# Calculus Bits and Pieces

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# 1 Arc Length

Consider a small DIFFERENTIAL piece of a CURVE $ds^2 = dx^2 + dy^2$. Then LENGTH OF THE CURVE on some interval is given by

$$L = \int_a^b ds. \tag{1}$$

Which intuitively says that LENGTH OF A CURVE ON AN INTERVAL A TO B IS GIVEN BY A SUM OF ITS DIFFERENTIAL SEGMENTS.

SUBSTITUTING $dx^2 + dy^2$ for $ds$ and REARRANGING yields two possibilities:

| Function | $ds$ | $L$ |
|---|---|---|
| $y = f(x)$ | $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ | $\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ |
| $x = f(y)$ | $\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$ | $\int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$ |

**FIVE STEP PROCEDURE TO DERIVE A FORMULA FOR ARC LENGTH OF A CURVE IN THE WILD:**

1. DO REMEMBER $L = \int_a^b ds$.
2. DO REMEMBER $ds = \sqrt{(dx^2 + dy^2)}$
3. FACTOR OUT $dx^2$ and USE IT AS DIFFERENTIAL if the function is $y = f(x)$
4. OR FACTOR OUT $dy^2$ and USE IT AS DIFFERENTIAL if the function is $x = f(y)$
5. THERE'S NO STEP NUMBER FIVE

## 2 Surface Area

If a DIFFERENTIAL segment $ds$ is REVOLVED around $x$ or $y$ axis it will produce a TRUNCATED CONE with a SURFACE AREA

$$2\pi r ds, \tag{2}$$

where $r$ is DISTANCE FROM THE CURVE TO THE AXIS OF REVOLUTION defined as

| Function | Axis | $r$ |
|---|---|---|
| $y = f(x)$ | $x$ | $f(x)$ |
| $x = f(y)$ | $x$ | $y$ |
| $x = f(y)$ | $y$ | $f(y)$ |
| $y = f(x)$ | $y$ | $x$ |

INTEGRATING $2\pi r ds$ yields area of the SURFACE PRODUCED BY REVOLVING A SEGMENT OF A CURVE ON THE INTERVAL $a$ TO $b$

$$A = 2\pi \int_a^b r\, ds. \tag{3}$$

REPLACING $r$ and $ds$ yields the FOUR FORMULAS FOR CALCULATING SURFACE AREAS:

| Function | Axis | $r$ | $A$ with $r$ replaced | $A$ with $ds$ replaced |
|---|---|---|---|---|
| $y = f(x)$ | $x$ | $f(x)$ | $A = 2\pi \int_a^b f(x)\, ds$ | $A = 2\pi \int_a^b f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ |
| $x = f(y)$ | $x$ | $y$ | $A = 2\pi \int_a^b y\, ds$ | $A = 2\pi \int_a^b y\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$ |
| $x = f(y)$ | $y$ | $f(y)$ | $A = 2\pi \int_a^b f(y)\, ds$ | $A = 2\pi \int_a^b f(y)\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$ |
| $y = f(x)$ | $y$ | $x$ | $A = 2\pi \int_a^b x\, ds$ | $A = 2\pi \int_a^b x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$ |

The above table can be reduced to just a few steps, allowing you to recreate it from minimum amount of information.

**Procedure to derive a formula for surface area in the wild:**

1. Do remember $A = 2\pi \int_a^b r\, ds$.
2. Replace $r$ with an expression of independent variable that would represent the distance between the curve and axis
3. Replace $ds$(same as arc length):

   i Do remember $ds = \sqrt{(dx^2 + dy^2)}$

   ii Factor out $dx^2$ and use it as differential if the function is $y = f(x)$

   iii OR Factor out $dy^2$ and use it as differential if the function is $x = f(y)$

# 3 Derivatives

## 3.1 Example: Motion of a Particle

We are given an equation describing law of motion of a particle $s(t) = t^4 - 6t^3 + 4t^2 + 3$ where $t >= 0$. To find its velocity we need to derivate the function $s(t)$ with respect to $t$:

$$s(t)' = 4 \times t^3 - 6 \times 3t^2 + 4 \times 2t^1 + 0 \times 3 \tag{4}$$

$$s(t)' = 4t^3 - 18t^2 + 8t^2 \tag{5}$$

To find the derivative $s(t)'$ above we made use of power rule from calculus which states:

$$(X^N)' = N \times x^{N-1} \tag{6}$$

By substituting 0 into equation (5) we can find the velocity of the particle at the $t = 0$:

$$s(0)' = 4 \times 0^3 - 18 \times 0^2 = 0 \tag{7}$$

The particle is at rest when $s(t)' = 0$. To find time $t$ when this happens we let $s(t)' = 0$ and solve for $t$:

$$s(x)' = 4t^3 - 18t^2 + 8t = 0 \tag{8}$$

This is a cubic equation, but we can solve it by factoring out $t$:

$$s(x)' = t(4t^2 - 18t + 8) = 0 \tag{9}$$

If the product of two factors equals zero then either or both factors must be zero for that to be true.

