The lattice of super-Belnap logics

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Introduction

The four-valued Belnap-Dunn logic $\mathcal{B}$ is a well-known logic for reasoning with incomplete and inconsistent information.

It was introduced by Nuel Belnap in 1977 as a "useful four-valued logic" or a logic of "how a computer should think".

Extensions of $\mathcal{B}$ will be called super-Belnap logics (following Rivieccio).

Examples: strong Kleene $\mathcal{K}$ and the Logic of Paradox $\mathcal{LP}$.

Our goal is to get a better view of the landscape of super-Belnap logics.

Truth and falsehood in $\mathcal{B}$

In the logic $\mathcal{B}$, truth values are computed in a perfectly classical way:

$\varphi \wedge \psi$ is true $\Leftrightarrow$ φ is true and ψ is true

$\varphi \wedge \psi$ is false $\Leftrightarrow$ φ is false or ψ is false

$\varphi \vee \psi$ is true $\Leftrightarrow$ φ is true or ψ is true

$\varphi \vee \psi$ is false $\Leftrightarrow$ φ is false and ψ is false

$-\varphi$ is true $\Leftrightarrow$ φ is false

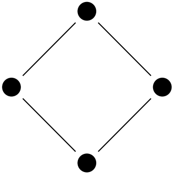
$-\varphi$ is false $\Leftrightarrow$ φ is true

. . . it's just that sentences may be both true and false or neither.

In other words, the truth and falsehood values are computed separately.

De Morgan algebras

De Morgan algebras are bounded distributive lattices with an order-inverting involution $-$. They form a variety DMA.



$\mathbf{DM_4}$

$\mathsf{DMA} = \mathbb{SP}(\mathbf{DM_4})$

$\mathbf{K_3}$

$\mathsf{KA} = \mathbb{SP}(\mathbf{K_3})$

$x \wedge -x \leq y \vee -y$

$\mathbf{B_2}$

$\mathsf{BA} = \mathbb{SP}(\mathbf{B_2})$

$x \wedge -x \leq y$

Logics of order

Each class of lattice-ordered algebras K naturally yields a logic of order:

$$\Gamma \vdash \varphi \text{ if } \bigwedge \overline{\Gamma} \leq \varphi \text{ holds in K for some finite } \overline{\Gamma} \subseteq \Gamma$$

The logic of order of DMA:	$\mathcal{B}$	(the Belnap-Dunn logic)
The logic of order of KA:	$\mathcal{K}_{\leq}$	(Kleene's logic of order)
The logic of order of BA:	$\mathcal{CL}$	(classical logic)

Logics of order are always self-extensional: $\varphi \dashv\vdash \psi \Rightarrow \chi(\varphi) \dashv\vdash \chi(\psi)$

Logics given by matrices

A matrix is an algebra $\mathbf{A}$ with a set of designated values $\mathcal{D} \subseteq \mathbf{A}$.

A matrix $(\mathbf{A}, \mathcal{D})$ is a model of a logic $\mathcal{L}$ in case for each $v : \mathbf{Fm} \to \mathbf{A}$:

if $\Gamma \vdash_{\mathcal{L}} \varphi$ and $v[\Gamma] \subseteq \mathcal{D}$, then $v(\varphi) \in \mathcal{D}$

A matrix is reduced if no non-trivial congruence on $\mathbf{A}$ preserves $\mathcal{D}$.

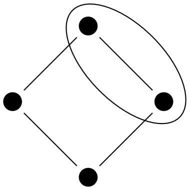
Each matrix $\mathbf{M}$ has a logically equivalent reduced matrix $\mathbf{M}/\theta$.

$\mathrm{Mod}\,\mathcal{L}$ = class of all models of $\mathcal{L}$

$\mathrm{Mod}^*\,\mathcal{L}$ = class of all reduced models of $\mathcal{L}$

Handling incomplete and contradictory information: $\mathcal{B}$

The Belnap-Dunn logic is given by the matrix $\mathbf{B_4}$:



$\mathbf{B_4}$

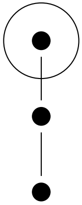
The truth values are: True, False, Neither, Both.

