Calculus Bits and Pieces

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1 Arc Length

Consider a small DIFFERENTIAL piece of a CURVE $ds^2 = dx^2 + dy^2$. Then LENGTH OF THE CURVE on some interval is given by

$$L = \int_a^b ds. \tag{1}$$

Which intuitively says that LENGTH OF A CURVE ON AN INTERVAL A TO B IS GIVEN BY A SUM OF ITS DIFFERENTIAL SEGMENTS.

SUBSTITUTING $dx^2 + dy^2$ for ds and REARRANGING yields two possibilities:

Function	ds	L
$y = f(x)$	$\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$	$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$
$x = f(y)$	$\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$	$\int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$

FIVE STEP PROCEDURE TO DERIVE A FORMULA FOR ARC LENGTH OF A CURVE IN THE WILD:

1. DO REMEMBER $L = \int_a^b ds$.
2. DO REMEMBER $ds = \sqrt{(dx^2 + dy^2)}$
3. FACTOR OUT dx^2 and USE IT AS DIFFERENTIAL if the function is $y = f(x)$
4. OR FACTOR OUT dy^2 and USE IT AS DIFFERENTIAL if the function is $x = f(y)$
5. THERE'S NO STEP NUMBER FIVE

2 Surface Area

If a DIFFERENTIAL segment ds is REVOLVED around x or y axis it will produce a TRUNCATED CONE with a SURFACE AREA

$$2\pi r ds, \tag{2}$$

where r is DISTANCE FROM THE CURVE TO THE AXIS OF REVOLUTION defined as

Function	Axis	r
$y = f(x)$	x	$f(x)$
$x = f(y)$	x	y
$x = f(y)$	y	$f(y)$
$y = f(x)$	y	x

INTEGRATING $2\pi r ds$ yields area of the SURFACE PRODUCED BY REVOLVING A SEGMENT OF A CURVE ON THE INTERVAL a TO b

$$A = 2\pi \int_a^b r\, ds. \tag{3}$$

REPLACING r and ds yields the FOUR FORMULAS FOR CALCULATING SURFACE AREAS:

Function	Axis	r	A with r replaced	A with ds replaced
$y = f(x)$	x	$f(x)$	$A = 2\pi \int_a^b f(x)\, ds$	$A = 2\pi \int_a^b f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$
$x = f(y)$	x	y	$A = 2\pi \int_a^b y\, ds$	$A = 2\pi \int_a^b y\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$
$x = f(y)$	y	$f(y)$	$A = 2\pi \int_a^b f(y)\, ds$	$A = 2\pi \int_a^b f(y)\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$
$y = f(x)$	y	x	$A = 2\pi \int_a^b x\, ds$	$A = 2\pi \int_a^b x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

The above table can be reduced to just a few steps, allowing you to recreate it from minimum amount of information.

Procedure to derive a formula for surface area in the wild:

1. Do remember $A = 2\pi \int_a^b r\,ds$.
2. Replace r with an expression of independent variable that would represent the distance between the curve and axis
3. Replace ds(same as arc length):
   - i Do remember $ds = \sqrt{(dx^2 + dy^2)}$
   - ii Factor out dx^2 and use it as differential if the function is $y = f(x)$
   - iii OR Factor out dy^2 and use it as differential if the function is $x = f(y)$

3 Techniques of Integration

3.1 Integration By Parts

Integration by parts is a technique that uses the following relation between two integrals:

$$\int u\,dv = uv + \int v\,du \tag{4}$$

The relation comes from product rule. Suppose we have a product of two functions $f(x)g(x)$. Differentiate both sides:

$$(f(x)g(x))' = f(x)'g(x) + f(x)g(x)' \tag{5}$$

Integrate both sides:

$$\int (f(x)g(x))'\,dx = \int f(x)'g(x)\,dx + \int f(x)g(x)'\,dx \tag{6}$$

$$f(x)g(x) = \int f(x)'g(x)\,dx + \int f(x)g(x)'\,dx \tag{7}$$

4 Derivatives

4.1 Example: Motion of a Particle

We are given an equation describing law of motion of a particle $s(t) = t^4 - 6t^3 + 4t^2 + 3$ where $t >= 0$. To find its velocity we need to derivate the function $s(t)$ with respect to t:

$$s(t)' = 4 \times t^3 - 6 \times 3t^2 + 4 \times 2t^1 + 0 \times 3 \tag{8}$$

$$s(t)' = 4t^3 - 18t^2 + 8t^2 \tag{9}$$

To find the derivative $s(t)'$ above we made use of power rule from calculus which states:

$$(X^N)' = N \times x^{N-1} \tag{10}$$

By substituting 0 into equation (9) we can find the velocity of the particle at the $t = 0$:

$$s(0)' = 4 \times 0^3 - 18 \times 0^2 = 0 \tag{11}$$

The particle is at rest when $s(t)' = 0$. To find time t when this happens we let $s(t)' = 0$ and solve for t:

$$s(x)' = 4t^3 - 18t^2 + 8t = 0 \tag{12}$$

This is a cubic equation, but we can solve it by factoring out t:

$$s(x)' = t(4t^2 - 18t + 8) = 0 \tag{13}$$

If the product of two factors equals zero then either or both factors must be zero for that to be true.

