

Single chain completeness and some related properties

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joint work with

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An FL_{ew} algebra is an algebra $\langle A, *, \rightarrow, \wedge, \vee, 0, 1 \rangle$ such that:

- ① $\langle A, \wedge, \vee, 0, 1 \rangle$ is a bounded lattice with minimum 0 and maximum 1.
- ② $\langle A, *, 1 \rangle$ is a commutative monoid.
- ③ $\langle *, \rightarrow \rangle$ forms a *residuated pair*: $z * x \leq y$ iff $z \leq x \rightarrow y$ for all $x, y, z \in A$. In particular, it holds that $x \rightarrow y = \max\{z \in A : z * x \leq y\}$.
- ④ An MTL-algebra is an FL_{ew} -algebra satisfying

$$(\text{Prelinearity}) \quad (x \rightarrow y) \vee (y \rightarrow x) = 1.$$

- The class of MTL-algebras forms a variety, called **MTL**. The logic corresponding to MTL-algebras is called **MTL**.
- An axiomatic extension of MTL is a logic obtained by adding other axioms to it.
- Every axiomatic extension of MTL is algebraizable in the sense of [Blok and Pigozzi, 1989], and hence every subvariety of **MTL** induces a logic.

Definition

Let L be an axiomatic extension of MTL. Then:

- L enjoys the *single chain completeness* (SCC) if there is an L -chain such that L is complete w.r.t. it.
- L enjoys the *finite strong single chain completeness* (FSSCC) if there is an L -chain such that L is finitely strongly complete w.r.t. it.
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Theorem ([Montagna, 2011])

Let L be an axiomatic extension of MTL. If L enjoys the FSSCC, then the SSCC holds.

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Let L be an axiomatic extension of MTL. If L enjoys the FSSCC, then the SSCC holds.

Problem ([Montagna, 2011])

Does the SCC implies the SSCC, in general?

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- L has the *deductive Maksimova's variable separation property* (DMVP) if, for all sets of formulas $\Gamma \cup \{\varphi\}$ and $\Sigma \cup \{\psi\}$ that have no variables in common, $\Gamma, \Sigma \vdash_L \varphi \vee \psi$ implies $\Gamma \vdash_L \varphi$ or $\Sigma \vdash_L \psi$.

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It is immediate to check that the DMVP implies the HC. Moreover

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Problem ([Galatos et al., 2007])

Find examples of substructural logics with DMVP, but without MVP.

The equivalence between HC and SCC

An FL_{ew} -algebra is said to be well connected whenever for every pair of elements x, y , if $x \sqcup y = 1$, then $x = 1$ or $y = 1$.

Theorem ([Galatos et al., 2007, Theorem 5.28])

Let L be an axiomatic extension of FL_{ew} . The following are equivalent:

- ❶ *L has the Halldén completeness.*
- ❷ *L is complete w.r.t. a well-connected FL_{ew} -algebra.*
- ❸ *L is meet irreducible (in the lattice of axiomatic extensions of FL_{ew}).*

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Theorem

Let L be a logic over MTL. The following are equivalent:

- ❶ *L has the Halldén completeness.*
- ❷ *L is complete w.r.t. an MTL-chain, that is L enjoys the SCC.*
- ❸ *L is meet irreducible (in the lattice of axiomatic extensions of MTL).*

Theorem ([Kihara, 2006, Theorem 6.9])

The following conditions are equivalent for every axiomatic extension L of MTL:

- *L has the DMVP.*
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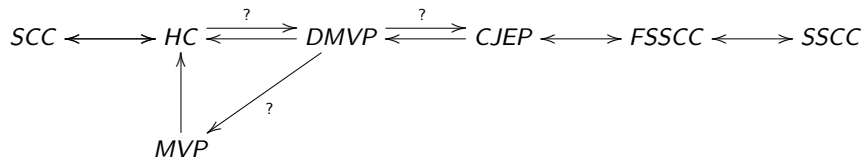
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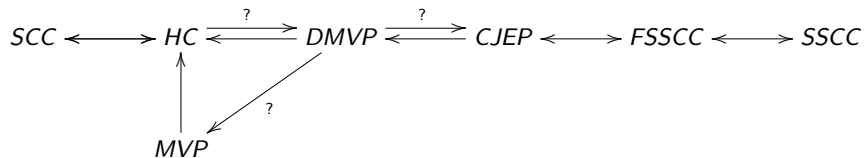
Corollary

Every axiomatic extension of MTL enjoying the SSCC has also the DMVP.

