Many-valued reasoning, from model theory to topological dynamics

Itaï BEN YAACOV

Université Claude Bernard Lyon 1
Institut Camille Jordan

ManyVal 2015

My original aim

Foundations for Model Theory of metric structures.
Necessarily oriented toward semantics.

My original aim

Foundations for Model Theory of metric structures.
Necessarily oriented toward semantics.

Nice by-product

"Fuzzy Topology", with applications in topological dynamics of Polish groups.

Outline

Chang and Keisler 1966: "Continuous model theory".

- Logic with truth values in arbitrary compact Hausdorff spaces.
- Logical connectives are arbitrary continuous functions.
- Quantifiers are continuous in the Vietoris topology:

$$\big(Qy\,\varphi(a,y)\big)^{\mathbf{M}} = Q\big\{\varphi(a,b)^{\mathbf{M}} : b \in \mathbf{M}\big\}.$$

Chang and Keisler 1966: "Continuous model theory".

- Logic with truth values in arbitrary compact Hausdorff spaces.
- Logical connectives are arbitrary continuous functions.
- Quantifiers are continuous in the Vietoris topology:

$$\big(Qy\,\varphi(a,y)\big)^{\mathbf{M}} = Q\big\{\varphi(a,b)^{\mathbf{M}} : b \in \mathbf{M}\big\}.$$

- As usual, a distinguished relation symbol for equality.

Beginnings

Chang and Keisler 1966: “Continuous model theory”.

- Logic with truth values in arbitrary compact Hausdorff spaces.
- Logical connectives are arbitrary continuous functions.
- Quantifiers are continuous in the Vietoris topology:

$$\left(Qy\,\varphi(a,y)\right)^{\mathbf{M}} = Q\left\{\varphi(a,b)^{\mathbf{M}} : b \in \mathbf{M}\right\}.$$

- As usual, a distinguished relation symbol for equality.
  CAVEAT: Requires distinguished truth valued for equal/unequal.

Chang and Keisler 1966: "Continuous model theory".

- Logic with truth values in arbitrary compact Hausdorff spaces.
- Logical connectives are arbitrary continuous functions.
- Quantifiers are continuous in the Vietoris topology:

$$\big(Qy\varphi(a,y)\big)^{\mathbf{M}} = Q\big\{\varphi(a,b)^{\mathbf{M}} : b \in \mathbf{M}\big\}.$$

- As usual, a distinguished relation symbol for equality.
  CAVEAT: Requires distinguished truth valued for equal/unequal.

Prove all the "usual" theorems: Łos, compactness, Löwenheim-Skolem, ...

Chang and Keisler 1966: "Continuous model theory".

- Logic with truth values in arbitrary compact Hausdorff spaces.
- Logical connectives are arbitrary continuous functions.
- Quantifiers are continuous in the Vietoris topology:

$$\big(Qy\,\varphi(a,y)\big)^{\mathbf{M}} = Q\{\varphi(a,b)^{\mathbf{M}} : b \in \mathbf{M}\}.$$

- As usual, a distinguished relation symbol for equality.
  CAVEAT: Requires distinguished truth valued for equal/unequal.

Prove all the "usual" theorems: Łos, compactness, Löwenheim-Skolem, ...

Very general indeed.

Maybe too general?

Any compact Hausdorff space X, embeds in $[0,1]^\lambda$, so any truth value $x \in X$ is coded by a sequence in $[0,1]$.

Maybe too general?

Any compact Hausdorff space X, embeds in $[0, 1]^{\lambda}$, so any truth value $x \in X$ is coded by a sequence in $[0, 1]$.
This suggests: essentially, Łukasiewicz logic.

Maybe too general?

Any compact Hausdorff space X, embeds in $[0,1]^{\lambda}$, so any truth value $x \in X$ is coded by a sequence in $[0,1]$.
This suggests: essentially, Łukasiewicz logic.

