# HÁJEK'S FUZZY PROBABILITY LOGIC REVISITED

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Building logics for reasoning under probabilistic uncertainty:

1. Boolean logic is the calculus for events $\varphi$
2. Łukasiewicz logic determines reasoning with probabilities $P\varphi$

## Features

- $P\varphi$ reads as `probably` $\varphi$
- probability of $\varphi$ is identified with truth value of $P\varphi$
- nesting of $P$ is not allowed

# HÁJEK'S LOGIC FP

### Definition

The language of FP is built over variables $\{X_1, X_2, \dots\}$ and

- connectives of Boolean logic
- connectives of Łukasiewicz logic
- symbol $P$ for the modality `probably`

Any formula $\varphi \in \mathrm{Fm}(\mathrm{FP})$ is either

1. **non-modal**: $\varphi \in \mathrm{Fm}(\mathrm{Bool})$ or
2. **modal**: $\varphi$ is built from atomic modal formulas $P\psi$ with $\psi \in \mathrm{Fm}(\mathrm{Bool})$ using the connectives of Ł

## Axioms

- axioms of Bool for non-modal formulas
- axioms of Ł for modal formulas
- axioms for the modality $P$:

$$
\begin{aligned}
&\textbf{H1} \quad P(\varphi) \to (P(\varphi \Rightarrow \psi) \to P\psi) \\
&\textbf{H2} \quad P(\varphi') \leftrightarrow \neg P\varphi \\
&\textbf{H3} \quad P(\varphi \vee \psi) \leftrightarrow [(P\varphi \to P(\varphi \wedge \psi)) \to P\psi]
\end{aligned}
$$

## Rules

- MP for both modal and non-modal formulas
- if $\varphi \in \mathrm{Fm}(\mathrm{Bool})$, then $\varphi \vdash P\varphi$

A **probability Kripke frame** is a structure

$$\mathbf{F} = \langle W, \mathcal{B}, \mu \rangle,$$

where $W$ is a set of worlds and $\mu \colon \mathcal{B} \to [0,1]$ is a state of a BA $\mathcal{B} \subseteq 2^W$:

$$\mu(a \vee b) = \mu(a) + \mu(b), \quad a \wedge b = \bot, \qquad \mu(\top) = 1$$

**Definition**

A Kripke model over $\mathbf{F}$ is a structure $\mathbf{K} = \langle \mathbf{F}, (e_w)_{w \in W} \rangle$ such that

- $e_w$ is a Boolean evaluation of non-modal formulas for each $w \in W$
- $\{w \in W \mid e_w(\varphi) = 1\} \in \mathcal{B}$, for each $\varphi \in \mathrm{Fm}(\mathrm{Bool})$

**Definition**

Let $\mathbf{K} = \langle \mathbf{F}, (e_w)_{w \in W} \rangle$ be a Kripke model over $\mathbf{F}$. If $\varphi \in \mathrm{Fm}(\mathrm{FP})$ is

- non-modal then $\|\varphi\|_{\mathbf{K}}^{w} := e_w(\varphi)$ for each $w \in W$,
- an atomic modal formula $P\psi$, then

$$\|P\psi\|_{\mathbf{K}} := s\big(\{w \in W \mid e_w(\psi) = 1\}\big),$$

- a non-atomic modal formula, then $\|\varphi\|_{\mathbf{K}}$ is computed by using the operations of $[0,1]_{MV}$

**Theorem (Hájek)**

Let $\Phi$ and $\Gamma$ be finite sets of non-modal and modal formulas, respectively, and let $\psi \in \mathrm{Fm}(\mathrm{FP})$. Then TFAE:

- $\Phi, \Gamma \vdash_{FP} \psi$
- $\|\psi\|_{\mathsf{K}} = 1$ for all Kripke models $\mathsf{K}$ over every probability Kripke frame satisfying $\Phi$ and $\Gamma$

# GENERALIZED STATES

1. Boolean algebra of events $\langle \mathcal{E}, \vee, \wedge, ', \top, \bot \rangle$
2. MV-algebra of probability degrees $\langle \mathcal{D}, \oplus, \odot, \neg, 1, 0 \rangle$

**Definition**

A mapping $s \colon \mathcal{E} \to \mathcal{D}$ is a generalized state if

- for every $a, b \in \mathcal{E}$ such that $a \wedge b = \bot$,

$$s(a \vee b) = s(a) \oplus s(b) \quad \text{and} \quad s(a) \odot s(b) = 0$$

- $s(\top) = 1$

**Example**

Any state of $\mathcal{E}$ is a generalized state $\mathcal{E} \to [0,1]$.

Let $\mathcal{S}(\mathcal{E})$ be the state space of $E$ and let $\mathcal{D} = \mathrm{C}(\mathcal{S}(\mathcal{E}))$.

**Example**

- For each $a \in \mathcal{E}$, the map $\hat{a} \colon P \in \mathcal{S}(\mathcal{E}) \mapsto P(a)$ is continuous affine.
- Let $s(a) = \hat{a}$. Then
$$s \colon E \to \mathrm{C}(\mathcal{S}(\mathcal{E}))$$
is a generalized state.

But $\mathcal{D}$ can even be non-semisimple...

**What is a probability that $n \in \mathbb{N}$ is drawn at random?**

Boolean algebra of events $\mathcal{E}$ is the finite-cofinite algebra:

$$\mathcal{E} = \{A \subseteq \mathbb{N} \mid A \text{ finite or cofinite}\}$$

Put

$$P(A) = \begin{cases} 0 & A \text{ finite,} \\ 1 & A \text{ cofinite.} \end{cases}$$

Then $P$ is a state of $\mathcal{E}$.

