

# Morita-equivalences for MV-algebras

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# Topoi

A **topos**  $\mathcal{E}$  is a universe which generalizes the classic universe of sets.

- $\mathcal{E}$  has all finite limits (= is finitely complete)
- $\mathcal{E}$  has all finite co-limits (= is finitely co-complete)
- $\mathcal{E}$  has exponentiation
- in  $\mathcal{E}$  we can classify subobjects ( $\mathcal{E}$  has a subobject classifier)

## Examples:

- Category of sets ( $\cong$  category of sheaves over  $\{*\}$ )
- Categories of sheaves on a topological spaces
- *Grothendieck topoi*: categories of sheaves on a site

# Models in topoi

In a Grothendieck topos we can consider models of every first-order theory. Given  $\Sigma$  a first-order signature, a  $\Sigma$ -*structure* in a topos  $\mathcal{E}$  is defined by the following data

- sorts  $\rightarrow$  **objects** in  $\mathcal{E}$
- function symbols  $\rightarrow$  **arrows** in  $\mathcal{E}$
- relation symbols  $\rightarrow$  **subobjects** in  $\mathcal{E}$

A  $\Sigma$ -structure  $M$  is a *model* of a theory  $\mathbb{T}$  over the signature  $\Sigma$  if every axiom of  $\mathbb{T}$  is valid in  $M$ .

# Problem

Let  $\mathbb{T}$  and  $\mathbb{S}$  be two geometric theories such that

$$\mathbb{T}\text{-mod}(\mathbf{Set}) \cong \mathbb{S}\text{-mod}(\mathbf{Set})$$

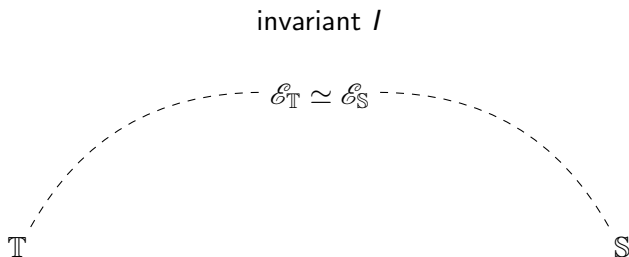
**Question:** Is it true that

$$\mathbb{T}\text{-mod}(\mathcal{E}) \cong \mathbb{S}\text{-mod}(\mathcal{E})$$

naturally in  $\mathcal{E}$ ?

If this is true we say that  $\mathbb{T}$  and  $\mathbb{S}$  are **Morita-equivalent** (equivalently, the two theories have the same classifying topos).

# The 'bridge' technique



T-characterization of  $I$

S-characterization of  $I$

If  $\mathbb{T}$  and  $\mathbb{S}$  are Morita-equivalent we can transfer properties and results from one theory to the other by using **topos-theoretic invariants** defined over the same classifying topos.

# Morita-equivalences for MV-algebras

- **Lift** to Morita-equivalences of two well-known categorical equivalences between classes of MV-algebras and classes of lattice-ordered abelian groups, namely
  - **Mundici's equivalence:**  
category of MV-algebras  $\simeq$  category of  $\ell$ -groups with strong unit
  - **Di Nola-Lettieri's equivalence:**  
category of perfect MV-algebras  $\simeq$  category of  $\ell$ -groups
- Application of the method 'toposes as bridges' to these Morita-equivalences
- **Construction** (by means of the investigation of certain classifying toposes) of a new class of (Morita-)equivalences containing in particular the one lifting Di Nola-Lettieri's equivalence

# Results in connection with Mundici's equivalence

- The theory of  $\ell$ -groups with strong unit is of **presheaf type** and in fact Morita-equivalent to an algebraic theory (namely that of MV-algebras)
- Bijective correspondence between the geometric theory of MV-algebras and the geometric theory of  $\ell$ -u groups (in spite of the fact that they are not bi-interpretable)
- Logical characterization of the finitely presentable  $\ell$ -u groups
- Form of compactness and completeness for the geometric theory of  $\ell$ -u groups (in spite of the infinitary nature of this theory);
- Sheaf-theoretic version of Mundici's equivalence

