall c[(a \leq c < b) \Rightarrow (\mathcal{M}, c) \Vdash_V \varphi]];$$

$$(\mathcal{M}, a) \Vdash_V \mathbf{N}\varphi \quad \Leftrightarrow \quad [(a \ Next \ b) \Rightarrow (\mathcal{M}, b) \Vdash_V \varphi].$$

Definition. *The logic $\mathcal{LTL}_{NT}$ is the set of all formulas which are valid at any model $\mathcal{M}$ with any valuation.*

A short comparison this logic with standard $\mathcal{LTL}$ will be given a bit later. The relation $\bigcup_{i \in N} R_i$ is evidently non-transitive, though any R_i is linear and transitive. Its action (more precisely - the whole interval $[i, m_i]$ itself) may be interpreted as the interval of time which agent i remember. In any time point

$i+k$ it might be the new agent, the same as the old (previous) one - just those who inspect. Being based at this interpretation, we may consider *interpretations of various aspects of knowledge* discussed above and their extended versions. E.g.

Examples :

$$(\mathcal{M}, a) \Vdash_V K\varphi \Leftrightarrow (\mathcal{M}, a) \Vdash_V \varphi \ \mathbf{U} \ [[\mathbf{N}^{m+1}\neg\varphi] \wedge [\mathbf{N}^m\varphi]].$$

Here K acts to say that knowledge codded by φ been achieved only m 'years' ago and holds true since then. This example works even in the linear temporal logic $\mathcal{LTL}$ itself.

$$(\mathcal{M}, a) \Vdash_V K_1\varphi \Leftrightarrow (\mathcal{M}, a) \Vdash_V \Box\neg\varphi \ \wedge \Diamond(\neg\varphi \wedge \mathbf{N}(K\varphi)).$$

Now K_1 determines that φ was wrong in all observable time in past, but before it has been time interval of length m, when φ was true (so to say it was a local temporal knowledge).

$$(\mathcal{M}, a) \Vdash_V K_2\varphi \Leftrightarrow (\mathcal{M}, a) \Vdash_V \Box^k\neg\varphi \ \wedge \Diamond^k(\neg\varphi \wedge \mathbf{N}(K\varphi)).$$

Here K_2 says that φ was wrong in subsequent k 'memorizable' intervals in time, but then it has been in past a local knowledge for a time interval of length m.

Even with these simple examples it is easy to imagine which wide possibilities for expression properties of knowledge in time perspective might be achieved via assumption that time could be non-transitive.

Now we turn to our main topic - solution of decidability and satisfiability problems for our logic $\mathcal{LTL}_{NT}$. It is immediately seen that the old standard techniques to solve these problems do not work because the accessibility relation is not transitive. We paly our favorite technique - via rules.

Recall that a (sequential) (inference) rule is an expression

$$\mathbf{r} := \frac{\varphi_1(x_1,\dots,x_n),\dots,\varphi_l(x_1,\dots,x_n)}{\psi(x_1,\dots,x_n)},$$

where $\varphi_1(x_1,\dots,x_n),\dots,\varphi_l(x_1,\dots,x_n)$ and $\psi(x_1,\dots,x_n)$ are formulas constructed out of letters (variables) $x_1,\dots,x_n$. Meaning of **r** is: $\psi(x_1,\dots,x_n)$ (which is called conclusion) follows (logically follows) from $\varphi_1(x_1,\dots,x_n)$, ..., $\varphi_l(x_1,\dots,x_n)$.

Definition. *A rule* $\mathbf{r}$ *is said to be* valid *in a model* $\mathcal{M}$ *if and only if the following holds:*

$$[\forall a\ ((\mathcal{M},a)\Vdash_V \bigwedge_{1\leq i\leq l} \varphi_i)] \Rightarrow [\forall a\ ((\mathcal{M},a)\Vdash_V \psi)].$$

Otherwise we say $\mathbf{r}$ *is refuted in* $\mathcal{M}$*, or refuted in* $\mathcal{M}$ *by* V*, and write* $\mathcal{M} \nVdash_V \mathbf{r}$*. A rule* $\mathbf{r}$ *is valid in a frame* $\mathcal{F}$ *(notation* $\mathcal{F} \Vdash \mathbf{r}$*) if it is valid in any model based at* $\mathcal{F}$*.*