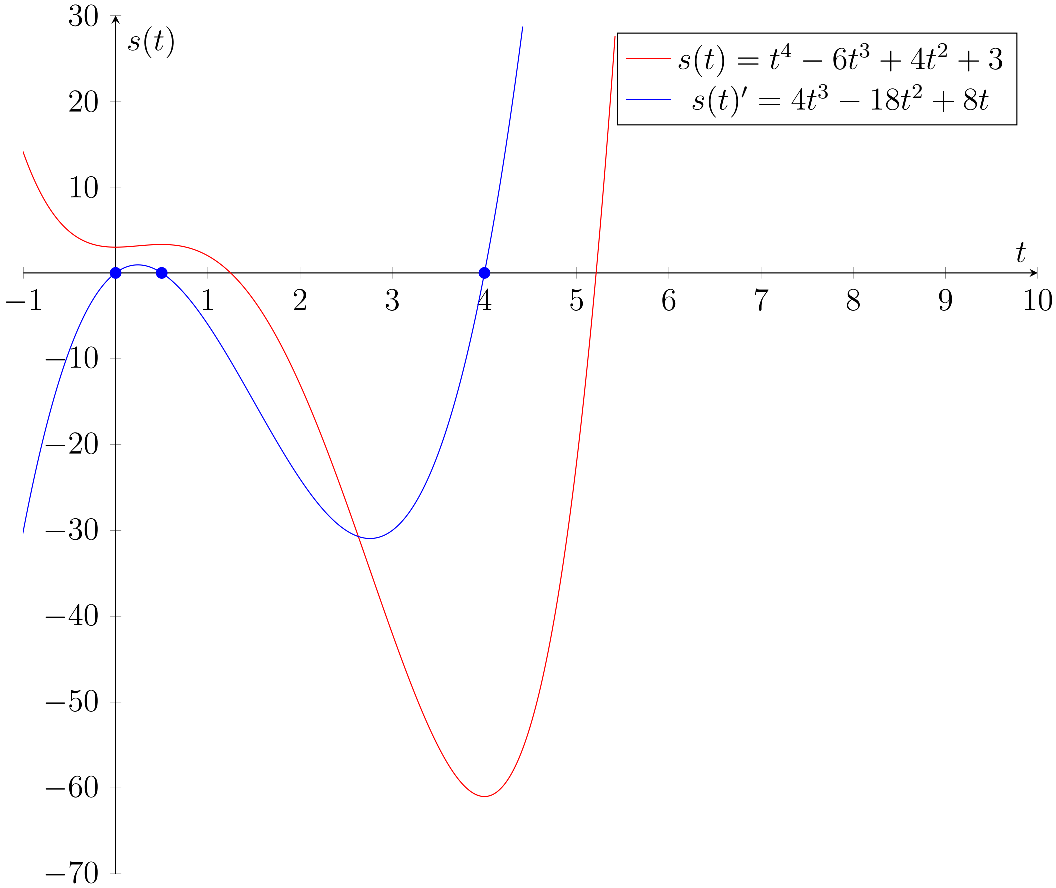
$$t = 0 \qquad 4t^2 - 18t + 8 = 0 \tag{14}$$

$$t_1 = 4, t_2 = \frac{1}{2} \tag{15}$$

The particle is at rest at $t = 0, \frac{1}{2}, 4$. We can evaluate $s(t)'$ on each of the time intervals to find signs of $s(t)'$(and whenever the particle's distance is *increasing* or *decreasing*):

Interval	$s(t)'$	Sign of $s(t)'$
$(-\infty, 0)$	$s(-1) = -30$	$-$
$(0, \frac{1}{2})$	$s(0.25) = 0.9375$	$+$
$(\frac{1}{2}, 4)$	$s(1) = -6$	$-$
$(4, +\infty)$	$s(5) = 90$	$+$

Plotting $s(t)$ and $s(t)'$ confirms the behavior of the function:



Examining the graph we see that $s(t)$ is flat whenever $s(t)'$ crosses the x axis, meaning the particle stays at rest at those points. We can also see that $s(t)$ is *decreasing* whether $s(t)'$ is below the x axis and *increasing* whenever $s(t)'$ is above the x axis.

