On modal expansions of left-continuous t-norm logics

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Summary of contents

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Fuzzy & modal logics

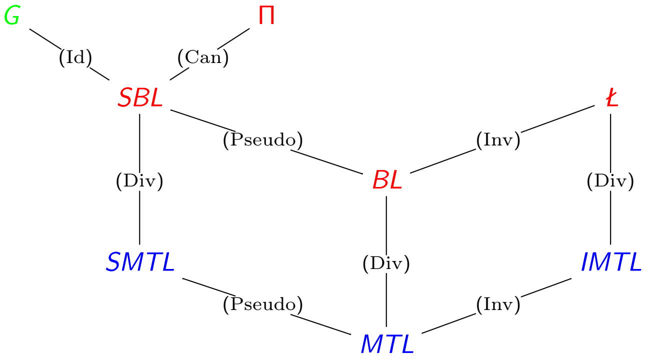
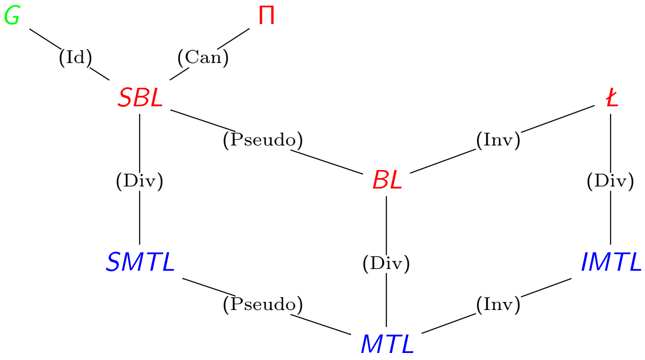


Figure: Hierarchy of t-norm based logics and extensions

Fuzzy & modal logics



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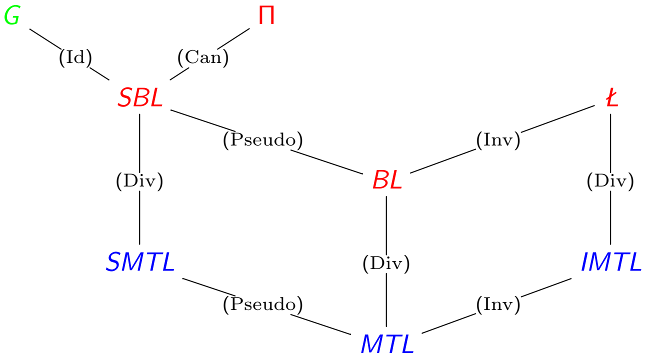

Figure: Hierarchy of t-norm based logics and extensions

- Reasoning over qualification of sentences (necessity, possibility...)
- Usual modalities: $\Box$, $\Diamond$. Kripke semantics.

Definition

$A \in MTL$-algebra. An A-**Kripke model** ($*$-Kripke model) $M = \langle W, R, e \rangle$ s.t. $[W \neq \emptyset]$ $[R : W \times W \to A]$ $[e : W \times Var \to A]$:

$$\begin{aligned} e(w, \Box\varphi) &:= \inf\{Rwv \Rightarrow e(v,\varphi) : v \in W\} \\ e(w, \Diamond\varphi) &:= \sup\{Rwv \odot e(v,\varphi) : v \in W\} \end{aligned}$$

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- **Finite residuated lattices** [Bou et. al]: Kripke models over finite residuated lattices, only $\Box$ operator. One constant in the language for each element in the algebra.
- **Gödel** [Caicedo-Rodríguez]: Kripke models over $[0,1]_G$, both $\Box$ and $\Diamond$. Strong standard completeness of Gödel logic and of characteristics of $[0,1]_G$-endomorphisms. **Finite model property**[+ Metcalfe-Rogger]

Summary of contents

Strong completeness

objective: for a left-continuous t-norm $*$ » an **(strong) axiomatization** of $[0,1]_*^{\mathbb{Q}_*}$: standard algebra of $*$ with Δ and canonical (rational) constants.

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Proposition [Cintula]

Let A be an expansion of an standard MTL-algebra with a non-continuous operation, and L_A a finitary axiomatic system for A. Then no finitary rational expansion of L_A enjoys the Pavelka-style completeness.

- Π logic + Δ op $\quad \dfrac{\{p \to \overline{c}\}_{c\in(0,1)_{\mathbb{Q}}}}{\neg p}, \quad \dfrac{\{\overline{c} \to p\}_{c\in(0,1)_{\mathbb{Q}}}}{\Delta p}$

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- General: an axiomatic system of $[0,1]_*$ validates a certain infinitary rule for each discontinuity point of the operations & is seminilinear $\Longrightarrow$ it is Pavelka complete.

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* many cases: uncountable infinitary rules

* how to know when a logic with infinitary rules is semilinear?

Δ, constants and semilinearity

Semilinearity Lemma

Let L be an implicative logic expanding MTL_Δ such that

- There is a countable amount of (infinitary) inference rules (with a finite number of variables each)
- The derivations of L are closed under $\vee$

Then L is semilinear.

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goal: finding a countable amount of infinitary rules for treating all the discontinuity points

A strong SC axiomatization

Takeuti and Titani's density rule can be adapted

$$D^\infty : \ \frac{\{(p \to \overline{c}) \vee (\overline{c} \to q)\}_{c \in \mathbb{Q}_*}}{(p \to q)}$$

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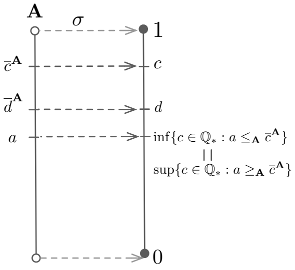
- Constants are dense on linearly ordered L_*^∞-algebras.