The probability that $n \in \mathbb{N}$ is drawn is 0.

**What is a probability that $n \in \mathbb{N}$ is drawn at random?**

Now we evaluate the events in Chang's algebra:

$$\mathcal{C} = \{0, \varepsilon, 2\varepsilon, \dots, 1-2\varepsilon, 1-\varepsilon, 1\}$$

Define:

$$s(A) = \begin{cases} |A|\varepsilon & A \text{ finite,} \\ 1 - |A'|\varepsilon & A \text{ cofinite.} \end{cases}$$

Then $s \colon \mathcal{E} \to \mathcal{C}$ is a generalized state.

The probability that $n \in \mathbb{N}$ is drawn is $\varepsilon$.

**Proposition**

A mapping $s\colon \mathcal{E} \to \mathcal{D}$ is a generalized state iff

1. $s(a \vee b) = s(a) \oplus s(b \wedge a')$
2. $s(a') = \neg s(a)$
3. $s(\top) = 1$

In case that $\mathcal{E} = \mathcal{D}$ the three identities are among axioms of internal states (Flaminio, Montagna).

# FP LOGIC WITH SEMANTICS BASED ON GENERALIZED STATES

### Axioms

- axioms of Bool for non-modal formulas
- axioms of Ł for modal formulas
- axioms for the modality $P$:

$$\begin{array}{ll} \textbf{A1} & P(\varphi \vee \psi) \leftrightarrow (P\varphi \oplus P(\psi \wedge \neg\varphi)) \\ \textbf{A2} & P(\varphi') \leftrightarrow \neg P\varphi \\ \textbf{A3} & P\overline{1} \end{array}$$

### Rules

- MP for both modal and non-modal formulas
- if $\varphi \in \mathrm{Fm}(\mathrm{Bool})$, then $\varphi \vdash P\varphi$

**Definition**

- A probabilistic structure is a triple $\mathbf{S} = \langle \mathcal{E}, \mathcal{D}, s \rangle$, where $s\colon \mathcal{E} \to \mathcal{D}$ is a generalized state.
- A probabilistic model is a pair $\mathbf{M} = \langle \mathbf{S}, v \rangle$, where $v$ is an $\mathcal{E}$-evaluation of non-modal formulas and $\mathbf{S}$ is a probab. structure.

If $\varphi \in \mathrm{Fm(FP)}$ is

1. non-modal, then $\|\varphi\|_{\mathbf{M}} := v(\varphi)$,
2. an atomic modal formula $P\psi$, then $\|P\psi\|_{\mathbf{M}} := s(v(\psi))$,
3. a non-atomic modal formula, then $\|\varphi\|_{\mathbf{M}}$ is computed by using the operations of $[0,1]_{MV}$.

### Theorem

Let $\Phi$ and $\Gamma$ be any sets of non-modal and modal formulas, respectively, and let $\psi \in \mathsf{Fm}(\mathsf{FP})$. Then TFAE:

- $\Phi, \Gamma \vdash_{FP} \psi$
- $\|\psi\|_{\mathbf{M}} = 1$ for all probabilistic models $\mathbf{M}$ satisfying $\Phi$ and $\Gamma$

### Corollary

FP is strongly complete with respect to all probabilistic models $\mathbf{M} = \langle \mathcal{E}, \mathcal{D}, s, v \rangle$ such that $\mathcal{D}$ is an MV-chain.

**Theorem**

Let $\Phi$ and $\Gamma$ be finite sets of non-modal and modal formulas, respectively, and let $\psi \in \mathrm{Fm}(\mathrm{FP})$. Then TFAE:

- $\Phi, \Gamma \vdash_{FP} \psi$
- $\|\psi\|_{\mathsf{M}} = 1$ for all probabilistic models $\mathsf{M} = \langle \mathcal{E}, [0,1], s, v \rangle$ satisfying $\Phi$ and $\Gamma$

- Let $V$ be the canonical basis of $\mathbb{R}^{2^n}$
- Let $\Delta_V \subseteq [0,1]^{2^n}$ be the convex hull of $V$

**Fact**

Let $\bar{\alpha}\colon V \to \{0,1\}$ be the BF corresponding to a non-modal formula $\alpha(X_1,\ldots,X_n)$. Then $\bar{\alpha}$ has a unique extension $f_{\bar{\alpha}} \in \mathcal{M}(\Delta_V)$ that is linear on each face of $\Delta_V$.

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1. $\mathcal{E} = \mathit{Free}_n(\mathrm{Bool}) \qquad v(X_i) = [X_i]$
2. $\mathcal{D} = \mathit{Free}_{2^n}(\text{Ł})/\equiv_P \quad \simeq \quad \mathcal{M}(\Delta_V)$
3. $s([\alpha]) = f_{\bar{\alpha}}$

**Theorem**

Let $\psi \in \mathrm{Fm}(\mathrm{FP})$. Then TFAE:

- $\vdash_{FP} \psi$
- $\|\psi\|_{\mathbf{M}} = 1$ for all finite probabilistic models $\mathbf{M} = \langle \mathcal{E}, [0,1], s, v \rangle$, that is, $\mathcal{E}$ is a finite BA.

- Development of more "algebraic" semantics for FP
- Algebraic framework for probabilistic structures $\langle \mathcal{E}, \mathcal{D}, s \rangle$
- FP over many-valued events — Łukasiewicz logic in both layers