On Possibilistic Gödel modal logics: an axiomatization and neighbourhood semantics

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On possibilistic modal logics defined over MTL-chains

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Preamble

P. Hájek, D. Harmancová, F. Esteva, P. Garcia, L. G.

On Modal Logics for Qualitative Possibility in a Fuzzy Setting

In *Proc. of the 94 Uncertainty in Artificial Intelligence Conference* (UAI'94), 278–285, Morgan Kaufmann, 1994

—MVDKD45: a modal logic over a finitely-valued Łukasiewicz logic to capture possibilistic reasoning on many-valued events—

Aim: generalize the approach to possibilistic modal logics over an arbitrary finite MTL-chain

Outline

- Motivation and related work
- $Pos(\mathbf{A}^c_\Delta)$: a possibilistic modal logic over a finite MTL-chain $\mathbf{A}$
- Open problems and conclusions

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Modal logics of belief

- Epistemic modal logics: reasoning about an agent's beliefs

  - non-modal (objective) φ: what is true in the actual world
  - modal $\Box\varphi$: what is believed by an agent

- Modal logic of belief KD45:

  - relational (Kripke) semantics $M = (W, R, e)$:
    epistemic state = $R \subseteq W \times W$ serial, euclidean and transitive

$$(M, w) \models \Box\varphi \text{ if } (M, w') \models \varphi \text{ for all } w' \text{ such that } (w, w') \in R$$

  - simpler but equivalent semantics $M = (W, E, e)$:
    epistemic state = non-empty $E \subseteq W$

$$(M, w) \models \Box\varphi \text{ if } (M, w') \models \varphi \text{ for all } w' \in E$$

From sets to graded epistemic states

- Graded epistemic states: enriching an epistemic state with a ranking of plausibility

$$\pi : W \to [0, 1]$$

  - $\pi(w) < \pi(w')$: w' is more plausible than w; $\exists w$ s.t. $\pi(w) = 1$
  - $\pi(w) = 0$: w is discarded
  - $\pi(w) = 1$: w is totally plausible

- From E's to π's: Possibilistic models $M = (W, \pi, e)$
  - to what extent all plausible worlds satisfy φ :

$$N(\varphi) = \inf\{1 - \pi(w) \mid (M, w) \models \varphi\}$$

  - to what extent there exists a plausible world sastifying φ:

$$\Pi(\varphi) = \sup\{\pi(w) \mid (M, w) \models \varphi\} = 1 - N(\neg\varphi)$$

- $(W, \pi, e) \equiv (M, \{E_\lambda\}_\lambda, e)$, nested set of KD45 models

SOME MODAL-LIKE FORMALISMS

Two-tiered logics, non-nested modalities:

- Possibilistic logic (Dubois-Prade et al., 80's)
  - Weighted formulas: $(\varphi, \lambda) \equiv \Box_\lambda \varphi$, where $\varphi \in \mathcal{L}_{CPL}, \lambda \in (0, 1]$.
  - Semantics: $N(\varphi) \geq \lambda$
  - Weighted MP: $\Box_\lambda \varphi, \Box_\mu(\varphi \to \psi) \vdash \Box_{\lambda \wedge \mu} \psi$

- Generalized possibilistic logic: Boolean combination of $\Box_\lambda \varphi$'s

  (K) $\Box_\lambda(\varphi \to \psi) \to (\Box_\lambda \varphi \to \Box_\lambda \psi)$

  (D) $\Diamond_1 \top$

  (Nes) $\Box_{\lambda_1} \varphi \to \Box_{\lambda_2} \varphi$, if $\lambda_1 \geq \lambda_2$

  Easily extendable to a full (multi-)modal KD45 system

- Fuzzy logic approach: truth-value($\Box\varphi$) $:= N(\varphi)$ (Hájek et al.)
  φ's are Boolean propositions, $\Box\varphi$'s are fuzzy propositions

  (Ł) Łukasiewicz logic axioms

  (K) $\Box(\varphi \to \psi) \to_{Ł} (\Box\varphi \to_{Ł} \Box\psi)$

  (D) $\Diamond \top$

Generalizing to the MV setting

$[0,1]_*$: MTL-algebra

- many-valued possibilistic Kripke models: $M = (W, \pi, e)$, where $e : W \times Var \to [0,1]$, and $\pi : W \to [0,1]$

