The strength of Łukasiewicz predicate logic

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ABSTRACT. The models of predicate Łukasiewicz logic with identity (Ł∀) as presented by Novák (1990, 1999) and Hájek (1998) may be seen as pseudometric structures with 1-Lipschitz operations and [0,1]-valued predicates, the pseudometric being the Lukasiewicz negation of the identity predicate, and the Lipschitz condition expressing the congruence axioms of identity. These structures are logically indistinguishable from their metric quotients. Therefore, Ł∀ appears as a fragment restricted to 1-Lipchitz models of first-order continuous logic (CL), introduced by Ben Yaacov and Usvyatsov (2008), with a switch in the distinguished truth value. However, our main result below, a Lindström theorem for Ł∀, shows that if we allow more general identity axioms in Ł∀ these two logics are essentially equivalent in the sense that they have the same axiomatizable classes.

The models of CL are complete metric structures with uniformly continuous operations and [0,1]-valued predicates, uniform continuity being a more liberal version of the identity congruence axioms. Metric completeness appears naturally and is a desideratum in most applications of CL, but the logic may be interpreted in incomplete structures, which are logically indistinguishable from their metric completions. Both logics satisfy on complete or incomplete metric structures of any fixed uniform continuity moduli:

1. Compactness
2. The (separable) Löwenheim-Skolem property
3. The witnesses property.

The third property means that any satisfiable sentence has a model M such that for any existential formula with parameters: $M \models \exists y \varphi(\overline{a}, y)$ implies $M \models \varphi(\overline{a}, b)$ for some b. This follows from ω_1-saturation of ultraproducts or from compactness and an elementary chain argument.

Call a [0,1]-valued logic on continuous metric structures *regular* if it is closed under Lukasiewicz connectives $\rightarrow$, $\neg$.

Theorem 1. *Any sentence of a regular logic extending* Ł∀ *and satisfying* (1, 2, 3) *is equivalent for 1-satisfaction to a countable theory of* Ł∀.

Thus, UCL, the closure of CL under uniform limits of definable predicates, has the same axiomatizable classes as Ł∀. Any sentence of Rational Łukasiewicz-Pavelka predicate logic (RŁ∀, Ł∀ enriched with rational constants) has the same models as a sentence of Ł∀.

But the full picture of a sentence φ is given by the map $M \longmapsto \varphi^M \in [0,1]$ sending structures to truth-values, while Theorem 1 gives account only of the class of structures attaining value 1 (or r for any given real $r \in [0,1]$ in a non uniform way). Moreover, Ł∀ is not able to approximate all continuous connnectives, not even the constant map $\frac{1}{2}$. Luckily, adding rational constants suffices to capture the full picture.

Theorem 2. *Any sentence of a regular logic extending* RŁ∀ *and satisfying* (1, 2, 3) *is the uniform limit of a sequence of sentences of* RŁ∀, *as maps.*

Condition (3) can be eliminated from the above results if we consider complete structures only, because for a compact logic L extending Ł∀ the following are equivalent:

- the witnesses property,
- each structure is equivalent in L to its metric completion.

Corollary. UCL *is the largest extension of* RŁ∀ *satisfying* (1) *and* (2) *on complete structures.*