- Truth values in $[0,1]$ (or in compact subsets of $\mathbf{R}$).
- Connectives: continuous functions $[0,1]^n \to [0,1]$ (or $\mathbf{R}^n \to \mathbf{R}$).
- Quantifiers are sup and inf.

$$\left(\inf_y \varphi(a,y)\right)^{\mathbf{M}} = \inf\left\{\varphi(a,b) : b \in \mathbf{M}\right\}.$$

Maybe too general?

Any compact Hausdorff space X, embeds in $[0,1]^\lambda$, so any truth value $x \in X$ is coded by a sequence in $[0,1]$.
This suggests: essentially, Łukasiewicz logic.

- Truth values in $[0,1]$ (or in compact subsets of $\mathbf{R}$).
- Connectives: continuous functions $[0,1]^n \to [0,1]$ (or $\mathbf{R}^n \to \mathbf{R}$).
- Quantifiers are sup and inf.

$$\left(\inf_y \varphi(a,y)\right)^{\mathbf{M}} = \inf\left\{\varphi(a,b) : b \in \mathbf{M}\right\}.$$

No loss of expressive power.

And yet, not general enough... (the problem with Equality)

Like Chang and Keisler, we may restrict the truth values of $(x = y)$ to two distinguished truth values: 1 for equality, 0 for inequality.

- Reflexivity: $(x = y) = 1$ iff $x = y$.
- Symmetry: $(x = y) = (y = x)$.
- Transitivity: $(x = z) \geq (x = y) + (y = z) - 1$.

And yet, not general enough... (the problem with Equality)

Like Chang and Keisler, we may restrict the truth values of $(x = y)$ to two distinguished truth values: 0 for equality, 1 for inequality.

- Reflexivity: $(x = y) = 0$ iff $x = y$.
- Symmetry: $(x = y) = (y = x)$.
- Transitivity: $(x = z) \leq (x = y) + (y = z)$.

And yet, not general enough... (the problem with Equality)

Like Chang and Keisler, we may restrict the truth values of $(x = y)$ to two distinguished truth values: 0 for equality, 1 for inequality.

- Reflexivity: $d(x, y) = 0$ iff $x = y$.
- Symmetry: $d(x, y) = d(y, x)$.
- Transitivity: $d(x, z) \leq d(x, y) + d(y, z)$.

But then, we might as well drop the restriction on truth values, and write $d(x, y)$ instead of $(x = y)$...

$\Longrightarrow$ Natural logic for metric structures.

- Predicate symbols and formulae take values in compact subsets of $\mathbf{R}$.
- Connectives are continuous $\mathbf{R}^n \to \mathbf{R}$.
  By Stone-Weierstrass: $+$, $\cdot$, $-$, $\mathbf{Q}$ (and maybe $|\cdot|$) suffice.
- Quantifiers are $\sup$ and $\inf$.
- Distinguished predicate symbol d for distance.
- Predicate and function symbols are required to be uniformly continuous wrt d.
  Consequently, every formula is uniformly continuous.

- Predicate symbols and formulae take values in compact subsets of $\mathbf{R}$.
- Connectives are continuous $\mathbf{R}^n \to \mathbf{R}$.
  By Stone-Weierstrass: $+$, $\cdot$, $-$, $\mathbf{Q}$ (and maybe $|\cdot|$) suffice.
- Quantifiers are $\sup$ and $\inf$.
- Distinguished predicate symbol d for distance.
- Predicate and function symbols are required to be uniformly continuous wrt d.
  Consequently, every formula is uniformly continuous.
- Structures are complete bounded metric spaces.

- Predicate symbols and formulae take values in compact subsets of $\mathbf{R}$.
- Connectives are continuous $\mathbf{R}^n \to \mathbf{R}$.
  By Stone-Weierstrass: $+$, $\cdot$, $-$, $\mathbf{Q}$ (and maybe $|\cdot|$) suffice.
- Quantifiers are $\sup$ and $\inf$.
- Distinguished predicate symbol d for distance.
- Predicate and function symbols are required to be uniformly continuous wrt d.
  Consequently, every formula is uniformly continuous.
- Structures are complete bounded metric spaces.
- CONVENTION: $0 \in \mathbf{R}$ is *True*.
  For positive truth values, $\forall = \sup$, $\exists = \inf$.