  - to what extent all plausible worlds satisfy φ:

  $$e(w, \Box\varphi) = N(\varphi) = \inf_{w'} \pi(w') \to e(w', \varphi)$$

  - to what extent some plausible world satisfies φ:

  $$e(w, \Diamond\varphi) = \Pi(\varphi) = \sup_{w'} \pi(w') * e(w', \varphi)$$

- If we define $R(w, w') = \pi(w')$, then $M = (W, R, e)$ is a mv-KD45 model:
  - seriality: $\exists w'$ s.t. $R(w, w') = 1$
  - transitivity: $R(w, w') * R(w', w'') \leq R(w, w'')$
  - euclidean: $R(w, w') * R(w, w'') \leq R(w', w'')$

Some modal-like accounts

Two-tiered (fuzzy) logics:

- L_1: logic of the φ's
  L_2: logic of the $\Box\varphi$'s and $\Diamond\varphi$'s
- So far, basically, two choices: both some variants of Łukasiewicz logic (Flaminio et al.) or Gödel logic (Dellunde et al.)
- For instance, for $L_1 = L_2 = Ł_k$ with truth-constants:

$$\begin{array}{ll} \text{(K)} & \Box(\psi \to_{Ł} \varphi) \to_{Ł} (\Box\psi \to_{Ł} \Box\varphi) \\ \text{(D)} & \Diamond\top \\ & \bar{r} \leftrightarrow_{Ł} \Box(\bar{r}, \text{ for } r \in \{0, 1/k, \ldots, (k-1)/k, 1\} \end{array}$$

- To get completeness, one usually requires L_2 to be strongly complete
- For uncertainty two-tiered logics: (Flaminio et al. 2011)
  For a general theory: (Cintula-Noguera, 2014)

Some modal accounts

A full fledged (fuzzy) modal logic: the $Ł_k$-based modal logic MVKD45 (Hájek et al., 94)

- Finite set of propositional variables $p_1 \ldots \wedge p_n$
- $M = (W, \pi, e)$, with $\pi : W \to \{0, 1/k, \ldots, (k-1)/k, 1\}$

$$e(w, \Box\varphi) = \min_{w'}\{\neg\pi(w') \vee e(w', \varphi)\}$$

$$e(w, \Diamond\varphi) = \max_{w'}\{\pi(w') \wedge e(w', \varphi)\}$$

- This particular semantics for $\Box$ (it is not based on $\to_{Ł}$) was chosen because of axiom (K)

Some modal accounts (2)

Notation: $(r)\varphi := (\bar{r} \leftrightarrow \varphi)^k$; **mec**: $(r_1)p_1 \wedge \ldots \wedge (r_n)p_n$

Axiomatics:

- Axioms of KD45:
  $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
  $\Box\varphi \leftrightarrow \Box\,\Box\varphi$
  $\Diamond\varphi \leftrightarrow \Box\,\Diamond\varphi$
  $\Diamond\top$
- $(r)\Box\varphi \leftrightarrow \Box(r)\Box\varphi$, $(r)\Diamond\varphi \leftrightarrow \Box(r)\Diamond\varphi$
- Possibilistic axioms:
  $((r)\Diamond\varphi \wedge E) \rightarrow (\leq r)(\varphi \wedge \Diamond E)$, with $E \in$ **mec**
  $(r)\Diamond\varphi \rightarrow \bigvee_{E\in\mathbf{mec}}(\geq r)\Diamond(E \wedge (r)(\varphi \wedge \Diamond E))$, with $r > 0$

Deductions rules are modus ponens and necessitation for $\Box$ (from φ infer $\Box\varphi$).

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The logic $Pos(\mathbf{A}^c_\Delta)$ defined

We fix:

- a *finite* MTL-chain $\mathbf{A} = (A, \wedge, \vee, *, \rightarrow, 0, 1)$
- finitely many propositional variables $Var = \{p_1, \ldots, p_n\}$

We let:

- $\Lambda(\mathbf{A}^c_\Delta)$ be the finitely-valued propositional logic of the MTL-chain $\mathbf{A}$ expanded with Δ and truth-constants $\bar{r}$ for $r \in A$:

  $\Gamma \models_{\mathbf{A}^c_\Delta} \varphi$ if for all $\mathbf{A}$evaluation e, if $e(\psi) = 1$ for all $\psi \in \Gamma$ then $e(\varphi) = 1$.

