Back and forth conditions for elementary equivalence in model theory of non-classical logics

Pilar Dellunde[1,2] Àngel García-Cerdaña[2,3] Carles Noguera[4]

[1]Department of Philosophy
Universitat Autònoma de Barcelona

[2]Artificial Intelligence Research Institute (IIIA)
Spanish National Research Council (CSIC)

[3]Department of Information and Communication Technologies
University Pompeu Fabra

[4]Institute of Information Theory and Automation
Czech Academy of Sciences

Model theory for fuzzy logics

First-order fuzzy logic is strong enough to support non-trivial formal mathematical theories

Mathematical concepts in such theories show gradual rather than bivalent structure

Examples:

- Skolem, Hájek (1960, 2005): naïve set theory over Ł
- Takeuti–Titani (1994): ZF-style fuzzy set theory in a system close to Gödel logic
- Restall (1995), Hájek–Paris–Shepherdson (2000): arithmetic with the truth predicate over Ł
- Hájek–Haniková (2003, 2011): ZF-style set theory over $\mathrm{HL}_{\triangle}$
- Novák (2004): Church-style fuzzy type theory over $\mathrm{IMTL}_{\triangle}$
- Běhounek–Cintula (2005): higher-order fuzzy logic

Model theory for fuzzy logics

Also theories about extra-mathematical gradual notions may be expressible in first-order fuzzy logics.

Fuzzy description logic studies tractable fragments of such formalisms.

Model theory studies the models of first-order theories. Over classical logic, it constitutes a major branch of Mathematical Logic, but over fuzzy logics is still in its first stages:

- G. Gerla, Di Nola: first-order logics of bounded semilattices (semantical approach)
- Novák: fuzzy logics with evaluated syntax
- Hájek, Cintula et al: basic notions of morphisms, diagram, elementary equivalence
- Dellunde: more on morphisms, ultraproducts

Model theory for fuzzy logics

Very basic issues are not yet well understood, such as classification of models in terms of their first-order properties, i.e. elementary equivalence.

Model theory for fuzzy logics

Very basic issues are not yet well understood, such as classification of models in terms of their first-order properties, i.e. elementary equivalence.

A much more ambitious project:

a model theory of non-classical logics

A general non-classical framework (Cintula, Noguera)

L algebraizable in the sense of Blok and Pigozzi

The algebras are ordered (by implication), therefore we can define:

- Usual classical syntax
- Semantics as in Mostowski, Rasiowa, Hájek tradition $\langle \boldsymbol{A}, \mathbf{M} \rangle$
- $\forall = \inf$ and $\exists = \sup$
- Notion of safe structure, where truth values of all formulas are defined.
- Notion of model: $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} \in \mathcal{F}^{\boldsymbol{A}}$ (where $\mathcal{F}^{\boldsymbol{A}} = \{a \mid \overline{1}^{\boldsymbol{A}} \leq_{\boldsymbol{A}} a\}$ is the filter of designated elements of the algebra $\boldsymbol{A}$).

We still have:

- axiomatic Hilbert-style presentation
- completeness theorem

(Elementary) substructure

$\langle \boldsymbol{B}, \mathbf{N} \rangle$ is a substructure of $\langle \boldsymbol{A}, \mathbf{M} \rangle$ if:

1. $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$.
2. $N \subseteq M$.
3. For each n-ary function symbol $F \in \mathcal{P}$, and elements $d_1, \ldots, d_n \in N$,

$$F_{\mathbf{N}}(d_1, \ldots, d_n) = F_{\mathbf{M}}(d_1, \ldots, d_n).$$

4. For each n-ary predicate $P \in \mathcal{P}$, $P_{\mathbf{N}}$ is the restriction of $P_{\mathbf{M}}$ to N.

(Elementary) substructure

$\langle \boldsymbol{B}, \mathbf{N} \rangle$ is a substructure of $\langle \boldsymbol{A}, \mathbf{M} \rangle$ if:

1. $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$.
2. $N \subseteq M$.
3. For each n-ary function symbol $F \in \mathcal{P}$, and elements $d_1, \ldots, d_n \in N$,

$$F_{\mathbf{N}}(d_1, \ldots, d_n) = F_{\mathbf{M}}(d_1, \ldots, d_n).$$

4. For each n-ary predicate $P \in \mathcal{P}$, $P_{\mathbf{N}}$ is the restriction of $P_{\mathbf{M}}$ to N.

Moreover, if both structures are safe, $\langle \boldsymbol{B}, \mathbf{N} \rangle$ is an elementary substructure of $\langle \boldsymbol{A}, \mathbf{M} \rangle$ if for every $\mathcal{P}$-formula $\varphi(x_1, \ldots, x_n)$, and elements $d_1, \ldots, d_n \in N$,

$$\|\varphi(d_1, \ldots, d_n)\|^{\boldsymbol{B}}_{\mathbf{N}} = \|\varphi(d_1, \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{M}}$$

