Back and forth conditions for elementary equivalence in model theory of non-classical logics*

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Elementary equivalence is a central notion in classical model theory that allows the classification of first-order structures and has received several useful characterizations, among others, in terms of systems of back-and-forth equivalence [6]. In the context of fuzzy predicate logics, the notion of elementary equivalence was defined in [5]. A few recent papers have contributed to the model theory of predicate versions of fuzzy and other non-classical logics logics [3, 2, 1, 4]. However, the understanding of the central notion of elementary equivalence is still far from its counterpart in classical model theory. This talk intends to provide some advances towards this goal.

Let L be a fixed algebraizable logic in a propositional language $\mathcal{L}$. The language $\mathcal{P} = \langle \mathbf{P}, \mathbf{F}, Ar_{\mathcal{P}}\rangle$ of a first-order extension of L is defined in the same way as in classical first-order logic. Also $\mathcal{P}$-terms and $\mathcal{P}$-formulae and other syntactical notions are defined as in classical logic.

A $\mathcal{P}$-*structure* is $\langle \boldsymbol{A}, \mathbf{M}\rangle$ where $\boldsymbol{A}$ is an L-algebra and $\mathbf{M} = \langle S, \langle P_{\mathbf{M}}\rangle_{P\in\mathbf{P}}, \langle f_{\mathbf{M}}\rangle_{f\in\mathbf{F}}\rangle$, where $M \neq \emptyset$; $P_{\mathbf{M}}$ is a function $S^n \to A$, for each n-ary predicate symbol $P \in \mathbf{P}$ with $n \geq 1$ and an element of A if P is a truth constant; $f_{\mathbf{M}}$ is a function $M^n \to M$ for each n-ary $f \in \mathbf{F}$ with $n \geq 1$ and an element of M if f is an object constant. An $\mathbf{M}$-*evaluation* is a mapping v which assigns to each variable an element from S. Given a variable x and $a \in M$, $\mathrm{v}[x{\to}a]$ is an $\mathbf{M}$-evaluation such that $\mathrm{v}[x{\to}a](x) = a$ and $\mathrm{v}[x{\to}a](y) = \mathrm{v}(y)$ for each $y \neq x$. We define *values* of $\mathcal{P}$-terms and *truth values* of $\mathcal{P}$-formulae in $\mathbf{M}$ for an evaluation v as:

$$
\begin{array}{rcll}
\|x\|^{\mathbf{M}}_{\mathrm{v}} & = & \mathrm{v}(x), & \\
\|f(t_1,\dots,t_n)\|^{\mathbf{M}}_{\mathrm{v}} & = & f_{\mathbf{M}}(\|t_1\|^{\mathbf{S}}_{\mathrm{v}},\dots,\|t_n\|^{\mathbf{M}}_{\mathrm{v}}), & \text{for } f \in \mathbf{F} \\
\|P(t_1,\dots,t_n)\|^{\mathbf{M}}_{\mathrm{v}} & = & P_{\mathbf{M}}(\|t_1\|^{\mathbf{M}}_{\mathrm{v}},\dots,\|t_n\|^{\mathbf{M}}_{\mathrm{v}}), & \text{for } P \in \mathbf{P} \\
\|c(\varphi_1,\dots,\varphi_n)\|^{\mathbf{M}}_{\mathrm{v}} & = & c^{\boldsymbol{A}}(\|\varphi_1\|^{\mathbf{M}}_{\mathrm{v}},\dots,\|\varphi_n\|^{\mathbf{M}}_{\mathrm{v}}), & \text{for } c \in \mathcal{L} \\
\|(\forall x)\varphi\|^{\mathbf{M}}_{\mathrm{v}} & = & \inf_{\leq^{\boldsymbol{A}}}\{\|\varphi\|^{\mathbf{M}}_{\mathrm{v}[x\to a]} \mid a \in M\}, & \\
\|(\exists x)\varphi\|^{\mathbf{M}}_{\mathrm{v}} & = & \sup_{\leq^{\boldsymbol{A}}}\{\|\varphi\|^{\mathbf{M}}_{\mathrm{v}[x\to a]} \mid a \in M\}. &
\end{array}
$$

We say that $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is *safe* iff $\|\varphi\|^{\mathbf{M}}_{\mathrm{v}}$ is always defined. $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is a *model* of a set of formulae Γ if it is safe and for every $\varphi \in \Gamma$ and every $\mathbf{M}$-evaluation v, $\|\varphi\|^{\mathbf{M}}_{\mathrm{v}} \in \mathcal{F}^{\boldsymbol{A}}$ (where $\mathcal{F}^{\boldsymbol{A}}$ is the filter of designated elements of the algebra $\boldsymbol{A}$). The semantical notion of consequence is defined in the usual way (every model of the premises is also a model of the conclusion) and general completeness theorems are proved in [1]) with respect to suitable Hilbert-style calculus.

*Dellunde is supported by the Spanish project MILESS (TIN2013-45039-P) and the grant 2014SGR-118 from Generalitat de Catalunya; García-Cerdaña by the MICINN project *Lógica algebraica y lógicas no clásicas* (MTM2011-25747), the CSIC project 201450E045 and the grant 2014SGR-788 from Generalitat de Catalunya; Noguera by project of Austrian Science Fund (FWF) I1897-N25 and Czech Science Foundation (GACR) 15-34650L.

