A Hennessy-Milner Property for Many-Valued Modal Logics

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Overview

MTL-Chains

Definition

An **MTL-chain** is an algebraic structure

$$\mathbf{A} = \langle A, \wedge, \vee, \cdot, \to, \bot, \top \rangle$$

satisfying

1. $\langle A, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice where $a \le b \Leftrightarrow a \wedge b = a$.
2. $\langle A, \cdot, \top \rangle$ is a commutative monoid.
3. $a \cdot b \le c$ if and only if $a \le b \to c$ for all $a, b, c \in A$
4. $\le$ is a linear order on A

$\mathbf{A}$ is **complete** iff $\bigwedge B$ and $\bigvee B$ exist in A for all $B \subseteq A$.

t-Norms

If the universe of **A** is the real unit interval $[0, 1]$, then the monoidal operation $\cdot$ is a **t-norm** with unit 1 and residual $\to$.

1. Łukasiewicz logic, $\cdot$ is the Łukasiewicz t-norm $\max(0, x + y - 1)$
2. Gödel logic, $\cdot$ is the minimum t-norm $\min(x, y)$
3. product logic, $\cdot$ is the product t-norm xy (multiplication)

Language and Formulas

Our language consists of

- constants $\bot$, $\top$
- a fixed countably infinite set Var of (propositional) variables, denoted $p, q, \ldots$
- binary connectives $\rightarrow$, $\wedge$, $\vee$, $\cdot$
- unary (modal) connectives $\Box$ and $\Diamond$

We define negation by $\neg\varphi := \varphi \rightarrow \bot$.
The set of **formulas** of this language is denoted by $\mathrm{Fm}_{\Box\Diamond}$, with arbitrary members denoted $\varphi, \psi, \chi, \ldots$. We also denote the set of (purely) **propositional formulas** by Fm.

Semantics

Definition

- A **(crisp) frame** is a pair $\langle W, R\rangle$ where $W \neq \emptyset$ is a set of **states** and $R \subseteq W \times W$ is a binary **accessibility relation**.
- For a complete MTL-chain **A**, a $\mathrm{K}(\mathbf{A})^{\mathsf{C}}$**-model** is a triple $\mathfrak{M} = \langle W, R, V\rangle$ where $\langle W, R\rangle$ is a frame and $V\colon \mathrm{Var} \times \mathrm{W} \to \mathrm{A}$ is a mapping, called a **valuation**. The valuation V is extended to $V\colon \mathrm{Fm}_{\Box\Diamond} \times \mathrm{W} \to \mathrm{A}$ by

$$
\begin{aligned}
V(\bot, w) = \bot, \quad V(\top, w) &= \top \\
V(\varphi \star \psi, w) &= V(\varphi, w) \star V(\psi, w) \qquad \text{for } \star \in \{\wedge, \vee, \to, \cdot\} \\
V(\Box\varphi, w) &= \bigwedge\{V(\varphi, v) : Rwv\} \\
V(\Diamond\varphi, w) &= \bigvee\{V(\varphi, v) : Rwv\}
\end{aligned}
$$

Modal Equivalence

Definition

Let $\mathbf{A}$ be a complete MTL-chain, $\mathfrak{M} = \langle W, R, V \rangle$ and $\mathfrak{M}' = \langle W', R', V' \rangle$ two $\mathrm{K}(\mathbf{A})^{\mathrm{C}}$-models.
$w \in W$ and $w' \in W'$ are **modally equivalent**, written $w \leftrightsquigarrow w'$, iff

$$V(\varphi, w) = V'(\varphi, w') \text{ for all } \varphi \in \mathrm{Fm}_{\Box\Diamond}$$

Bisimulation

Definition

A non-empty binary relation $Z \subseteq W \times W'$ is a **bisimulation** between $\mathfrak{M}$ and $\mathfrak{M}'$ iff the following conditions are satisfied:

1. If wZw', then $V(p, w) = V'(p, w')$ for all $p \in \mathrm{Var}$.
2. If wZw' and Rwv, then there exists $v' \in W'$ such that vZv' and $R'w'v'$ (the forth condition).
3. If wZw' and $R'w'v'$, then there exists $v \in W$ such that vZv' and Rwv (the back condition).

We say that $w \in W$ and $w' \in W'$ are **bisimilar**, written $w \equiv w'$, iff there exists a bisimulation Z between $\mathfrak{M}$ and $\mathfrak{M}'$ such that wZw'.

Bisimilarity implies Modal Equivalence

Lemma

Let $\mathfrak{M} = \langle W, R, V \rangle$ *and* $\mathfrak{M}' = \langle W', R', V' \rangle$ *be* $\mathrm{K}(\mathbf{A})^{\mathsf{C}}$*-models. If* $w \in W$ *and* $w' \in W'$ *are bisimilar, then they are modally equivalent.*

The Hennessy-Milner Property

Definition

A class $\mathcal{K}$ of $\mathrm{K}(\mathbf{A})^{\mathrm{C}}$-models has the **Hennessy-Milner property** if for any models $\mathfrak{M} = \langle W, R, V \rangle$ and $\mathfrak{M}' = \langle W', R', V' \rangle$ in $\mathcal{K}$, whenever the states $w \in W$ and $w' \in W'$ are modally equivalent, they are bisimilar.

