

A POSSIBILISTIC GÖDEL MODAL LOGIC: AXIOMATIZATION AND NEIGHBOURHOOD SEMANTICS

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In approximate reasoning, sometimes one needs to simultaneously deal with both fuzziness of propositions and modalities, for instance one may try to assign a degree of truth to propositions like “John is possibly tall” or “John is certainly tall”, where “John is tall” is considered as a genuine fuzzy proposition. In this sense, modal extensions of fuzzy logics can be considered as a suitable tool to model not only vagueness (in the sense of graded properties) but also other kinds of information features like uncertainty or belief.

Possibility theory is with no doubt one of the main theories, alternative to Probability theory, for modelling and reasoning under uncertainty, by means of the use of pairs of dual possibility and necessity measures. Furthermore, Possibility theory has been formalized by many authors within a classical modal framework (see [8, 15, 13, 12]).

Therefore, it is natural to consider a combination of many-valued logics and modal logic in order to be capable of dealing with possibilistic uncertainty and vagueness in the same representation language. The idea here is that, in a many-valued modal framework, intermediate truth values assigned to a (pure) propositional formula φ (denoting a gradual property) are interpreted as partial degrees of truth, while truth values assigned to a modal formula $\Box\varphi$ or $\Diamond\varphi$ are interpreted as degrees of necessity and possibility respectively. However, the problem we have to face is the search for a syntactical characterization of many-valued modal logics that may work in most of the cases. Unfortunately, the well known Kripke semantics does not work out well because in many cases the K axiom is not valid, and a general method to axiomatize the many-valued modal logics arising from these semantics is not known. Indeed, it turns out that the only logics axiomatized in the literature are those where the underlying (propositional) many-valued logic is the one corresponding to either a finite Heyting algebra [9, 10], the standard (infinite) Gödel algebra [3] or a finite residuated algebra [2], in particular a finite Łukasiewicz linearly ordered algebra [14].

In order to overcome this difficulty we propose an alternative semantics which is a generalization of the classical *neighborhood semantics* [4]. More concretely, we will consider Possibilistic modal logics defined by neighborhood frames (possibly with many-valued neighborhood functions) where each world follows the rules of the logic G_{\sim} , i.e. Gödel logic expanded with an involutive negation, being the same for every world.

In fact, we will bring to this framework a graded version of the classical modal logic of belief KD45, also related to the possibilistic uncertainty model [6]. It has to be noted that similar fuzzy uncertainty logics have been proposed but either with a restricted language [5, 11] or over finite MTL chains [1]. In this context, we show that neighborhood models provide an alternative semantics for possibilistic reasoning uncertainty over Gödel events. Furthermore, we will introduce a complete axiomatization of this semantics. Finally, we will establish the connection between our generalization of neighborhood semantics and the well known Kripke semantics.

This contribution will elaborate on two previous workshop papers by the same authors [16, 17].

REFERENCES

- [1] F. Bou, F. Esteva and L. Godo. On possibilistic modal logics defined over MTL-chains, in Petr Hájek on Mathematical Fuzzy Logic. Outstanding Contributions to Logic no. 6, Franco Montagna (ed.), Springer, pp. 225-244, 2015.
- [2] F. Bou, F. Esteva, L. Godo, and R.O. Rodríguez. On the Minimum Many-Valued Modal Logic over a Finite Residuated Lattice. *Journal of Logic and Computation* 21(5): 739-790, 2011.
- [3] X. Caicedo and R.O. Rodríguez. Bi-Modal Gödel Logic over $[0,1]$ -valued Kripke frames. *Journal of Logic and Computation* 25(1): 37-55, 2015.
- [4] B. F. Chellas *Modal Logic: an introduction*. Cambridge University Press. 1980.
- [5] P. Dellunde, L. Godo, E. Marchioni. Extending possibilistic logic over $\text{Gi}_{\frac{1}{2}}$ del logic. *Int. J. Approx. Reasoning* 52(1): 63-75 (2011)
- [6] D. Dubois and H. Prade. An introduction to possibilistic and fuzzy logics. In P. Smets, A. Mamdani, D. Dubois, and H. Prade, editors, *Non-Standard Logics for Automatic Reasoning*. Academic Press, 1988.
- [7] D. Dubois and H. Prade. Fuzzy sets in approximate reasoning, part 1: Inference with possibility distributions. *Fuzzy Sets and Systems*, 40:143-202, 1991.
- [8] L. Fariñas del Cerro and A. Herzig. A modal analysis of possibility theory. In: Ph. Jorrand, J. Kelemen (Eds.), *Fundamentals of Artificial Intelligence Research (FAIR-91)*, Lecture Notes in Computer Science, Vol. 535, Springer, Berlin, pp. 11-18, 1991.
- [9] M. Fitting. Many valued modal logics. *Fundamenta Informaticae* 15, pp. 325-254, 1991.
- [10] M. Fitting. Many valued modal logics, II. *Fundamenta Informaticae* 17, pp. 55-73, 1992.
- [11] T. Flaminio, L. Godo and E. Marchioni On the Logical Formalization of Possibilistic Counterparts of States over n -Valued Lukasiewicz Events. *J. Log. Comput.* 21(3): 429-446, 2011.
- [12] J. Y. Halpern. *Reasoning about Uncertainty*. The MIT Press, 2003.
- [13] P. Hájek. Possibilistic Logic as Interpretability Logic. In *Proceedings IPMU'94*. July 4-8. Paris, France. Springer. 1994.
- [14] G. Hansoul and B. Teheux. Extending Łukasiewicz Logics with a Modality: Algebraic Approach to Relational Semantics. *Studia Logica*, vol. 101, no. 3, pp. 505-545, 2013.
- [15] G. Klir and D. Harmanec. On Modal logic Interpretation of Possibility Theory. *International Journal of Uncertainty Fuzziness and Knowledge-Based Systems*. Vol. 2-237-245. 1994.
- [16] R.O. Rodríguez and L. Godo Modal uncertainty logics with fuzzy neighborhood semantics. In *Working Notes of WL4AI-2013*, Beijing, China. August 3, 2013.
- [17] R.O. Rodríguez and L. Godo On the fuzzy modal logics of belief $KD45(\mathcal{A})$ and $Prob(L_n)$: axiomatization and neighbourhood semantics. In *Working Notes of WL4AI-2015*, Buenos Aires, Argentina. July 27, 2015.