Geometric description of projective MV-algebras

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ManyVal – 2015

Preliminaries

Projective Algebras

Let $\mathcal{K}$ be a class of algebras. An algebra $A \in \mathcal{K}$ is **projective** if:

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A \arrow[dr] & \\
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Preliminaries

If $\mathcal{V}$ is a variety (equational class) of algebras then $A \in \mathcal{V}$ is projective iff there exists a cardinal κ such that

$$\begin{tikzcd} A \arrow[rr, hook, bend right] & & \mathbf{Free}_{\mathcal{V}}(\kappa) \arrow[ll, two heads, bend right] \end{tikzcd}$$

Preliminaries
MV-algebras

An **MV-algebra** is an algebraic structure $(M, \oplus, \neg, 0)$ where:

- $(M, \oplus, 0)$ is a commutative monoid,
- $\neg$ is a unary operation,
- $\neg$ and $\oplus$ satisfy the following
  - $\neg\neg x = x$,
  - $x \oplus \neg 0 = \neg 0$, and
  - $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

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Standard example: $[0,1]_{MV} = ([0,1], \oplus, \neg, 0)$ where

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Standard example: $[0,1]_{MV} = ([0,1], \oplus, \neg, 0)$ where

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Chang's completeness theorem states that $[0,1]_{MV}$ generates the variety of MV-algebras.

Main Result

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Theorem

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  (i) *P is contractible,*

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  (iii) *for each $v \in P \cap \mathbb{Q}^n$ there exist $w \in \mathbb{Z}^n$ and $\varepsilon > 0$ such that the convex segment $\mathrm{conv}(v, v + \varepsilon(w - v))$ is contained in P ($\Leftrightarrow$ P is strongly regular $\Leftrightarrow$ P is anchored), and*

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  (iv) *$A \cong \{f \restriction_P \mid f\colon [0,1]^n \to [0,1] \text{ is a McNaughton map}\}$.*

Main Result
Duality

A **rational polyhedron** $P \subseteq [0,1]^n$ is the pointset union of finitely many convex set with rational vertices.

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That is, there exists a finite set $K \subseteq [0,1]^n \cap \mathbb{Q}^n$ and a $L \subseteq \mathcal{P}(K)$ such that

$$P = \bigcup \{\mathrm{conv}(S) \mid S \in L\}.$$

Main Result

Duality

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Theorem

Let $P \subseteq [0,1]^n$. Then TFAE:

(i) *P is a rational polyhedron,*

(ii) *$P = f^{-1}(1)$ for some $f \in \mathcal{M}([0,1]^n)$.*

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Theorem

Let $f \in \mathcal{M}([0,1]^n)$ and $P = f^{-1}(1)$, then

$$\mathcal{M}([0,1]^n)/\langle f \approx 1\rangle \cong \mathcal{M}(P) = \{g \restriction_P \mid g \in \mathcal{M}([0,1]^n)\}$$

Main Result

Duality

Theorem (Marra, Spada – 2012)

The category of finitely presented MV-algebras is dually equivalent to the category of rational polyhedra with $\mathbb{Z}$*-maps.*

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Definition

A map $\eta\colon [0,1]^n \to [0,1]^m$ is called a **$\mathbb{Z}$-map** if it satisfies the following conditions:

(i) η is continuous,

(ii) piecewise (affine) linear, and each linear piece of η has integer coefficients.

Main Result

Dual of Projective

$$
\begin{tikzcd}
A \arrow[rr, hook, bend right] & & \mathbf{Free}_{MV}(n) \arrow[ll, two heads, bend right]
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Main Result

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$$\begin{tikzcd} A \arrow[rr, bend right, hook] & & \mathbf{Free}_{MV}(n) \arrow[ll, bend right, two heads] \end{tikzcd}$$

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L.M. Cabrer

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Projective Algebras
MV-algebras

Main Result
Duality
Previous Results
Sketch of the proof

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Dual of Projective

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\begin{tikzcd}
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$$
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\end{tikzcd}
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$$
\begin{tikzcd}
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Main Result

Dual Problem

Presented with $P \subseteq [0,1]^n$, find necessary and sufficient conditions for P to be a $\mathbb{Z}$-retract of $[0,1]^n$.