# Results in connection with Di Nola-Lettieri's equivalence

- The theory of perfect MV-algebras is of **presheaf type** and in fact Morita-equivalent to an algebraic theory (namely that of  $\ell$ -groups)
- Three levels of partial bi-interpretability for
  - **irreducible formulas**
  - **geometric sentences**
  - **imaginaries**
- the finitely presentable models of the theory of perfect MV-algebras are finitely presentable as objects in the variety generated by Chang's algebra
- **Representation result:** every non-trivial finitely generated MV-algebra in the variety generated by Chang's MV-algebra is a finite direct product of perfect MV-algebras



# Equivalences for local MV-algebras in varieties

- The theory of local MV-algebras is **NOT** of presheaf type
- The theory of local MV-algebras in an arbitrary proper subvariety of MV-algebras **IS** of presheaf type
- The theory of local MV-algebras in an arbitrary proper subvariety of MV-algebras is **Morita-equivalent** to a theory extending that of lattice-ordered abelian groups
- the finitely presentable models of the theory of local MV-algebras in an arbitrary proper subvariety are finitely presentable also with respect to the variety
- **Representation result:** every finitely generated MV-algebra in an arbitrary proper subvariety of MV-algebras is a finite direct product of local MV-algebras

## Categorical equivalences

- Mundici's equivalence:  $\Gamma : \mathbb{L}_u\text{-mod}(\mathbf{Set}) \cong \mathbf{MV}\text{-mod}(\mathbf{Set})$
- Di Nola-Lettieri's equivalence:  $\Delta : \mathbb{P}\text{-mod}(\mathbf{Set}) \cong \mathbb{L}\text{-mod}(\mathbf{Set})$

## Lifts

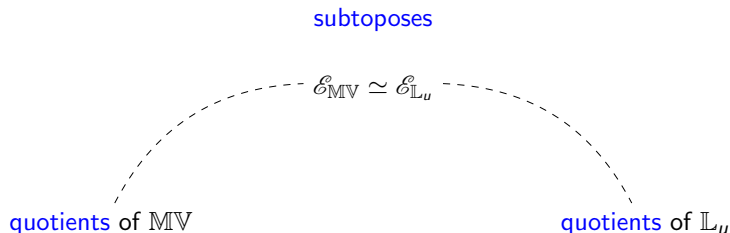
- $\Gamma_{\mathcal{E}} : \mathbb{L}_u\text{-mod}(\mathcal{E}) \cong \mathbf{MV}\text{-mod}(\mathcal{E})$
- $\Delta_{\mathcal{E}} : \mathbb{P}\text{-mod}(\mathcal{E}) \cong \mathbb{L}\text{-mod}(\mathcal{E})$

for every Grothendieck topos  $\mathcal{E}$ , naturally in  $\mathcal{E}$

## Morita-equivalences

- $\mathbf{MV}$  is Morita-equivalent to  $\mathbb{L}_u$
- $\mathbb{P}$  is Morita-equivalent to  $\mathbb{L}$

# Bijjective correspondence between quotients



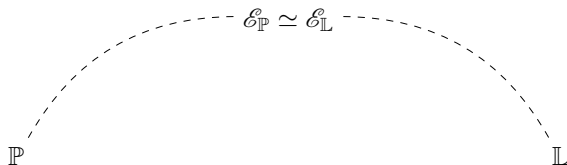
## Theorem

*Every quotient of the theory  $\mathbf{MV}$  is Morita-equivalent to a quotient of the theory  $\mathbf{L}_u$ , and conversely. These Morita-equivalences are the restrictions of the one between  $\mathbf{MV}$  and  $\mathbf{L}_u$ .*

This result is non-trivial since the two theories are **not** bi-interpretable.

# Partial bi-interpretations

irreducible objects  
subterminal objects  
coherent objects



$\mathbb{P}$ -irreducible formulas  
geometric sentences over  $\Sigma_{\mathbb{P}}$   
imaginaries for  $\mathbb{P}$

$\mathbb{L}$ -irreducible formulas  
geometric sentences over  $\Sigma_{\mathbb{L}}$   
imaginaries for  $\mathbb{L}$

These bi-interpretations are interesting since we do **not** have bi-interpretability at the level of the coherent syntactic categories of the two theories.

