ON QUOTIENTS OF THE GEOMETRIC THEORY OF LOCAL MV-ALGEBRAS

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In this work we continue the study of categorical equivalences arising in the field of many-valued logic started in the papers [3] and [2].

The literature for this class of algebras includes several categorical equivalences with categories of lattice-ordered abelian groups (ℓ-groups, for short). The most important are Mundici's equivalence and Di Nola-Lettieri's equivalence. The former was presented by Mundici in [6] and establishes the categorical equivalence between the whole category of MV-algebras and the category of lattice-ordered abelian groups with a distinguished element, called *strong unit*. In particular the so-called Mundici's functor Γ associates, for every ℓ-group G with strong unit u the MV-algebra built on the interval $[0, u]$. Further, in [7], Di Nola and Lettieri proved that the category of perfect MV-algebras is equivalent to the whole category of ℓ-groups .

In [2], we generalized the latter categorical equivalence exending it to categories of models no longer in the topos **Set** of sets but in an arbitrary Grothendieck topos. Indeed, Grothendieck toposes can be seen as generalized universes of sets which are very rich in terms of categorical structure; indeed, they are closed under arbitrary limits and arbitrary colimits. This richness allows to consider models of any kind of theory in them. The generalization of the categorical equivalence, together with the fact that the functor are defined by only using geometric construction, i.e. finite limits and arbitrary colimits, yields a Morita-equivalence between the theory $\mathbb{P}$ of perfect MV-algebras and the theory $\mathbb{L}$ of ℓ-groups and allow us to apply the bridge technique of [1] to transfer properties and results from one theory to the other, obtaining new insights on the theories which are not visible by using classical techniques. The central idea of this technique is the nature of Grothendieck topos that allows different representations. Decks of bridges are topos-theoretic invariants, i.e. properties and constructions preserved by categorical equivalences, while the aches of bridges are site-characterization of these invariants.

Among the results that we obtained with this technique, we mention three different levels of bi-interpretabilitity between the theory $\mathbb{P}$ and the theory $\mathbb{L}$ and a representation theorem for the finitely presentable objects of Changs variety as finite products of perfect MV-algebras. Given the fact that perfect MV-algebras are exactly the local MV-algebras in the variety generated by Changs algebra, it is natural to wonder whether analogues of Di Nola-Lettieri's equivalence exist for local MV-algebras in a given proper subvariety of MV-algebras. A complete characterization of subvarieties of the whole variety of MV-algebras **MV** was presented by Komory in [5]. He proved that every proper subvariety is generated by a finite number (possibly zero) of finite MV-chains and a finite number (possibly zero) of Komori's chains.

Let V be an arbitrary subvariety of MV-algebras, we study the class of local MV-algebras in V but we proceed in the opposite way with respect to the case of perfect MV-algebras. Indeed, we first prove that the the theory of local MV-algebras in V is of presheaf type and then we establish a Morita-equivalence with a theory that extends the one of ℓ-groups. Changing the variety we have different theories. These theories are always of presheaf type but they are algebraic if and only if the variety is generated by a single Komori's chain. It is interesting to observe that globally the theory of local MV-algebras is not of presheaf type.

Furthermore, studying the classifying topos of these theories of local MV-algebras, we can represent the finitely presented objects in every proper subvariety as a finite product of finitely presentable models of the theory of local MV-algebras in the variety.

The interest in local MV-algebras is justified by their use in representation results for MV-algebras. Indeed, Chang Representation Theorem states that every MV-algebra is a subdirect product of a MV-chains, that are in particular local MV-algebras. Further, Filipoiu and Georgescu proved in [4] that every MV-algebra can be seen as the algebra of global sections of a sheaf whose stalks are local MV-algebras.

References

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