Morita-equivalences for MV-algebras

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Topoi

A topos $\mathcal{E}$ is a universe which generalizes the classic universe of sets.

- $\mathcal{E}$ has all finite limits (= is finitely complete)
- $\mathcal{E}$ has all finite co-limits (= is finitely co-complete)
- $\mathcal{E}$ has exponentiation
- in $\mathcal{E}$ we can classify subobjects ($\mathcal{E}$ has a subobject classifier)

Examples:

- Category of sets ($\cong$ category of sheaves over $\{*\}$)
- Categories of sheaves on a topological spaces
- *Grothendieck topoi*: categories of sheaves on a site

Models in topoi

In a Grothendieck topos we can consider models of every first-order theory. Given Σ a first-order signature, a Σ-*structure* in a topos $\mathcal{E}$ is defined by the following data

- sorts $\rightarrow$ objects in $\mathcal{E}$
- function symbols $\rightarrow$ arrows in $\mathcal{E}$
- relation symbols $\rightarrow$ subobjects in $\mathcal{E}$

A Σ-structure M is a *model* of a theory $\mathbb{T}$ over the signature Σ if every axiom of $\mathbb{T}$ is valid in M.

Problem

Let $\mathbb{T}$ and $\mathbb{S}$ be two geometric theories such that

$$\mathbb{T}\text{-mod}(\mathbf{Set}) \cong \mathbb{S}\text{-mod}(\mathbf{Set})$$

Question: Is it true that

$$\mathbb{T}\text{-mod}(\mathcal{E}) \cong \mathbb{S}\text{-mod}(\mathcal{E})$$

naturally in $\mathcal{E}$?

If this is true we say that $\mathbb{T}$ and $\mathbb{S}$ are Morita-equivalent (equivalently, the two theories have the same classifying topos).

The ‘bridge’ technique

invariant I

$\mathscr{E}_{\mathbb{T}} \simeq \mathscr{E}_{\mathbb{S}}$

$\mathbb{T}$ $\mathbb{S}$

$\mathbb{T}$-characterization of I $\mathbb{S}$-characterization of I

If $\mathbb{T}$ and $\mathbb{S}$ are Morita-equivalent we can transfer properties and results from one theory to the other by using topos-theoretic invariants defined over the same classifying topos.

Morita-equivalences for MV-algebras

- Lift to Morita-equivalences of two well-known categorical equivalences between classes of MV-algebras and classes of lattice-ordered abelian groups, namely
  - Mundici's equivalence:

    category of MV-algebras $\simeq$ category of ℓ-groups with strong unit
  - Di Nola-Lettieri's equivalence:

    category of perfect MV-algebras $\simeq$ category of ℓ-groups
- Application of the method 'toposes as bridges' to these Morita-equivalences
- Construction (by means of the investigation of certain classifying toposes) of a new class of (Morita-)equivalences containing in particular the one lifting Di Nola-Lettieri's equivalence

Results in connection with Mundici's equivalence

- The theory of ℓ-groups with strong unit is of presheaf type and in fact Morita-equivalent to an algebraic theory (namely that of MV-algebras)
- Bijective correspondence between the geometric theory of MV-algebras and the geometric theory of ℓ-u groups (in spite of the fact that they are not bi-interpretable)
- Logical characterization of the finitely presentable ℓ-u groups
- Form of compactness and completeness for the geometric theory of ℓ-u groups (in spite of the infinitary nature of this theory);
- Sheaf-theoretic version of Mundici's equivalence

Results in connection with Di Nola-Lettieri's equivalence

- The theory of perfect MV-algebras is of presheaf type and in fact Morita-equivalent to an algebraic theory (namely that of ℓ-groups)
- Three levels of partial bi-interpretability for
  - irreducible formulas
  - geometric sentences
  - imaginaries
- the finitely presentable models of the theory of perfect MV-algebras are finitely presentable as objects in the variety generated by Chang's algebra
- Representation result: every non-trivial finitely generated MV-algebra in the variety generated by Chang's MV-algebra is a finite direct product of perfect MV-algebras

Equivalences for local MV-algebras in varieties

- The theory of local MV-alebras is NOT of presheaf type
- The theory of local MV-algebras in an arbitrary proper subvariety of MV-algebras IS of presheaf type
- The theory of local MV-algebras in an arbitrary proper subvariety of MV-algebras is Morita-equivalent to a theory extending that of lattice-ordered abelian groups
- the finitely presentable models of the theory of local MV-algebras in an arbitrary proper subvariety are finitely presentable also with respect to the variety
- Representation result: every finitely generated MV-algebra in an abitrary proper subvariety of MV-algebras is a finite direct product of local MV-algebras

