Single chain completeness and some related properties

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joint work with

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$\mathrm{FL_{ew}}$-algebras and MTL-algebras

An $\mathrm{FL_{ew}}$ algebra is an algebra $\langle A, *, \rightarrow, \wedge, \vee, 0, 1\rangle$. such that:

1. $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice with minimum 0 and maximum 1.
2. $\langle A, *, 1\rangle$ is a commutative monoid.
3. $\langle *, \rightarrow\rangle$ forms a *residuated pair*: $z * x \leq y$ iff $z \leq x \rightarrow y$ for all $x, y, z \in A$. In particular, it holds that $x \rightarrow y = \max\{z \in A : z * x \leq y\}$.
4. An MTL-algebra is an $\mathrm{FL_{ew}}$-algebra satisfying

(Prelinearity) $$(x \rightarrow y) \vee (y \rightarrow x) = 1.$$

- The class of MTL-algebras forms a variety, called $\mathbb{MTL}$. The logic corresponding to MTL-algebras is called MTL.
- An axiomatic extension of MTL is a logic obtained by adding other axioms to it.
- Every axiomatic extension of MTL is algebraizable in the sense of [Blok and Pigozzi, 1989], and hence every subvariety of $\mathbb{MTL}$ induces a logic.

Single chain completeness properties

Definition

Let L be an axiomatic extension of MTL. Then:

- L enjoys the *single chain completeness* (SCC) if there is an L-chain such that L is complete w.r.t. it.
- L enjoys the *finite strong single chain completeness* (FSSCC) if there is an L-chain such that L is finitely strongly complete w.r.t. it.
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Theorem ([Montagna, 2011])

Let L be an axiomatic extension of MTL. If L enjoys the FSSCC, then the SSCC holds.

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Let L be an axiomatic extension of MTL. If L enjoys the FSSCC, then the SSCC holds.

Problem ([Montagna, 2011])

Does the SCC implies the SSCC, in general?

Some logical properties

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- L has the *deductive Maksimova's variable separation property* (DMVP) if, for all sets of formulas $\Gamma \cup \{\varphi\}$ and $\Sigma \cup \{\psi\}$ that have no variables in common, $\Gamma, \Sigma \vdash_L \varphi \vee \psi$ implies $\Gamma \vdash_L \varphi$ or $\Sigma \vdash_L \psi$.

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Problem ([Galatos et al., 2007])

Find examples of substructural logics with DMVP, but without MVP.

The equivalence between HC and SCC

An $\mathrm{FL_{ew}}$-algebra is said to be well connected whenever for every pair of elements x, y, if $x \sqcup y = 1$, then $x = 1$ or $y = 1$.

Theorem ([Galatos et al., 2007, Theorem 5.28])

Let L be an axiomatic extension of FL_{ew}. The following are equivalent:

1. *L has the Halldén completeness.*
2. *L is complete w.r.t. a well-connected FL_{ew}-algebra.*
3. *L is meet irreducible (in the lattice of axiomatic extensions of FL_{ew}).*

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It is easy to check that an MTL-algebra is well-connected if and only if it is totally ordered. Hence we obtain the following result.

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It is easy to check that an MTL-algebra is well-connected if and only if it is totally ordered. Hence we obtain the following result.

Theorem

Let L be a logic over MTL. The following are equivalent:

1. *L has the Halldén completeness.*
2. *L is complete w.r.t. an MTL-chain, that is L enjoys the SCC.*
3. *L is meet irreducible (in the lattice of axiomatic extensions of MTL).*

DMVP and SSCC

Theorem ([Kihara, 2006, Theorem 6.9])

The following conditions are equivalent for every axiomatic extension L of MTL:

- *L has the DMVP.*
- *All pairs of subdirectly irreducible L-algebras are jointly embeddable into an L-chain.*
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We say that an axiomatic extension L of MTL has joint embedding property w.r.t. the class of L-chains (CJEP), if every pair of L-chains is jointly embeddable into an L-chain.

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Theorem ([Montagna, 2011])

Let L be an axiomatic extension of MTL. Then L enjoys the SSCC iff it has the CJEP.

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Theorem ([Montagna, 2011])

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Corollary

Every axiomatic extension of MTL enjoying the SSCC has also the DMVP.

