A Hennessy-Milner Property for Many-Valued Modal Logics*

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In this work, we investigate and characterize underlying algebras of many-valued modal logics admitting an analogue of the Hennessy-Milner property (modal equivalence coincides with bisimilarity) for their image-finite models. Modal equivalence between two states means in this context that each formula takes the same value in both states; the definition of a bisimulation matches the classical notion except that variables must take the same value in bisimilar states. Informally, the goal is to determine whether the language of a many-valued modal logic is expressive enough to distinguish image-finite models.

We restrict our attention here to many-valued modal logics defined over a single *complete MTL-chain* (where MTL stands for monoidal t-norm logic), an algebraic structure $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \to, \bot, \top \rangle$ satisfying

1. $\langle A, \wedge, \vee, \bot, \top \rangle$ is a complete chain (where $a \leq b$ if and only if $a \wedge b = a$).
2. $\langle A, \cdot, \top \rangle$ is a commutative monoid.
3. $a \cdot b \leq c$ if and only if $a \leq b \to c$ for all $a, b, c \in A$.

In particular, if A is the real unit interval $[0, 1]$, then the monoidal operation $\cdot$ is a left-continuous t-norm with unit 1 and residual $\to$, and $\mathbf{A}$ is called *standard*. Such algebras provide semantics for Łukasiewicz logic, Gödel logic, and product logic when $\cdot$ is the Łukasiewicz t-norm $\max(0, x + y - 1)$, the minimum t-norm $\min(x, y)$, or the product t-norm xy (multiplication), respectively.

Given a fixed complete MTL-chain $\mathbf{A}$, the many-valued modal logic $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$ is defined for formulas $\mathrm{Fm}_{\Box\Diamond}$ built inductively using the operations of $\mathbf{A}$, the modal operators $\Box$ and $\Diamond$, and a countably infinite set of variables Var. A *(crisp) frame* is a pair $\langle W, R \rangle$ where W is a non-empty set of *states* and $R \subseteq W \times W$ is a binary *accessibility relation* on W. A $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-*model* is a triple $\mathfrak{M} = \langle W, R, V \rangle$ where $\langle W, R \rangle$ is a frame and $V \colon \mathrm{Var} \times \mathrm{W} \to \mathrm{A}$ is a mapping, called a *valuation*. The valuation V is extended to $V \colon \mathrm{Fm}_{\Box\Diamond} \times \mathrm{W} \to \mathrm{A}$ by

$$\begin{aligned} V(\bot, w) &= \bot & V(\top, w) &= \top \\ V(\varphi \wedge \psi, w) &= V(\varphi, w) \wedge V(\psi, w) & V(\varphi \vee \psi, w) &= V(\varphi, w) \vee V(\psi, w) \\ V(\varphi \cdot \psi, w) &= V(\varphi, w) \cdot V(\psi, w) & V(\varphi \to \psi, w) &= V(\varphi, w) \to V(\psi, w) \\ V(\Box\varphi, w) &= \bigwedge\{V(\varphi, v) : Rwv\} & V(\Diamond\varphi, w) &= \bigvee\{V(\varphi, v) : Rwv\}. \end{aligned}$$

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A formula $\varphi \in \mathrm{Fm}_{\Box\Diamond}$ is *valid* in a $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-model $\mathfrak{M} = \langle W, R, V \rangle$ if $V(\varphi, w) = 1$ for all $w \in W$, and $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-valid if it is valid in all $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models.

For two $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models $\mathfrak{M} = \langle W, R, V \rangle$ and $\mathfrak{M}' = \langle W', R', V' \rangle$, we will say that $w \in W$ and $w' \in W'$ are *modally equivalent* if $V(\varphi, w) = V'(\varphi, w')$ for all $\varphi \in \mathrm{Fm}_{\Box\Diamond}$. A non-empty binary relation $Z \subseteq W \times W'$ will be called a *bisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$ if the following conditions are satisfied:

1. If wZw', then $V(p, w) = V'(p, w')$ for all $p \in \mathrm{Var}$.
2. If wZw' and Rwv, then there exists $v' \in W'$ such that vZv' and $R'w'v'$ (the forth condition).
3. If wZw' and $R'w'v'$, then there exists $v \in W$ such that vZv' and Rwv (the back condition).

We say that $w \in W$ and $w' \in W'$ are *bisimilar* if there exists a bisimulation Z between $\mathfrak{M}$ and $\mathfrak{M}'$ such that wZw'. A class $\mathcal{K}$ of $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models has the *Hennessy-Milner property* if for any models $\mathfrak{M} = \langle W, R, V \rangle$ and $\mathfrak{M}' = \langle W', R', V' \rangle$ in $\mathcal{K}$, whenever the states $w \in W$ and $w' \in W'$ are modally equivalent, they are bisimilar.

We present a property of MTL-chains – the so-called distinguishing formula property – that characterizes precisely those complete MTL-chains $\mathbf{A}$ for which the class of image-finite $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models has the Hennessy-Milner property. This characterization can then be used to obtain a precise classification in the setting of finite and standard *BL-chains* (MTL-chains satisfying $a \cdot (a \to b) = a \wedge b$ for all $a, b \in A$). Let $\mathbf{Ł_{n+1}}$ ($n \in \mathbb{N}$) and $\mathbf{Ł}$ denote the $n+1$-element Łukasiewicz chain and the standard Łukasiewicz chain over $[0,1]$, respectively, and let $\mathbf{Ł^h_{n+1}}$ and $\mathbf{Ł^h}$ denote their hoop reducts. The operation $\oplus$ denotes the ordinal sum construction for hoop reducts of BL-chains.

Theorem 1. *The following are equivalent for any finite BL-chain* $\mathbf{A}$*:*

(1) *The class of image-finite* $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$*-models has the Hennessy-Milner property.*
(2) $\mathbf{A}$ *is isomorphic to* $\mathbf{Ł_{n+1}}$ *or* $\mathbf{Ł^h_{n+1}} \oplus \mathbf{Ł^h_{m+1}}$ *for some* $m, n \in \mathbb{N}$.

Theorem 2. *The following are equivalent for any standard BL-chain* $\mathbf{A}$*:*

(1) *The class of image-finite* $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$*-models has the Hennessy-Milner property.*
(2) $\mathbf{A}$ *is isomorphic to* $\mathbf{Ł}$ *or* $\mathbf{Ł^h} \oplus \mathbf{Ł^h}$.