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# Feeding Many Values to Light-weight Description Logics

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# Logic-based Knowledge Representation

relevant names from the domain:

- concepts (sets)
  Hippo, Female, Jedi
- roles (binary relations)
  carries, hasFather

constraints on these names

- Jedis carry lightsabers
- Lightsabers are weapons

deduce consequences

- Jedis carry weapons

## Basic Terminology

$N_C$, $N_R$ disjoint sets: concept names and role names

## ($\mathcal{EL}$-)Concepts

Built by induction

- every $A \in N_C$ is a concept
- if $C, D$ are concepts and $r \in N_R$ then

| | | |
|---|---|---|
| $\top$ | (top) | |
| $C \sqcap D$ | (conjunction) | $\wedge$ |
| $\exists r.C$ | (existential restriction) | $\Diamond_r$ |

  are concepts

## Examples

$\mathsf{Hippo} \sqcap \mathsf{Female}$

$\exists \mathsf{carries}.\top$

How to handle imprecise concepts?

Strong

Graceful

Rich

## Degrees of Membership

Chain $\mathcal{C} \subseteq [0,1]$

- "infinite" $[0,1]$
- "finite;" "n-valued" $0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1$

t-norm $\otimes$

binary operator over $\mathcal{C}$

- associative
- commutative
- monotonic
- unit 1

residuum $\Rightarrow$

$$z \otimes x \leq y \quad \text{iff} \quad z \leq x \Rightarrow y$$

| | $x \otimes y$ | $x \Rightarrow y$ |
|---|---|---|
| Gödel | $\min\{x, y\}$ | $\begin{cases} 1 & x \leq y \\ y & \text{otherwise} \end{cases}$ |
| Łukasiewicz | $\max\{x + y - 1, 0\}$ | $\min\{1 - x + y, 1\}$ |
| Product | $x \cdot y$ | $\begin{cases} 1 & x \leq y \\ y/x & \text{otherwise} \end{cases}$ |

Concepts, roles interpreted as fuzzy sets, binary relations

$$\begin{aligned} C^{\mathcal{I}}(x) &\in \mathcal{C} \\ r^{\mathcal{I}}(x, y) &\in \mathcal{C} \end{aligned}$$

$\mathrm{Strong}^{\mathcal{I}}(x) = 0.8 \quad \longrightarrow$ x is strong with degree 0.8

$\mathrm{hasFriend}^{\mathcal{I}}(x,y) = 0.9 \quad \longrightarrow$ x and y are friends with degree 0.9

Intuition: degree of "C-ness" of the individual

Degree 0: not at all
1: totally
all kinds of shades of gray in between

Interpretations $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$

non-empty domain $\Delta^{\mathcal{I}}$

interpretation function $\cdot^{\mathcal{I}}$

- $A^{\mathcal{I}} : \Delta^{\mathcal{I}} \to \mathcal{C}$ for all $A \in N_C$ (sets)
- $r^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to \mathcal{C}$ for all $r \in N_R$ (binary relations)

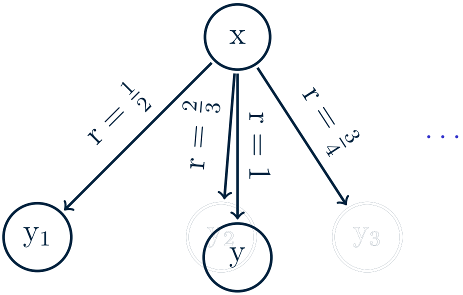
$\cdot^{\mathcal{I}}$ is extended to concepts as follows:

$$\begin{aligned}
(\top)^{\mathcal{I}}(x) &:= 1 \\
(C \sqcap D)^{\mathcal{I}}(x) &:= C^{\mathcal{I}}(x) \otimes D^{\mathcal{I}}(x) \\
(\exists r.C)^{\mathcal{I}}(x) &:= \max_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, y) \otimes C^{\mathcal{I}}(y)
\end{aligned}$$

