Geometric description of Projective MV-algebras

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The variety of MV-algebras is the algebraic semantic of Łukasiewicz infinite valued logic. An *MV-algebra* is an algebraic structure $(M, \oplus, \neg, 0)$, where $(M, \oplus, 0)$ is a commutative monoid, $\neg$ is a unary operation satisfying $\neg\neg x = x$, $x \oplus \neg 0 = \neg 0$, and $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

The aim of this paper is to study the structure of finitely generated projective MV-algebras. An MV-algebra M is *(regular) projective* if whenever $\psi\colon A \to B$ is a surjective homomorphism and $\phi\colon M \to B$ is a homomorphism, there is a homomorphism $\theta\colon M \to A$ such that $\phi = \psi \circ \theta$. We provide a geometric description of finitely generated projective MV-algebras, thus solving the sixth problem in the list of open problems presented by Mundici in [8, Section 20.3]. Our result is based on the connection between finitely presented MV-algebras and rational polyhedra. This connection is a consequence of Chang's completeness theorem and McNaughton's representation of free MV-algebras (see [7] for a detailed account of this connection).

To present our result we need the following definitions. A simplex S is said to be *rational* if the coordinates of its vertices are rational numbers. A set $P \subseteq \mathbb{R}^n$ is said to be a *rational polyhedron* if there are rational simplexes $T_1, \ldots, T_l \subseteq \mathbb{R}^n$ such that $P = T_1 \cup \cdots \cup T_l$. Given a rational polyhedron $P \subseteq [0,1]^n$, we let $\mathcal{M}(P)$ denote the set of all continuous functions $f\colon P \to [0,1]$ having the following property: there are finitely many linear (in the affine sense) polynomials $p_1, \ldots, p_m$ with integer coefficients, such that for all $x \in [0,1]^n$ there is $i \in \{1, \ldots, m\}$ with $f(x) = p_i(x)$. The set $\mathcal{M}(P)$ carries a natural structure of MV-algebra, where the operations are defined pointwise from $([0,1]; \oplus, \neg, 0)$, where $x \oplus y = \min\{x + y, 1\}$ and $\neg x = 1 - x$.

In [2] and [3] the author and D. Mundici proved that finitely generated projective MV-algebras are isomorphic to $\mathcal{M}(P)$ where the rational polyhedron $P \subseteq [0,1]^n$ is a special kind of retract of $[0,1]^n$ (for some $n = 1, 2, \ldots$) called $\mathbb{Z}$-retract. A rational polyhedron P is a *$\mathbb{Z}$-retract* of a cube $[0,1]^n$ if there exists $f\colon [0,1]^n \to [0,1]^n$ such that $f \circ f = f$ (i.e., f is a retraction), for each projection map $\pi_i\colon [0,1]^n \to [0,1]$, the map $\pi_i \circ f$ is in $\mathcal{M}([0,1]^n)$, and $P = f([0,1]^n)$.

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As continuous retractions of cubes, $\mathbb{Z}$-retracts are *contractible*, that is, homotopically equivalent to a point (see for example [5, Chapter 0]). Another important property of $\mathbb{Z}$-retracts is that they are *strongly regular* (see [3, Definition 3.1]). In [1, Theorem 4.17], we observed how strong regularity is connected with the notion of anchored polytopes defined by Jeřábek in [6]. In this paper we will first present a simple description of strongly regular polyhedron (based on Jeřábek definition of anchored polytope). More precisely, we prove that a rational polyhedron P is strongly regular if and only if for each $v \in P \cap \mathbb{Q}^n$ there exist $w \in \mathbb{Z}^n$ and $\varepsilon > 0$ such that the convex segment $\operatorname{conv}(v, v + \varepsilon(w - v))$ is contained in P. The main result of this paper is the following.

Theorem 1. *Let A be an MV-algebra. The following are equivalent:*

(i) A is finitely generated and projective.

(ii) there exist $n \in \{1, 2, \ldots\}$ and a rational polyhedron $P \subseteq [0,1]^n$ such that the following conditions hold:

- *(a) $A \cong \mathcal{M}(P)$,*
- *(b) P is contractible,*
- *(c) $P \cap \{0,1\}^n \neq \emptyset$, and*
- *(d) for each $v \in P \cap \mathbb{Q}^n$ there exist $w \in \mathbb{Z}^n$ and $\varepsilon > 0$ such that the convex segment $\operatorname{conv}(v, v + \varepsilon(w - v))$ is contained in P.*

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