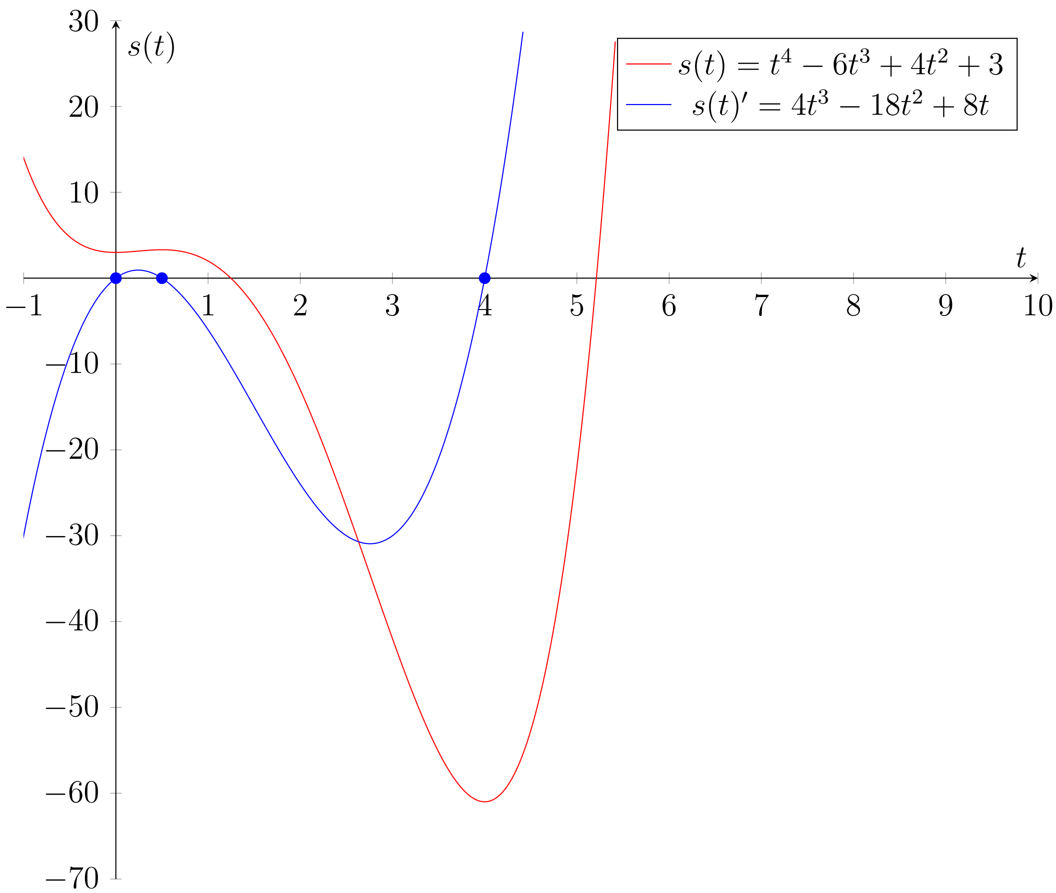
$$t = 0 \qquad\qquad 4t^2 - 18t + 8 = 0 \tag{10}$$

$$t_1 = 4, t_2 = \frac{1}{2} \tag{11}$$

The particle is at rest at $t = 0, \frac{1}{2}, 4$. We can evaluate $s(t)'$ on each of the time intervals to find signs of $s(t)'$(and whenever the particle's distance is *increasing* or *decreasing*):

| Interval | $s(t)'$ | Sign of $s(t)'$ |
|---|---|---|
| $(-\infty, 0)$ | $s(-1) = -30$ | $-$ |
| $(0, \frac{1}{2})$ | $s(0.25) = 0.9375$ | $+$ |
| $(\frac{1}{2}, 4)$ | $s(1) = -6$ | $-$ |
| $(4, +\infty)$ | $s(5) = 90$ | $+$ |

Plotting $s(t)$ and $s(t)'$ confirms the behavior of the function:




Examining the graph we see that $s(t)$ is flat whenever $s(t)'$ crosses the $x$ axis, meaning the particle stays at rest at those points. We can also see that $s(t)$ is *decreasing* whether $s(t)'$ is below the $x$ axis and *increasing* whenever $s(t)'$ is above the $x$ axis.