Definition

A $\mathrm{K}(\mathbf{A})^{\mathrm{C}}$-model is **image-finite** iff $R[w] = \{v \in W : Rwv\}$ is finite for each $w \in W$.

The Hennessy-Milner Property for Image-Finite Models

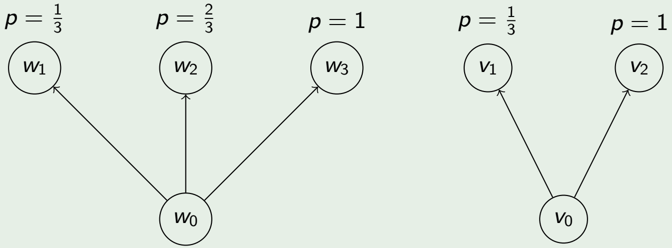
Question: For which $\mathbf{A}$ does the class of image-finite $\mathbf{K}(\mathbf{A})^{\mathbf{C}}$-models have the Hennessy-Milner property?

Failure of H.M. property

Example

Consider the four-valued Gödel modal logic $\mathrm{K}(\mathbf{G_4})^{\mathrm{C}}$ where

$$\mathbf{G_4} = \langle\{0, \tfrac{1}{3}, \tfrac{2}{3}, 1\}, \min, \max, \min, \to_{\mathrm{G}}, 0, 1\rangle$$



w_0 and v_0 are modally equivalent, but not bisimilar.

Failure of H.M. property

So the class of image-finite $\mathrm{K}(\mathbf{G_4})^{\mathrm{C}}$-models does not have the Hennessy-Milner property.

A Sufficient Condition

Lemma

Suppose that for any distinct $a, b \in A$, there is a one-variable propositional formula $\psi_{a,b}(p) \in \mathrm{Fm}$ such that $\psi_{a,b}[a] = \top$ and $\psi_{a,b}[b] \neq \top$. Then the class of image-finite $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$-models has the Hennessy-Milner property.

Example: Three-Valued Łukasiewicz Logic

Example

Consider the algebra $\mathbf{Ł_3}$ for three-valued Łukasiewicz logic. We can distinguish values using the following formulas:

$$\psi_{1,0} = (p \leftrightarrow \top), \qquad \psi_{1,\frac{1}{2}} = (p \cdot p), \qquad \psi_{\frac{1}{2},0} = (\neg p \to p)$$

$$\psi_{0,1} = (p \leftrightarrow \bot), \qquad \psi_{\frac{1}{2},1} = (p \to \neg p), \qquad \psi_{0,\frac{1}{2}} = (\neg p \cdot \neg p).$$

By the previous lemma, the class of image-finite $\mathrm{K}(\mathbf{Ł_3})^{\mathrm{C}}$-models has the Hennessy-Milner property.

A Necessary and Sufficient Condition

Definition

- Let $\vec{a} \in A^n$ and $\vec{C} = (\vec{c}_1, \ldots, \vec{c}_n) \in A^{n \times n}$. A formula $\psi(p_1, \ldots, p_n) \in \mathrm{Fm}$ is an $\vec{a}/\vec{C}$**-distinguishing formula** if

$$\psi[\vec{a}] > \bigvee_{i=1}^{n} \psi[\vec{c}_i] \quad \text{or} \quad \psi[\vec{a}] < \bigwedge_{i=1}^{n} \psi[\vec{c}_i]$$

- **A** has the **distinguishing formula property** if for all $n \in \mathbb{N}$, $\vec{a} \in A^n$, and $\vec{C} = (\vec{c}_1, \ldots, \vec{c}_n) \in A^{n \times n}$ such that $\vec{a} \neq \vec{c}_i$ for $i \in \{1, \ldots, n\}$, there is an $\vec{a}/\vec{C}$-distinguishing formula.

Characterization Theorem

Theorem

The following are equivalent for any complete MTL-chain $\mathbf{A}$*:*

(1) *The class of image-finite* $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$*-models has the Hennessy-Milner property.*

(2) $\mathbf{A}$ *has the distinguishing formula property.*

BL-Chains

Definition

An MTL-chain is a **BL-chain** iff $a \wedge b = a \cdot (a \to b) = b \cdot (b \to a)$ for all $a, b \in A$ (divisibility).

Remark

- In the case where $A = [0,1]$, the monoidal operation $\cdot$ is a **continuous t-norm** and **A** is called a **standard BL-chain**.
- for all $a, b \in A$:

$$a \wedge b = a \cdot (a \to b)$$
$$a \vee b = ((a \to b) \to b) \wedge ((b \to a) \to a)$$

$\wedge, \vee$ can therefore be dropped from our language.

MV-Chains

Definition

An **MV-chain** is a BL-chain satisfying $\neg\neg a = a$ for all $a \in A$.