Categorical equivalences

- Mundici's equivalence: $\Gamma : \mathbb{L}_u\text{-mod}(\mathbf{Set}) \cong \mathbb{MV}\text{-mod}(\mathbf{Set})$
- Di Nola-Lettieri's equivalence: $\Delta : \mathbb{P}\text{-mod}(\mathbf{Set}) \cong \mathbb{L}\text{-mod}(\mathbf{Set})$

Lifts

- $\Gamma_{\mathcal{E}} : \mathbb{L}_u\text{-mod}(\mathcal{E}) \cong \mathbb{MV}\text{-mod}(\mathcal{E})$
- $\Delta_{\mathcal{E}} : \mathbb{P}\text{-mod}(\mathcal{E}) \cong \mathbb{L}\text{-mod}(\mathcal{E})$

for every Grothendieck topos $\mathcal{E}$, naturally in $\mathcal{E}$

Morita-equivalences

- $\mathbb{MV}$ is Morita-equivalent to $\mathbb{L}_u$
- $\mathbb{P}$ is Morita-equivalent to $\mathbb{L}$

Bijective correspondence between quotients

subtoposes

$\mathscr{E}_{\mathbb{MV}} \simeq \mathscr{E}_{\mathbb{L}_u}$

quotients of $\mathbb{MV}$ quotients of $\mathbb{L}_u$

Theorem

Every quotient of the theory $\mathbb{MV}$ is Morita-equivalent to a quotient of the theory $\mathbb{L}_u$, and conversely. These Morita-equivalences are the restrictions of the one between $\mathbb{MV}$ and $\mathbb{L}_u$.

This result is non-trivial since the two theories are **not** bi-interpretable.

Partial bi-interpretations

irreducible objects
subterminal objects
coherent objects

$\mathscr{E}_{\mathbb{P}} \simeq \mathscr{E}_{\mathbb{L}}$

$\mathbb{P}$ $\mathbb{L}$

$\mathbb{P}$-irreducible formulas
geometric sentences over $\Sigma_{\mathbb{P}}$
imaginaries for $\mathbb{P}$

$\mathbb{L}$-irreducible formulas
geometric sentences over $\Sigma_{\mathbb{L}}$
imaginaries for $\mathbb{L}$

These bi-interpretations are interesting since we do **not** have bi-interpretability at the level of the coherent syntactic categories of the two theories.

Representation result

Theorem

Every finitely generated non-trivial MV-algebra in Chang's variety is a direct product of a finite family of finitely generated perfect MV-algebras.

Proof. Recall that the theory $\mathbb{P}$ is a quotient of the theory of Chang's variety obtained by adding the sequent

$$\top \vdash_x (2x)^2 = 0 \vee (2x)^2 = 1$$

This sequent generates the topology J associated with $\mathbb{P}$. If $\mathcal{A}$ is a finitely generated MV-algebra in Chang's variety and $\{a_1, \ldots, a_n\}$ is generating system of $\mathcal{A}$ then the final algebras in the following diagram (which generates a J-covering cosieve) are perfect MV-algebras.

Representation result

$$
\begin{tikzcd}
 & & & & (\mathcal{A}/_{((2a_1)^2)}/\dots)/_{([\dots[(2a_n)^2]\dots])} \\
 & & \mathcal{A}/_{((2a_1)^2)}/_{([(2a_2)^2])} & \cdots \arrow[ur] \arrow[dr] & \\
 & & & & (\mathcal{A}/_{((2a_1)^2)}/\dots)/_{(\neg[\dots[(2a_n)^2]\dots])} \\
 & \mathcal{A}/_{((2a_1)^2)} \arrow[uur] \arrow[dr] & & & \\
 & & \mathcal{A}/_{((2a_1)^2)}/_{(\neg[(2a_2)^2])} & \cdots & \\
\mathcal{A} \arrow[uur] \arrow[ddr] & & & & \\
 & & \mathcal{A}/_{(\neg(2a_1)^2)}/_{([(2a_2)^2])} & \cdots & \\
 & \mathcal{A}/_{(\neg(2a_1)^2)} \arrow[ur] \arrow[ddr] & & & \\
 & & & & (\mathcal{A}/_{(\neg(2a_1)^2)}/\dots)/_{([\dots[(2a_n)^2]\dots])} \\
 & & \mathcal{A}/_{(\neg(2a_1)^2)}/_{(\neg[(2a_2)^2])} & \cdots \arrow[ur] \arrow[dr] & \\
 & & & & (\mathcal{A}/_{(\neg(2a_1)^2)}/\dots)/_{(\neg[\dots[(2a_n)^2]\dots])}
\end{tikzcd}
$$

From perfect to local MV-algebras

$$\mathit{Perfect} = \mathit{Local} \cap V(S_1^{\omega})$$

We proved that $\mathbb{P}$ is a theory of presheaf type which is Morita-equivalent to the theory $\mathbb{L}$. It is natural to wonder if there is a theory axiomatizing

$$\mathit{Local} \cap V$$

which it is also of presheaf type and Morita-equivalent to a theory extending the theory $\mathbb{L}$.