For any formula φ, we can transform φ into the rule $x \to x/\varphi$ and employ a technique of reduced normal forms for inference rules as follows. We start from self-evident

Lemma. *For any formula* φ*,* φ *is a theorem of* $\mathcal{LTL}_{NT}$ *(that is* $\varphi \in \mathcal{LTL}_{NT}$*) iff the rule* $(x \to x/\varphi)$ *is valid in any frame* $\mathcal{F}$ *.*

Definition. *A rule* $\mathbf{r}$ *is said to be in* reduced normal form *if* $\mathbf{r} = \varepsilon/x_1$ *where*

$$\varepsilon := \bigvee_{1\leq j\leq l} [\bigwedge_{1\leq i\leq n} x_i^{t(j,i,0)} \wedge \bigwedge_{1\leq i\leq n} (\mathbf{N}x_i)^{t(j,i,1)} \wedge$$

$$\bigwedge_{1\leq i,k\leq n, i\neq k} (x_i \mathbf{S} x_k)^{t(j,i,k,1)}]$$

and, for any formula α *above,* $\alpha^0 := \alpha$, $\alpha^1 := \neg\alpha$.

Definition. *Given a rule* $\mathbf{r_{nf}}$ *in reduced normal form,* $\mathbf{r_{nf}}$ *is said to be a normal reduced form for a rule* $\mathbf{r}$ *iff, for any frame* $\mathcal{F}$ *for* $\mathcal{LTL}_{NT}$*,*

$$\mathcal{F} \Vdash \mathbf{r} \quad \Leftrightarrow \quad \mathcal{F} \Vdash \mathbf{r_{nf}}.$$

Theorem. *There exists an algorithm running in (single) exponential time, which, for any given rule* $\mathbf{r}$*, constructs its normal reduced form* $\mathbf{r_{nf}}$.

Here we will need a simple modification of models for $\mathcal{LTL}_{NT}$ introduced earlier. Let as earlier $\mathcal{F} := \langle N, \leq, \mathrm{Next}, \bigcup_{i \in N} R_i \rangle$,

where each R_i is the standard linear order ($\leq$) on the interval $[i, m_i]$, where $m_i \in N, m_i > i$ and $m_{i+1} > m_i$, as before, and yet $t(i) := m_i$. If $a\ Next\ b$ we will write $Next(a) = b$.

For any natural number r, consider the following frame $\mathcal{F}(N(r))$ based at the initial interval of the frame $\mathcal{F}$:

$$\mathcal{F}(N(r)) := \langle N(r), \leq, \mathsf{Next}, \bigcup_{i\in N} R_i\rangle,$$

where $r > g \geq t^2(0)$, the base set $N(r)$ of this frame is

$$N(r) := [0, t(0)] \cup [t(0), t^2(0)] \cup \ldots \cup$$

$$[t^g(0), t^{g+1}(0)] \cup, \ldots, \cup [t^r(0), t^{r+1}(0)],$$

where the relations R_i and $Next$ act on this frame exactly as at $\mathcal{F}$ but (i) $Next(t^{r+1}(0)) := t^g(0)$ and (ii) R_i acts on $[t^r(0), t^{r+1}(0)]$ as if the next interval for $[t^r(0), t^{r+1}(0)]$ would be $[t^g(0), t^{g+1}(0)]$. The valuation V on such finite frame might be defined as before, and we may extend it to formulas with $\mathbf{U}$ and $\mathbf{N}$ similar as before.