Main Result
Previous Results

Theorem (C., Mundici - 2011)

Let A be a finitely generated and projective MV-algebra. Then there exist $n \in \{1, 2, \ldots\}$ and a rational polyhedron $P \subseteq [0, 1]^n$ such that

(i) *P is contractible,*

(ii) *$P \cap \{0, 1\}^n \neq \emptyset$,*

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(iii) **strongly regular:** *P has a regular triangulation Δ such that for every maximal simplex $T \in \Delta$, the denominators of the vertices of T are coprime, and*

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Regular Simplexes

For v in $\mathbb{Q}^n$ we let **den**(v) denote the least common denominator of the coordinates of v.

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The vector $\widetilde{v} = \mathrm{den}(v)(v, 1) \in \mathbb{Z}^{n+1}$ is called the **homogeneous correspondent** of v.

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For v in $\mathbb{Q}^n$ we let **den(v)** denote the least common denominator of the coordinates of v.

The vector $\widetilde{v} = \mathrm{den}(v)(v, 1) \in \mathbb{Z}^{n+1}$ is called the **homogeneous correspondent** of v.

A simplex $S \subseteq [0, 1]^n$ is called **regular** if the set of homogeneous correspondents of its vertices is part of a basis of the free Abelian group $\mathbb{Z}^{n+1}$.

Theorem (C., Mundici - 2011)

Let $P \subseteq [0,1]^n$ be a rational polyhedron such that

(i) *P is collapsible,*

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Then $\mathcal{M}(P)$ is projective.

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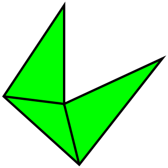
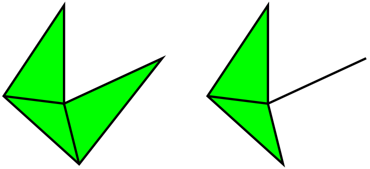
Then $\mathcal{M}(P)$ is projective.



Main Result
Previous Results



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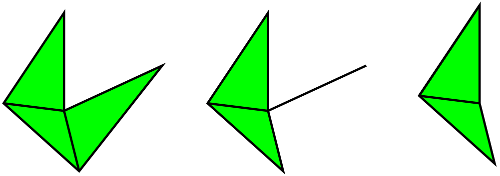

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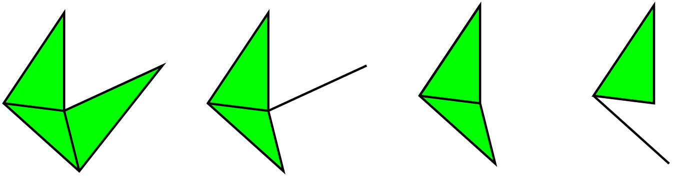

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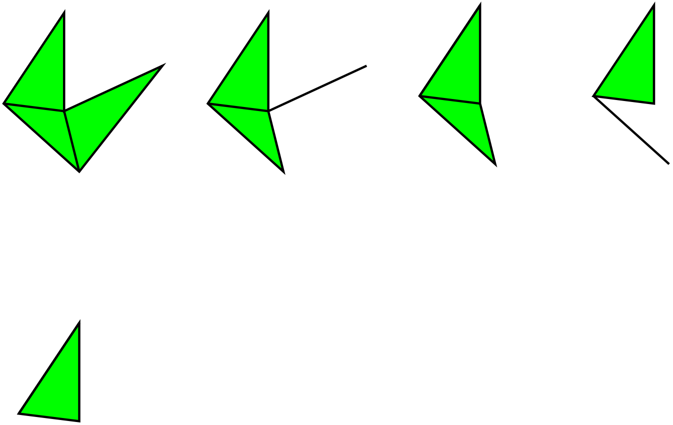