Model Theory for metric structures in First Order Continuous Logic

- Predicate symbols and formulae take values in compact subsets of $\mathbf{R}$.
- Connectives are continuous $\mathbf{R}^n \to \mathbf{R}$.
  By Stone-Weierstrass: $+$, $\cdot$, $-$, $\mathbf{Q}$ (and maybe $|\cdot|$) suffice.
- Quantifiers are $\sup$ and $\inf$.
- Distinguished predicate symbol d for distance.
- Predicate and function symbols are required to be uniformly continuous wrt d.
  Consequently, every formula is uniformly continuous.
- Structures are complete bounded metric spaces.
- CONVENTION: $0 \in \mathbf{R}$ is *True*.
  For positive truth values, $\forall = \sup$, $\exists = \inf$.

Superseded previous formalisms such as Henson's True/False logic of "approximate satisfaction" in Banach space structures.
Stability theory etc. generalise painlessly.

Example

The probability algebra associated to a probability space (Ω, Σ, μ) is the Boolean algebra $\overline{\Sigma} = \Sigma/\ker\mu$, equipped with the induced measure function $\overline{\mu}\colon \overline{\Sigma} \to [0, 1]$.

Example

The probability algebra associated to a probability space (Ω, Σ, μ) is the Boolean algebra $\overline{\Sigma} = \Sigma / \ker \mu$, equipped with the induced measure function $\overline{\mu}\colon \overline{\Sigma} \to [0, 1]$. Axioms:

Boolean algebra axioms $\forall \bar{x}\, \sigma = \tau$	$\sup_{\bar{x}} d\big(\sigma(\bar{x}), \tau(\bar{x})\big) \qquad [\ = 0\ ]$
Finite additivity	$\sup_{x,y} \lvert \mu(x) + \mu(y) - \mu(x \wedge y) - \mu(x \vee y) \rvert$
$\mu(0) = 0$	$\mu(0)$
$\mu(1) = 1$	$\mu(1) - 1$
Atomless	$\sup_{x} \inf_{y} \lvert \mu(x \wedge y) - \mu(x)/2 \rvert$

Example

The probability algebra associated to a probability space (Ω, Σ, μ) is the Boolean algebra $\overline{\Sigma} = \Sigma/\ker\mu$, equipped with the induced measure function $\overline{\mu}\colon \overline{\Sigma} \to [0,1]$. Axioms:

Boolean algebra axioms $\forall\bar{x}\,\sigma = \tau$	$\sup_{\bar{x}} d\big(\sigma(\bar{x}), \tau(\bar{x})\big) \qquad [\ = 0\ ]$
Finite additivity	$\sup_{x,y} \big\lvert\mu(x) + \mu(y) - \mu(x \wedge y) - \mu(x \vee y)\big\rvert$
$\mu(0) = 0$	$\mu(0)$
$\mu(1) = 1$	$\mu(1) - 1$
Atomless	$\sup_{x} \inf_{y} \big\lvert\mu(x \wedge y) - \mu(x)/2\big\rvert$

The models of **Pr** $= \{$all axioms but last$\}$ are the probability algebras.
The models of **APr** $= \{$all axioms$\}$ are the atomless probability algebras.

Example

The probability algebra associated to a probability space (Ω, Σ, μ) is the Boolean algebra $\overline{\Sigma} = \Sigma / \ker \mu$, equipped with the induced measure function $\overline{\mu} \colon \overline{\Sigma} \to [0, 1]$. Axioms:

Boolean algebra axioms $\forall \bar{x}\, \sigma = \tau$	$\sup_{\bar{x}} d\big(\sigma(\bar{x}), \tau(\bar{x})\big) \qquad [\ = 0\ ]$
Finite additivity	$\sup_{x,y} \lvert \mu(x) + \mu(y) - \mu(x \wedge y) - \mu(x \vee y) \rvert$
$\mu(0) = 0$	$\mu(0)$
$\mu(1) = 1$	$\mu(1) - 1$
Atomless	$\sup_{x} \inf_{y} \lvert \mu(x \wedge y) - \mu(x)/2 \rvert$

The models of **Pr** $= \{$all axioms but last$\}$ are the probability algebras.
The models of **APr** $= \{$all axioms$\}$ are the atomless probability algebras.
These are elementary classes. **APr** is a complete theory, with quantifier elimination.