An intermediate picture

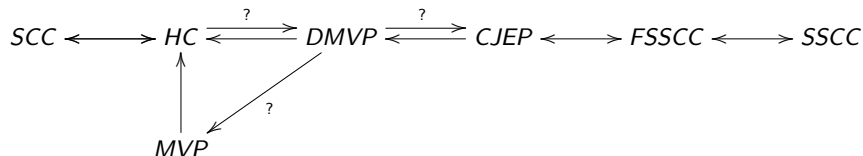


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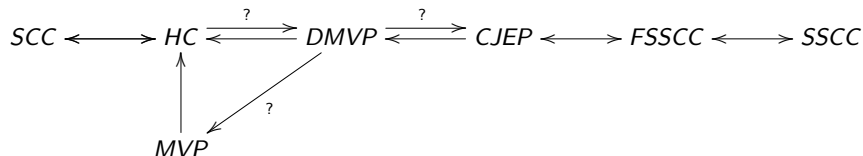
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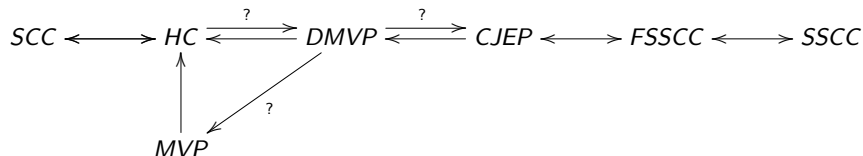
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- 1 Does exist an axiomatic extension of MTL having the DMVP, but for which the MVP fails?
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- 1 Does exist an axiomatic extension of MTL having the DMVP, but for which the MVP fails?
- 2 Are the DMVP and SSCC equivalent, for the axiomatic extensions of MTL?
- 3 Are the SCC and the SSCC equivalent, for the axiomatic extensions of MTL?

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DMVP and MVP: a solution to the open problem

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Let L be an axiomatic extension of MTL whose chains are isomorphic to ordinal sums of bounded and subdirectly irreducible semihoops. Then the DMVP and the SSCC are equivalent for L .

The SCC does not imply the SSCC: a counterexample - 1

Theorem ([Montagna, 2011])

Let L be an axiomatic extension of BL that is complete w.r.t. a finite chain. Then L enjoys the SSCC.

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Definition

Let us call \mathcal{Q} the MTL-chain (WNM-chain) $\langle \{0, a, b, c, 1\}, *, \Rightarrow, \min, \max, 0, 1 \rangle$, with $0 < a < b < c < 1$ and such that:

| x | $\sim x$ |
|-----|----------|
| 0 | 1 |
| a | c |
| b | a |
| c | a |
| 1 | 0 |

$$x * y = \begin{cases} 0 & \text{if } x \leq \sim y \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$

Let us call \mathbb{Q} the variety generated by \mathcal{Q} , and let Q be the corresponding logic.

The SCC does not imply the SSCC: a counterexample - 2

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- Observe that $\mathcal{Q} \models \bigvee_{i < 5} (x_i \rightarrow x_{i+1})$, and since $\mathbf{V}(\mathcal{Q}) = \mathbb{Q}$, by [Cintula et al., 2009, Proposition 4.18] every Q -chain cannot have more than five elements.



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- Notice that the three element Gödel-chain \mathbf{G}_3 belongs to \mathbb{Q} . Just take the quotient of \mathcal{Q} over the filter $\{c, 1\}$.



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- Notice that the three element Gödel-chain \mathbf{G}_3 belongs to \mathbb{Q} . Just take the quotient of \mathcal{Q} over the filter $\{c, 1\}$.
- Observe that $\mathbf{G}_3 \not\hookrightarrow \mathcal{Q}$.
- From these fact it is easy to check that there is no \mathcal{Q} -chain \mathcal{C} such that $\mathcal{Q} \hookrightarrow \mathcal{C}$ and $\mathbf{G}_3 \hookrightarrow \mathcal{C}$. Hence the DMVP fails.



The SCC does not imply the SSCC: a counterexample - 3

Theorem

- *For the axiomatic extensions of MTL, the HC does not necessarily imply the DMVP.*
- *For the axiomatic extensions of MTL, the SCC does not necessarily imply the SSCC (or the FSSCC).*

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Interestingly, \mathcal{Q} provides a counterexample of minimum cardinality.

