On MV-algebras with convexity operators

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ManyVal'15 - Les Diablerets

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joint work with Tommaso Flaminio

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- Convex combinations appear in any area of mathematics as well as in other sciences,
- MV-algebras are a good candidate to define in abstract way: Di Nola and Leuștean - MV-algebras with scalar product,
- characterization of states on MV-algebras: states are natural generalization of probability on boolean algebras and they are convex combinations of homomorphisms.

MV-algebras

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- Unit interval of lattice ordered groups with strong unit,

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- Unit interval of lattice ordered groups with strong unit,
- A generalization of boolean algebras.

The standard model

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- $x \oplus y = \min(x+y, 1)$, $x^* = 1 - x$;
- is the set of truth values for Łukasiewicz logic;
- it is closed for product between real numbers!

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Relation with boolean algebras

$B(A) = \{x \mid x \oplus x = x\}$ is a boolean algebra

$B(A)$ is called boolean skeleton of A.

Homomorphisms and ideals

- A, B MV-algebras, $h : A \to B$ such that
  $h(0) = 0$,
  $h(x \oplus y) = h(x) \oplus h(y)$,
  $h(x^*) = h(x)^*$.
- $I \subseteq A$, ideal if
  $0 \in I$,
  $x, y \in I$ implies $x \oplus y \in I$,
  $x \in I$, $y \leq x$ then $y \in I$.

MV-algebras via groups

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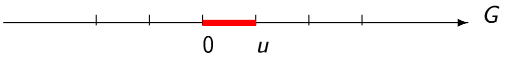
$(G, +, 0, \leq)$ is ℓ-*group*

- if $(G, +, 0)$ group,
- $(G, \leq)$ lattice,
- $x \leq y$ implies $x + z \leq y + z$ for any x, y, $z \in G$.

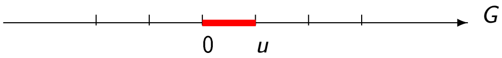
ℓu-groups

$u \in G$ is a **strong unit**: $u \geq 0$, for any $x \in G$ there is $n \geq 1$ s.t. $x \leq nu$.

An abelian ℓ-group with strong unit is an ℓu-group.



For any $x \geq 0$ in G there are $x_1, \dots, x_n \in [0, u]$ s.t. $x = x_1 + \cdots + x_n$.



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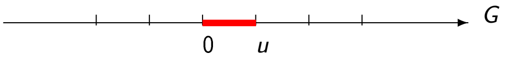
$$x \oplus y = (x +_G y) \wedge u, \quad x^* = u -_G x$$



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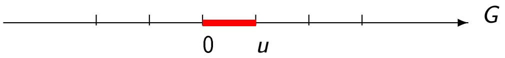

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Categorical equivalence. Mundici, 1986

For any MV-algebra A there exists a ℓu-group (G, u) such that $A \simeq [0, u]_G$.

The category of MV-algebras is equivalent with the category of Abelian lattice-ordered groups with strong unit with unit preserving homomorphism.

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- they form a variety,
- categorical equivalence with ℓu-rings,
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Riesz MV-algebras (Di Nola, Leuștean, 2014)

- they form a variety,
- categorical equivalence with Riesz Spaces (vector lattices) with strong unit,
- $\mathbb{RMV} = HSP([0,1]_{RMV})$.

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For example

In $[0,1]$ $x + y$ is defined when $x +_{\mathbb{R}} y \leq 1$!

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states are convex combinations of homomorphism

Convex combinations in MV-algebras

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We want to find a set of axioms that captures

for any $x, y, \alpha \in [0,1]$

$$\alpha x + (1 - \alpha) y$$

MV-algebras with convexity operators (CMV-algebras)

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MV-algebra **A** together with a family of binary operators $\mathcal{C} = \{cc_\alpha\}_{\alpha \in [0,1]}$ on A such that

(C1) $cc_0(x, y) = y$;

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(C4) $cc_\alpha(cc_\beta(x, y), z) = cc_{\alpha\beta}(x, cc_\gamma(y, z))$, with γ arbitrary if $\alpha = \beta = 1$ and $\gamma = \frac{\alpha(1-\beta)}{1-\alpha\beta}$ otherwise;

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(C5) For all $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$, $cc_\alpha(x, 0) + cc_\beta(x, 0)$ is defined and it coincides with $cc_{\alpha+\beta}(x, 0)$;

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(C7) $cc_\alpha(x, y)^* = cc_\alpha(x^*, y^*)$.

Examples

- Let $[0,1]_{PMV}$ be the standard PMV-algebra and, for every $x, y, \alpha \in [0,1]$, define

$$cc_\alpha(x,y) = \alpha x \oplus (1-\alpha)y.$$

In this case $\oplus$ is the usual sum between real numbers!