Let D be the *total distance* traveled by the particle after 6 seconds. To calculate D we need to sum up distances traveled by the particle on the intervals $(0, \frac{1}{2})$, $(\frac{1}{2}, 4)$, $(4, 6)$:

$$D_{6sec} = |f(\frac{1}{2}) - f(0)| + |f(4) - f(\frac{1}{2})| + |f(6) - f(4)| \tag{16}$$

$$D_{6sec} = 0.3125 + 64.3125 + 208 = 272.625 \tag{17}$$

By differentiating $s(t)$ one more time we get expression for acceleration of the particle, from which we can also find $s(0)''$:

$$s(x)'' = 12t^2 - 36t + 8 \tag{18}$$

$$s(0)'' = 12 \times 0^2 - 36 \times 0 + 8 = 8 \tag{19}$$

To perform sign analysis of $s(t)''$ we need to know the values of t where it crosses the x axis. Let $s(t)'' = 0$ and solve for t:

$$s(x)'' = 12t^2 - 36t + 8 = 0 \tag{20}$$

$$4(3t^2 - 9t + 2) = 0 \tag{21}$$

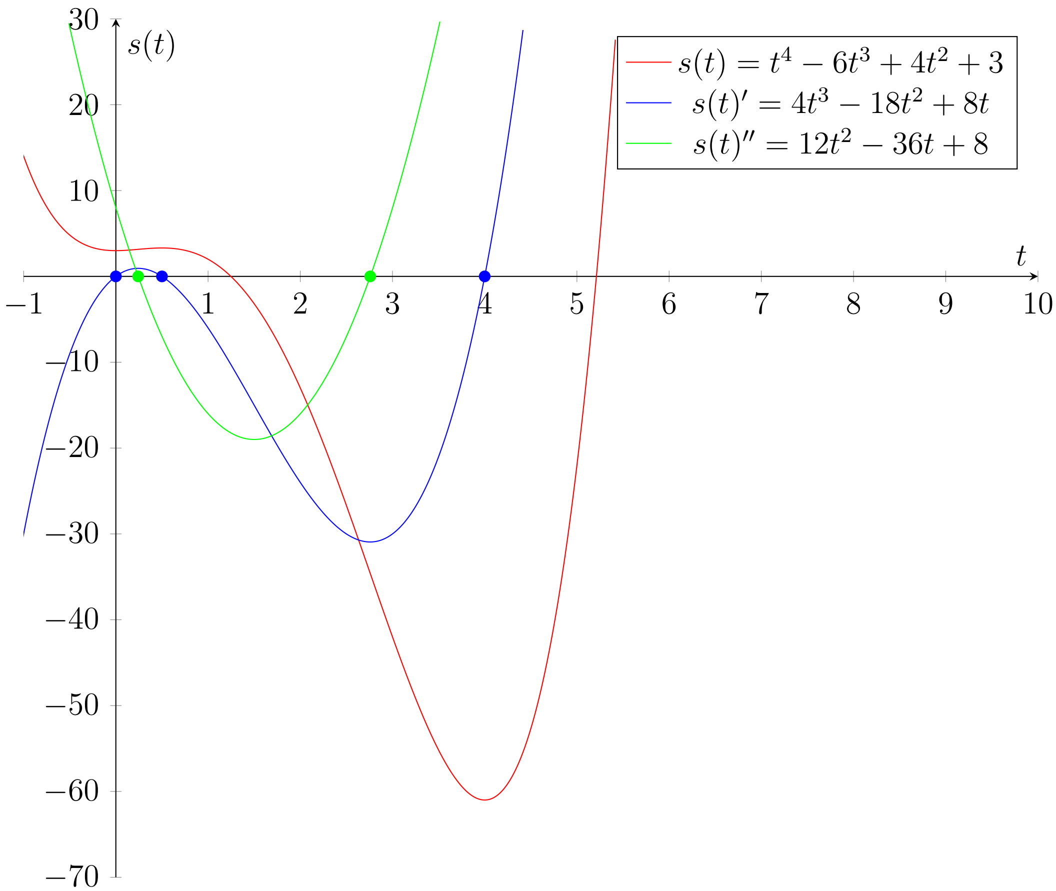
$$t_1 = \frac{9 + \sqrt{57}}{6} \approx 2.76, t_2 = \frac{9 - \sqrt{57}}{6} \approx 0.24 \tag{22}$$

Velocity of the particle is not changing at $t = 0.24, 2.76$. We can evaluate $s(t)''$ on each of the time intervals to find signs of $s(t)''$(and whether the particle is *accelerating* or *decelerating* on each of them):

Interval	$s(t)''$	Sign of $s(t)''$
$(-\infty, \frac{9-\sqrt{57}}{6})$	$s(-1) = 56$	$+$
$(\frac{9-\sqrt{57}}{6}, \frac{9+\sqrt{57}}{6})$	$s(1) = -16$	$-$
$(\frac{9+\sqrt{57}}{6}, +\infty)$	$s(3) = 8$	$+$

The particle is *accelerating* on the intervals $(-\infty, 0.24)$ and $(2.76, +\infty)$. The particle is *decelerating* on $(0.24, 2.76)$.

The graph confirms this behavior:



Examining the graph we see that $s(t)'$ is flat whenever $s(t)''$ is crossing the x axis, meaning the particle is not accelerating at those points. We also see that the particle is *accelerating* whenever $s(t)''$ is above the x axis and *decelerating* whenever $s(t)''$ is below the x axis.

5 Multiple Integrals

5.1 Surface Area

$$A = \iint\limits_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA \tag{23}$$

Or

$$A = \iint\limits_D \sqrt{f_x(x, y)^2 + f_y(x, y)^2 + 1}\, dA \tag{24}$$

1. Compute partial derivatives f_x and f_y.
2. Evaluate double integral above