$(\exists r.\top)^{\mathcal{I}}(x) = 1$ with witnessed interpretations means

x has an r successor with degree 1

(not true in general interpretations)



A general concept inclusion (GCI) is of the form

$$\langle \mathrm{C} \sqsubseteq \mathrm{D} : \mathrm{c} \rangle \qquad \mathrm{C}, \mathrm{D} \text{ concepts, } \mathrm{c} \in \mathcal{C}$$

$$\begin{aligned} \langle \mathrm{Tourist} &\sqsubseteq \exists \mathrm{carries}.\mathrm{Camera} : 1 \rangle \\ \langle \mathrm{Man} \sqcap \exists \mathrm{hasSon}.\mathrm{Handsome} &\sqsubseteq \mathrm{Handsome} : 0.8 \rangle \end{aligned}$$

A general TBox is a finite set of GCIs

$\mathcal{I}$ satisfies $\langle \mathrm{C} \sqsubseteq \mathrm{D} : \mathrm{c} \rangle$ if
$\mathrm{C}^{\mathcal{I}}(\mathrm{x}) \Rightarrow \mathrm{D}^{\mathcal{I}}(\mathrm{x}) \geq \mathrm{c}$ for all $\mathrm{x} \in \Delta^{\mathcal{I}}$

$\mathcal{I}$ is a model of $\mathcal{T}$ if it satisfies all GCIs in $\mathcal{T}$

Given a TBox $\mathcal{T}$

A is p-subsumed by B $\quad$ ($\mathrm{A} \sqsubseteq^{\mathrm{p}}_{\mathcal{T}} \mathrm{B}$)

iff

$$\mathrm{A}^{\mathcal{I}}(\mathrm{x}) \Rightarrow \mathrm{B}^{\mathcal{I}}(\mathrm{x}) \geq \mathrm{p}$$

- for all $\mathrm{x} \in \Delta^{\mathcal{I}}$
- in all models of $\mathcal{T}$

# The simple Gödel semantics

TBoxes can be normalized

$$\langle \mathrm{A} \sqsubseteq \mathrm{B} : \mathrm{c} \rangle \qquad \langle \mathrm{A}_1 \sqcap \mathrm{A}_2 \sqsubseteq \mathrm{B} : \mathrm{c} \rangle$$

$$\langle \exists \mathrm{r}.\mathrm{A} \sqsubseteq \mathrm{B} : \mathrm{c} \rangle \qquad \langle \mathrm{A} \sqsubseteq \exists \mathrm{r}.\mathrm{B} : \mathrm{c} \rangle$$

(at most one constructor per axiom)

$$\langle \mathrm{A} \sqsubseteq \mathrm{B} \sqcap \mathrm{C} : 0.8 \rangle \rightsquigarrow \begin{cases} \langle \mathrm{A} \sqsubseteq \mathrm{B} : 0.8 \rangle \\ \langle \mathrm{A} \sqsubseteq \mathrm{C} : 0.8 \rangle \end{cases}$$

Make the logical closure of the normalized TBox (w.r.t. concept names)

| Premise | Axiom | | Consequence |
|---|---|---|---|
| $\langle A \sqsubseteq B : c\rangle$ | $\langle B \sqsubseteq C : d\rangle$ | $\rightsquigarrow$ | $\langle A \sqsubseteq C : \min\{c, d\}\rangle$ |
| $\langle A \sqsubseteq B_1 : c_1\rangle$<br>$\langle A \sqsubseteq B_2 : c_2\rangle$ | $\langle B_1 \sqcap B_2 \sqsubseteq C : d\rangle$ | $\rightsquigarrow$ | $\langle A \sqsubseteq C : \min\{c_1, c_2, d\}\rangle$ |
| $\langle A \sqsubseteq A_1 : c\rangle$ | $\langle A_1 \sqsubseteq \exists r.B : d\rangle$ | $\rightsquigarrow$ | $\langle A \sqsubseteq \exists r.B : \min\{c, d\}\rangle$ |
| $\langle A \sqsubseteq \exists r.B : c\rangle$<br>$\langle B \sqsubseteq B_1 : d\rangle$ | $\langle \exists r.B_1 \sqsubseteq C : e\rangle$ | $\rightsquigarrow$ | $\langle A \sqsubseteq C : \min\{c, d, e\}\rangle$ |