Homomorphisms

The pair $\langle f, g\rangle$ is an homomorphism from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ into $\langle \boldsymbol{B}, \mathbf{N}\rangle$ (safe $\mathcal{P}$-structures) if

1) $f : \boldsymbol{A} \to \boldsymbol{B}$ is an homomorphism of L-algebras
2) $g : M \to N$ is a mapping from M into N
3) for every n-ary $F \in \mathcal{P}$ and $d_1, \ldots, d_n \in M$,

$$g(F_{\mathbf{M}}(d_1, \ldots, d_n)) = F_{\mathbf{N}}(g(d_1), \ldots, g(d_n)).$$

4) For every n-ary predicate symbol $P \in \mathcal{P}$, and $d_1, \ldots, d_n \in M$,

$$P_{\mathbf{M}}(d_1, \ldots, d_n) \in \mathcal{F}^{\boldsymbol{A}} \;\Rightarrow\; P_{\mathbf{N}}(g(d_1), \ldots, g(d_n)) \in \mathcal{F}^{\boldsymbol{B}}.$$

Homomorphisms

The pair $\langle f, g\rangle$ is an homomorphism from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ into $\langle \boldsymbol{B}, \mathbf{N}\rangle$ (safe $\mathcal{P}$-structures) if

1) $f : \boldsymbol{A} \to \boldsymbol{B}$ is an homomorphism of L-algebras
2) $g : M \to N$ is a mapping from M into N
3) for every n-ary $F \in \mathcal{P}$ and $d_1, \dots, d_n \in M$,

$$g(F_{\mathbf{M}}(d_1, \dots, d_n)) = F_{\mathbf{N}}(g(d_1), \dots, g(d_n)).$$

4) For every n-ary predicate symbol $P \in \mathcal{P}$, and $d_1, \dots, d_n \in M$,

$$P_{\mathbf{M}}(d_1, \dots, d_n) \in \mathcal{F}^{\boldsymbol{A}} \Rightarrow P_{\mathbf{N}}(g(d_1), \dots, g(d_n)) \in \mathcal{F}^{\boldsymbol{B}}.$$

It is a σ-homomorphism if f preserves all the existing infima and suprema.
It is a strong homomorphism if for every n-ary predicate symbol $P \in \mathcal{P}$ and $d_1, \dots, d_n \in M$, $f(P_{\mathbf{M}}(d_1, \dots, d_n)) = P_{\mathbf{N}}(g(d_1), \dots, g(d_n))$.

Elementary homomorphisms

A homomorphism from $\langle \boldsymbol{A}, \mathbf{M} \rangle$ into $\langle \boldsymbol{B}, \mathbf{N} \rangle$ (safe $\mathcal{P}$-structures) $\langle f, g \rangle$ is elementary if for each $\mathcal{P}$-formula $\varphi(x_1, \ldots, x_n)$ and elements $d_1, \ldots, d_n \in M$,

$$f(\|\varphi(d_1, \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{M}}) = \|\varphi(g(d_1), \ldots, g(d_n))\|^{\boldsymbol{B}}_{\mathbf{N}}$$

A useful lemma

Lemma 1

Let $\langle f, g\rangle$ be a strong σ-homomorphism from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ into $\langle \boldsymbol{B}, \mathbf{N}\rangle$ (safe $\mathcal{P}$-structures) such that g is onto. Then $\langle f, g\rangle$ is elementary and for every $\mathcal{P}$-sentence σ,

$$\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} \in \mathcal{F}^{\boldsymbol{A}} \;\Leftrightarrow\; \|\sigma\|^{\boldsymbol{B}}_{\mathbf{N}} \in \mathcal{F}^{\boldsymbol{B}}.$$

Three notions of elementary equivalence – 1

Let $\langle \boldsymbol{A}, \mathbf{M} \rangle$ and $\langle \boldsymbol{B}, \mathbf{N} \rangle$ be safe $\mathcal{P}$-structures. We say that they are:

1. Elementarily equivalent (in symbols: $\langle \boldsymbol{A}, \mathbf{M} \rangle \equiv \langle \boldsymbol{B}, \mathbf{N} \rangle$) if, for every $\mathcal{P}$-sentence σ, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} \in \mathcal{F}^{\boldsymbol{A}} \Leftrightarrow \|\sigma\|^{\boldsymbol{B}}_{\mathbf{N}} \in \mathcal{F}^{\boldsymbol{B}}$.