An intermediate picture

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\begin{tikzcd}
SCC \arrow[r, leftrightarrow] & HC \arrow[r, shift left, "?"] & DMVP \arrow[l, shift left] \arrow[r, shift left, "?"] \arrow[dl, "?"] & CJEP \arrow[l, shift left] \arrow[r, leftrightarrow] & FSSCC \arrow[r, leftrightarrow] & SSCC \\
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1. Does exist an axiomatic extension of MTL having the DVMP, but for which the MVP fails?
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DMVP and MVP: a solution to the open problem

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Let L be an axiomatic extension of MTL not extending SMTL. Then the MVP fails, for L.

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The three-valued Łukasiewicz logic enjoys the DMVP, whilst the MVP fails.

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Let L be an axiomatic extension of MTL whose chains are isomorphic to ▸ ordinal sums *of bounded and subdirectly irreducible semihoops. Then the DMVP and the SSCC are* ▸ equivalent *for L.*

The SCC does not imply the SSCC: a counterexample - 1

Theorem ([Montagna, 2011])

Let L be an axiomatic extension of BL that is complete w.r.t. a finite chain. Then L enjoys the SSCC.

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Definition

Let us call $\mathcal{Q}$ the MTL-chain (WNM-chain) $\langle\{0, a, b, c, 1\}, *, \Rightarrow, \min, \max, 0, 1\rangle$, with $0 < a < b < c < 1$ and such that:

x	$\sim x$
0	1
a	c
b	a
c	a
1	0

$$x * y = \begin{cases} 0 & \text{if } x \leq \sim y \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$

Let us call $\mathbb{Q}$ the variety generated by $\mathcal{Q}$, and let Q be the corresponding logic.

The SCC does not imply the SSCC: a counterexample - 2

Proposition

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Proof (sketch).

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- Observe that $\mathcal{Q} \models \bigvee_{i<5}(x_i \to x_{i+1})$, and since $\mathbf{V}(\mathcal{Q}) = \mathbb{Q}$, by [Cintula et al., 2009, Proposition 4.18] every Q-chain cannot have more than five elements.

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- Notice that the three element Gödel-chain $\mathbf{G}_3$ belongs to $\mathbb{Q}$. Just take the quotient of $\mathcal{Q}$ over the filter $\{c, 1\}$.

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- Observe that $\mathbf{G}_3 \not\hookrightarrow \mathcal{Q}$.

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- Notice that the three element Gödel-chain $\mathbf{G}_3$ belongs to $\mathbb{Q}$. Just take the quotient of $\mathcal{Q}$ over the filter $\{c, 1\}$.
- Observe that $\mathbf{G}_3 \not\hookrightarrow \mathcal{Q}$.
- From these fact it is easy to check that there is no Q-chain $\mathcal{C}$ such that $\mathcal{Q} \hookrightarrow \mathcal{C}$ and $\mathbf{G}_3 \hookrightarrow \mathcal{C}$. Hence the DMVP fails.

□

The SCC does not imply the SSCC: a counterexample - 3

Theorem

- *For the axiomatic extensions of MTL, the HC does not necessarily imply the DMVP.*
- *For the axiomatic extensions of MTL, the SCC does not necessarily imply the SSCC (or the FSSCC).*

The SCC does not imply the SSCC: a counterexample - 3

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Interestingly, $\mathcal{Q}$ provides a counterexample of minimum cardinality.

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Interestingly, $\mathcal{Q}$ provides a counterexample of minimum cardinality.

Theorem

Let $\mathcal{A}$ be a non-trivial MTL-chain with less than five elements, and let $\mathbb{L} = \mathbf{V}(\mathcal{A})$. Then L enjoys the SSCC.

A general picture

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Extensions of MTL expanded with Δ

For every axiomatic extension L of MTL, we denote with L_Δ its expansion with an operator Δ satisfying the following axioms:

$(\Delta 1)$	$\Delta(\varphi) \vee \neg\Delta(\varphi)$.
$(\Delta 2)$	$\Delta(\varphi \vee \psi) \to (\Delta(\varphi) \vee \Delta(\psi))$.
$(\Delta 3)$	$\Delta(\varphi) \to \varphi$.
$(\Delta 4)$	$\Delta(\varphi) \to \Delta(\Delta(\varphi))$.
$(\Delta 5)$	$\Delta(\varphi \to \psi) \to (\Delta(\varphi) \to \Delta(\psi))$.

and the following additional inference rule: $\frac{\varphi}{\Delta\varphi}$.
An MTL_Δ-chain is an MTL-chain expanded with an operation δ, interpreting Δ, such that, for every element x, $\delta(x) = 1$ if $x = 1$, whilst $\delta(x) = 0$ if $x < 1$.