In the general framework that we are considering the usual classical notion of elementary equivalence can be generalized in three different meaningful ways:

Definition 1. *Let $\langle \boldsymbol{A}, \mathbf{M}\rangle$ and $\langle \boldsymbol{B}, \mathbf{N}\rangle$ be safe $\mathcal{P}$-structures. We say that they are:*

1 Elementarily equivalent *(in symbols: $\langle \boldsymbol{A}, \mathbf{M}\rangle \equiv \langle \boldsymbol{B}, \mathbf{N}\rangle$) if, for every $\mathcal{P}$-sentence σ, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} \in \mathcal{F}^{\boldsymbol{A}}$ if and only if $\|\sigma\|^{\boldsymbol{B}}_{\mathbf{N}} \in \mathcal{F}^{\boldsymbol{B}}$.*

Assume now that $\boldsymbol{A} \subseteq \boldsymbol{B}$.

2 Filter-strongly elementarily equivalent *(in symbols: $\langle \boldsymbol{A}, \mathbf{M}\rangle \equiv^{fs} \langle \boldsymbol{B}, \mathbf{N}\rangle$) if, for every $\mathcal{P}$-sentence σ, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} \in \mathcal{F}^{\boldsymbol{A}}$ if and only if $\|\sigma\|^{\boldsymbol{B}}_{\mathbf{N}} \in \mathcal{F}^{\boldsymbol{B}}$ and, in this case, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\sigma\|^{\boldsymbol{B}}_{\mathbf{N}}$.*

3 Strongly elementarily equivalent *(in symbols: $\langle \boldsymbol{A}, \mathbf{M}\rangle \equiv^{s} \langle \boldsymbol{B}, \mathbf{N}\rangle$) if, for every $\mathcal{P}$-sentence σ, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\sigma\|^{\boldsymbol{B}}_{\mathbf{N}} \in A$.*

The first two notions are clearly equivalent for logics with weakening, because then $\mathcal{F}^{\boldsymbol{A}} = \mathcal{F}^{\boldsymbol{B}} = \{\overline{1}^{\boldsymbol{A}}\}$. We can show counterexamples separating the three notions in general.

Definition 2 (Finitely isomorphic structures)**.** *Two safe $\mathcal{P}$-structures $\langle \boldsymbol{A}, \mathbf{M}\rangle$, $\langle \boldsymbol{B}, \mathbf{N}\rangle$ are said to be* finitely isomorphic, *written $\langle \boldsymbol{A}, \mathbf{M}\rangle \cong_f \langle \boldsymbol{B}, \mathbf{N}\rangle$ if there is a sequence $\langle I_n \mid n \in \mathsf{N}\rangle$ such that:*

1. *Every I_n is a non-empty set of partial isomorphisms from $\langle \boldsymbol{A}, \mathbf{M}\rangle$ to $\langle \boldsymbol{B}, \mathbf{N}\rangle$.*
2. *For each $n \in \mathsf{N}$, $I_{n+1} \subseteq I_n$.*
3. *(Forth-property I) For every $\langle p, r\rangle \in I_{n+1}$ and $d \in M$, there is $\langle p, r'\rangle \in I_n$ such that $r \subseteq r'$ and $d \in dom(r')$.*
4. *(Back-property I) For every $\langle p, r\rangle \in I_{n+1}$ and $e \in N$, there is $\langle p, r'\rangle \in I_n$ such that $r \subseteq r'$ and $e \in rg(r')$.*
5. *(Forth-property II) For every $\langle p, r\rangle \in I_{n+1}$ and $a \in A$, there is $\langle p', r\rangle \in I_n$ such that $p \subseteq p'$ and $a \in dom(p')$.*
6. *(Back-property II) For every $\langle p, r\rangle \in I_{n+1}$ and $b \in B$, there is $\langle p', r\rangle \in I_n$ such that $p \subseteq p'$ and $b \in rg(p')$.*

This notion of finitely isomorphic structures gives a sufficient condition for (strong) elementary equivalence as stated by following theorem, which we can prove by translating $\mathcal{P}$-structures into classical 2-sorted structures.

Theorem 3. *Let $\mathcal{P}$ be a finite predicate language and $\langle \boldsymbol{A}, \mathbf{M}\rangle$, $\langle \boldsymbol{B}, \mathbf{N}\rangle$ be safe $\mathcal{P}$-structures. Then:*

$$\langle \boldsymbol{A}, \mathbf{M}\rangle \cong_f \langle \boldsymbol{B}, \mathbf{N}\rangle \quad \Rightarrow \quad \langle \boldsymbol{A}, \mathbf{M}\rangle \equiv \langle \boldsymbol{B}, \mathbf{N}\rangle.$$

We will show a counterexample for the reverse implication and consider the possibility of strengthening the result with strong elementary equivalence. Aiming to obtain characterizations of elementary equivalence we will also consider other forms of back and forth systems, involving a suitable notion of the nested rank of a formula and in terms of partial relative relations.

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