Outline

Type spaces

Let T be a (complete) theory (e.g., $\mathbf{APr}$). Say $\mathbf{M} \vDash T$ is a model and $\bar{a} \in \mathbf{M}^n$. The type of $\bar{a}$, call it $p = \mathrm{tp}(\bar{a})$ is the map

$$p\colon \varphi \mapsto \varphi^p = \varphi(\bar{a}),$$

where φ varies over formulae with n free variables.

Let T be a (complete) theory (e.g., **APr**). Say $\mathbf{M} \models T$ is a model and $\bar{a} \in \mathbf{M}^n$. The type of $\bar{a}$, call it $p = \mathrm{tp}(\bar{a})$ is the map

$$p\colon \varphi \mapsto \varphi^p = \varphi(\bar{a}),$$

where φ varies over formulae with n free variables.
The space of all types of n-tuples in models of T is denoted $\mathrm{S}_n(T)$. We equip it with the minimal topology in which every map

$$\overline{\varphi}\colon p \to \varphi^p$$

is continuous.

Type spaces

Let T be a (complete) theory (e.g., $\mathbf{APr}$). Say $\mathbf{M} \models T$ is a model and $\bar{a} \in \mathbf{M}^n$.
The type of $\bar{a}$, call it $p = \mathrm{tp}(\bar{a})$ is the map

$$p \colon \varphi \mapsto \varphi^p = \varphi(\bar{a}),$$

where φ varies over formulae with n free variables.
The space of all types of n-tuples in models of T is denoted $\mathrm{S}_n(T)$. We equip it with the minimal topology in which every map

$$\overline{\varphi} \colon p \to \varphi^p$$

is continuous.
The Compactness Theorem for continuous logic asserts that $\mathrm{S}_n(T)$ is compact.

Theorem (Folklore?)

Say T is a complete countable theory in classical (binary) logic and $p \in \mathrm{S}_n(T)$. TFAE:

- *Every $\mathbf{M} \vDash T$ contains a realisation of p (p cannot be omitted)*
- *The type p is isolated in $\mathrm{S}_n(T)$ (by a formula = topologically): near p, the topology is discrete.*

Theorem (Folklore?)

Say T is a complete countable theory in classical (binary) logic and $p \in \mathrm{S}_n(T)$. TFAE:

- *Every $\mathbf{M} \vDash T$ contains a realisation of p (p cannot be omitted)*
- *The type p is isolated in $\mathrm{S}_n(T)$ (by a formula = topologically): near p, the topology is discrete.*

False in continuous logic: the type $\mathrm{tp}(0) \in \mathrm{S}_1(\mathbf{APr})$ is not topologically isolated.

Omitting types

Theorem (Folklore?)

Say T is a complete countable theory in classical (binary) logic and $p \in S_n(T)$. TFAE:

- *Every $\mathbf{M} \vDash T$ contains a realisation of p (p cannot be omitted)*
- *The type p is isolated in $S_n(T)$ (by a formula = topologically): near p, the topology is discrete.*

False in continuous logic: the type $\mathrm{tp}(0) \in S_1(\mathbf{APr})$ is not topologically isolated. No type is.
And every type is isolated by "$\varphi(x) = 0$".

Theorem (Folklore?)

Say T is a complete countable theory in classical (binary) logic and $p \in \mathrm{S}_n(T)$. TFAE:

- *Every $\mathbf{M} \vDash T$ contains a realisation of p (p cannot be omitted)*
- *The type p is isolated in $\mathrm{S}_n(T)$ (by a formula = topologically): near p, the topology is discrete.*

False in continuous logic: the type $\mathrm{tp}(0) \in \mathrm{S}_1(\mathbf{APr})$ is not topologically isolated. No type is.
And every type is isolated by "$\varphi(x) = 0$".