Let $D$ be the *total distance* traveled by the particle after 6 seconds. To calculate $D$ we need to sum up distances traveled by the particle on the intervals $(0, \frac{1}{2})$, $(\frac{1}{2}, 4)$, $(4, 6)$:

$$D_{6sec} = |f(\frac{1}{2}) - f(0)| + |f(4) - f(\frac{1}{2})| + |f(6) - f(4)| \tag{12}$$

$$D_{6sec} = 0.3125 + 64.3125 + 208 = 272.625 \tag{13}$$

By differentiating $s(t)$ one more time we get expression for acceleration of the particle, from which we can also find $s(0)''$:

$$s(x)'' = 12t^2 - 36t + 8 \tag{14}$$

$$s(0)'' = 12 \times 0^2 - 36 \times 0 + 8 = 8 \tag{15}$$

To perform sign analysis of $s(t)''$ we need to know the values of $t$ where it crosses the $x$ axis. Let $s(t)'' = 0$ and solve for $t$:

$$s(x)'' = 12t^2 - 36t + 8 = 0 \tag{16}$$

$$4(3t^2 - 9t + 2) = 0 \tag{17}$$

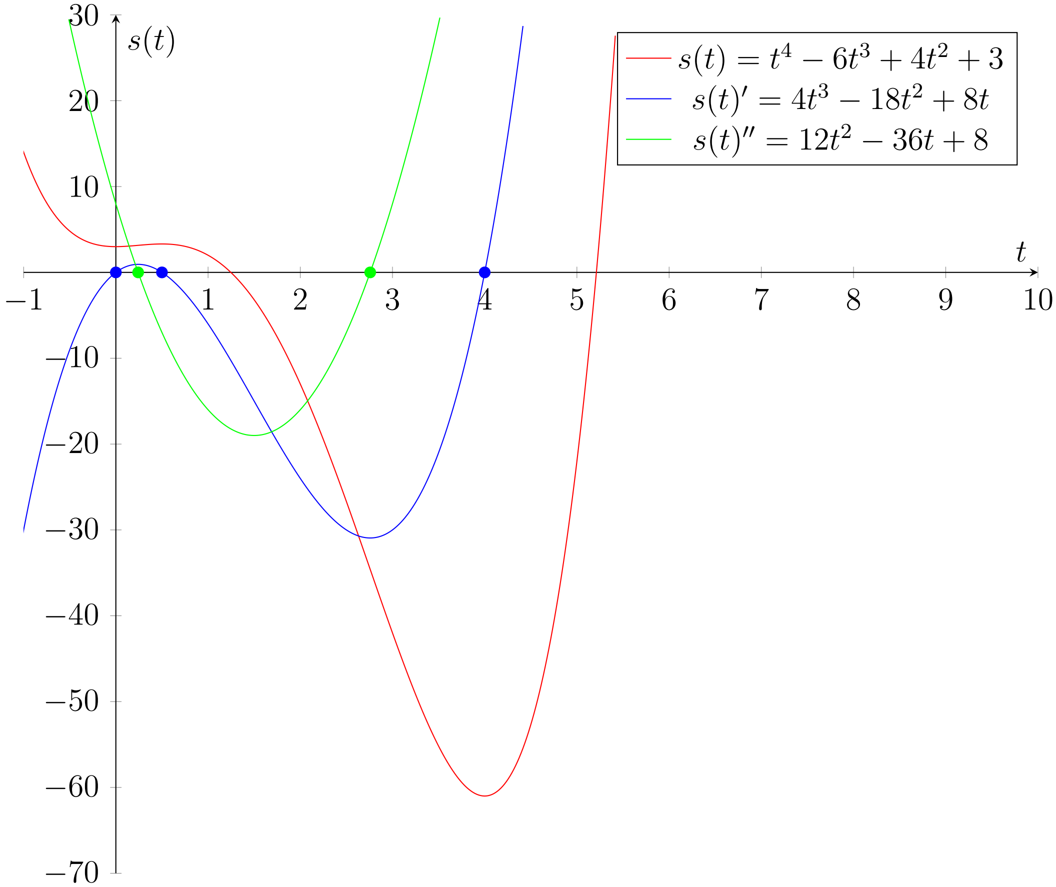
$$t_1 = \frac{9 + \sqrt{57}}{6} \approx 2.76, t_2 = \frac{9 - \sqrt{57}}{6} \approx 0.24 \tag{18}$$

Velocity of the particle is not changing at $t = 0.24, 2.76$. We can evaluate $s(t)''$ on each of the time intervals to find signs of $s(t)''$(and whether the particle is *accelerating* or *decelerating* on each of them):

| Interval | $s(t)''$ | Sign of $s(t)''$ |
|---|---|---|
| $(-\infty, \frac{9-\sqrt{57}}{6})$ | $s(-1) = 56$ | $+$ |
| $(\frac{9-\sqrt{57}}{6}, \frac{9+\sqrt{57}}{6})$ | $s(1) = -16$ | $-$ |
| $(\frac{9+\sqrt{57}}{6}, +\infty)$ | $s(3) = 8$ | $+$ |

The particle is *accelerating* on the intervals $(-\infty, 0.24)$ and $(2.76, +\infty)$. The particle is *decelerating* on $(0.24, 2.76)$.

The graph confirms this behavior:




Examining the graph we see that $s(t)'$ is flat whenever $s(t)''$ is crossing the $x$ axis, meaning the particle is not accelerating at those points. We also see that the particle is *accelerating* whenever $s(t)''$ is above the $x$ axis and *decelerating* whenever $s(t)''$ is below the $x$ axis.