Hilbert-style axiomatization by Font (1997).

$$\mathrm{Mod}^* \mathcal{B} \subsetneq \{(\mathbf{A}, \mathcal{D}) \mid \mathbf{A} \in \mathrm{DMA}, \mathcal{D} \text{ lattice filter on } \mathbf{A}\} \subsetneq \mathrm{Mod}\, \mathcal{B}$$

Handling incomplete information: $\mathcal{K}$

Consider the matrix $\mathbf{K_3}$:



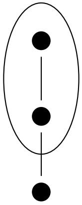
The truth values are: True, False, Neither.

This logic extends $\mathcal{B}$ by the rule of resolution: $p \vee q, -q \vee r \vdash p \vee r$.

This is Stephen C. Kleene's strong three-valued logic $\mathcal{K}$ (1938).

Handling contradictory information: $\mathcal{LP}$

Consider the matrix $\mathbf{LP_3}$:



The truth values are: True, False, Both.

This logic extends $\mathcal{B}$ by the law of the excluded middle: $\emptyset \vdash p \vee -p$.

This is Graham Priest's Logic of Paradox $\mathcal{LP}$ (1979).

The picture so far

$$
\begin{tikzcd}[column sep=small]
 & \mathcal{TRIV} \arrow[d, no head] & \\
 & \mathcal{CL} \arrow[dl, no head] \arrow[dr, no head] & \\
\mathcal{LP} \arrow[dr, no head] & & \mathcal{K} \arrow[dl, no head] \\
 & \mathcal{K}_{\leq} \arrow[d, no head] & \\
 & \mathcal{B} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathcal{TRIV} \arrow[d, dash] & \\
& \mathcal{CL} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{LP} \arrow[dr, dash] & & \mathcal{K} \arrow[dl, dash] \\
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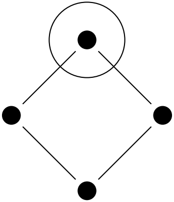
& \mathcal{B} &
\end{tikzcd}
$$

Until recently, these were the only known super-Belnap logics.

No coincidence: these are the only well-behaved super-Belnap logics.

Preserving exact truth: $\mathcal{ETL}$

Changing the designated values of $\mathbf{B_4}$ yields the matrix $\mathbf{ETL_4}$:



This logic extends $\mathcal{B}$ by the disjunctive syllogism: $p, -p \vee q \vdash q$.

This is the Exactly True Logic introduced by Pietz and Rivieccio (2013).

Ex contradictione quodlibet

Consider the logics $\mathcal{ECQ}_n$ extending $\mathcal{B}$ by the rules:

$$(p_1 \wedge -p_1) \vee \ldots \vee (p_n \wedge -p_n) \vdash \emptyset \qquad (ECQ_n)$$

Define $\mathcal{ETL}_n = \mathcal{ETL} \vee \mathcal{ECQ}_n$ ($\mathcal{ECQ} = \mathcal{ECQ}_1$ and $\mathcal{ETL} = \mathcal{ETL}_1$).

These form an infinite increasing chain (Rivieccio 2012):

$$\mathcal{ETL} \subsetneq \mathcal{ETL}_2 \subsetneq \ldots \subsetneq \mathcal{ETL}_\omega$$

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Contrary to popular opinion, $p, -p \vdash \emptyset$ is not *ex contradictione quodlibet*:

$\chi_2 = (p_1 \wedge -p_1) \vee (p_2 \wedge -p_2)$ is a contradiction, yet $\chi_2 \nvdash_{\mathcal{ECQ}} \emptyset$.

Explosive extensions

Explosive rules are rules of the form $\Gamma \vdash \emptyset$.

Explosive rules are dual to axiomatic rules of the form $\emptyset \vdash \varphi$.

Explosive extensions are extensions by explosive rules.

$\mathsf{Exp}_{\mathcal{B}}\,\mathcal{L}$ shall be the least explosive extension of $\mathcal{B}$ below $\mathcal{L}$.