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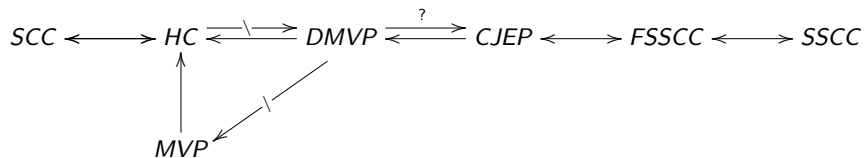
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Theorem

Let \mathcal{A} be a non-trivial MTL-chain with less than five elements, and let $\mathbb{L} = \mathbf{V}(\mathcal{A})$. Then L enjoys the SSCC.

A general picture



Extensions of MTL expanded with Δ

For every axiomatic extension L of MTL, we denote with L_Δ its expansion with an operator Δ satisfying the following axioms:

- ($\Delta 1$) $\Delta(\varphi) \vee \neg \Delta(\varphi).$
- ($\Delta 2$) $\Delta(\varphi \vee \psi) \rightarrow (\Delta(\varphi) \vee \Delta(\psi)).$
- ($\Delta 3$) $\Delta(\varphi) \rightarrow \varphi.$
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and the following additional inference rule: $\frac{\varphi}{\Delta\varphi}.$

An MTL_Δ -chain is an MTL-chain expanded with an operation δ , interpreting Δ , such that, for every element x , $\delta(x) = 1$ if $x = 1$, whilst $\delta(x) = 0$ if $x < 1$.

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We can define the notions of SCC, FSSCC and SSCC in a similar way to the MTL case.

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and the following additional inference rule: $\frac{\varphi}{\Delta\varphi}.$

An MTL_Δ -chain is an MTL-chain expanded with an operation δ , interpreting Δ , such that, for every element x , $\delta(x) = 1$ if $x = 1$, whilst $\delta(x) = 0$ if $x < 1$.

We can define the notions of SCC, FSSCC and SSCC in a similar way to the MTL case.

Theorem







Let L be an axiomatic extension of MTL. Then the SCC, FSSCC and SSCC are equivalent, for L_Δ and $L_\Delta \forall$.

- Does the DMVP implies the SSCC, in general?

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- Do MTL and IMTL enjoy the SCC?
- How about the first-order case? Are there some logical properties characterizing the SCC and the SSCC? Note that the SCC for an axiomatic extension L of MTL do not necessarily implies the SCC for L^\forall . As shown in [Montagna, 2011] BL is a counterexample.

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APPENDIX

Definition

A *semihoop* is a structure $\mathcal{A} = \langle A, *, \sqcap, \Rightarrow, 1 \rangle$ such that $\langle A, \sqcap, 1 \rangle$ is an inf-semilattice with upper bound 1, $*$ is a binary operation on A with unit 1, and \Rightarrow is a binary operation such that:

- $x \leq y$ iff $x \Rightarrow y = 1$,
- $(x * y) \Rightarrow z = x \Rightarrow (y \Rightarrow z)$.

A *bounded* semihoop is a semihoop with a minimum element; conversely, an *unbounded* hoop is a hoop without minimum.

- A hoop is a semihoop satisfying $x * (x \Rightarrow y) = y * (y \Rightarrow x)$.
- A Wajsberg hoop is a hoop satisfying $x \Rightarrow (x \Rightarrow y) = y \Rightarrow (y \Rightarrow x)$.

- Let $\langle I, \leq \rangle$ be a totally ordered set with minimum 0. For all $i \in I$, let \mathcal{A}_i be a totally ordered semihoop such that for $i \neq j$, $A_i \cap A_j = \{1\}$, and assume that \mathcal{A}_0 is bounded.
- Then $\bigoplus_{i \in I} \mathcal{A}_i$ (the *ordinal sum* of the family $(\mathcal{A}_i)_{i \in I}$) is the structure whose base set is $\bigcup_{i \in I} A_i$, whose bottom is the minimum of \mathcal{A}_0 , whose top is 1, and whose operations are

$$\begin{array}{c}
 A_j \\
 | \\
 A_i
 \end{array}
 \quad
 \begin{aligned}
 x \rightarrow y &= \begin{cases} x \rightarrow^{\mathcal{A}_i} y & \text{if } x, y \in A_i \\ y & \text{if } \exists i > j (x \in A_i \text{ and } y \in A_j) \\ 1 & \text{if } \exists i < j (x \in A_i \setminus \{1\} \text{ and } y \in A_j) \end{cases} \\
 x * y &= \begin{cases} x *^{\mathcal{A}_i} y & \text{if } x, y \in A_i \\ x & \text{if } \exists i < j (x \in A_i \setminus \{1\}, y \in A_j) \\ y & \text{if } \exists i < j (y \in A_i \setminus \{1\}, x \in A_j) \end{cases}
 \end{aligned}$$

- As a consequence, if $x \in A_i \setminus \{1\}$, $y \in A_j$ and $i < j$ then $x < y$.