Three notions of elementary equivalence – 1

Let $\langle \boldsymbol{A}, \mathbf{M} \rangle$ and $\langle \boldsymbol{B}, \mathbf{N} \rangle$ be safe $\mathcal{P}$-structures. We say that they are:

1. Elementarily equivalent (in symbols: $\langle \boldsymbol{A}, \mathbf{M} \rangle \equiv \langle \boldsymbol{B}, \mathbf{N} \rangle$) if, for every $\mathcal{P}$-sentence σ, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} \in \mathcal{F}^{\boldsymbol{A}} \Leftrightarrow \|\sigma\|^{\boldsymbol{B}}_{\mathbf{N}} \in \mathcal{F}^{\boldsymbol{B}}$.

Assume now that $\boldsymbol{A} \subseteq \boldsymbol{B}$.

2. Filter-strongly elementarily equivalent (in symbols: $\langle \boldsymbol{A}, \mathbf{M} \rangle \equiv^{fs} \langle \boldsymbol{B}, \mathbf{N} \rangle$) if, for every $\mathcal{P}$-sentence σ, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} \in \mathcal{F}^{\boldsymbol{A}} \Leftrightarrow \|\sigma\|^{\boldsymbol{B}}_{\mathbf{N}} \in \mathcal{F}^{\boldsymbol{B}}$ and, in this case, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\sigma\|^{\boldsymbol{B}}_{\mathbf{N}}$.

Three notions of elementary equivalence – 1

Let $\langle \boldsymbol{A}, \mathbf{M} \rangle$ and $\langle \boldsymbol{B}, \mathbf{N} \rangle$ be safe $\mathcal{P}$-structures. We say that they are:

1. Elementarily equivalent (in symbols: $\langle \boldsymbol{A}, \mathbf{M} \rangle \equiv \langle \boldsymbol{B}, \mathbf{N} \rangle$) if, for every $\mathcal{P}$-sentence σ, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} \in \mathcal{F}^{\boldsymbol{A}} \Leftrightarrow \|\sigma\|^{\boldsymbol{B}}_{\mathbf{N}} \in \mathcal{F}^{\boldsymbol{B}}$.

Assume now that $\boldsymbol{A} \subseteq \boldsymbol{B}$.

2. Filter-strongly elementarily equivalent (in symbols: $\langle \boldsymbol{A}, \mathbf{M} \rangle \equiv^{fs} \langle \boldsymbol{B}, \mathbf{N} \rangle$) if, for every $\mathcal{P}$-sentence σ, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} \in \mathcal{F}^{\boldsymbol{A}} \Leftrightarrow \|\sigma\|^{\boldsymbol{B}}_{\mathbf{N}} \in \mathcal{F}^{\boldsymbol{B}}$ and, in this case, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\sigma\|^{\boldsymbol{B}}_{\mathbf{N}}$.
3. Strongly elementarily equivalent (in symbols: $\langle \boldsymbol{A}, \mathbf{M} \rangle \equiv^{s} \langle \boldsymbol{B}, \mathbf{N} \rangle$) if, for every $\mathcal{P}$-sentence σ, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\sigma\|^{\boldsymbol{B}}_{\mathbf{N}} \in A$.

Three notions of elementary equivalence – 1

Let $\langle \boldsymbol{A}, \mathbf{M} \rangle$ and $\langle \boldsymbol{B}, \mathbf{N} \rangle$ be safe $\mathcal{P}$-structures. We say that they are:

1. Elementarily equivalent (in symbols: $\langle \boldsymbol{A}, \mathbf{M} \rangle \equiv \langle \boldsymbol{B}, \mathbf{N} \rangle$) if, for every $\mathcal{P}$-sentence σ, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} \in \mathcal{F}^{\boldsymbol{A}} \Leftrightarrow \|\sigma\|^{\boldsymbol{B}}_{\mathbf{N}} \in \mathcal{F}^{\boldsymbol{B}}$.

Assume now that $\boldsymbol{A} \subseteq \boldsymbol{B}$.