$\mathsf{Exp}_{\mathcal{B}}$ takes $\mathcal{L}$ and forgets all the non-explosive rules.

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Examples:

$$\mathrm{Exp}_{\mathcal{B}}\,\mathcal{CL} = \mathcal{ECQ}_{\omega}$$
$$\mathrm{Exp}_{\mathcal{ETL}}\,\mathcal{CL} = \mathcal{ETL}_{\omega}$$
$$\mathrm{Exp}_{\mathcal{B}}\,\mathcal{ETL}_{n} = \mathcal{ECQ}_{n}$$

Some completeness theorems

The operator $\mathrm{Exp}_{\mathcal{B}}$ is useful for proving completeness:

$$\mathrm{Log}\, \Pi_{i\in I}\mathbf{A}_i = \bigcap_{i\in I} \mathrm{Log}\, \mathbf{A}_i \cup \bigcup_{i\in I} \mathrm{Exp}_{\mathcal{B}}\, \mathrm{Log}\, \mathbf{A}_i$$

We can now immediately compute:

$$\mathrm{Log}\, \mathbf{B_2} \times \mathbf{B_4} = (\mathcal{CL} \cap \mathcal{B}) \cup \mathrm{Exp}_{\mathcal{B}}\, \mathcal{CL} \cup \mathrm{Exp}_{\mathcal{B}}\, \mathcal{B} = \mathcal{B} \cup \mathcal{ECQ}_\omega \cup \mathcal{B} = \mathcal{ECQ}_\omega$$
$$\mathrm{Log}\, \mathbf{B_2} \times \mathbf{ETL_4} = (\mathcal{CL} \cap \mathcal{ETL}) \cup \mathrm{Exp}_{\mathcal{ETL}}\, \mathcal{CL} \cup \mathrm{Exp}_{\mathcal{ETL}}\, \mathcal{ETL} = \mathcal{ETL}_\omega$$
$$\mathrm{Log}\, \mathbf{ETL_4} \times \mathbf{B_4} = (\mathcal{ETL} \cap \mathcal{B}) \cup \mathrm{Exp}_{\mathcal{B}}\, \mathcal{ETL} \cup \mathrm{Exp}_{\mathcal{B}}\, \mathcal{B} = \mathcal{ECQ}$$

Sidenote: paraconsistent logics

Paraconsistent logics are usually understood as logics which do not satisfy *ex contradictione quodlibet*, understood as $p, -p \vdash \emptyset$.

If we reject this reading of ECQ, how do we understand paraconsistency?

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Proposal:

> A logic is paraconsistent if it has no non-trivial explosive extension.

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Is Łukasiewicz paraconsistent?

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Is Łukasiewicz paraconsistent?

After all, $(p_1 \wedge -p_1) \oplus (p_2 \wedge -p_2) \nvdash_{\text{Ł}} \emptyset$.

Lattices of super-Belnap logics

The lattice of (finitary) super-Belnap logics: $\operatorname{Ext}_{(\omega)} \mathcal{B}$

The lattice of (finitary) explosive extensions of $\mathcal{B}$: $\operatorname{Exp} \operatorname{Ext}_{(\omega)} \mathcal{B}$

$\operatorname{Exp}_{\mathcal{B}}$ is an interior operator on $\operatorname{Ext}_{\omega} \mathcal{B}$.

Proposition

$\operatorname{Exp} \operatorname{Ext}_{(\omega)} \mathcal{L}$ *is a distributive sublattice of* $\operatorname{Ext}_{(\omega)} \mathcal{L}$.

Proposition

$\operatorname{Ext}_{(\omega)} \mathcal{B}$ *is non-modular:* $(\mathcal{LP} \cap \mathcal{ETL}) \vee \mathcal{ECQ} < (\mathcal{LP} \vee \mathcal{ECQ}) \cap \mathcal{ETL}$.