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(Δ4) $\Delta(\varphi) \to \Delta(\Delta(\varphi))$.

(Δ5) $\Delta(\varphi \to \psi) \to (\Delta(\varphi) \to \Delta(\psi))$.

and the following additional inference rule: $\frac{\varphi}{\Delta\varphi}$.
An MTL_Δ-chain is an MTL-chain expanded with an operation δ, interpreting Δ, such that, for every element x, $\delta(x) = 1$ if $x = 1$, whilst $\delta(x) = 0$ if $x < 1$.

We can define the notions of SCC, FSSCC and SSCC in a similar way to the MTL case.

For every axiomatic extension L of MTL, we denote with L_Δ its expansion with an operator Δ satisfying the following axioms:

(Δ1)	$\Delta(\varphi) \vee \neg\Delta(\varphi)$.
(Δ2)	$\Delta(\varphi \vee \psi) \to (\Delta(\varphi) \vee \Delta(\psi))$.
(Δ3)	$\Delta(\varphi) \to \varphi$.
(Δ4)	$\Delta(\varphi) \to \Delta(\Delta(\varphi))$.
(Δ5)	$\Delta(\varphi \to \psi) \to (\Delta(\varphi) \to \Delta(\psi))$.

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An MTL_Δ-chain is an MTL-chain expanded with an operation δ, interpreting Δ, such that, for every element x, $\delta(x) = 1$ if $x = 1$, whilst $\delta(x) = 0$ if $x < 1$.

We can define the notions of SCC, FSSCC and SSCC in a similar way to the MTL case.

Theorem

Let L be an axiomatic extension of MTL. Then the SCC, FSSCC and SSCC are equivalent, for L_Δ and $L_\Delta\forall$.

Open problems

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- Do MTL and IMTL enjoy the SCC?
- How about the first-order case? Are there some logical properties characterizing the SCC and the SSCC? Note that the SCC for an axiomatic extension L of MTL do not necessarily implies the SCC for L$\forall$. As shown in [Montagna, 2011] BL is a counterexample.

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APPENDIX

Definition

A *semihoop* is a structure $\mathcal{A} = \langle A, *, \sqcap, \Rightarrow, 1\rangle$ such that $\langle A, \sqcap, 1\rangle$ is an inf-semilattice with upper bound 1, $*$ is a binary operation on A with unit 1, and $\Rightarrow$ is a binary operation such that:

- $x \leq y$ iff $x \Rightarrow y = 1$,
- $(x * y) \Rightarrow z = x \Rightarrow (y \Rightarrow z)$.

A *bounded* semihoop is a semihoop with a minimum element; conversely, an *unbounded* hoop is a hoop without minimum.

- A hoop is a semihoop satisfying $x * (x \Rightarrow y) = y * (y \Rightarrow x)$.
- A Wajsberg hoop is a hoop satisfying $x \Rightarrow (x \Rightarrow y) = y \Rightarrow (y \Rightarrow x)$.

back

- Let $\langle I, \leq \rangle$ be a totally ordered set with minimum 0. For all $i \in I$, let $\mathcal{A}_i$ be a totally ordered semihoop such that for $i \neq j$, $A_i \cap A_j = \{1\}$, and assume that $\mathcal{A}_0$ is bounded.
- Then $\bigoplus_{i \in I} \mathcal{A}_i$ (the *ordinal sum* of the family $(\mathcal{A}_i)_{i \in I}$) is the structure whose base set is $\bigcup_{i \in I} A_i$, whose bottom is the minimum of $\mathcal{A}_0$, whose top is 1, and whose operations are

$$x \to y = \begin{cases} x \to^{\mathcal{A}_i} y & \text{if } x, y \in A_i \\ y & \text{if } \exists i > j (x \in A_i \text{ and } y \in A_j) \\ 1 & \text{if } \exists i < j (x \in A_i \setminus \{1\} \text{ and } y \in A_j) \end{cases}$$

$$x * y = \begin{cases} x *^{\mathcal{A}_i} y & \text{if } x, y \in A_i \\ x & \text{if } \exists i < j (x \in A_i \setminus \{1\},\ y \in A_j) \\ y & \text{if } \exists i < j (y \in A_i \setminus \{1\}, x \in A_j) \end{cases}$$

- As a consequence, if $x \in A_i \setminus \{1\}$, $y \in A_j$ and $i < j$ then $x < y$.