Theorem (Henson)

Say T is a complete countable theory in continuous logic and $p \in \mathrm{S}_n(T)$. TFAE:

- *Every $\mathbf{M} \vDash T$ contains a realisation of p.*
- *Near p, the topology coincides with the distance*

$$\partial(q, r) = \inf \left\{ d(\bar{a}, \bar{b}) : \mathrm{tp}(\bar{a}) = q \, , \, \mathrm{tp}(\bar{b}) = r \right\}.$$

Stability

- A classical ($\{0,1\}$-valued) theory T is κ-stable if, after naming κ constants, $|S_1(T)| \leq \kappa$.
- It is κ-categorical it if has a unique model of cardinality κ.

Theorem (Morley)

- $\aleph_1$*-categorical* $\implies$ $\aleph_0$*-stable* $\implies$ *existence of prime models over sets.*
- $\aleph_1$*-categorical* $\iff$ κ*-categorical, for any* $\kappa \geq \aleph_1$.

Stability

- A classical ($\{0,1\}$-valued) theory T is κ-stable if, after naming κ constants, $|S_1(T)| \leq \kappa$.
- It is κ-categorical it if has a unique model of cardinality κ.

Theorem (Morley)

- *$\aleph_1$-categorical $\implies$ $\aleph_0$-stable $\implies$ existence of prime models over sets.*
- *$\aleph_1$-categorical $\iff$ κ-categorical, for any $\kappa \geq \aleph_1$.*

Many Hilbert spaces of any cardinal $\kappa^{\aleph_0}$. But: unique Hilbert space of density character $\kappa > \aleph_0$. But #2: Type spaces in Hilbert spaces have size $2^{\aleph_0}$.

Stability

- A classical ($\{0,1\}$-valued) theory T is κ-stable if, after naming κ constants, $|S_1(T)| \leq \kappa$.
- It is κ-categorical it if has a unique model of cardinality κ.

Theorem (Morley)

- *$\aleph_1$-categorical $\Longrightarrow$ $\aleph_0$-stable $\Longrightarrow$ existence of prime models over sets.*
- *$\aleph_1$-categorical $\Longleftrightarrow$ κ-categorical, for any $\kappa \geq \aleph_1$.*

Many Hilbert spaces of any cardinal $\kappa^{\aleph_0}$. But: unique Hilbert space of density character $\kappa > \aleph_0$. But #2: Type spaces in Hilbert spaces have size $2^{\aleph_0}$.

- A metric theory T is κ-stable if, after naming κ constants, the density character of $(S_1(T), \partial)$ is $\leq \kappa$.
- It is κ-categorical it if has a unique model of density character κ.

Stability

- A classical ($\{0,1\}$-valued) theory T is κ-stable if, after naming κ constants, $|S_1(T)| \leq \kappa$.
- It is κ-categorical it if has a unique model of cardinality κ.

Theorem (Morley)

- *$\aleph_1$-categorical $\Longrightarrow$ $\aleph_0$-stable $\Longrightarrow$ existence of prime models over sets.*
- *$\aleph_1$-categorical $\Longleftrightarrow$ κ-categorical, for any $\kappa \geq \aleph_1$.*

Many Hilbert spaces of any cardinal $\kappa^{\aleph_0}$. But: unique Hilbert space of density character $\kappa > \aleph_0$. But #2: Type spaces in Hilbert spaces have size $2^{\aleph_0}$.

- A metric theory T is κ-stable if, after naming κ constants, the density character of $(S_1(T), \partial)$ is $\leq \kappa$.
- It is κ-categorical it if has a unique model of density character κ.

Theorem (B.)

Morley's Theorem for metric theories.

Topometric spaces

- One keeps encountering objects carrying "similar structure".
- At some point, one wants to give them a name.

- One keeps encountering objects carrying “similar structure”.
- At some point, one wants to give them a name.

