The lattice of super-Belnap logics

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We study the lattice $\operatorname{Ext}\mathcal{B}$ of extensions of the four-valued Belnap-Dunn logic $\mathcal{B}$, called super-Belnap logics following [4]. In particular, we settle the question of how many finitary super-Belnap logics there are, and describe a connection between the worlds of super-Belnap logics and finite graphs.

The logic which we call $\mathcal{B}$ here was introduced by Belnap [1] following related work by Dunn [2] as a logic for dealing with potentially inconsistent and incomplete information. Although the Belnap-Dunn logic has attracted much attention from researchers in logic and computer science, there has been little systematic research into its extensions other than classical logic $\mathcal{CL}$. The most prominent among those which have been studied so far are Priest's Logic of Paradox $\mathcal{LP}$ and Kleene's strong three-valued logic $\mathcal{K}$, which have been used by philosophers in gappy and glutty accounts of truth. More recently, Exactly True Logic $\mathcal{ETL}$ was introduced by Pietz and Rivieccio [3].

The explicit study of $\operatorname{Ext}\mathcal{B}$ was only initiated by Rivieccio [4], who showed that there are infinitely many finitary super-Belnap logics. (The sublattice of finitary super-Belnap logics will be denoted $\operatorname{Ext}_\omega\mathcal{B}$.) We improve on this result and also shed some light on the lattice structure of $\operatorname{Ext}\mathcal{B}$.

Theorem 1. $\operatorname{Ext}_\omega\mathcal{B}$ *is a non-modular lattice. It contains a distributive sublattice which has the cardinality of the continuum.*

Proposition 2. $\operatorname{Ext}\mathcal{B}$ *has a unique atom.* $\mathcal{K}$ *has a largest proper sublogic among extensions of* $\mathcal{ETL}$.

Theorem 3. *The only non-trivial protoalgebraic super-Belnap logic is* $\mathcal{CL}$. *The only non-trivial Fregean super-Belnap logic is* $\mathcal{CL}$. *The only non-trivial self-extensional super-Belnap logics are* $\mathcal{B}$, $\mathcal{LP}\cap\mathcal{K}$, $\mathcal{CL}$.

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It turns out that the lattice of super-Belnap logics with the finite model property may be described in terms of finite graphs (possibly with loops). In the following theorem, by a homomorphic (epimorphic) class of graphs we mean a class of graphs closed under (surjective) homomorphisms.

Theorem 4. *The lattice of homomorphic classes of finite graphs dually embeds into* $\mathrm{Ext}_\omega \mathcal{B}$*. The lattice of epimorphic classes of finite graphs closed under finite disjoint unions is dually isomorphic to the lattice of logics with the finite model property in the interval* $[\mathcal{ETL}, \mathcal{ETL}_\omega]$*.*

Corollary 5. *There is a continuum of finitary explosive extensions of* $\mathcal{B}$*. There is a continuum of antivarieties of de Morgan algebras.*

The above results are not sensitive to the presence or absence of the constants $\top$ and $\bot$ in the logic. By contrast, adopting a multiple-conclusion perspective changes the picture dramatically: the only non-trivial multiple-conclusion super-Belnap logics are the multiple-conclusion versions of the logics $\mathcal{B}$, $\mathcal{LP} \cap \mathcal{K}$, $\mathcal{LP}$, $\mathcal{K}$, and $\mathcal{CL}$.

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References

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