2. Filter-strongly elementarily equivalent (in symbols: $\langle \boldsymbol{A}, \mathbf{M} \rangle \equiv^{fs} \langle \boldsymbol{B}, \mathbf{N} \rangle$) if, for every $\mathcal{P}$-sentence σ, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} \in \mathcal{F}^{\boldsymbol{A}} \Leftrightarrow \|\sigma\|^{\boldsymbol{B}}_{\mathbf{N}} \in \mathcal{F}^{\boldsymbol{B}}$ and, in this case, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\sigma\|^{\boldsymbol{B}}_{\mathbf{N}}$.
3. Strongly elementarily equivalent (in symbols: $\langle \boldsymbol{A}, \mathbf{M} \rangle \equiv^{s} \langle \boldsymbol{B}, \mathbf{N} \rangle$) if, for every $\mathcal{P}$-sentence σ, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\sigma\|^{\boldsymbol{B}}_{\mathbf{N}} \in A$.

- Clearly, $\equiv$ and $\equiv^{fs}$ are the same notion for logics with weakening, because then $\mathcal{F}^{\boldsymbol{A}} = \mathcal{F}^{\boldsymbol{B}} = \{\overline{1}^{\boldsymbol{A}}\}$.

Three notions of elementary equivalence – 2

- Consider a predicate language with only one monadic predicate P and $\langle \boldsymbol{A}, \mathbf{M} \rangle$ and $\langle \boldsymbol{A}, \mathbf{N} \rangle$ with both domains the set of all natural numbers, and $\boldsymbol{A}$ the standard uninorm given by:

$$x \,\&^{\boldsymbol{A}}\, y = \begin{cases} \min\{x, y\}, & \text{if } x \leq 1 - y, \\ \max\{x, y\}, & \text{if } x > 1 - y. \end{cases}$$

$\mathcal{F}^{\boldsymbol{A}} = [\frac{1}{2}, 1]$

$$P_{\mathbf{M}}(n) = \begin{cases} \frac{4}{5} - \frac{1}{n}, & \text{if } n \geq 2, \\ 0, & \text{if } 0 \leq n \leq 1. \end{cases}$$

$$P_{\mathbf{N}}(n) = \begin{cases} \frac{3}{5} - \frac{1}{n}, & \text{if } n \geq 2, \\ 0, & \text{if } 0 \leq n \leq 1. \end{cases}$$

$\|(\exists x)P(x)\|_{\mathbf{M}} = \frac{4}{5}$ and $\|(\exists x)P(x)\|_{\mathbf{M}} = \frac{3}{5}$, but taking a strong σ-homomorphism $\langle f, Id \rangle$ with $f(\frac{4}{5} - \frac{1}{n}) = \frac{3}{5} - \frac{1}{n}$ and applying the lemma, we obtain $\langle \boldsymbol{A}, \mathbf{M} \rangle \equiv \langle \boldsymbol{A}, \mathbf{N} \rangle$

Three notions of elementary equivalence – 3

- Consider a predicate language with only one monadic predicate P and take $\langle[0,1]_{\mathrm{G}}, \mathbf{M}\rangle$ and $\langle[0,1]_{\mathrm{G}}, \mathbf{N}\rangle$, both with the set of natural numbers as domain.

$$P_{\mathbf{M}}(n) = \begin{cases} \frac{3}{4} - \frac{1}{n} & \text{if } n \geq 2, \\ 0 & 0 \leq n \leq 1. \end{cases}$$

$$P_{\mathbf{N}}(n) = \begin{cases} \frac{1}{2} - \frac{1}{n} & \text{if } n \geq 2, \\ 0 & 0 \leq n \leq 1. \end{cases}$$

$\|(\exists x)P(x)\|_{\mathbf{M}} = \frac{3}{4}$ and $\|(\exists x)P(x)\|_{\mathbf{N}} = \frac{1}{2}$. Taking f as any non-decreasing bijection such that $f(\frac{3}{4}) = \frac{1}{2}$, $f(1) = 1$, $f(0) = 0$, and for every $n \in$ N, $f(\frac{3}{4} - \frac{1}{n}) = \frac{1}{2} - \frac{1}{n}$, and applying again the lemma we obtain $\langle[0,1]_{\mathrm{G}}, \mathbf{M}\rangle \equiv \langle[0,1]_{\mathrm{G}}, \mathbf{N}\rangle$.