# Representation result

## Theorem

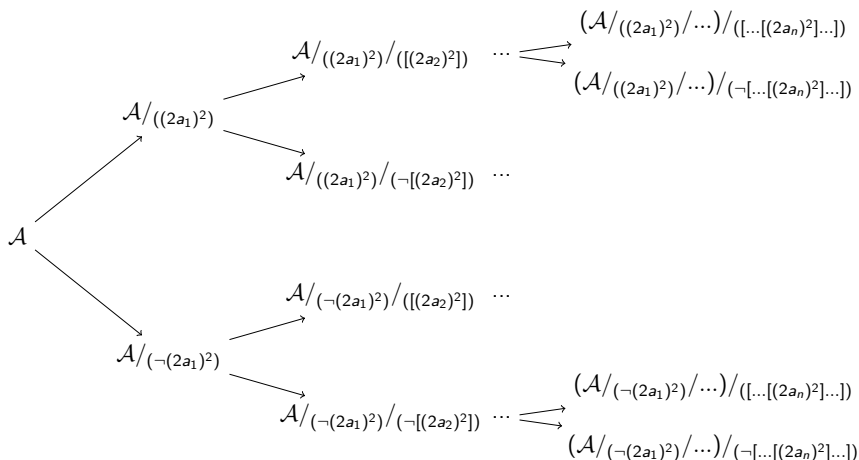
*Every finitely generated non-trivial MV-algebra in Chang's variety is a direct product of a finite family of finitely generated perfect MV-algebras.*

*Proof.* Recall that the theory  $\mathbb{P}$  is a quotient of the theory of Chang's variety obtained by adding the sequent

$$\top \vdash_x (2x)^2 = 0 \vee (2x)^2 = 1$$

This sequent generates the topology  $J$  associated with  $\mathbb{P}$ . If  $\mathcal{A}$  is a finitely generated MV-algebra in Chang's variety and  $\{a_1, \dots, a_n\}$  is generating system of  $\mathcal{A}$  then the final algebras in the following diagram (which generates a  $J$ -covering cosieve) are perfect MV-algebras.

# Representation result



# From perfect to local MV-algebras

$$Perfect = Local \cap V(S_1^\omega)$$

We proved that  $\mathbb{P}$  is a theory of presheaf type which is Morita-equivalent to the theory  $\mathbb{L}$ . It is natural to wonder if there is a theory axiomatizing

$$Local \cap V$$

which it is also of presheaf type and Morita-equivalent to a theory extending the theory  $\mathbb{L}$ .

# From perfect to local MV-algebras

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We proved that  $\mathbb{P}$  is a theory of presheaf type which is Morita-equivalent to the theory  $\mathbb{L}$ . It is natural to wonder if there is a theory axiomatizing

$$\text{Local} \cap V$$

which it is also of presheaf type and Morita-equivalent to a theory extending the theory  $\mathbb{L}$ .

$$\text{Local} \cap MV$$

## Theorem

*The geometric theory of local MV-algebras is not of presheaf type.*



# Local MV-algebras in a proper variety $V$

## Komori's theorem

An arbitrary proper subvariety of MV-algebras is of the form

$$V = V(\{S_i\}_{i \in I}, \{S_j^\omega\}_{j \in J})$$

where  $S_i = \Gamma(\mathbb{Z}, i)$  are simple MV-algebras,  $S_j^\omega = \Gamma(\mathbb{Z} \times_{lex} \mathbb{Z}, (j, 0))$  are called Komori chains and  $I$  and  $J$  are finite subset of  $\mathbb{N}$ .

We set:

- $n = \text{l.c.m.}(I, J)$  (invariant w.r.t. the generators)
- $\mathbb{T}_V$ : theory of the variety  $V$
- $\mathbb{L}_{oc_V}$ : theory of local MV-algebra in  $V$

**First axiomatization for  $\mathbb{L}_{oc_V}$ :**

- axioms of  $\mathbb{T}_V$
- $\sigma_n : \top \vdash_x ((n+1)x)^2 = 0 \vee ((n+1)x)^2 = 1$

We call  $J_1$  the Grothendieck topology associated with this axiomatization.

