Logics of graded predicates

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This talk has two different perspectives/starting points. The first one is simple and purely mathematical: I offer an abstract treatment of numerous results on 'logics of graded predicates' existing in the literature. The second one is more controversial and is more concerned about nature of the whole enterprise. Disregarding the second perspective, the abstract of my talk could be simply the following:

> Classical predicate logic interprets n-ary predicates as mappings from the n-th power of a given domain into the two-valued boolean algebra 2. The idea of replacing 2 by a more general structure is very natural and was shown to lead to a very interesting mathematics: prime examples are the Boolean-valued or Heyting-valued *models of set theory* (or even more general models proposed e.g. by Takeuti–Titani [11], Titani [12], or Hájek–Haniková [6]). There exists a stream of research, by Rasiowa–Sikorski [9], Horn [7], Rasiowa [8], Hájek [5] to give just a few names, studying *logics* where predicates can take values in lattices (with additional operators) from a certain class. In this talk I present first a general framework [4] for studying predicate logics where the mentioned class of lattices is the equivalent algebraic semantics of a propositional logic algebraizable in the sense of Blok and Pigozzi [1] and then some recent results on Skolem and Herbrand theorem for some of these logics [2, 3].

But even a purely-mathematically-minded reader could ask: 'why should I care and even if I would, where do I get a propositional logic to start with in the first place?'

The top to bottom approach

Let me start by an informal definition of graded predicates. A unary predicate (let us restrict to unary ones for now) is *graded* if it admits a (usually natural) 'classical' interpretation as a binary predicate (usually a function) between objects and a set of admissible 'grades' (e.g. 'tall' as a function assigning to every person its height, 'red' assigns to objects their distance from the point $\langle 255, 0, 0\rangle$ in the RGB space, 'intelligent' assigns to each person its IQ, etc.).

We could analyze statements such as 'all tall people are intelligent' in purely classical first-order multi-sorted logic. But we could also consider rescaling all those sets of admissible grades into one fixed scale and see its values as truth-degrees of unary predicates 'tall', 'red', or 'intelligent' and use a non-classical logic given by that scale.[1] Is this a suitable linguistical analysis of graded predicates? Of course not, but neither is using multi-sorted classical logic nor the usual approach which gets rid of graduality by 'breaking' the predicate into several classical ones (or even just one), e.g. 'tall' = $\langle$'shorter than 1.5m', 'between 1.5m and 2m', 'taller than 2m'$\rangle$. But that is not my point: I want to build the logic of graded predicates starting from the above informal idea and advocate the *usefulness* of the proposed approach in 'direct' treating of graded predicates especially in artificial reasoning scenarios and eventually (and hopefully) also in the natural ones.

I start building my logic by fixing the syntax: I consider first-order language consisting of function and predicate symbols and formulas built from the atomic ones using quantifiers $\forall$ and $\exists$ and *at least* the following binary logical connectives $\wedge$, $\vee$, $\to$ and a truth constant $\overline{1}$. Then I am ready to propose a natural first-order semantics based on arbitrary lattice-ordered scale of

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[1] Of course there is a strong connection to degree-theoretic approach to vagueness, see e.g. [10], but I do not want to presuppose any particular 'nature' of the grades.

grades (ordered because we do want to say that it is more the case that 'Light is fast' than it is the case that 'Shinkansen is fast' and the lattice operations are natural candidates for conjunction and disjunction). The next step is a bit more 'controversial': logic is (in my opinion) the study of consequence and the usual semantical way of presenting consequence is that of preservation of certain quality disregarding the actual 'state of the affairs', i.e., under arbitrary 'valuation', if all premises are 'OK' so is the conclusion. In order to accommodate this I consider as 'OK elements' all grades greater than or equal to $\overline{1}$ (note that this decision allows the rules $\varphi, \psi \models \varphi \wedge \psi$, $\varphi \wedge \psi \models \varphi$, and $\models \overline{1}$ to be 'valid'). Thus for any scale of grades (or a class of such scales) I can define the logic $\models$ of graded predicts taking values in that scale.

Final and the most controversial design choice is the following restriction on the behaviour of implication: 'the implication between two formulas is 'OK' whenever the second one is more 'OK' than the first one; i.e., the following has to hold for each scale of grades:

$$1 \leq x \to y \qquad \text{iff} \qquad x \leq y$$

This last design choice is motivated by requirement for validity of *modus ponens* and has a strong mathematical consequence: if we fix a quasivariety of such lattices, it is an equivalent algebraic semantics of an algeraizable logic which coincide with my semantically given logic of graded predicates if I consider language consisting solely of nullary predicates.

Structure of the talk

After a general introduction and explaining the motivation sketched above in details I present a natural framework for the study of logics where predicates can take values in a lattice (with additional operators) from a given class satisfying certain minimal conditions. For each such logic I first describe its 'propositional' part and then use it to give an axiomatization of the full first-order logic. Then I localize the resulting logics in the logical landscape and review their known basic properties concentrating on Skolem and Herbrand theorems.

At the end I would like to invite everyone to start a discussion about the role these logics can play in analysing and performing reasoning in the formalized contexts involving graded predicates and (potentially) in the study of natural reasoning scenarios and in the process of their transformation into the formalized ones.

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