Löwenheim–Skolem theorems

Observation: if $\boldsymbol{A} \subseteq \boldsymbol{B}$ and there is an elementary homomorphism $\langle Id_{\boldsymbol{A}}, g \rangle$ from $\langle \boldsymbol{A}, \mathbf{M} \rangle$ to $\langle \boldsymbol{B}, \mathbf{N} \rangle$, then $\langle \boldsymbol{A}, \mathbf{M} \rangle \equiv^{s} \langle \boldsymbol{B}, \mathbf{N} \rangle$.

Löwenheim–Skolem theorems

Observation: if $\boldsymbol{A} \subseteq \boldsymbol{B}$ and there is an elementary homomorphism $\langle Id_{\boldsymbol{A}}, g\rangle$ from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{B}, \mathbf{N}\rangle$, then $\langle \boldsymbol{A}, \mathbf{M}\rangle \equiv^{s} \langle \boldsymbol{B}, \mathbf{N}\rangle$.

Aim: Given a model, obtain bigger and smaller strongly elementarily equivalent models.

P. Dellunde, À. García-Cerdaña, and C. Noguera. Löwenheim–Skolem theorems for non-classical first-order algebraizable logics. To appear in *Logic Journal of the IGPL*.

Downward Löwenheim–Skolem theorem

Theorem 2 (Classical)

Let $\mathcal{P}$ be a predicate language and $\mathbf{M}$ a $\mathcal{P}$-structure. For each subset $Z \subseteq M$, and κ a cardinal such that $\max\{\omega, |\mathcal{P}|, |Z|\} \leq \kappa \leq |M|$, there is an elementary substructure $\mathbf{N}$ of $\mathbf{M}$ such that $|N| = \kappa$ and $Z \subseteq N$.

Downward Löwenheim–Skolem theorem

Theorem 2 (Classical)

Let $\mathcal{P}$ be a predicate language and $\mathbf{M}$ a $\mathcal{P}$-structure. For each subset $Z \subseteq M$, and κ a cardinal such that $\max\{\omega, |\mathcal{P}|, |Z|\} \leq \kappa \leq |M|$, there is an elementary substructure $\mathbf{N}$ of $\mathbf{M}$ such that $|N| = \kappa$ and $Z \subseteq N$.

Theorem 3 (Non-classical)

Take a safe $\mathcal{P}$-structure $\langle \mathbf{A}, \mathbf{M} \rangle$ and assume that every subset of A definable with parameters in $\langle \mathbf{A}, \mathbf{M} \rangle$ has infimum and supremum. Then, for every $Z \subseteq M$ and every cardinal κ such that

$$\max\{\omega, |\mathcal{P}|, |Z|, p(\mathbf{A})\} \leq \kappa \leq |M|,$$

there is a safe $\mathcal{P}$-structure $\langle \mathbf{A}, \mathbf{N} \rangle$ which is an elementary substructure of $\langle \mathbf{A}, \mathbf{M} \rangle$ such that $|N| = \kappa$ and $Z \subseteq N$.

Upward Löwenheim–Skolem theorem

Theorem 4 (Classical)

Let $\mathcal{P}$ be a predicate language, $\mathbf{M}$ an infinite $\mathcal{P}$-structure, and κ a cardinal such that $\max\{|\mathcal{P}|, |M|\} \leq \kappa$. Then there is an elementary extension $\mathbf{N}$ of $\mathbf{M}$ such that $|N| = \kappa$.

Upward Löwenheim–Skolem theorem

Theorem 4 (Classical)

Let $\mathcal{P}$ be a predicate language, $\mathbf{M}$ an infinite $\mathcal{P}$-structure, and κ a cardinal such that $\max\{|\mathcal{P}|, |M|\} \leq \kappa$. Then there is an elementary extension $\mathbf{N}$ of $\mathbf{M}$ such that $|N| = \kappa$.

Theorem 5 (Non-classical)

Let $\mathcal{P}$ be an equality-free language. For every infinite safe $\mathcal{P}$-structure $\langle \boldsymbol{A}, \mathbf{M} \rangle$ and every cardinal κ with $\max\{|\mathcal{P}|, |M|\} \leq \kappa$, there is a safe $\mathcal{P}$-structure $\langle \boldsymbol{A}, \mathbf{N} \rangle$ of cardinality κ and an elementary embedding from $\langle \boldsymbol{A}, \mathbf{M} \rangle$ to $\langle \boldsymbol{A}, \mathbf{N} \rangle$.