Consider the MV-chains

$$\begin{aligned} \mathbf{Ł_{n+1}} &= \langle \{0, \tfrac{1}{n}, \ldots, \tfrac{n-1}{n}, 1\}, \cdot_{Ł}, \to_{Ł}, 0, 1 \rangle \quad (n \in \mathbb{Z}^+) \\ \mathbf{Ł}_\infty &= \langle [0,1], \cdot_{Ł}, \to_{Ł}, 0, 1 \rangle \end{aligned}$$

where $x \cdot_{Ł} y = \min(1, x + y - 1)$ and $x \to_{Ł} y = \max(0, 1 - x + y)$. Every finite MV-chain $\mathbf{A}$ is isomorphic to $\mathbf{Ł}_{|A|}$ and every standard MV-chain is isomorphic to $\mathbf{Ł}_\infty$.

In general, in the algebras of Łukasiewicz logic, we may distinguish between rational values in $[0,1]$ using unary **McNaughton functions**:

Lemma

For each $\alpha \in \mathbb{Z}^+ \cup \{\infty\}$, the class of image-finite $\mathsf{K}(\mathbf{Ł}_\alpha)^{\mathsf{C}}$-models has the Hennessy-Milner property.

Hoops

Definition

A **hoop** is an algebraic structure $\mathbf{H} = \langle H, \cdot, \to, \top \rangle$ such that $\langle H, \cdot, \top \rangle$ is a commutative monoid and for all $a, b, c \in H$:

1. $a \to a = \top$.
2. $a \cdot (a \to b) = b \cdot (b \to a)$.
3. $a \to (b \to c) = (a \cdot b) \to c$.

$a \leq b :\Leftrightarrow a \to b = \top$.
If $\leq$ is linear, then $\mathbf{H}$ is a **linearly ordered hoop** (**o-hoop**).
An o-hoop is **standard** if $H = [0, 1]$.

Hoops

Definition

- The o-hoop **A** is called **cancellative** iff $a \to (a \cdot b) = b$ for all $a, b \in A$
- **A** is the **hoop reduct** of an MV-chain $\langle A, \cdot, \to, \bot, \top \rangle$ if $\mathbf{A} = \langle A, \cdot, \to, \top \rangle$.

Ordinal Sum of o-hoops

Definition

Let I be a linearly ordered set with bottom element i_0 and suppose that $\mathbf{A}_i = \langle A_i, \cdot_i, \to_i, \top \rangle$ is a non-trivial o-hoop for each $i \in I$. Suppose that $A_i \cap A_j = \{\top\}$ for $i \neq j$ and that $\mathbf{A}_{i_0}$ has a bottom element $\bot$. Then the **(bounded) ordinal sum** of $(\mathbf{A}_i)_{i \in I}$ is defined as

$$\bigoplus_{i \in I} \mathbf{A}_i = \langle \bigcup_{i \in I} A_i, \cdot, \to, \bot, \top \rangle$$

$$x \cdot y \quad = \quad \begin{cases} x \cdot_i y & \text{if } x, y \in A_i \\ x & \text{if } x \in A_i \setminus \{\top\},\ y \in A_j, \text{ and } i < j \\ y & \text{if } y \in A_i \setminus \{\top\},\ x \in A_j, \text{ and } i < j \end{cases}$$

and $\to$ is the residual.

The Structure of BL-Chains

Theorem (Aglianò and Montagna)

Every non-trivial BL-chain is the unique ordinal sum of a family of o-hoops each of which is either the hoop reduct of an MV-chain or a cancellative o-hoop.

H.M. Property

Lemma

Suppose that $\mathbf{A}$ *is the ordinal sum of a family of (non-trivial) o-hoops* $(\mathbf{A}_i)_{i\in I}$. *If* $|I| \geq 3$ *or* $\mathbf{A}_i$ *is cancellative for some* $i \in I$, *then the class of image-finite* $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$*-models does not have the Hennessy-Milner property.*

Lemma

Let $\mathbf{A} = \mathbf{Ł}^{\mathbf{h}}_{\alpha} \oplus \mathbf{Ł}^{\mathbf{h}}_{\beta}$ *with* $\alpha, \beta \in \mathbb{Z}^+ \cup \{\infty\}$. *Then the class of image-finite* $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$*-models has the Hennessy-Milner property.*

Characterization Theorems

Theorem

The following are equivalent for any finite BL-chain $\mathbf{A}$*:*

(1) *The class of image-finite* $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$*-models has the Hennessy-Milner property.*

(2) $\mathbf{A}$ *is isomorphic to* $\mathbf{Ł_{n+1}}$ *or* $\mathbf{Ł^h_{n+1}} \oplus \mathbf{Ł^h_{m+1}}$ *for some* $m, n \in \mathbb{N}$*.*

Theorem

The following are equivalent for any standard BL-chain $\mathbf{A}$*:*

(1) *The class of image-finite* $\mathsf{K}(\mathbf{A})^{\mathsf{C}}$*-models has the Hennessy-Milner property.*

(2) $\mathbf{A}$ *is isomorphic to* $\mathbf{Ł}_\infty$ *or* $\mathbf{Ł}^{\mathbf{h}}_\infty \oplus \mathbf{Ł}^{\mathbf{h}}_\infty$*.*

Thank you for your attention!