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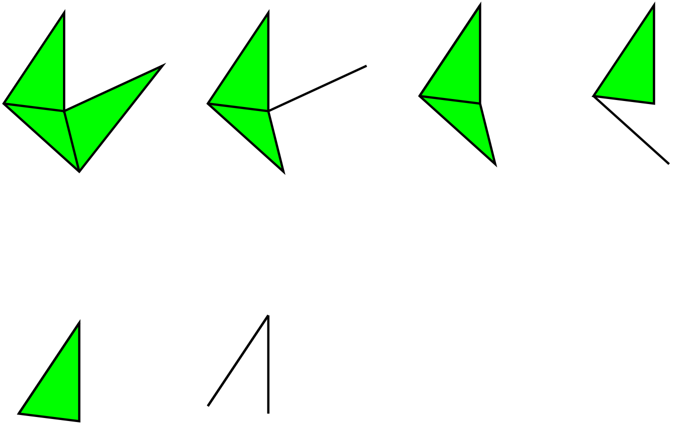

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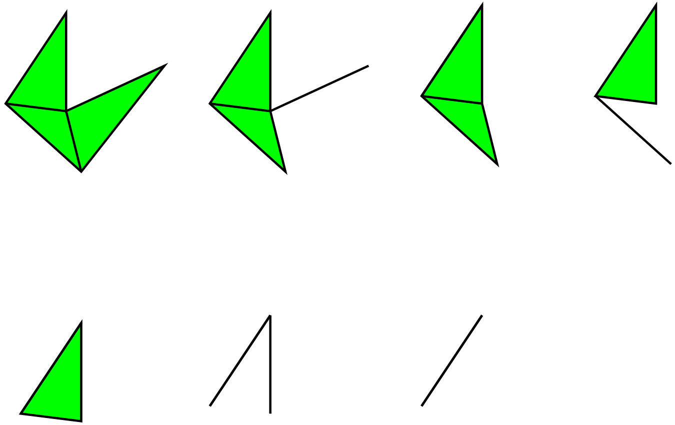

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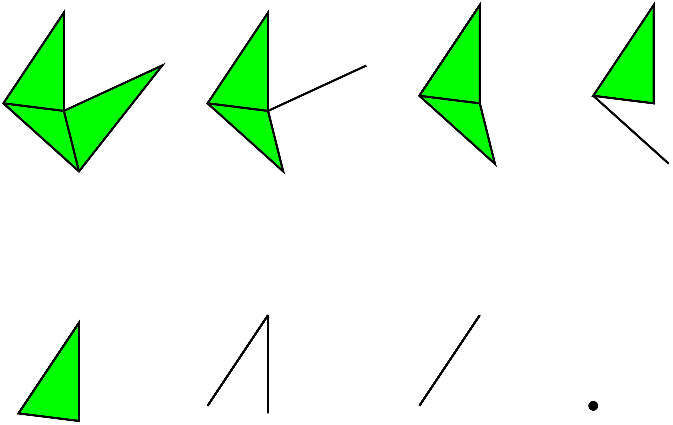

Theorem

Let A be an MV-algebra. Then TFAE:

(a) *A is finitely generated and projective,*

(b) *there exist $n \in \{1, 2, \ldots\}$ and a rational polyhedron $P \subseteq [0,1]^n$ such that*
   - (i) *P is contractible,*
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   - (iv) *$A \cong \{f \restriction_P \mid f\colon [0,1]^n \to [0,1]$ is a McNaughton map$\}$.*

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Sketch of the proof

Input: A rational polyhedron $P \subseteq [0,1]^n$

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Input: A rational polyhedron $P \subseteq [0,1]^n$

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Output:

(a) $k \in \mathbb{Z}$,

(b) $Q \subseteq [0,1]^k$ a collapsible rational polyhedron, and

(c) $\mathbb{Z}$-maps $\eta\colon [0,1]^k \to P$ and $\iota\colon P \to [0,1]^k$, such that

$$\eta \circ \iota = \mathrm{Id}_P.$$

Main Result

Sketch of the proof

$$
\begin{tikzcd}
P \arrow[dd, bend right, tail] \arrow[dr, tail] \arrow[dd, phantom, "\mathbb{Z}"] & \\
& {[0,1]^n} \arrow[ul, two heads, bend right] \arrow[dl, tail, bend left, "PL"'] \\
Q \arrow[uu, bend right, tail] \arrow[d, bend right, tail] \arrow[d, phantom, "\mathbb{Z}"] & \\
{[0,1]^k} \arrow[u, bend right, tail] &
\end{tikzcd}
$$

Main Result
Sketch of the proof : 10

Input: A contractible rational polyhedron $P \subseteq [0,1]^n$.