Failure of the Upward L–S Th. for logics with equality

- Take $\mathrm{G}_{\triangle}$ assume that the language contains a unary predicate P and an equality symbol $\approx$.

Failure of the Upward L–S Th. for logics with equality

- Take $\mathrm{G}_{\triangle}$ assume that the language contains a unary predicate P and an equality symbol $\approx$.
- Take a semantics of models $\langle [0,1]_{\mathrm{G}_{\triangle}}, \mathbf{M}\rangle$, where $\approx$ is interpreted as classical equality.

Failure of the Upward L–S Th. for logics with equality

- Take $\mathrm{G}_{\triangle}$ assume that the language contains a unary predicate P and an equality symbol $\approx$.
- Take a semantics of models $\langle [0,1]_{\mathrm{G}_{\triangle}}, \mathbf{M}\rangle$, where $\approx$ is interpreted as classical equality.
- Consider $\chi = (\forall x)(\forall y)(\neg\triangle(x \approx y) \to \neg\triangle(P(x) \leftrightarrow P(y)))$ that codifies the fact that P is interpreted as an injective mapping from the domain to the algebra of truth-values.

Failure of the Upward L–S Th. for logics with equality

- Take $\mathrm{G}_\triangle$ assume that the language contains a unary predicate P and an equality symbol $\approx$.
- Take a semantics of models $\langle [0,1]_{\mathrm{G}_\triangle}, \mathbf{M}\rangle$, where $\approx$ is interpreted as classical equality.
- Consider $\chi = (\forall x)(\forall y)(\neg\triangle(x \approx y) \to \neg\triangle(P(x) \leftrightarrow P(y)))$ that codifies the fact that P is interpreted as an injective mapping from the domain to the algebra of truth-values.
- Therefore, $\langle [0,1]_{\mathrm{G}}, \mathbf{M}\rangle$ is a model of χ if and only if $|M| \leq 2^{\aleph_0}$, and hence the upward theorem does not hold.

Many-sorted classical first-order logic

Many-sorted predicate language: $\langle \mathcal{S}, Pred, Func, Ar, Sort \rangle$, where $\mathcal{S}$ is a non-empty set of *sorts*, Ar is the arity function and $Sort$ is a function that maps each n-ary $R \in Pred$ to a sequence of n sorts and each n-ary $F \in Func$ to a sequence of $n+1$ sorts.

Many-sorted structure: $\mathbf{M} = \langle M, \langle R^{\mathbf{M}} \rangle_{R \in Pred}, \langle F^{\mathbf{M}} \rangle_{f \in Func} \rangle$, where M is a family of non-empty domains $\{S(M) \mid S \in \mathcal{S}\}$; for each n-ary $R \in Pred$, if $Sort(R) = \langle S_1, \ldots, S_n \rangle$, $R^{\mathbf{M}} \subseteq S_1(M) \times \ldots \times S_n(M)$; for each n-ary $F \in Func$, if $Sort(F) = \langle S_1, \ldots, S_n, S \rangle$, $F^{\mathbf{M}}$ is a function from $S_1(M) \times \ldots \times S_n(M)$ to $S(M)$.

By the cardinality $|M|$ of $\mathbf{M}$ we mean the sum of the cardinalities of the sets $\{S(M) \mid S \in \mathcal{S}\}$.

Translation to two-sorted structures

P. Cintula, F. Esteva, J. Gispert, L. Godo, F. Montagna and C. Noguera, Distinguished Algebraic Semantics For T-Norm Based Fuzzy Logics: Methods and Algebraic Equivalencies, *Annals of Pure and Applied Logic* 160(1):53–81, 2009.

Given a $\mathcal{P}$-structure $\langle \boldsymbol{B}, \mathbf{M} \rangle$, we build a 2-sorted structure $\boldsymbol{B}_{\mathbf{M}}$:

- The universe of sort 1 is B and the universe of sort 2 is M.
- The symbols $\approx_i$ are interpreted as crisp equality in the corresponding sorts.
- For each propositional n-ary connective λ, define $\lambda^{\boldsymbol{B}_{\mathbf{M}}}$ as $\lambda^{\boldsymbol{B}}$.
- For each n-ary functional symbol $F \in Func$, define $F^{\boldsymbol{B}_{\mathbf{M}}}$ as $F_{\mathbf{M}}$.
- For each n-ary relational symbol $R \in Pred$, define $R^{\boldsymbol{B}_{\mathbf{M}}}$ as $R_{\mathbf{M}}$.