Then $([0,1]_{MV}, \mathcal{C})$ is a CMV-algebra, with $\mathcal{C} = \{cc_\alpha\}_{\alpha\in[0,1]}$

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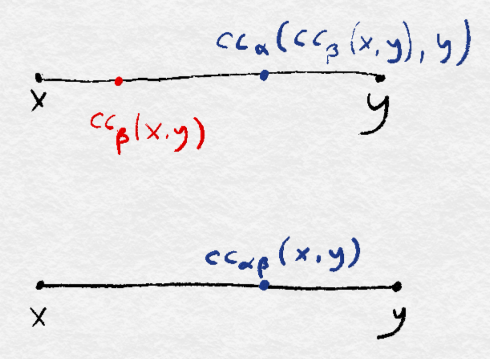
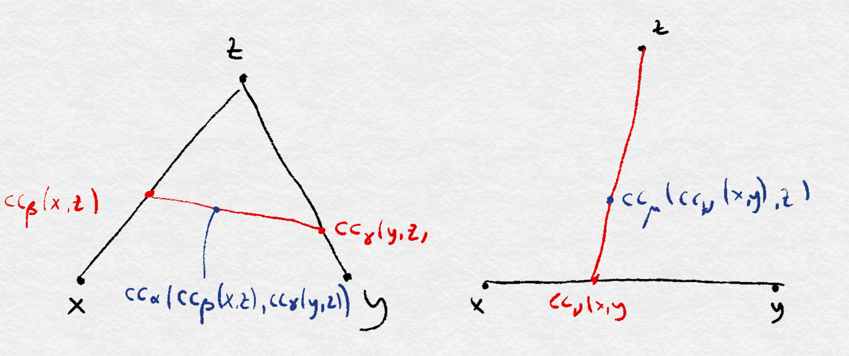
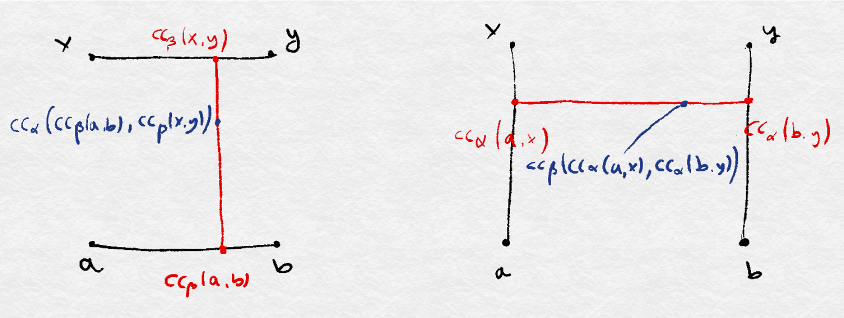


(iii) $cc_\alpha(cc_\beta(x,z), cc_\gamma(y,z)) = cc_\mu(cc_\nu(x,y),z)$, with $\mu = \alpha\beta + (1-\alpha)\gamma$ and $\nu = \frac{\alpha\beta}{\mu}$;



(iv) $cc_\alpha(cc_\beta(a,b), cc_\beta(x,y)) = cc_\beta(cc_\alpha(a,x), cc_\alpha(b,y))$;



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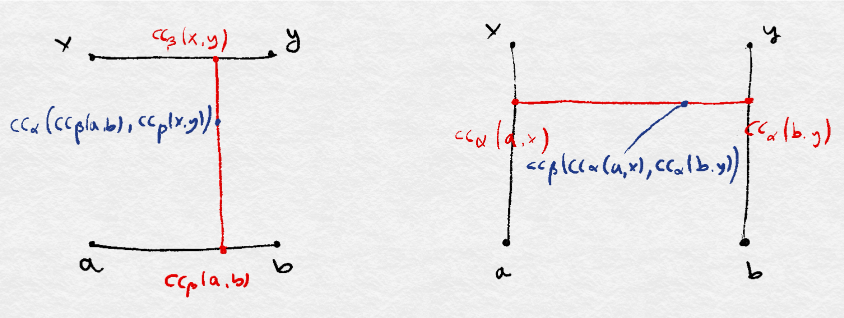

(v) If $\alpha \leq \beta$, then $cc_\alpha(x,0) \leq cc_\beta(x,0)$.

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(viii) $cc_\alpha(x,0) \leq x$ and $cc_\alpha(0,y) \leq y$.

The main result

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$\mathbb{CMV}$ and $\mathbb{RMV}$ are term-wise equivalent

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- $\mathbb{CMV} = HSP([0, 1])$.

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$\{\pi_1, \ldots, \pi_k\}$ generates the free MV-algebra over k generators $F_{MV}(k)$,

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$F_{MV}(k)$ is not closed for convex combinations!

We consider the Riesz Hull and we get

$$R(F_{MV}(k)) \cong [0,1] \otimes F_{MV}(k) \cong F_{RMV}(k),$$

$F_{RMV}(k)$ is the algebra of piecewise linear functions with real coefficients.

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Goal

To write a states $s : [0,1]^k \to [0,1]$ as a term in $F_{RMV}(k)$!

Notation

for every $\alpha_1, \ldots, \alpha_{k-1} \in [0,1]$ we set $[\alpha_i] = \alpha_1, \ldots, \alpha_{k-1}$ and for every $f_1, \ldots, f_k \in F_{RMV}(k)$, we write

$$\boxed{cc_{[\alpha_i]}(f_1, \ldots, f_k)}$$
$$=$$
$$cc_{\alpha_1}(f_1, cc_{\alpha_2}(f_2, \ldots cc_{\alpha_{k-1}}(f_{k-1}, f_k) \ldots)).$$

And finally...

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For every $\alpha_1, \ldots, \alpha_{k-1} \in [0,1]$,

$$cc_{[\alpha_i]}(\pi_1, \ldots, \pi_k) \text{ is a state of } [0,1]^k.$$

And finally...

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(by the term-wise equivalence)

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(by the term-wise equivalence)

For every state $s \in S([0, 1]^k)$, there exists $f \in (F_{RMV}(k), \mathcal{C})$ such that, for every $x \in A$,

$$s(x) = f(x).$$

And finally...

For every $\alpha_1, \ldots, \alpha_{k-1} \in [0,1]$,

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$(S([0,1]^k) = co(H([0,1]^k)) +$ the fact that projections generate $F_{RMV}(k))$

THANK YOU!