A is p-subsumed by B iff we can deduce

$$\langle A \sqsubseteq B : q \rangle$$

for some $q \geq p$

decides all atomic subsumptions in quadratic time

# Beyond Idempotency . . .

## No normalization

$$\langle \mathrm{A} \sqsubseteq \mathrm{B} \sqcap \mathrm{C} : 0.8 \rangle \rightsquigarrow \begin{cases} \langle \mathrm{A} \sqsubseteq \mathrm{B} : 0.8 \rangle \\ \langle \mathrm{A} \sqsubseteq \mathrm{C} : 0.8 \rangle \end{cases}$$

## Logical deduction

$$\left. \begin{array}{l} \langle \mathrm{A} \sqsubseteq \mathrm{B} : 1 \rangle \\ \langle \mathrm{A} \sqsubseteq \mathrm{C} : 1 \rangle \end{array} \right\} \rightsquigarrow \langle \mathrm{A} \sqsubseteq \mathrm{B} \sqcap \mathrm{C} : ??? \rangle$$

# A look to Łukasiewicz

$\mathcal{ALC}$ is $\mathcal{EL}$ extended with negation constructor $\neg$

(multimodal K with universal modality)

Reasoning in $\mathcal{ALC}$

- finitely valued: ExpTime-complete
- infinitely valued: undecidable

(* if negation is involutive)

**Observation**

any extension of $\mathcal{EL}$ that simulates (classical) disjuction is at least as expressive as (classical) $\mathcal{ALC}$

## Disjunction in 3-valued Ł

$\text{Ł}_3$:

| $\otimes$ | 0 | 1/2 | 1 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1/2 | 0 | 0 | 1/2 |
| 1 | 0 | 1/2 | 1 |

$x \otimes y \geq 1/2$ iff $x = 1$ or $y = 1$

$$
\begin{aligned}
\mathrm{A}_1 \sqcap \mathrm{A}_2 \sqsubseteq \mathrm{B} \quad &\rightsquigarrow \quad \langle \mathrm{A}_1 \sqcap \mathrm{A}_2 \sqsubseteq \mathrm{B} : 1 \rangle \\
\exists \mathrm{r}.\mathrm{A} \sqsubseteq \mathrm{B} \quad &\rightsquigarrow \quad \langle \exists \mathrm{r}.\mathrm{A} \sqsubseteq \mathrm{B} : 1 \rangle \\
\mathrm{A} \sqsubseteq \exists \mathrm{r}.\mathrm{B} \quad &\rightsquigarrow \quad \langle \mathrm{A} \sqsubseteq \exists \mathrm{r}.\mathrm{B} \sqcap \exists \mathrm{r}.\mathrm{B} : \tfrac{1}{2} \rangle \\
\mathrm{A} \sqsubseteq \mathrm{B}_1 \sqcup \mathrm{B}_2 \quad &\rightsquigarrow \quad \langle \mathrm{A} \sqsubseteq \mathrm{B}_1 \sqcap \mathrm{B}_2 : \tfrac{1}{2} \rangle
\end{aligned}
$$

$\mathrm{A}_0$ is subsumed by $\mathrm{B}_0$ in classical $\mathcal{ELU}$ iff

$\mathrm{A}_0 \sqcap \mathrm{A}_0$ is 1-subsumed by $\mathrm{B}_0 \sqcap \mathrm{B}_0$ in Ł$_3$-$\mathcal{EL}$

