Single chain completeness and some related properties

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Extended abstract

In [2] Franco Montagna investigated two interesting properties of the axiomatic extensions of MTL, the single chain completeness (SCC) and the strong single chain completeness (SSCC).

Definition 1. *An axiomatic extension L of MTL enjoys the SCC if there is an L-chain $\mathcal{A}$ s.t. L is complete w.r.t. $\mathcal{A}$, whilst L enjoys the SSCC if there is an L-chain $\mathcal{A}$ s.t. L is strongly complete w.r.t. $\mathcal{A}$.*

Clearly the SSCC implies the SCC, whilst the converse implication has been left as an open problem.

Our aim is to solve this problem, and for doing this we will show that SCC and SSCC are strongly related to some logical properties, such as Halldén completeness and variable separation properties. Such notions have been established for a long time in the area of modal and superintuitionistic logics, and more recently they have been studied also for substructural logic, in [1].

Definition 2. *Let L be an axiomatic extension of MTL. Then,*

- *L has the* Halldén completeness *(HC) if for every formulas φ, ψ with no variables in common, $\vdash_L \varphi \vee \psi$ implies that $\vdash_L \varphi$ or $\vdash_L \psi$.*
- *L has the* deductive Maksimova's variable separation property *(DMVP) if, for all sets of formulas $\Gamma \cup \{\varphi\}$ and $\Sigma \cup \{\psi\}$ that have no variables in common, $\Gamma, \Sigma \vdash_L \varphi \vee \psi$ implies $\Gamma \vdash_L \varphi$ or $\Sigma \vdash_L \psi$.*
- *L has the* Maksimova's variable separation property *(MVP) if, for all formulas $\varphi_1 \to \varphi_2$ and $\psi_1 \to \psi_2$ with no variables in common, $\vdash_L (\varphi_1 \wedge \psi_1) \to (\varphi_2 \vee \psi_2)$ implies $\vdash_L \varphi_1 \to \varphi_2$ or $\vdash_L \psi_1 \to \psi_2$.*

Clearly the DMVP implies the HC. Moreover, in [1] it is left as an open problem to find examples of substructural logics in which the DMVP holds, and the MVP fails.

In this talk we will tackle these problems, by showing the following results.

- For every axiomatic extension of MTL the HC is equivalent to the SCC, whilst the SSCC implies the DMVP.
- For every n-contractive axiomatic extension of BL, the SSCC is equivalent to the DMVP.
- There is an axiomatic extension of WNM in which the HC holds, whilst the DMVP fails. Hence the SCC does not necessarily imply the SSCC, and this solves the problem left open by Franco Montagna.
- Let S be the class of all the axiomatic extensions of MTL not extending SMTL. Then the MVP fails, for every $\mathrm{L} \in S$; however, there are some logics in S (as BL, NM, Ł...) enjoying the DMVP (since the SSCC holds, for them). This solves an open problem discussed in [1].
- For every axiomatic extension of MTL expanded with Δ, SCC and SSCC coincide. This holds even in the first-order case.

References

[1] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono, *Residuated lattices: An algebraic glimpse at substructural logics*, Studies in Logic and The Foundations of Mathematics, vol. 151, Elsevier, 2007. 1, 2

[2] F. Montagna, *Completeness with respect to a chain and universal models in fuzzy logic*, Arch. Math. Log. **50** (2011), no. 1-2, 161–183, doi:10.1007/s00153-010-0207-6. 1