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$$\Downarrow$$

Whitehead's Theorem combined with the Simplicial Approximation Lemma.

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Whitehead's Theorem combined with the Simplicial Approximation Lemma.

$$\Downarrow$$

Output: A PL-retraction $\eta_{10} : [0,1]^n \to P$.

Main Result

Sketch of the proof : 9

Input: A PL-retraction $\eta_{10}\colon [0,1]^n \to P$.

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Input: A PL-retraction $\eta_{10} \colon [0,1]^n \to P$.

$$\Downarrow$$

Since $[0,1]^n$ is collapsible, applying finitely many stellar subdivisions to a collapsible triangulation of $[0,1]^n$:

Main Result
Sketch of the proof : 9

Input: A PL-retraction $\eta_{10} \colon [0,1]^n \to P$.

$$\Downarrow$$

Since $[0,1]^n$ is collapsible, applying finitely many stellar subdivisions to a collapsible triangulation of $[0,1]^n$:

$$\Downarrow$$

Output: A collapsible triangulation Δ_9 of $[0,1]^n$ such that η_{10} is linear on each simplex of Δ_9.

Main Result
Sketch of the proof : 8

Input: A collapsible triangulation Δ_9 of $[0,1]^n$.

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Suitable stellar subdivisions.

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$$\Downarrow$$

Suitable stellar subdivisions.

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Output: A collapsible triangulation Δ_8 of $[0,1]^n$
$\{S \in \Delta_8 \mid S \subseteq P\}$ is a triangulation of P.

Main Result
Sketch of the proof : 7

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Extending Beynon result about existence of rational triangulations for rational polyhedra.

Main Result

Sketch of the proof : 7

Input: A collapsible triangulation Δ_8 of $[0,1]^n$
$\{S \in \Delta_8 \mid S \subseteq P\}$ is a triangulation of P.

$$\Downarrow$$

Extending Beynon result about existence of rational triangulations for rational polyhedra.

$$\Downarrow$$

Output: A collapsible triangulation Δ_7 of $[0,1]^n$ such that $\{S \in \Delta_7 \mid S \subseteq P\}$ is a rational triangulation of P.

Main Result

Sketch of the proof : 6

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$$\Downarrow$$

Affine version of the desingularization process by stellar subdivisions (but only to $\{S \in \Delta_7 \mid S \subseteq P\}$).

Main Result

Sketch of the proof : 6

Input: A collapsible triangulation Δ_7 of $[0,1]^n$ such that $\{S \in \Delta_7 \mid S \subseteq P\}$ is a rational triangulation of P.

$$\Downarrow$$

Affine version of the desingularization process by stellar subdivisions (but only to $\{S \in \Delta_7 \mid S \subseteq P\}$).

$$\Downarrow$$

Output: A collapsible triangulation Δ_6 of $[0,1]^n$ such that $\{S \in \Delta_6 \mid S \subseteq P\}$ is a regular triangulation of P.

Main Result

Sketch of the proof : 5

Input: A collapsible triangulation Δ_6 of $[0,1]^n$ such that $\{S \in \Delta_6 \mid S \subseteq P\}$ is a regular triangulation of P, and the PL-retraction $\eta_{10} \colon [0,1]^n \to P$.

Main Result

Sketch of the proof : 5

Input: A collapsible triangulation Δ_6 of $[0,1]^n$ such that $\{S \in \Delta_6 \mid S \subseteq P\}$ is a regular triangulation of P, and the PL-retraction $\eta_{10} \colon [0,1]^n \to P$.

$$\Downarrow$$

For each v vertex of Δ_6 we define

$$\eta_5(v) = \begin{cases} v & \text{if } v \in P; \\ \eta_{10}(v) & \text{if } \notin P. \end{cases}$$

Main Result

Sketch of the proof : 5

Input: A collapsible triangulation Δ_6 of $[0,1]^n$ such that $\{S \in \Delta_6 \mid S \subseteq P\}$ is a regular triangulation of P, and the PL-retraction $\eta_{10} \colon [0,1]^n \to P$.