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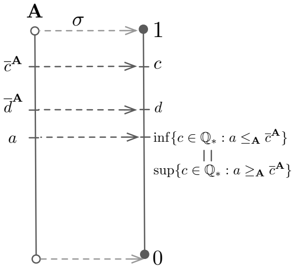

Strong Standard Completeness

$$\Gamma \vdash_{L_*^\infty} \varphi$$

iff

$$\Gamma \models_{[0,1]_*^{\mathbb{Q}_*}} \varphi$$

Conjunctive rules

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Conjunctive rules: for $x \in [0,1]$

$$R_x^{\infty} : \frac{\{(p \to \overline{c}) \wedge (\overline{d} \to q)\}_{d \in [0,x)_{\mathbb{Q}_*}, c \in (x,1]_{\mathbb{Q}_*}}}{p \to q}$$

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Lemma

Let A be a $MTL_*^{\mathbb{Q}_*}$-chain. Then the following are equivalent:

1. A validates D^∞,
2. A validates R_x^∞ for all $x \in [0, 1]$,
3. for all $a < b \in A$, there is $c \in \mathbb{Q}_*$ such that $a < \overline{c}^A < b$.

t-norms with a conjunctive axiomatization

Definition

A t-norm $*$ **accepts a conjunctive axiomatization** when there exists an axiomatic system L such that:

- L is strongly complete with respect to $[0,1]_*^{\mathbb{Q}_*}$,
- L extends $MTL_*^{\mathbb{Q}_*}$,
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$\surd$ Left-continuous t-norms with a countable number of discontinuity points on the diagonal of its residuum.

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Kripke Semantics

In what follows, $*$ **is a left-continuous t-norm accepting a conjunctive axiomatization.**

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- $\mathbb{SM}_*$: class of all **crisp** $[0,1]_*^{\mathbb{Q}_*}$-Kripke models.
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Local modal logic: $\Gamma \Vdash_{\mathbb{M}} \varphi$ iff for any $M \in \mathbb{M}$,

$$\text{for any } w \in W, \text{ if } e(w, \Gamma) = \{1\} \text{ then } e(w, \varphi) = 1.$$

Global modal logic: $\Gamma \Vdash_{\mathbb{M}}^{g} \varphi$ iff for any $M \in \mathbb{M}$,

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Axiomatic systems: K_* and K_*^g

- **Local modal logic**: K_* is the expansion of L_*^∞ with

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K &: \Box(p \to q) \to (\Box p \to \Box q)\\
\Box 1 &: \Box(\overline{c} \to p) \leftrightarrow (\overline{c} \to \Box p)\\
\Box 2 &: \Delta \Box p \to \Box \Delta p\\
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Some important properties of K_*

1. $\Gamma \vdash_{K_*} \varphi$ if and only if $(\Gamma \cup Th(K_*))^\# \vdash_{L_*^\infty} \varphi^\#$
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2. $\Gamma \vdash_{K_*} \varphi$ implies that $\Box\Gamma \vdash_{K_*} \Box\varphi$.

Canonical model for the local logic

The **canonical model of** K_* **is the** $[0,1]^{\mathbb{Q}_*}_*$**-Kripke model** $M^*_c = \langle W^*_c, R^*_c, e^*_c \rangle$ with:

- $W^*_c = \{h \in Hom(Fm^\#, [0,1]^{\mathbb{Q}_*}_*) : h(Th(K_*)^\#) = \{1\}\}$
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2. Let $v \in W_c^*$ and φ be such that $w(\varphi^{\#}) = 1$ for all $w \in W_c^*$ with $R_c vw$. Then $v(\varphi_\Box) = 1$.

Completeness of the local modal logic

Truth Lemma

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Strong standard completeness of K_*

For any $\Gamma \cup \{\varphi\} \subseteq Fm$, the following are equivalent

1. $\Gamma \vdash_{K_*} \varphi$,
2. $\Gamma \Vdash_{\mathbb{SM}_*} \varphi$,
3. $\Gamma \Vdash_{\mathbb{KM}_*} \varphi$.

- soundness is easy to check (wrt 3.)
- If $\Gamma \nvdash_{K_*} \varphi$ then $(\Gamma \cup Th(K_*))^{\#} \nvdash_{L_*^\infty} \varphi^{\#}$. Non-modal $\mathcal{SSC}$ provides a world w from the canonical model where $e_c^*(w, \Gamma) = w(\Gamma^{\#}) = \{1\}$ and $e_c^*(w, \varphi) = w(\varphi^{\#}) < 1$.

The global modal logic

Instead of a unique canonical model, one model for each set of formulas.

For Γ a set of formulas, the Γ-**canonical model of** K_*^g **is the** $[0,1]_*^{\mathbb{Q}_*}$-**Kripke model** $M_c^*[\Gamma] = \langle W_c^*[\Gamma], R_c^*[\Gamma], e_c^*[\Gamma]\rangle$ with:

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Open problems

- Axiomatic system for logics arising from an arbitrary left-continuous t-norm.
- Modal expansion of *MTL*-logics without rational constants.
- Axiomatic system of the finitary modal logics -evaluated over infinite-valued algebras-.
- Logics arising from Kripke models evaluated over *MTL*-algebras whose accessibility relation is no longer crisp.
- Decidability and complexity problems.
- ...

Thank you!