The lattice Ext $\mathcal{B}$

$$
\begin{tikzcd}[column sep=small]
 & & \mathcal{CL} \arrow[dl, dash] \arrow[dr, dash] & & & & \\
 & \mathcal{LP} \vee \mathcal{ECQ} \arrow[dl, dash] \arrow[dr, dash] & & \mathcal{K} \arrow[dl, dash] \arrow[dr, dash] & & & \\
\mathcal{LP} \arrow[dr, dash] & & \mathcal{K}_{\leq} \vee \mathcal{ECQ} \arrow[dl, dash] \arrow[dr, dash] & & \mathcal{ETL}_{\omega} \arrow[dl, dash] \arrow[dr, dash] & & \\
 & \mathcal{K}_{\leq} \arrow[dr, dash] & & \mathcal{ECQ}_{\omega} \arrow[dl, dash] \arrow[dr, dash] & & \cdots \arrow[dr, dash] & \\
 & & \mathcal{LP} \cap \mathcal{ECQ}_{\omega} \arrow[dr, dash] & & \cdots \arrow[dr, dash] & & \mathcal{ETL} \arrow[dl, dash] \\
 & & & \cdots \arrow[dr, dash] & & \mathcal{ECQ} \arrow[dl, dash] & \\
 & & & & \mathcal{LP} \cap \mathcal{ECQ} \arrow[d, dash] & & \\
 & & & & \mathcal{B} & &
\end{tikzcd}
$$

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 & & \mathcal{CL} \arrow[dl, dash] \arrow[dr, dash] & & & & \\
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 & \mathcal{K}_{\leq} \arrow[dr, dash] & & \mathcal{ECQ}_{\omega} \arrow[dl, dash] \arrow[dr, dash] & & \cdots \arrow[dr, dash] & \\
 & & \mathcal{LP} \cap \mathcal{ECQ}_{\omega} \arrow[dr, dash] & & \cdots \arrow[dr, dash] & & \mathcal{ETL} \arrow[dl, dash] \\
 & & & \cdots \arrow[dr, dash] & & \mathcal{ECQ} \arrow[dl, dash] & \\
 & & & & \mathcal{LP} \cap \mathcal{ECQ} \arrow[d, dash] & & \\
 & & & & \mathcal{B} & &
\end{tikzcd}$$

$\mathcal{L} \supseteq \mathcal{LP} \cap \mathcal{ECQ}$ or $\mathcal{L} = \mathcal{B}$

The lattice Ext $\mathcal{B}$

$$
\begin{tikzcd}[column sep=small]
 & & \mathcal{CL} \arrow[dl, dash] \arrow[dr, dash] & & & & \\
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 & \mathcal{K}_{\leq} \arrow[dr, dash] & & \mathcal{ECQ}_{\omega} \arrow[dl, dash] \arrow[dr, dash] & & \cdots \arrow[dr, dash] & \\
 & & \mathcal{LP} \cap \mathcal{ECQ}_{\omega} \arrow[dr, dash] & & \cdots \arrow[dr, dash] & & \mathcal{ETL} \arrow[dl, dash] \\
 & & & \cdots \arrow[dr, dash] & & \mathcal{ECQ} \arrow[dl, dash] & \\
 & & & & \mathcal{LP} \cap \mathcal{ECQ} \arrow[d, dash] & & \\
 & & & & \mathcal{B} & &
\end{tikzcd}
$$

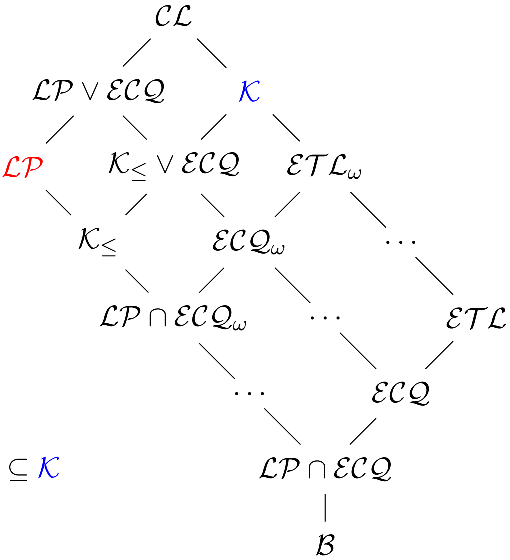
$\mathcal{L} \supseteq \mathcal{ECQ}$ or $\mathcal{L} \subseteq \mathcal{LP}$