# The Grothendieck topology $J_1$

The covering are finite multicompositions of diagrams of this form.

$$\begin{array}{c}
 \mathcal{A} / ((n+1)x)^2 \\
 \nearrow \\
 \mathcal{A} \\
 \searrow \\
 \mathcal{A} / (\neg((n+1)x)^2)
 \end{array}$$

## Proposition

The Grothendieck topology  $J_1$  is subcanonical.

## Definition

A Grothendieck topology  $J$  on a category  $\mathcal{C}$  is subcanonical if the arrows in a  $J$ -covering of an element  $c \in \mathcal{C}$  form a limit diagrams with respect to the diagram consisting of all morphisms between them over  $c$ .

# Cartesianization and finitely presentable models

## Theorem

*Every cartesian sequent that is provable in the theory  $\mathbb{L}oc_V$  is also provable in the theory  $\mathbb{T}_V$ .*

## Proposition

The radical of every algebra  $\mathcal{A}$  in the variety  $V$  is defined by the following equation.

$$Rad(\mathcal{A}) = \{x \in A \mid ((n+1)x)^2 = 0\}$$

By using this we can obtain the following result.

## Proposition

Every finitely presentable model of  $\mathbb{L}oc_V$  is finitely presentable in  $V$ .

# Rigidity and theories of presheaf type

## Definition

A Grothendieck topology  $J$  on a category  $\mathcal{C}$  is **rigid** if every object  $c \in \mathcal{C}$  has a  $J$ -covering generated by  $J$ -irreducible objects.

Given a rigid topology  $J$  on a small category  $\mathcal{C}$  we have an equivalence

$$\mathbf{Sh}(\mathcal{C}, J) \cong [\mathcal{D}^{op}, \mathbf{Set}]$$

where  $\mathcal{D}$  is the full subcategory of  $\mathcal{C}$  of the  $J$ -irreducible objects.

## Theorem (Caramello)

*Let  $\mathbb{T}'$  be a quotient of a theory of presheaf type  $\mathbb{T}$  corresponding to a Grothendieck topology  $J$  on the category  $f.p.\mathbb{T}\text{-mod}(\mathbf{Set})^{op}$  under the Duality Theorem for subtoposes. Suppose that  $\mathbb{T}'$  is itself of presheaf type. Then every finitely presentable  $\mathbb{T}'$ -model is finitely presentable also as a  $\mathbb{T}$ -model if and only if the topology  $J$  is rigid.*

It follows that the theory  $\mathbb{Loc}_V$  is of presheaf type if and only if the associated Grothendieck topology is rigid.

# Local MV-algebras of finite rank

A local MV-algebra  $\mathcal{A}$  is said to be of *finite rank* if there is an isomorphism

$$\phi_{\mathcal{A}} : \mathcal{A}/\text{Rad}(\mathcal{A}) \rightarrow S_m$$

and  $m$  is the *rank* of  $\mathcal{A}$ . For every  $d \in S_m$ ,  $\phi_{\mathcal{A}}^{-1}(d)$  is called *radical class* of  $\mathcal{A}$ .

## Theorem (Di Nola-Esposito-Gerla)

*Every local MV-algebra in  $V$  has finite rank and its rank divides the rank of one of the generators of  $V$ .*

## Second axiomatization for $\mathbb{L}oc_V$ :

- axioms of  $\mathbb{T}_V$
- $\rho_n : \top \vdash_x \bigvee_{d=0}^n x \in \text{Fin}_d$ , where  $\text{Fin}_d$  is the formula describing  $\phi^{-1}(d)$

## Proposition

The following sequents are provable in  $\mathbb{T}_V$ .