From perfect to local MV-algebras

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$$Local \cap MV$$

Theorem

The geometric theory of local MV-algebras is not of presheaf type.

Local MV-algebras in a proper variety V

Komori's theorem

An arbitrary proper subvariety of MV-algebras is of the form

$$V = V(\{S_i\}_{i\in I}, \{S_j^\omega\}_{j\in J})$$

where $S_i = \Gamma(\mathbb{Z}, i)$ are simple MV-algebras, $S_j^\omega = \Gamma(\mathbb{Z} \times_{lex} \mathbb{Z}, (j, 0))$ are called Komori chains and I and J are finite subset of $\mathbb{N}$.

We set:

- $n = \text{l.c.m.}(I, J)$ (invariant w.r.t. the generators)
- $\mathbb{T}_V$: theory of the variety V
- $\mathbb{L}oc_V$: theory of local MV-algebra in V

First axiomatization for $\mathbb{L}oc_V$:

- axioms of $\mathbb{T}_V$
- $\sigma_n : \top \vdash_x ((n+1)x)^2 = 0 \vee ((n+1)x)^2 = 1$

We call J_1 the Grothendieck topology associated with this axiomatization.

The Grothendieck topology J_1

The covering are finite multicompositions of diagrams of this form.

$$\begin{tikzcd} & \mathcal{A}/(((n+1)x)^2) \\ \mathcal{A} \arrow[ur] \arrow[dr] & \\ & \mathcal{A}/(\neg((n+1)x)^2) \end{tikzcd}$$

Proposition

The Grothendieck topology J_1 is subcanonical.

Definition

A Grothendieck topology J on a category $\mathcal{C}$ is subcanonical if the arrows in a J-covering of an element $c \in \mathcal{C}$ form a limit diagrams with respect to the diagram consisting of all morphisms between them over c.

Cartesianization and finitely presentable models

Theorem

Every cartesian sequent that is provable in the theory $\mathbb{L}oc_V$ is also provable in the theory $\mathbb{T}_V$.

Proposition

The radical of every algebra $\mathcal{A}$ in the variety V is a defined by the following equation.

$$Rad(\mathcal{A}) = \{x \in A \mid ((n+1)x)^2 = 0\}$$

By using this we can obtain the following result.

Proposition

Every finitely presentable model of $\mathbb{L}oc_V$ is finitely presentable in V.

Rigidity and theories of presheaf type

Definition

A Grothendieck topology J on a category $\mathcal{C}$ is rigid if every object $c \in \mathcal{C}$ has a J-covering generated by J-irreducible objects.

Given a rigid topology J on a small category $\mathcal{C}$ we have an equivalence

$$\mathbf{Sh}(\mathcal{C}, J) \cong [\mathcal{D}^{op}, \mathbf{Set}]$$

where $\mathcal{D}$ is the full subcategory of $\mathcal{C}$ of the J-irreducible objects.

Theorem (Caramello)

Let $\mathbb{T}'$ be a quotient of a theory of presheaf type $\mathbb{T}$ corresponding to a Grothendieck topology J on the category $f.p.\mathbb{T}\text{-}mod(\mathbf{Set})^{op}$ under the Duality Theorem for subtoposes. Suppose that $\mathbb{T}'$ is itself of presheaf type. Then every finitely presentable $\mathbb{T}'$-model is finitely presentable also as a $\mathbb{T}$-model if and only if the topology J is rigid.

It follows that the theory $\mathbb{L}oc_V$ is of presheaf type if and only if the associated Grothendieck topology is rigid.

Local MV-algebras of finite rank

A local MV-algebra $\mathcal{A}$ is said to be of *finite rank* if there is an isomorphism

$$\phi_{\mathcal{A}} : \mathcal{A}/Rad(\mathcal{A}) \to S_m$$

and m is the *rank* of $\mathcal{A}$. For every $d \in S_m$, $\phi_{\mathcal{A}}^{-1}(d)$ is called *radical class* of $\mathcal{A}$.

Theorem (Di Nola-Esposito-Gerla)

Every local MV-algebra in V has finite rank and its rank divides the rank of one of the generators of V.