The lattice Ext $\mathcal{B}$

$$
\begin{tikzcd}[column sep=tiny]
 & & \mathcal{CL} \arrow[dl, dash] \arrow[dr, dash] & & & & \\
 & \color{blue}{\mathcal{LP} \vee \mathcal{ECQ}} \arrow[dl, dash] \arrow[dr, dash] & & \mathcal{K} \arrow[dl, dash] \arrow[dr, dash] & & & \\
\mathcal{LP} \arrow[dr, dash] & & \mathcal{K}_{\leq} \vee \mathcal{ECQ} \arrow[dl, dash] \arrow[dr, dash] & & \mathcal{ETL}_{\omega} \arrow[dl, dash] \arrow[dr, dash] & & \\
 & \mathcal{K}_{\leq} \arrow[dr, dash] & & \mathcal{ECQ}_{\omega} \arrow[dl, dash] \arrow[dr, dash] & & \cdots \arrow[dr, dash] & \\
 & & \mathcal{LP} \cap \mathcal{ECQ}_{\omega} \arrow[dr, dash] & & \cdots \arrow[dr, dash] & & \color{red}{\mathcal{ETL}} \arrow[dl, dash] \\
 & & & \cdots \arrow[dr, dash] & & \mathcal{ECQ} \arrow[dl, dash] & \\
 & & & & \mathcal{LP} \cap \mathcal{ECQ} \arrow[d, dash] & & \\
 & & & & \mathcal{B} & &
\end{tikzcd}
$$

$\mathcal{L} \supseteq \color{red}{\mathcal{ETL}}$ or $\mathcal{L} \subseteq \color{blue}{\mathcal{LP} \vee \mathcal{ECQ}}$

The lattice Ext $\mathcal{B}$



$\mathcal{L} \supseteq \mathcal{LP}$ or $\mathcal{L} \subseteq \mathcal{K}$

The lattice $\mathrm{Ext}\,\mathcal{ETL}$

$$
\begin{tikzcd}[column sep=small]
 & & \mathcal{CL} \arrow[d, dash] & & & \\
 & & \mathcal{K} \arrow[d, dash] & & & \\
 & & \mathcal{ETL}_\omega^+ \arrow[dl, dash] \arrow[dr, dash] & & & \\
 & \mathcal{ETL}_\omega \arrow[dr, dash] & & \cdots \arrow[dr, dash] & & \\
 & & \cdots \arrow[dr, dash] & & \mathcal{ETL}_2^+ \arrow[dl, dash] \arrow[dr, dash] & \\
 & & & \mathcal{ETL}_2 \arrow[dr, dash] & & \mathcal{ETL}^+ \arrow[dl, dash] \\
 & & & & \mathcal{ETL} &
\end{tikzcd}
$$

$\mathcal{ETL}_n^+ : \chi_n \vee q, -q \vee r \vdash r$

$\chi_n = (p_1 \wedge -p_1) \vee \ldots \vee (p_n \wedge -p_n)$

The lattice Ext $\mathcal{ETL}$

$\mathcal{L} \supseteq \mathcal{K}$ or $\mathcal{L} \subseteq \mathcal{ETL}^+_\omega$