Definition

A topometric space is a triplet (X, τ, ∂) where

- (X, τ) is a topological space.
- ∂ is a distance on X refining τ, and lower semi-continuous wrt. τ.

CONVENTION: topological vocabulary applies to τ, metric vocabulary to ∂, except when qualified.

Example

- Type spaces $\mathrm{S}_n(T)$, $\mathrm{S}_n(A)$, as defined earlier.
- Local type spaces $\mathrm{S}_\varphi(\mathbf{M})$ (useful for a single stable formula φ).
- $(E^*, w^*, \|\cdot\|)$, for a Banach space E (the unit ball is moreover compact).
- Polish groups (later).
- Samuel compatification (maybe).

Topometric spaces

- One keeps encountering objects carrying "similar structure".
- At some point, one wants to give them a name.

Definition

A topometric space is a triplet (X, τ, ∂) where

- (X, τ) is a topological space.
- ∂ is a distance on X refining τ, and lower semi-continuous wrt. τ.

CONVENTION: topological vocabulary applies to τ, metric vocabulary to ∂, except when qualified.

Example

- Type spaces $\mathrm{S}_n(T)$, $\mathrm{S}_n(A)$, as defined earlier.
- Local type spaces $\mathrm{S}_\varphi(\mathbf{M})$ (useful for a single stable formula φ).
- $(E^*, w^*, \|\cdot\|)$, for a Banach space E (the unit ball is moreover compact).
- Polish groups (later).
- Samuel compatification (maybe).

Can be thought of as "fuzzy topology".

Outline

Definition

- A Polish group is a topological group G which is separable and completely metrisable.

Polish groups

Definition

- A Polish group is a topological group G which is separable and completely metrisable.
- Equivalently: $G \cong \mathrm{Aut}(\mathbf{M})$ for a (complete) separable structure $\mathbf{M}$, with point-wise convergence.

Polish groups

Definition

- A Polish group is a topological group G which is separable and completely metrisable.
- Equivalently: $G \cong \mathrm{Aut}(\mathbf{M})$ for a (complete) separable structure $\mathbf{M}$, with point-wise convergence.

For many interesting countable (discrete) structures, there are co-meagre conjugation classes in $\mathrm{Aut}(\mathbf{M})$ (and even $\mathrm{Aut}(\mathbf{M})^n$...). This has strong consequences (e.g., automatic continuity).

Definition

- A Polish group is a topological group G which is separable and completely metrisable.
- Equivalently: $G \cong \mathrm{Aut}(\mathbf{M})$ for a (complete) separable structure $\mathbf{M}$, with point-wise convergence.

For many interesting countable (discrete) structures, there are co-meagre conjugation classes in $\mathrm{Aut}(\mathbf{M})$ (and even $\mathrm{Aut}(\mathbf{M})^n$...). This has strong consequences (e.g., automatic continuity).

If $\mathbf{M}$ is a metric structure, then most often all conjugation classes of $\mathrm{Aut}(\mathbf{M})$ are meagre (e.g., ℓ^2, $\mathbf{APr}$, $\mathbf{U}_1$).
But we are missing part of the structure: the distance ∂ of uniform convergence. This makes $(\mathrm{Aut}(\mathbf{M}), \text{p-w conv.}, \partial)$ a topometric group.

Definition

- A Polish group is a topological group G which is separable and completely metrisable.
- Equivalently: $G \cong \mathrm{Aut}(\mathbf{M})$ for a (complete) separable structure $\mathbf{M}$, with point-wise convergence.

For many interesting countable (discrete) structures, there are co-meagre conjugation classes in $\mathrm{Aut}(\mathbf{M})$ (and even $\mathrm{Aut}(\mathbf{M})^n$...). This has strong consequences (e.g., automatic continuity).

If $\mathbf{M}$ is a metric structure, then most often all conjugation classes of $\mathrm{Aut}(\mathbf{M})$ are meagre (e.g., ℓ^2, $\mathbf{APr}$, $\mathbf{U}_1$).
But we are missing part of the structure: the distance ∂ of uniform convergence. This makes $\left(\mathrm{Aut}(\mathbf{M}), \text{p-w conv.}, \partial\right)$ a topometric group.