Language of $Pos(\mathbf{A}^c_\Delta)$:

- expand that of $\Lambda(\mathbf{A}^c_\Delta)$ with $\Box$ and $\Diamond$
- abbreviations: $(r)\varphi := \Delta(\bar{r} \leftrightarrow \varphi)$; **mec**: $(r_1)p_1 \wedge \ldots \wedge (r_n)p_n$

THE LOGIC $Pos(\mathbf{A}^c_\Delta)$ DEFINED: SEMANTICS

A-valued possibilistic Kripke frame: $F = \langle W, \pi \rangle$,
where $W \neq \emptyset$ and $\pi : W \longrightarrow A$ s.t. $\exists w \in W$ with $\pi(w) = 1$

A-valued possibilistic Kripke model: $K = \langle W, e, \pi \rangle$
where $\langle W, \pi \rangle$ is an **A**-valued poss. frame and $e : W \times Var \longrightarrow A$

- $\| \cdot \|_{K,w}$ as usual for variables, connectives and truth-constants
- $$\|\Diamond\varphi\|_K = \max_{u \in W}\{\pi(u) * \|\varphi\|_{K,u}\}$$
  $$\|\Box\varphi\|_K = \min_{u \in W}\{\pi(u) \rightarrow \|\varphi\|_{K,u}\}$$
- $(K, w) \models \varphi$ if $\|\varphi\|_{K,w} = 1$

Logical consequence: $\Gamma \models \varphi$, for any $K = (W, e, \pi)$ and $w \in W$, if $(K, w) \models \psi$ for every $\psi \in \Gamma$, then $(K, w) \models \varphi$

The logic $Pos(\mathbf{A}^c_\Delta)$ defined: semantics

$K = (W, e, \pi)$ is reduced when for any $w, w' \in W$, if $e(w, \cdot) = e(w', \cdot)$ then $w = w'$, and hence $\pi(w) = \pi(w')$.

Since *Var* and *A* are finite, only finitely-many reduced models. And we can restrict ourselves to reduced models:

- For any model K there is a reduced model K' such that $\|\varphi\|_K = \|\varphi\|_{K'}$ for any formula φ

Hence:

- Modulo semantical equivalence, there are only a finite number of different formulas

THE LOGIC $Pos(\mathbf{A}^c_\Delta)$ DEFINED: SEMANTICS

Some further remarks:

- $\Box$ and $\Diamond$ are not dual: $\Diamond\varphi$ is not equivalent to $\neg\Box\neg\varphi$, unless $\mathbf{A}$ is an IMTL-algebra
- But, due to the presence of truth-constants, they are indeed inter-definable:

  - $\Box\varphi$ as $\bigwedge_{r\in A}(\Diamond(r)\varphi \to \bar{r})$

  - $\Diamond\varphi$ as
  $(>0)\Box\neg\varphi \wedge \left(\bigwedge_{r\in A\setminus 0}\left((1)\Box(\varphi\to\bar{r}) \wedge (<1)\Box(\varphi\to\overline{r^-})\right)\to\bar{r}\right)$
- Axiom (K) is not valid in general
  This is in contrast with Hájek's et al. MVKD45

THE LOGIC $Pos(\mathbf{A}^c_\Delta)$ DEFINED: SYNTAX

- Recall $Pos(\mathbf{A}^c_\Delta)$ is defined on top of $\Lambda(\mathbf{A}^c_\Delta)$

$\Lambda(\mathbf{A}^c_\Delta)$ can be axiomatized, relatively to one for $\Lambda(\mathbf{A})$ with only MP as rule, by adding:

- axioms and necessitation rule for Δ
- book-keeping axioms:

$$
\begin{aligned}
&(\bar{r} \& \bar{s}) \leftrightarrow \overline{r \odot s} \\
&(\bar{r} \to \bar{s}) \leftrightarrow \overline{r \Rightarrow s} \\
&(\bar{r} \wedge \bar{s}) \leftrightarrow \overline{\min(r, s)} \\
&\Delta\bar{r} \leftrightarrow \overline{\Delta r},
\end{aligned}
$$

- witnessing axiom:

$$\bigvee_{r \in A}(\varphi \leftrightarrow \bar{r}).$$

THE LOGIC $Pos(\mathbf{A}^c_\Delta)$ DEFINED: SYNTAX

The logic $Pos(\mathbf{A}^c_\Delta)$ is syntactically defined by:

- Axioms and rules of $\Lambda(\mathbf{A}^c_\Delta)$
- Axioms from KD45:

  (4) $\Box\varphi \leftrightarrow \Box\,\Box\varphi$
  (5) $\Diamond\varphi \leftrightarrow \Box\,\Diamond\varphi$
  (D) $\Diamond\top$

  (4') $(r)\Box\varphi \leftrightarrow \Box(r)\Box\varphi$, for each $r \in A$
  (5') $(r)\Diamond\varphi \leftrightarrow \Box(r)\Diamond\varphi$, for each $r \in A$