$$
\begin{tikzcd}[column sep=small]
 & \mathcal{CL} \arrow[d, dash] & & & \\
 & \mathcal{K} \arrow[d, dash] & & & \\
 & \mathcal{ETL}^+_\omega \arrow[dl, dash] \arrow[dr, dash] & & & \\
\mathcal{ETL}_\omega \arrow[dr, dash] & & \cdots \arrow[dr, dash] & & \\
 & \cdots \arrow[dr, dash] & & \mathcal{ETL}^+_2 \arrow[dl, dash] \arrow[dr, dash] & \\
 & & \mathcal{ETL}_2 \arrow[dr, dash] & & \mathcal{ETL}^+ \arrow[dl, dash] \\
 & & & \mathcal{ETL} &
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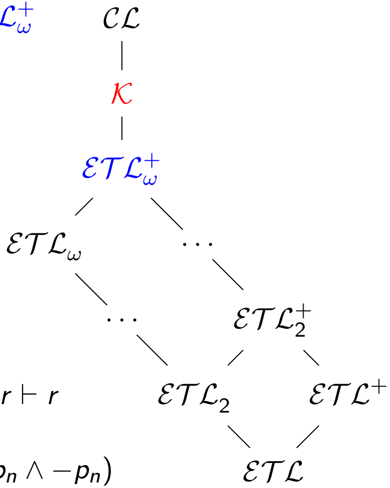

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$\chi_n = (p_1 \wedge -p_1) \vee \ldots \vee (p_n \wedge -p_n)$

Well-behaved super-Belnap logics are scarce

The following are natural properties for a logic to satisfy:

proof by cases:	$\varphi \vdash \chi \;\&\; \psi \vdash \chi \Rightarrow \varphi \vee \psi \vdash \chi$
contraposition:	$\varphi \vdash \psi \Rightarrow -\psi \vdash -\varphi$
self-extensionality:	$\varphi \dashv\vdash \psi \Rightarrow \chi(\varphi) \dashv\vdash \chi(\psi)$
protoalgebraicity:	$\varphi, \varphi \Rightarrow \psi \vdash \psi \;\&\; \emptyset \vdash \varphi \Rightarrow \varphi$

The following super-Belnap logics have these properties:

proof by cases:	$\mathcal{B}$, $\mathcal{K}_{\leq}$, $\mathcal{CL}$, $\mathcal{K}$, $\mathcal{LP}$
contraposition:	$\mathcal{B}$, $\mathcal{K}_{\leq}$, $\mathcal{CL}$
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self-extensionality:	$\mathcal{B}$, $\mathcal{K}_{\leq}$, $\mathcal{CL}$
protoalgebraicity:	$\mathcal{CL}$

Most super-Belnap logics are not very well-behaved.

Extensions of $\mathcal{ETL}$

Let us restrict our attention to $\mathrm{Ext}_\omega \mathcal{ETL}$ now. Rivieccio proved:

$$\mathrm{Mod}^* \mathcal{ETL} \subsetneq \{(\mathbf{A}, \{\top\}) \mid \mathbf{A} \in \mathrm{DMA}\} \subsetneq \mathrm{Mod}\, \mathcal{ETL}$$

Special quasiequation $= \top \approx \gamma_1 \ \& \ \dots \ \& \ \top \approx \gamma_n \Rightarrow \top \approx \varphi$

Special antiequation $= \top \not\approx \gamma$

Finitary extensions of $\mathcal{ETL}$ = special quasivarieties of DMAs

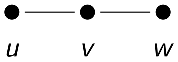
Finitary explosive extensions of $\mathcal{ETL}$ = special antivarieties of DMAs

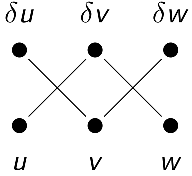
Our results about $\mathrm{Ext}_\omega \mathcal{ETL}$ will have algebraic corollaries.

From graphs to models of $\mathcal{ETL}$

For us, graphs will be finite undirected graphs, possibly with loops.

To each graph G, we assign the two matrices γG and μG.





$$G = (X, R)$$
$$-U = X \setminus R[U]$$

$$\mathcal{F} = (W, \leq \delta)$$
$$-U = W \setminus \delta[U]$$

$$\gamma G = (\mathcal{P}(X), \cap, \cup, X, \emptyset, -)$$
$$\mathcal{D} = \{X\}$$

$$\mu G = (\mathcal{P}_{\leq}(W), \cap, \cup, W, \emptyset, -)$$
$$\mathcal{D} = \{W\}$$

Example: ex contradictione and graph colourings

The finite reduced models of $\mathcal{ETL}$ are precisely the matrices $\mathbf{B_2}^n \times \boldsymbol{\mu} G$.