Theorem (Berenstein, Melleray, B.)

- *For $\mathbf{M} = \ell^2, \mathbf{APr}, \mathbf{U}_1$: co-meagre ∂-closures of conjugation classes.*
- *Various consequences.*
- *... (also Tsankov) $\Longrightarrow$ automatic continuity for $\mathrm{Aut}(\ell^2)$, $\mathrm{Aut}(\mu)$.*

Theorem (Kuratowski-Ulam)

Let X, Y be Polish space, $\pi\colon X \to Y$ continuous and open, and $A \subseteq X$ Baire-measurable (e.g., Borel). Let $X_y = \pi^{-1}(y)$ and $A_y = A \cap X_y$. TFAE:

- *A is co-meagre.*
- *The set $\{y \in Y : A_y \text{ is co-meagre in } X_y\}$ is co-meagre in Y.*

Theorem (Kuratowski-Ulam)

Let X, Y be Polish space, $\pi\colon X \to Y$ continuous and open, and $A \subseteq X$ Baire-measurable (e.g., Borel). Let $X_y = \pi^{-1}(y)$ and $A_y = A \cap X_y$. TFAE:

- *A is co-meagre.*
- *The set $\{y \in Y : A_y \text{ is co-meagre in } X_y\}$ is co-meagre in Y.*

What if (Y, τ, ∂) is a Polish topometric space? We face "fuzzy sets".

Theorem (Kuratowski-Ulam)

Let X, Y be Polish space, $\pi\colon X \to Y$ continuous and open, and $A \subseteq X$ Baire-measurable (e.g., Borel). Let $X_y = \pi^{-1}(y)$ and $A_y = A \cap X_y$. TFAE:

- *A is co-meagre.*
- *The set $\{y \in Y : A_y \text{ is co-meagre in } X_y\}$ is co-meagre in Y.*

What if (Y, τ, ∂) is a Polish topometric space? We face “fuzzy sets”.

- The fibre X_y is no longer a set, but a function: $X_y(x) = \partial(\pi x, y)$. $X_y(x) = 0$ means that x is “entirely” in the fibre.
- Then we might as well let A be a function too: $A(x) = 0$ means that x is entirely in the set, otherwise $A(x) > 0$.
  The relative fibre: $A_y(x) = A(x) + \partial(\pi x, y)$.

Polish topometric spaces

Theorem (Kuratowski-Ulam)

Let X, Y be Polish space, $\pi\colon X \to Y$ continuous and open, and $A \subseteq X$ Baire-measurable (e.g., Borel). Let $X_y = \pi^{-1}(y)$ and $A_y = A \cap X_y$. TFAE:

- *A is co-meagre.*
- *The set $\{y \in Y : A_y \text{ is co-meagre in } X_y\}$ is co-meagre in Y.*

What if (Y, τ, ∂) is a Polish topometric space? We face "fuzzy sets".

- The fibre X_y is no longer a set, but a function: $X_y(x) = \partial(\pi x, y)$. $X_y(x) = 0$ means that x is "entirely" in the fibre.
- Then we might as well let A be a function too: $A(x) = 0$ means that x is entirely in the set, otherwise $A(x) > 0$.
  The relative fibre: $A_y(x) = A(x) + \partial(\pi x, y)$.

Theorem (Melleray, B.)

Say $A\colon X \to [0, \infty]$ is a Baire-measurable, $\pi\colon (X, \tau_X) \to (Y, \tau_Y, \partial_Y)$ is continuous and topometrically open. (And that Y is adequate). TFAE:

- *A is co-meagre (i.e., $\{x : A(x) = 0\}$ is).*
- *The set $\{y \in Y : A_y \text{ is co-meagre in } X_y\}$ is co-meagre in Y.*
  (For an appropriate notion of a "fuzzy set" being co-meagre in another.)

Thank you.