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An axiomatic extension L of MTL is said to be n -contractive ($n \geq 2$), whenever $L \vdash \varphi^{n-1} \rightarrow \varphi^n$.

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Theorem ([Bianchi and Montagna, 2011])

Every n -contractive BL-chain is isomorphic to an ordinal sum of finite MV-chains, each of them having at most n elements.

Proof.

We already know that the SSCC implies the DMVP, so assume that L has the DMVP. Let \mathcal{A}, \mathcal{B} be two L -chains: we show that the CJEP (and hence the SSCC) holds.

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We already know that the SSCC implies the DMVP, so assume that \mathbb{L} has the DMVP. Let \mathcal{A}, \mathcal{B} be two \mathbb{L} -chains: we show that the CJEP (and hence the SSCC) holds. Since the variety (let us call it \mathbb{L}) of \mathbb{L} -algebras is closed under ordinal sum, then $\mathcal{A} \oplus \mathbf{2} \in \mathbb{L}$ and $\mathcal{B} \oplus \mathbf{2} \in \mathbb{L}$.

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Proof.

We already know that the SSCC implies the DMVP, so assume that L has the DMVP. Let \mathcal{A}, \mathcal{B} be two L -chains: we show that the CJEP (and hence the SSCC) holds. Since the variety (let us call it \mathbb{L}) of L -algebras is closed under ordinal sum, then $\mathcal{A} \oplus \mathbf{2} \in \mathbb{L}$ and $\mathcal{B} \oplus \mathbf{2} \in \mathbb{L}$. Clearly $\mathcal{A} \hookrightarrow \mathcal{A} \oplus \mathbf{2}$, $\mathcal{B} \hookrightarrow \mathcal{B} \oplus \mathbf{2}$ and $\mathcal{A} \oplus \mathbf{2}, \mathcal{B} \oplus \mathbf{2}$ are both subdirectly irreducible, having both a coatom. By the DMVP we have that there is an L -chain \mathcal{C} such that $\mathcal{A} \oplus \mathbf{2} \hookrightarrow \mathcal{C}$ and $\mathcal{B} \oplus \mathbf{2} \hookrightarrow \mathcal{C}$. Then, $\mathcal{A} \hookrightarrow \mathcal{A} \oplus \mathbf{2} \hookrightarrow \mathcal{C}$, $\mathcal{B} \hookrightarrow \mathcal{B} \oplus \mathbf{2} \hookrightarrow \mathcal{C}$ and we conclude that L has also the CJEP and SSCC. □

Axiomatization of MTL

The basic connective are $\{\wedge, \&, \rightarrow, \perp\}$ (formulas built inductively: a theory is a set of formulas). Useful derived connectives are the following ones:

(negation) $\neg\varphi \stackrel{\text{def}}{=} \varphi \rightarrow \perp$

$$\text{(disjunction)} \quad \varphi \vee \psi \stackrel{\text{def}}{=} ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$$
$$(top) \quad \top \stackrel{\text{def}}{=} \neg \perp$$

MTL can be axiomatized by using these axioms and modus ponens: $\frac{\varphi \quad \varphi \rightarrow \psi}{\psi}$.

$$(A1) \quad (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$$
$$(A2) \quad (\varphi \& \psi) \rightarrow \varphi$$
$$(A3) \quad (\varphi \& \psi) \rightarrow (\psi \& \varphi)$$
$$(A4) \quad (\varphi \wedge \psi) \rightarrow \varphi$$
$$(A5) \quad (\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$$
$$(A6) \quad (\varphi \& (\varphi \rightarrow \psi)) \rightarrow (\psi \wedge \varphi)$$
$$(A7a) \quad (\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \& \psi) \rightarrow \chi)$$
$$(A7b) \quad ((\varphi \& \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$$
$$(A8) \quad ((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$$
$$(A9) \quad \perp \rightarrow \varphi$$

Sketch of the proof.

If L has the SSCC, then the DMVP holds. Assume that L has the DMVP, and let \mathcal{A} is an L chain.

◀ back

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