$$x \in \text{Fin}_d \wedge y \in \text{Fin}_b \vdash_{x,y} x \oplus y \in \text{Fin}_{d \oplus b}$$

$$x \in \text{Fin}_d \vdash_x \neg x \in \text{Fin}_{n-d}$$

# Main result

## Theorem

*The theory  $\mathbb{L}oc_V$  is of presheaf type.*

*Proof.* We call  $J_2$  the topology associated with the second axiomatization which is obtained by finite multicompositions of diagrams of this form

$$\begin{array}{ccc}
 & \mathcal{A}/(x \in Fin_0(\mathcal{A})) & \\
 & \vdots & \\
 \mathcal{A} & \begin{array}{c} \nearrow \\ \longrightarrow \\ \searrow \end{array} & \mathcal{A}/(x \in Fin_d(\mathcal{A})) \\
 & \vdots & \\
 & \mathcal{A}/(x \in Fin_n(\mathcal{A})) & 
 \end{array}$$

where  $\mathcal{A}$  is a finitely presentable algebra in  $V$ . If we choose at each step one of the generators of the algebra  $\mathcal{A}$ , the codomain algebras of the resulting diagram are local MV-algebras. Hence  $J_2$  is rigid and  $\mathbb{L}oc_V$  is of presheaf type.

# Representation result

The two axiomatizations for  $\mathbb{L}oc_V$  are equivalent whence  $J_1 = J_2$ . This implies that  $J_1$  is rigid and therefore the following result.

## Theorem

*Every finitely generated MV-algebra in  $V$  is a finite product of local MV-algebras.*

# Representation theorem for algebras of finite ranks

## Theorem (Di Nola-Esposito-Gerla)

*Every local MV-algebra in  $V$  is of finite rank and its rank divides one of the ranks of the generators of  $V$ . Further, any local MV-algebra of finite rank is of the form*

$$\Gamma(\mathbb{Z} \times_{\ell\text{ex}} G, (k, g))$$

*where  $G$  is an  $\ell$ -group,  $g \in G$  and  $k$  is the rank of the algebra.*



# Extension of the theory $\mathbb{L}$

Let  $\mathbb{G}_{(I,J)}$  be the theory whose signature is the one of  $\ell$ -groups to which we add an arbitrary constant and a 0-ary predicate  $R_k$  for each divisor  $k$  of the least common multiple of the numbers in  $I$  and  $J$ . The axioms of this theory are

- axioms of  $\mathbb{L}$
- $(\top \vdash R_1)$ ;
- $(R_k \vdash R_{k'})$ , for each  $k'$  which divides  $k$ ;
- $(R_k \wedge R_{k'} \vdash R_{l.c.m.(k,k')})$ , for any  $k, k'$ ;
- $(R_k \vdash_g g = 0)$ , for every  $k \in \delta(I) \setminus \delta(J)$ ;
- $(R_k \vdash \perp)$ , for any  $k \notin \delta(I) \cup \delta(J)$ .

where we indicate with  $\delta(I)$  and  $\delta(J)$  respectively the sets of divisors of the numbers in  $I$  and  $J$ .

The theory  $\mathbb{G}_{(I,J)}$  is of presheaf type and the models of  $\mathbb{G}_{(I,J)}$  in **Set** can be identified with the triples  $(G, g, k)$ , where  $G$  is an  $\ell$ -group,  $g \in G$  and  $k \in \delta(I) \cup \delta(J)$ .

# New Morita-equivalences

Let  $V = V(\{S_i\}_{i \in I}, \{S_j^\omega\}_{j \in J})$  be an arbitrary proper subvariety of MV-algebras.

## Theorem

*The categories of set-based models of the theories  $\mathbb{Loc}_V$  and  $\mathbb{G}_{(I,J)}$  are equivalent.*

$$\bullet \Lambda_{(I,J)} : \mathbb{Loc}_V\text{-mod}(\mathbf{Set}) \rightarrow \mathbb{G}_{(I,J)}\text{-mod}(\mathbf{Set})$$

$$\Lambda_{(I,J)}(\mathcal{A}) := (G, g, k)$$

for every  $\mathcal{A} \simeq \Gamma(\mathbb{Z} \times_{lex} G, (k, g))$  local MV-algebra  $\mathbb{Loc}_V\text{-mod}(\mathbf{Set})$

$$\bullet M_{(I,J)} : \mathbb{G}_{(I,J)}\text{-mod}(\mathbf{Set}) \rightarrow \mathbb{Loc}_V\text{-mod}(\mathbf{Set})$$

$$M_{(I,J)}(G, g, k) := \Gamma(\mathbb{Z} \times_{lex} G, (k, g))$$

for every set-based model of  $\mathbb{G}_{(I,J)}$

# References



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