The matrices $\boldsymbol{\gamma} G$ are almost never models of $\mathcal{B}$, much less $\mathcal{ETL}$...

...nonetheless, $\boldsymbol{\gamma} G$ agrees with $\boldsymbol{\mu} G$ on $\Gamma \vdash \varphi$ for φ positive, Γ simple.

The language of the Belnap-Dunn logic can be used to talk about graphs.

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Example:

$$(p_1 \wedge -p_1) \vee \ldots \vee (p_n \wedge p_n) \nvdash_{\boldsymbol{\mu} G} \emptyset \Leftrightarrow G \text{ is } n\text{-colourable}$$

Proposition

The logics $\mathcal{ETL}_n$ ($n \geq 2$) are complete w.r.t. the class of matrices $\boldsymbol{\mu} G$ such that G is not n-colourable.

The homomorphism order on finite graphs

The class of finite graphs may be ordered by homomorphisms:

$$G \leq H \text{ if and only if there is a graph homomorphism } h : G \to H$$

This yields the homomorphism order on finite graphs.

Theorem

Each countable pre-order embeds into the hom order on finite graphs.

(Usually formulated in terms of partial orders instead of pre-orders.)

Restricting to loopless graphs $=$ dropping the top equivalence class

Restricting to graphs with edges $=$ dropping the bottom equivalence class

$\mathrm{Ext}_\omega \mathcal{ETL}$ and $\mathrm{Exp}\,\mathrm{Ext}_\omega \mathcal{ETL}$

Theorem

The interval $[\mathcal{ETL}, \mathcal{ETL}_\omega] \subseteq \mathrm{Ext}_\omega \mathcal{ETL}$ *is dually isomorphic to the lattice of classes of graphs closed under finite disjoint unions, surjective graph homomorphisms, and vertex reductions.*

Vertex reduction means contracting all the outgoing edges of a vertex.

Theorem

The lattice $\mathrm{Exp}\,\mathrm{Ext}_\omega \mathcal{ETL}$ *is dually isomorphic to the lattice of classes of graphs (with a non-empty set of edges) closed under homomorphisms.*

Corollary

There is a continuum of finitary explosive extensions of $\mathcal{ETL}$ *(and of* $\mathcal{B}$*). The lattice* $\mathrm{Ext}_\omega \mathcal{B}$ *contains infinite increasing and decreasing chains.*

Explosive extensions of $\mathcal{B}$

The lattice $\operatorname{Exp}\operatorname{Ext}_\omega \mathcal{B}$ is essentially the same as $\operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ETL}$.

$$\operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ETL} = \operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ECQ} = \mathcal{LP} \cap \operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ECQ}.$$

via $\mathcal{L} \mapsto \operatorname{Exp}_{\mathcal{B}} \mathcal{L}$ and $\mathcal{L} \mapsto \mathcal{ETL} \vee \mathcal{L}$
via $\mathcal{L} \mapsto \mathcal{LP} \cap \mathcal{L}$ and $\mathcal{L} \mapsto \mathcal{ECQ} \vee \mathcal{L}$

$$\operatorname{Exp}\operatorname{Ext}_\omega \mathcal{B} = \{\mathcal{B}\} \cup \operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ECQ} = \{\mathcal{B}\} \cup \operatorname{Exp}\operatorname{Ext}_\omega \mathcal{ETL}.$$

Each of the intervals $[\mathcal{B}, \mathcal{LP}]$, $[\mathcal{ECQ}, \mathcal{LP} \vee \mathcal{ECQ}]$ and $\operatorname{Ext}_\omega \mathcal{ETL}$ thus contains a distributive sublattice of the cardinality of the continuum.

Example: from graph theory to super-Belnap logics

The girth of a (loopless) graph is the length of its shortest cycle.

Theorem (Erdős, 1959)

The class of non-n-colourable (loopless) graphs has unbounded girth.

Corollary

The logics $\mathcal{ETL}_n$ are not complete w.r.t. a finite set of finite matrices.

Proof.