Translation to two-sorted structures

Lemma 6

For each $\mathcal{P}$-formula $\varphi(v_1, \dots, v_n)$, there is a 2-sorted formula $E_\varphi(v_1, \dots, v_n, x)$ such that, for every $\mathcal{P}$-structure $\langle \boldsymbol{B}, \mathbf{M} \rangle$, and each $d_1, \dots, d_n \in M$,

$$\|\varphi(d_1, \dots, d_n)\|^{\boldsymbol{B}}_{\mathbf{M}} = b \quad \textit{if and only if} \quad \boldsymbol{B}_{\mathbf{M}} \models E_\varphi(d_1, \dots, d_n, b).$$

Corollary 7

A $\mathcal{P}$-structure $\langle \boldsymbol{B}, \mathbf{M} \rangle$ is safe if and only if, for every $\mathcal{P}$-formula $\varphi(v_1, \dots, v_n)$,

$$\boldsymbol{B}_{\mathbf{M}} \models (\forall v_1, \dots, v_n)(\exists! x) E_\varphi(v_1, \dots, v_n, x).$$

Löwenheim–Skolem Theorems (via 2-sorted structures)

Theorem 8

Let $\langle \boldsymbol{B}, \mathbf{M} \rangle$ be a safe $\mathcal{P}$-structure. Then, for every $Z \subseteq M$, every $X \subseteq B$ and every cardinal κ such that $\max\{|\mathcal{P}|, \omega, |Z|, |X|\} \leq \kappa \leq \max\{|B|, |M|\}$, there is a safe $\mathcal{P}$-structure $\langle \boldsymbol{A}, \mathbf{O} \rangle$ which is an elementary substructure of $\langle \boldsymbol{B}, \mathbf{M} \rangle$ such that $|A| + |O| = \kappa$, $Z \subseteq O$, and $X \subseteq A$.

Theorem 9

Let $\langle \boldsymbol{A}, \mathbf{M} \rangle$ be a safe infinite $\mathcal{P}$-structure and κ a cardinal such that $\max\{|\mathcal{P}|, |A|, |M|\} \leq \kappa$. Then there is a safe $\mathcal{P}$-structure $\langle \boldsymbol{B}, \mathbf{N} \rangle$ such that $\langle \boldsymbol{A}, \mathbf{M} \rangle$ is an elementary substructure of $\langle \boldsymbol{B}, \mathbf{N} \rangle$ and $|B| + |N| = \kappa$.

Finitely isomorphic 2-sorted structures

Two 2-sorted structures $\mathbf{M}$ and $\mathbf{N}$ are said to be finitely isomorphic, written $\mathbf{M} \cong_f \mathbf{N}$ if there is a sequence $\langle I_n \mid n \in \mathrm{N} \rangle$ with the following properties:

1. Each I_n is a non-empty set of partial isomorphisms from $\mathbf{M}$ to $\mathbf{N}$.
2. For each $n \in \mathrm{N}$, $I_{n+1} \subseteq I_n$.
3. (Forth property) For each $n \in \mathrm{N}$, $p \in I_{n+1}$, and $a \in S_1(M) \cup S_2(M)$, there is a mapping $q \in I_n$ such that $p \subseteq q$ and $a \in dom(q)$.
4. (Back property) For each $n \in \mathrm{N}$, $p \in I_{n+1}$, and $b \in S_1(N) \cup S_2(N)$, there is a mapping $q \in I_n$ such that $p \subseteq q$ and $b \in rg(q)$.

Theorem 10 (Fraïssé)

Let $\mathbf{M}$ and $\mathbf{N}$ be 2-sorted structures. Then:

$$\mathbf{M} \equiv \mathbf{N} \quad \Leftrightarrow \quad \mathbf{M} \cong_f \mathbf{N}.$$

Finitely isomorphic non-classical structures

Two $\mathcal{P}$-structures $\langle \boldsymbol{A}, \mathbf{M}\rangle$, $\langle \boldsymbol{B}, \mathbf{N}\rangle$ are said to be finitely isomorphic, written $\langle \boldsymbol{A}, \mathbf{M}\rangle \cong_f \langle \boldsymbol{B}, \mathbf{N}\rangle$ if there is a sequence $\langle I_n \mid n \in \mathrm{N}\rangle$ with the following properties:

