Stochastic optimization for large scale optimal transport

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Presentation of OT

Kantorovitch Formulation of regularized OT :

$$
\begin{aligned}
W_\varepsilon(\mu,\nu) &= \min_{\pi \in \Pi(\mu,\nu)} \int_{\mathcal{X},\mathcal{Y}} c(x,y)\mathrm{d}\pi(x,y) + \varepsilon \mathrm{KL}(\pi || \mu \otimes \nu) && (\mathcal{P}_\varepsilon) \\
&= \max_{(u,v)\in \mathcal{C}(\mathcal{X})\times\mathcal{C}(\mathcal{Y})} \int_{\mathcal{X}} u(x)\mathrm{d}\mu(x) + \int_{\mathcal{Y}} v(y)\mathrm{d}\nu(y) - \iota^{\varepsilon}_{U_c}(u,v) && (\mathcal{D}_\varepsilon) \\
&= \max_{v\in\mathcal{C}(\mathcal{Y})} H_\varepsilon(v) \triangleq \int_{\mathcal{X}} v^{c,\varepsilon}(x)\mathrm{d}\mu(x) + \int_{\mathcal{Y}} v(y)\mathrm{d}\nu(y) - \varepsilon && (\mathcal{S}_\varepsilon)
\end{aligned}
$$

Sinkhorn updates : $u^{\ell+1} = \frac{\mu}{Kv^{\ell}}; v^{\ell+1} = \frac{\nu}{K^T v^{\ell+1}} \rightarrow O(n^2)$

No general solver in the semi-discrete case.

What about large scale ?

Discrete OT :

$$W_\varepsilon(\mu,\nu) = \max_{\boldsymbol{v}\in\mathbb{R}^J} \sum_{i=1}^{I} \overline{h}_\varepsilon(x_i,\boldsymbol{v})\boldsymbol{\mu}_i$$

Gradient aggregation algorithms (SAG and SAGA)

$$v_{k+1} = v_k + \texttt{step} * \left(\nabla f_i - \nabla f_{[i]} + \frac{1}{I}\sum_i \nabla f_{[i]}\right)$$

Semi-discrete OT :

$$W_\varepsilon(\mu,\nu) = \max_{v}\ \mathbb{E}_X[h_\varepsilon(X,v)]$$

Stochastic gradient ascent

$$v_{k+1} = v_k + \frac{\texttt{step}}{\sqrt{k}} * \nabla f_i$$

Theoretical analysis

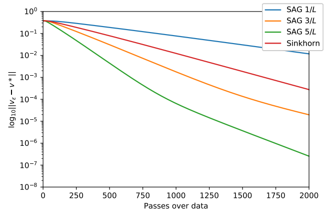
Convergence rates

Stochastic gradient $O(1/\sqrt{k})$ for non strongly convex, $O(1/k)$ for strongly convex

SAG and SAGA $O(1/k)$ for non strongly convex, linear for strongly convex (at the expense of storing gradients)

Numerical findings - Discrete OT

Synthetic data



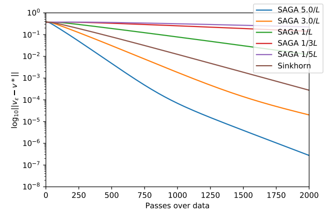
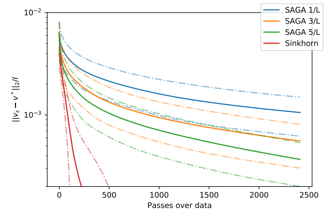
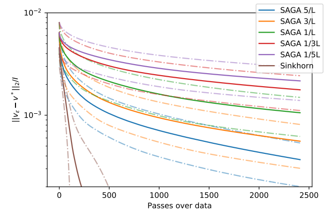
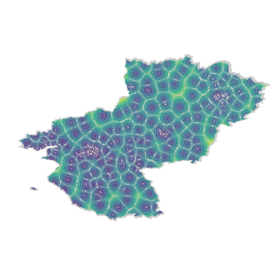


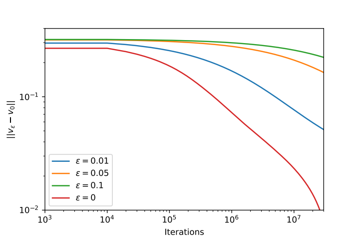
Image retrieval task

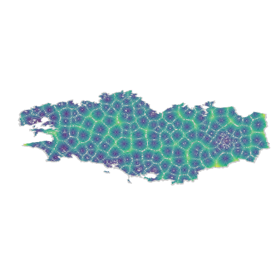


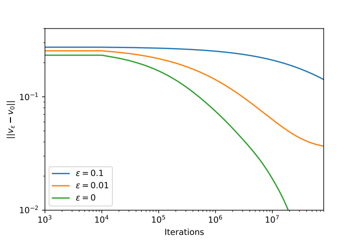


Numerical findings - Semi-discrete OT









Critics

- These methods should be tested in a much larger scale setting to show their real benefit
- Benefits over Sinkhorn of SAG and SAGA wasn't consistently observed and seems to depend on the parameters and structure of the problem

Perspective

- Applications of OT to problems with scales of the order of 10^6 and above
- Applications of semi-discrete OT to high dimensional problems with “exotic” cost functions