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For each v vertex of Δ_6 we define

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$$\Downarrow$$

Output: A PL-retraction $\eta_5 \colon [0,1]^n \to P$ such that η_5 is linear on each simplex of Δ_6.

Main Result

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Input: A PL-retraction $\eta_5\colon [0,1]^n \to P$ such that η_5 is linear on each simplex of Δ_6.

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Suitable stellar subdivisions (only on $S \in \Delta_6$ such that $S \not\subseteq P$).

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Sketch of the proof : 4

Input: A PL-retraction $\eta_5 \colon [0,1]^n \to P$ such that η_5 is linear on each simplex of Δ_6.

$$\Downarrow$$

Suitable stellar subdivisions (only on $S \in \Delta_6$ such that $S \not\subseteq P$).

$$\Downarrow$$

Output: A triangulation Δ_4 of $[0,1]^n$ such that for each n-simplex $S \in \Delta_4$, there exists $T_S \in \Delta_4$ with $\eta_5(S) \subseteq T_S$.

Main Result

Sketch of the proof : 3

Input: A triangulation Δ_4 of $[0,1]$ such that for each n-simplex $S \in \Delta_4$, there exists $T_S \in \Delta_4$ with $\eta_5(S) \subseteq T_S$.

Main Result
Sketch of the proof : 3

Input: A triangulation Δ_4 of $[0, 1]$ such that for each n-simplex $S \in \Delta_4$, there exists $T_S \in \Delta_4$ with $\eta_5(S) \subseteq T_S$.

$$\Downarrow$$

Modifying η_5 only on the vertices of Δ_4 that are not sent to vertices in Δ_4.

Main Result
Sketch of the proof : 3

Input: A triangulation Δ_4 of $[0,1]$ such that for each n-simplex $S \in \Delta_4$, there exists $T_S \in \Delta_4$ with $\eta_5(S) \subseteq T_S$.

$$\Downarrow$$

Modifying η_5 only on the vertices of Δ_4 that are not sent to vertices in Δ_4.

$$\Downarrow$$

Output: A PL-retract $\eta_3 \colon [0,1]^n \to P$ linear on each simplex of Δ_4 such that $\eta_3(v) \in \mathbb{Q}^n$ for each $v \in \Delta_4$.

Main Result

Sketch of the proof : 2

Input: A PL-retract $\eta_3 \colon [0,1]^n \to P$ linear on each simplex of Δ_4 such that $\eta_3(v) \in \mathbb{Q}^n$ for each $v \in \Delta_4$.

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Input: A PL-retract $\eta_3 \colon [0,1]^n \to P$ linear on each simplex of Δ_4 such that $\eta_3(v) \in \mathbb{Q}^n$ for each $v \in \Delta_4$.

$$\Downarrow$$

For each n-simplex $S \in \Delta_4$ with $\eta_5(S) \subseteq T_S$, there exists a rational point $x_S \in T_S$ such that $\mathrm{den}(x_S)$ is coprime with $\mathrm{den}(\eta_5(v))$ for each vertex v of S.

Main Result

Sketch of the proof : 2

Input: A PL-retract $\eta_3\colon [0,1]^n \to P$ linear on each simplex of Δ_4 such that $\eta_3(v) \in \mathbb{Q}^n$ for each $v \in \Delta_4$.

$$\Downarrow$$

For each n-simplex $S \in \Delta_4$ with $\eta_5(S) \subseteq T_S$, there exists a rational point $x_S \in T_S$ such that $\mathrm{den}(x_S)$ is coprime with $\mathrm{den}(\eta_5(v))$ for each vertex v of S.

$$\Downarrow$$

Output: A triangulation Δ_2 and a PL-retract $\eta_2\colon [0,1]^n \to P$ linear on each simplex of Δ_2 such that for each n-simplex $S \in \Delta_2$, $\gcd(\{\eta_2(v) \mid v \text{ vertex of } S\}) = 1$ (by strong regularity of P).