Subsumption in $Ł_n$-$\mathcal{EL}$ is ExpTime-complete

(for arbitrary non-idempotent finite chains)

reduction is not in normal form

$$\langle A \sqsubseteq \exists r.B \sqcap \exists r.B : \tfrac{1}{2} \rangle \qquad \langle A \sqsubseteq B_1 \sqcap B_2 : \tfrac{1}{2} \rangle$$

complexity for normalized TBoxes unknown

Any model of the axiom

$$\langle \mathrm{C} \sqcap \mathrm{C} \sqsubseteq \mathrm{C} \sqcap \mathrm{C} \sqcap \mathrm{C} : 1 \rangle$$

satisfies

$$\mathrm{C}^{\mathcal{I}}(\mathrm{x}) \in [0, \tfrac{1}{2}] \cup \{1\}$$

if we add

$$\langle \top \sqsubseteq \mathrm{C} : \tfrac{1}{2} \rangle$$

then

$$\mathrm{C}^{\mathcal{I}}(\mathrm{x}) \in \{\tfrac{1}{2}, 1\}$$

(same trick)

Subsumption in Ł-$\mathcal{EL}$ is ExpTime-hard

(for any t-norm “containing” Ł)

but no upper bound known

reduction is not in normal form

$$\langle \mathrm{C} \sqcap \mathrm{C} \sqsubseteq \mathrm{C} \sqcap \mathrm{C} \sqcap \mathrm{C} : 1 \rangle$$

dirty theoretical (and modeling) tricks

adding a bit of expressivity, possible to simulate
involutive negation over $[\frac{1}{2}, 1]$

Intuition:
Ł-$\mathcal{ALC}$ compressed to the second half of interval

Undecidability would be very surprising

non-idempotent t-norm without nilpotent elements

product

# Subsumption

Given a TBox $\mathcal{T}$

A is positively-subsumed by B $\quad (\mathrm{A} \sqsubseteq_{\mathcal{T}}^{>0} \mathrm{B})$

iff

$$\mathrm{A}^{\mathcal{I}}(\mathrm{x}) \Rightarrow \mathrm{B}^{\mathcal{I}}(\mathrm{x}) > 0$$

- for all $\mathrm{x} \in \Delta^{\mathcal{I}}$
- in all models of $\mathcal{T}$

Recall for product

$$x \Rightarrow y = 0 \qquad \text{iff} \qquad x > 0 \text{ and } y = 0$$

$A \sqsubseteq_{\mathcal{T}}^{>0} B$ means

an element that belongs (partially) to A
belongs (partially) to B

# Just Forget

For positive subsumption, axiom degrees are irrelevant

$$\mathcal{T} = \begin{array}{llcll} \langle & \mathrm{A} & \sqsubseteq & \mathrm{B} & : 0.9 \rangle \\ \langle & \exists \mathrm{r}.(\mathrm{A} \sqcap \mathrm{C}) & \sqsubseteq & \mathrm{D} \sqcap \exists \mathrm{s}.\mathrm{C} & : 0.4 \rangle \\ \langle & \exists \mathrm{r}.\exists \mathrm{s}.\mathrm{B} & \sqsubseteq & \exists \mathrm{s}.\exists \mathrm{r}.\mathrm{A} & : 0.5 \rangle \\ \langle & \mathrm{B} & \sqsubseteq & \mathrm{C} & : 0 \rangle \end{array}$$

$$\mathrm{A} \sqsubseteq_{\mathcal{T}}^{>0} \mathrm{B} \qquad \text{iff} \qquad \mathrm{A} \sqsubseteq_{\mathcal{T}_\mathrm{G}}^{1} \mathrm{B}$$

(in 2-valued semantics)

# Summary

| | G | finitely-valued | including Ł | strict | other |
|---|---|---|---|---|---|
| $\sqsubseteq^{\mathrm{p}}_{\mathcal{T}}$ | P | ExpTime-c | ExpTime-h | ?? | ?? |
| $\sqsubseteq^{>0}_{\mathcal{T}}$ | P | ExpTime-c | ?? | P | coNP-h |