If it were, then the class of non-n-colourable graphs would be the closure of a finite set of graphs under disjoint unions, surjective homomorphisms, and vertex reductions. But these operations all preserve or decrease girth. □

Example: a non-finitary super-Belnap logic

For each graph H (with some edges), there is a formula φ_H such that:

$$\boldsymbol{\mu} G \text{ satisfies } \varphi_H \vdash \emptyset \Leftrightarrow G \nleq H.$$

Let $\mathcal{L}_{\mathsf{K}}$ extend $\mathcal{ETL}$ by the rule $\{\varphi_H \mid H \in \mathsf{K}\} \vdash \emptyset$. Then:

$$\boldsymbol{\mu} G \notin \operatorname{Mod} \mathcal{L}_{\mathsf{K}} \Leftrightarrow G \leq H \text{ for all } H \in \mathsf{K}.$$

Theorem

There is a non-finitary super-Belnap logic.

Proof.

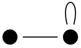
Consider a free countably generated meet-semilattice embedded into the hom order on finite graphs. Let K be its antichain of generators. If $\mathcal{L}_{\mathsf{K}}$ is finitary, then $\mathcal{L}_{\mathsf{K}} = \mathcal{L}_{\overline{\mathsf{K}}}$ for some finite $\overline{\mathsf{K}} \subseteq \mathsf{K}$. Thus $\operatorname{Mod} \mathcal{L}_{\mathsf{K}} = \operatorname{Mod} \mathcal{L}_{\overline{\mathsf{K}}}$, hence the finite meet of $\overline{\mathsf{K}}$ would have to be below each $H \in \mathsf{K}$. $\square$

Example: completeness of $\mathcal{ETL}^+_\omega$

Recall that $\mathcal{ETL}^+_n$ extends $\mathcal{B}$ by the rule $\chi_n \vee q, -q \vee r \vdash r$.

G is weakly n-colourable if there is a partial n-colouring which leaves some vertices U uncoloured such that not every vertex of G is a neighbour of U.

We may observe: $\boldsymbol{\mu} G \in \operatorname{Mod} \mathcal{ETL}^+_n \Leftrightarrow G$ not weakly n-colourable.

Let $G_2 =$  and let $\mathbf{ETL_8}$ be the 8-element matrix $\boldsymbol{\mu} G_2$.

Proposition

$\mathcal{ETL}^+_\omega = \operatorname{Log} \mathbf{ETL_8}$.

Proof.

Each graph which is not weakly n-colourable for any n is a surjective homomorphic image of a disjoint union of copies of G_2. $\square$

Conclusion

To sum up, $\mathrm{Ext}_\omega \mathcal{B}$ is a non-modular lattice of cardinality $2^{\aleph_0}$ which splits into three parts, namely $[\mathcal{B}, \mathcal{LP}]$, $[\mathcal{ECQ}, \mathcal{LP} \vee \mathcal{ECQ}]$, and $\mathrm{Ext}_\omega \mathcal{ETL}$.

Conclusion

To sum up, $\mathrm{Ext}_\omega \mathcal{B}$ is a non-modular lattice of cardinality $2^{\aleph_0}$ which splits into three parts, namely $[\mathcal{B}, \mathcal{LP}]$, $[\mathcal{ECQ}, \mathcal{LP} \vee \mathcal{ECQ}]$, and $\mathrm{Ext}_\omega \mathcal{ETL}$.

The notion of an explosive extension and the explosive part operator $\mathrm{Ext}_\mathcal{B}$ are useful in the study of super-Belnap logics.

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To sum up, $\mathrm{Ext}_\omega \mathcal{B}$ is a non-modular lattice of cardinality $2^{\aleph_0}$ which splits into three parts, namely $[\mathcal{B}, \mathcal{LP}]$, $[\mathcal{ECQ}, \mathcal{LP} \vee \mathcal{ECQ}]$, and $\mathrm{Ext}_\omega \mathcal{ETL}$.

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Perhaps somewhat surprisingly, graph-theoretic methods may be used to obtain non-trivial results about super-Belnap logics.

Thank you for your attention.