- Possibilistic axioms (for each $r \in A$):

  (NΠ) $\Box(\varphi \to \overline{r}) \leftrightarrow (\Diamond\varphi \to \overline{r})$
  (Π1) $((r)\Diamond\varphi \wedge E) \to (\leq r)(\varphi \& \Diamond E)$, with $E \in \mathbf{mec}$
  (Π2) $(r)\Diamond\varphi \to \bigvee_{E\in\mathbf{mec}}(\geq r)\Diamond(E \wedge (r)(\varphi\&\Diamond E))$, with $r > 0$

- The monotonicity rule for $\Box$:
  if $\varphi \to \psi$ is a theorem, infer $\Box\varphi \to \Box\psi$

About (Π1) and (Π2)

Recall:
- $\Pi(\varphi) = \max_w \pi(w) \wedge \|\varphi\|_w$
- a m.e.c E represents a possible world w_E, so $\Pi(E) = \pi(w_E)$

(Π1) $((r)\Diamond\varphi \wedge E) \to (\leq r)(\varphi \& \Diamond E)$, with $E \in \mathbf{mec}$

If $\Pi(\varphi) = r$ then $\pi(w) \wedge \|\varphi\|_w \leq r$

(Π2) $(r)\Diamond\varphi \to \bigvee_{E \in \mathbf{mec}} (\geq r)\Diamond(E \wedge (r)(\varphi \& \Diamond E))$, with $r > 0$

If $\Pi(\varphi) = r$ then there is w such that $\pi(w) \wedge \|\varphi\|_w = r$

Towards completeness

B-formulas: propositional combinations of formulas of the type $\Delta\varphi$, in particular $(r)\varphi$.

- $T \cup \{\psi\} \vdash \varphi$ iff $T \vdash \Delta\psi \to \varphi$

A theory T is a set of B-formulas.

Define $T \approx T'$ if for each r and φ, $T \vdash (r)\Diamond\varphi$ iff $T' \vdash (r)\Diamond\varphi$.

Let T be complete and consistent. Then:

- for every φ there exists a unique r such that $T \vdash (r)\varphi$
- there is a unique $E_T \in \mathbf{mec}$ such that $T \vdash E_T$
- for any φ, if $T \vdash (r)\Diamond\varphi$ then the following conditions hold:
  - (a) For any $T' \approx T$, $T' \vdash (\leq r)(\varphi \& \Diamond E_{T'})$
  - (b) There is $T_\varphi \approx T$ and $E_\varphi \in \mathbf{mec}$ such that $T_\varphi \vdash (r)(\varphi \& \Diamond E_\varphi)$

Canonical model

Assume $\Gamma \not\models \varphi$ and let T_0 be a complete theory extending Γ and $T_0 \not\models \varphi$.

Define the possibilistic Kripke model

$$K_0 = (W_0, e_0, \pi_0)$$

where

- $W_0 = \{T_0\} \cup \{T_\varphi \mid T_\varphi \approx T_0, \varphi \text{ formula}\}$
- $e_0 : W_0 \times Var \to A$ is defined by $e_0(T, p) = r$ whenever $T \vdash (r)p$
- $\pi_0 : W_0 \to A$ is defined by $\pi_0(T) = r$ if $T \vdash (r)\Diamond E_T$.

Truth Lemma: For each ψ, r and $T \in W_0$,

$$T \vdash (r)\psi \text{ iff } \|\psi\|_{T,K_0} = r.$$

Completeness

$Pos(\mathbf{A}^c_\Delta)$ is strongly complete w.r.t. the class of A-valued possibilistic Kripke frames, that is, the following are equivalent:

(1) $\Gamma \vdash \varphi$

(2) $\Gamma \models \varphi$

(3) For any *reduced* possibilistic Kripke model $K = (W, e, \pi)$ and $w \in W$, $\Gamma \models_{w,K} \varphi$.

This trivially implies that $Pos(\mathbf{A}^c_\Delta)$ is decidable.

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Conclusions and open problems

- Hájek et al.'s MVKD45 has been generalized to a wide family of logics over finite MTL-chains
- With a slightly different semantics:
  min replaced by $*$, $max(1-x, y)$ replaced by $\rightarrow$

Open Problems

- Extension to infinite-valued logics ?
- Ongoing work using neighborhood semantics
- In the many-valued case: KD45 semantics = Possibilistic semantics?

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"On possibilistic modal logics defined over MTL-chains"
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Announcement

iPMU 2016

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Submission deadline: January 8
Notification of acceptance: March 1
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