Second axiomatization for $\mathbb{L}oc_V$:

- axioms of $\mathbb{T}_V$
- $\rho_n : \top \vdash_x \bigvee_{d=0}^{n} x \in Fin_d$, where Fin_d is the formula describing $\phi^{-1}(d)$

Proposition

The following sequents are provable in $\mathbb{T}_V$.

$$x \in Fin_d \wedge y \in Fin_b \vdash_{x,y} x \oplus y \in Fin_{d \oplus b}$$

$$x \in Fin_d \vdash_x \neg x \in Fin_{n-d}$$

Main result

Theorem

The theory $\mathbb{L}oc_V$ is of presheaf type.

Proof. We call J_2 the topology associated with the second axiomatization which is obtained by finite multicompositions of diagrams of this form

$$
\begin{array}{ccc}
 & & \mathcal{A}/(x \in Fin_0(\mathcal{A})) \\
 & \nearrow & \vdots \\
\mathcal{A} & \longrightarrow & \mathcal{A}/(x \in Fin_d(\mathcal{A})) \\
 & \searrow & \vdots \\
 & & \mathcal{A}/(x \in Fin_n(\mathcal{A}))
\end{array}
$$

where $\mathcal{A}$ is a finitely presentable algebra in V. If we choose at each step one of the generators of the algebra $\mathcal{A}$, the codomain algebras of the resulting diagram are local MV-algebras. Hence J_2 is rigid and $\mathbb{L}oc_V$ is of presheaf type.

Representation result

The two axiomatizations for $\mathbb{L}oc_V$ are equivalent whence $J_1 = J_2$. This implies that J_1 is rigid and therefore the following result.

Theorem

Every finitely generated MV-algebra in V is a finite product of local MV-algebras.

Representation theorem for algebras of finite ranks

Theorem (Di Nola-Esposito-Gerla)

Every local MV-algebra in V is of finite rank and its rank divides one of the ranks of the generators of V. Further, any local MV-algebra of finite rank is of the form

$$\Gamma(\mathbb{Z} \times_{lex} G, (k, g))$$

where G is an ℓ-group, $g \in G$ and k is the rank of the algebra.

Extension of the theory $\mathbb{L}$

Let $\mathbb{G}_{(I,J)}$ be the theory whose signature is the one of ℓ-groups to which we add an arbitrary constant and a 0-ary predicate R_k for each divisor k of the least common multiple of the numbers in I and J. The axioms of this theory are

- axioms of $\mathbb{L}$
- $(\top \vdash R_1)$;
- $(R_k \vdash R_{k'})$, for each k' which divides k;
- $(R_k \wedge R_{k'} \vdash R_{l.c.m.(k,k')})$, for any k, k';
- $(R_k \vdash_g g = 0)$, for every $k \in \delta(I) \setminus \delta(J)$;
- $(R_k \vdash \bot)$, for any $k \notin \delta(I) \cup \delta(J)$.

where we indicate with $\delta(I)$ and $\delta(J)$ respectively the sets of divisors of the numbers in I and J.

The theory $\mathbb{G}_{(I,J)}$ is of presheaf type and the models of $\mathbb{G}_{(I,J)}$ in **Set** can be identified with the triples (G, g, k), where G is an ℓ-group, $g \in G$ and $k \in \delta(I) \cup \delta(J)$.

New Morita-equivalences

Let $V = V(\{S_i\}_{i\in I}, \{S_j^{\omega}\}_{j\in J})$ be an arbitrary proper subvariety of MV-algebras.

Theorem

The categories of set-based models of the theories $\mathbb{L}oc_V$ and $\mathbb{G}_{(I,J)}$ are equivalent.

- $\Lambda_{(I,J)} : \mathbb{L}oc_V\text{-mod}(\mathbf{Set}) \to \mathbb{G}_{(I,J)}\text{-mod}(\mathbf{Set})$

$$\Lambda_{(I,J)}(\mathcal{A}) := (G, g, k)$$

  for every $\mathcal{A} \simeq \Gamma(\mathbb{Z} \times_{lex} G, (k, g))$ local MV-algebra $\mathbb{L}oc_V\text{-mod}(\mathbf{Set})$
- $M_{(I,J)} : \mathbb{G}_{(I,J)}\text{-mod}(\mathbf{Set}) \to \mathbb{L}oc_V\text{-mod}(\mathbf{Set})$

$$M_{(I,J)}(G, g, k) := \Gamma(\mathbb{Z} \times_{lex} G, (k, g))$$

  for every set-based model of $\mathbb{G}_{(I,J)}$

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