Main Result

Sketch of the proof : 1

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Using canonical representation of abstract simplicial complexes.

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Using canonical representation of abstract simplicial complexes.

$$\Downarrow$$

Output: A rational polyhedron $Q \subseteq [0,1]^k$ that admits a regular triangulation Δ_1 simplicially isomorphic to Δ_2 (given by f) and such that $\mathrm{den}(v) = \mathrm{den}(\eta_2(f(v)))$ for each v vertex of Δ_1.

Main Result

Sketch of the proof : 0

By construction:

- Δ_1 is strongly regular,
- Δ_1 is collapsible,
- $Q \cap \{0,1\}^k \neq \emptyset$.

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The map $e_i/\mathrm{den}(\eta_2(v_i)) \mapsto \eta_2(v_i)$ extends to $\mathbb{Z}$-map form Q onto P.

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The map $e_i/\mathrm{den}(\eta_2(v_i)) \mapsto \eta_2(v_i)$ extends to $\mathbb{Z}$-map form Q onto P.

Also, the map $v_i \mapsto e_i/\mathrm{den}(v_i)$ extends to a one-one $\mathbb{Z}$-map from P into Q. $\square$

Thank you for your attention!

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Main Result

Toolkit

Theorem (C., Mundici - 2011)

Let $P \subseteq [0,1]^n$ be a rational polyhedron such that

(i) *P is collapsible,*

(ii) *$P \cap \{0,1\}^n \neq \emptyset$, and*

(iii) *for each $v \in P \cap \mathbb{Q}^n$ there exist $w \in \mathbb{Z}^n$ and $\varepsilon > 0$ such that the convex segment $\mathrm{conv}(v, v + \varepsilon(w - v))$ is contained in P.*

Then $\mathcal{M}(P)$ is projective.

Theorem (Whitehead's Theorem–contractible case)

Let $P \subseteq [0,1]^n$ be a polyhedron. If P is contractible, then P is a deformation retract *of $[0,1]^n$, that is, there exists a retraction $f\colon [0,1]^n \to P$ homotopically equivalent to the identity on $[0,1]^n$ relative to P.*

Theorem (Relative Simplicial Approximation)

Let $P \subseteq Q \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}^m$ be polyhedra and $f\colon Q \to R$ be a continuous map such that $f\restriction_P$ is a piecewise linear. Then there exists a piecewise linear map $g\colon Q \to R$ homotopically equivalent to f such that f and agrees with g on P (in symbols, $g\restriction_P = f\restriction_P$).

Main Result

Toolkit: Technicalities

Proposition

Let $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ be polyhedra, and let Δ and ∇ be triangulations of P and Q, respectively. If $\eta\colon P \to Q$ is a piecewise linear map compatible with Δ, there exists a stellar subdivision Δ' of Δ such that

(i) *for each $S \in \Delta'$, there exists $T \in \nabla$ with $\eta(S) \subseteq T$, and*

(ii) *if $S \in \Delta$ is such that there exists $T \in \nabla$ with $\eta(S) \subseteq T$, then $S \in \Delta'$.*

Proposition

Let $P \subseteq Q \subseteq \mathbb{R}^n$ be such that P is a rational polyhedron and Q a convex polyhedron. Let Δ be a triangulation of Q such that the simplicial complex $\Delta_P = \{S \in \Delta \mid S \subseteq P\}$ is a triangulation of P. Then there is triangulation ∇ of Q such that:

(i) *the simplicial complex $\nabla_P = \{S \in \nabla \mid S \subseteq P\}$ is a rational triangulation of P, and*

(ii) *∇ and Δ are simplicially isomorphic.*

Main Result

Toolkit: Technicalities

Proposition

Let S be a simplex such that for each $v \in S \cap \mathbb{Q}^n$ there exist $w \in \mathbb{Z}^n$ and $\varepsilon > 0$ such that the convex segment $\mathrm{conv}(v, v + \varepsilon(w - v))$ is contained in S.
For each $k = 1, 2, \ldots$ there exists a rational point $v \in S$ such that $\gcd(k, \mathrm{den}(v)) = 1$.