1. Every I_n is a non-empty set of partial isomorphisms from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{B}, \mathbf{N}\rangle$.
2. For each $n \in \mathrm{N}$, $I_{n+1} \subseteq I_n$.
3. (Forth-property I) For every $\langle p, r\rangle \in I_{n+1}$ and $m \in M$, there is a mapping s such that $r \subseteq s$, $m \in dom(s)$ and $\langle p, s\rangle \in I_n$.
4. (Back-property I) For every $\langle p, r\rangle \in I_{n+1}$ and $n \in N$, there is a mapping s such that $r \subseteq s$, $n \in rg(s)$ and $\langle p, s\rangle \in I_n$.
5. (Forth-property II) For every $\langle p, r\rangle \in I_{n+1}$ and $a \in A$, there is a mapping q such that $p \subseteq q$, $a \in dom(q)$ and $\langle q, r\rangle \in I_n$.
6. (Back-property II) For every $\langle p, r\rangle \in I_{n+1}$ and $b \in B$, there is a mapping q such that $p \subseteq q$, $b \in rg(q)$ and $\langle q, r\rangle \in I_n$.

Back-and-forth is a sufficient condition for elementary equivalence...

Theorem 11

Let $\mathcal{P}$ be a finite predicate language. Let $\langle \boldsymbol{A}, \mathbf{M}\rangle$, $\langle \boldsymbol{B}, \mathbf{N}\rangle$ be safe $\mathcal{P}$-structures. The following holds:

$$\langle \boldsymbol{A}, \mathbf{M}\rangle \cong_f \langle \boldsymbol{B}, \mathbf{N}\rangle \quad \Rightarrow \quad \langle \boldsymbol{A}, \mathbf{M}\rangle \equiv \langle \boldsymbol{B}, \mathbf{N}\rangle.$$

Furthermore, if $\boldsymbol{A} \subseteq \boldsymbol{B}$, then we have:

$$\langle \boldsymbol{A}, \mathbf{M}\rangle \cong_f \langle \boldsymbol{B}, \mathbf{N}\rangle \quad \Rightarrow \quad \langle \boldsymbol{A}, \mathbf{M}\rangle \equiv^s \langle \boldsymbol{B}, \mathbf{N}\rangle.$$

...but it is not necessary!

Lemma 12

If $\langle \boldsymbol{A}, \mathbf{M}\rangle \cong_f \langle \boldsymbol{B}, \mathbf{N}\rangle$ *and* $\langle \boldsymbol{A}, \mathbf{M}\rangle$ *is finite, then* $\langle \boldsymbol{A}, \mathbf{M}\rangle \cong \langle \boldsymbol{B}, \mathbf{N}\rangle$.

Let $\mathcal{P}$ be a finite predicate language. Let $\langle \boldsymbol{B}_2, \mathbf{M}\rangle$ be a classical first-order $\mathcal{P}$-structure. Now take an infinite L-algebra $\boldsymbol{A}$. Since $\boldsymbol{B}_2 \subseteq \boldsymbol{A}$, we can also see $\langle \boldsymbol{B}_2, \mathbf{M}\rangle$ as a structure over $\boldsymbol{A}$. Clearly $\langle \boldsymbol{B}_2, \mathbf{M}\rangle \equiv^s \langle \boldsymbol{A}, \mathbf{M}\rangle$ but it is not true that $\langle \boldsymbol{B}_2, \mathbf{M}\rangle \cong_f \langle \boldsymbol{A}, \mathbf{M}\rangle$.

Conclusions

- Model theory for fuzzy logics is well motivated but underdeveloped.
- Some results might as well be carried out in the much wider framework of first-order algebraizable logics.
- Such research is not trivial due to the failure of classical properties (witnessing, compactness, ...).
- The classical notion of elementary equivalence splits into three different non-classical notions.
- L–S theorems can be obtained by direct proofs or from many-sorted classical structures with different pros and cons.
- The classical back-and-forth characterizations of elementary equivalence cannot be directly imported (via 2-sorted structures).

Conclusions

- Model theory for fuzzy logics is well motivated but underdeveloped.
- Some results might as well be carried out in the much wider framework of first-order algebraizable logics.
- Such research is not trivial due to the failure of classical properties (witnessing, compactness, ...).
- The classical notion of elementary equivalence splits into three different non-classical notions.
- L–S theorems can be obtained by direct proofs or from many-sorted classical structures with different pros and cons.
- The classical back-and-forth characterizations of elementary equivalence cannot be directly imported (via 2-sorted structures).