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Definition

An axiomatic extension L of MTL is said to be n-contractive ($n \geq 2$), whenever $L \vdash \varphi^{n-1} \to \varphi^{n}$.

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Theorem ([Bianchi and Montagna, 2011])

Every n-contractive BL-chain is isomorphic to an ordinal sum of finite MV-chains, each of them having at most n elements.

back

Proof.

We already know that the SSCC implies the DMVP, so assume that L has the DMVP. Let $\mathcal{A}$, $\mathcal{B}$ be two L-chains: we show that the CJEP (and hence the SSCC) holds.

back

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Axiomatization of MTL

The basic connective are $\{\wedge, \&, \rightarrow, \bot\}$ (formulas built inductively: a theory is a set of formulas). Useful derived connectives are the following ones:

(negation)	$\neg\varphi \stackrel{\text{def}}{=} \varphi \rightarrow \bot$
(disjunction)	$\varphi \vee \psi \stackrel{\text{def}}{=} ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$
(top)	$\top \stackrel{\text{def}}{=} \neg\bot$

MTL can be axiomatized by using these axioms and modus ponens: $\frac{\varphi \quad \varphi\rightarrow\psi}{\psi}$.

(A1)	$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
(A2)	$(\varphi \& \psi) \rightarrow \varphi$
(A3)	$(\varphi \& \psi) \rightarrow (\psi \& \varphi)$
(A4)	$(\varphi \wedge \psi) \rightarrow \varphi$
(A5)	$(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$
(A6)	$(\varphi \& (\varphi \rightarrow \psi)) \rightarrow (\psi \wedge \varphi)$
(A7a)	$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \& \psi) \rightarrow \chi)$
(A7b)	$((\varphi \& \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
(A8)	$((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
(A9)	$\bot \rightarrow \varphi$

Sketch of the proof.

If L has the SSCC, then the DMVP holds. Assume that L has the DMVP, and let $\mathcal{A}$ is an L chain.

back

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If L has the SSCC, then the DMVP holds. Assume that L has the DMVP, and let $\mathcal{A}$ is an L chain. If $\mathcal{A}$ has a last component, then it is subdirectly irreducible, since every component is subdirectly irreducible by hypothesis.

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If L has the SSCC, then the DMVP holds. Assume that L has the DMVP, and let $\mathcal{A}$ is an L chain. If $\mathcal{A}$ has a last component, then it is subdirectly irreducible, since every component is subdirectly irreducible by hypothesis. If $\mathcal{A}$ does not have a last component, then it is infinite, it can be shown that $\mathcal{A} \oplus \mathbf{2}$ is an L-chain.

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DMVP and SSCC, bounded semihoop case

Sketch of the proof.

If L has the SSCC, then the DMVP holds. Assume that L has the DMVP, and let $\mathcal{A}$ is an L chain. If $\mathcal{A}$ has a last component, then it is subdirectly irreducible, since every component is subdirectly irreducible by hypothesis. If $\mathcal{A}$ does not have a last component, then it is infinite, it can be shown that $\mathcal{A} \oplus \mathbf{2}$ is an L-chain. In particular, since $\mathcal{A} \hookrightarrow \mathcal{A} \oplus \mathbf{2}$ it is sufficient to show that if $\mathcal{A} \oplus \mathbf{2} \not\models \varphi$, then $\mathcal{A} \not\models \varphi$. So, an L-chain is subdirectly irreducible or it can be embedded into a subdirectly irreducible L-chain. Since by the DMVP every pair of subdirectly irreducible L-algebras can be embedded into some L-chain, we have that L enjoys the CJEP and the SSCC. □

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