Lemma. *For any given rule* $\mathbf{r_{nf}}$ *in reduced normal form, if* $\mathbf{r_{nf}}$ *is refuted in a frame of* $\mathcal{F}$ *then* $\mathbf{r_{nf}}$ *can be refuted in some finite model* $\mathcal{F}(N(r))$ *(where* $r \in N$*) by a valuation* V *where the size of the frame* $\mathcal{F}(N(r))$ *is effectively computable from the size of the rule of* $\mathbf{r_{nf}}$ *(is at most* $[(n*l)*l^{(n*l)}*(n*l)!]+l^{(n*l)}$, *where* l *is the number of disjuncts in* $\mathbf{r_{nf}}$ *and* n *is the number of its letters).*

Lemma. *If a rule* $\mathbf{r_{nf}}$ *in reduced normal form is refuted in a model described in the lemma above then* $\mathbf{r_{nf}}$ *is not valid in* $\mathcal{LTL}_{NT}$, *i.e there is a standard frame* $\mathcal{F}$ *refuting* $\mathbf{r_{nf}}$*.*

Using these Lemmas we immediately derive:

Theorem. *Logic $\mathcal{LTL}_{NT}$ is decidable; the satisfiability problem for $\mathcal{LTL}_{NT}$ is decidable: for any formula we can compute if it is satisfiable and of yes to compute a valuation satisfying this formula in a finite model of kind $\mathcal{F}(N(r))$.*

But, what is with admissibility?

Logics with Uniform Bound for Intransitivity, Comparison

We consider some variation of $\mathcal{LTL}_{NT}$ - its extension, the logic generated by models with uniformly bounded measure of non-transitivity.

Definition. *A non-transitive possible-worlds linear frame $\mathcal{F}$ with uniform non-transitivity m is a particular case of frames for $\mathcal{LTL}_{NT}$:*

$$\mathcal{F} := \langle N, \leq, \mathsf{Next}, \bigcup_{i \in N} R_i \rangle,$$

where each R_i is the standard linear order ($\leq$) on the interval $[i, i+m]$, where ($m \geq 1$), and m is a fixed natural number (measure of intransitivity).

Definition. *The logic $\mathcal{LTL}_{NT}(m)$ is the set of all formulas which are valid at any model $\mathcal{M}$ with the measure of intransitivity m.*

It seems that to consider and discuss such logic is reasonable, since we may put limitations on the size of time intervals that agents (experts) may introspect in future (or to remember in past). First immediate, easy observation about $\mathcal{LTL}_{NT}(m)$ is

Proposition. *Logic $\mathcal{LTL}_{NT}(m)$ is decidable.*

Proof is **trivial** since for verification if a formula of temporal degree k is a theorem of $\mathcal{LTL}_{NT}(m)$ we will need to check it on only initial part of the frames consisting only $k+1$ subsequent intervals of length at most m each. Q.E.D.

Proposition. $\mathcal{LTL} \nsubseteq \mathcal{LTL}_{NT}$ and $\mathcal{LTL} \nsubseteq \mathcal{LTL}_{NT}(m)$ for all m.

Proof is evident since $\Box p \to \Box\Box P \ \in \ \mathcal{LTL}$.

Poposition. $\mathcal{LTL}_{NT}(m) \nsubseteq \mathcal{LTL}$ for all m.

Proof is evident since

$$(\bigwedge_{i \leq m} \mathbf{N}^i p \ \to \ \Box p) \ \in \mathcal{LTL}_{NT}(m).$$

Nonetheless, the following holds:

Theorem. $\mathcal{LTL}_{NT} \subset \mathcal{LTL}$.

Proof. This is not evident but it easy follows from our ....

As for admissibility we fortunately have:

Theorem. *For any m, the linear temporal logic with UNIFORM non-transitivity $\mathcal{LTL}_{NT}(m)$ is decidable w.r.t. admissibility of inference rules.*

Proof: Non-trivial ... though with tolerable length ...

Open problems

(i) Decidability of $\mathcal{LTL}_{NT}$ itself w.r.t. admissible inference rules.

(ii) Decidability w.r.t. admissible rules for the variant of $\mathcal{LTL}_{NT}(m)$ with non-uniform intransitivity.

(iii) The problems of axiomatization for $\mathcal{LTL}_{NT}$ and for $\mathcal{LTL}_{NT}(m)$.

(iv) it looks reasonable to extend our approach to linear logics with linear non-transitive but continues time.

(v) Multi-agent approach to suggested framework when any $n \in N$ would be represented by a cluster (circle) with n agent's knowledge relations K_i is also open and interesting.