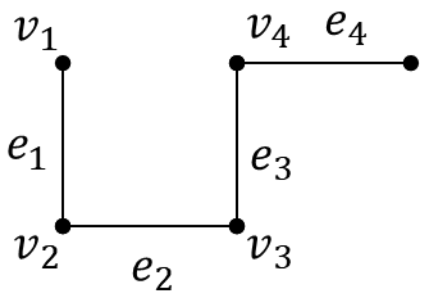
Matching, Covering and Greedy coloring algorithm

Matching/Independent edge set

Let G be a graph with no self loop. A subset M of the edge set E of a graph G is called a matching in G if no two edges in M are adjacent in G.



G

Consider the graph in the figure, then $E(G) = \{e_1, e_2, e_3, e_4\}$. Here $M_1 = \{e_1, e_3\}$, $M_2 = \{e_2, e_4\}$, $M_3 = \{e_1, e_4\}$ are matching. But $M_4 = \{e_2, e_3, e_4\}$ is not a matching.

M-Saturated vertex

Let M be a matching in a graph G. A vertex ‘v’ is said to be M-saturated if there is an edge in M incident with ‘v’. Otherwise ‘v’ is said to be M-unsaturated.

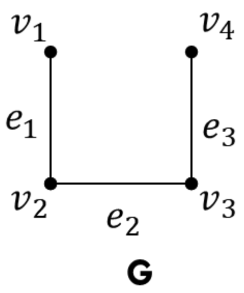
In the above example v_1, v_2, v_3, v_4 are M1-saturated and v_5 is unsaturated with respect to M1.

Now consider the matching M2, the vertices v_2, v_3, v_4, v_5 are M2 –saturated and $v_1$is unsaturated with respect to M2.

Perfect matching/Complete matching

A matching M in the graph G is said to be perfect if it saturates every vertex of G.

Consider the graph G, here $M_1 = \{e_1, e_3\}$ is a perfect matching as it saturates all the vertices v_1, v_2, v_3, v_4 of G.



A *maximal matching* is a matching to which no more edge can be added. For example, in a complete graph of 3 vertices, any single edge is a maximal matching. A graph may have many different maximal matchings. Among these the one with maximum number of edges is called the largest maximal matching. The number of edges in a largest maximal matching is called the *matching number* of the graph.

In the above figure both (a) and (b) are maximal matchings of the same graph. The figure in (b) is the largest maximal matching and hence the matching number is 2. (b) is also a complete matching.

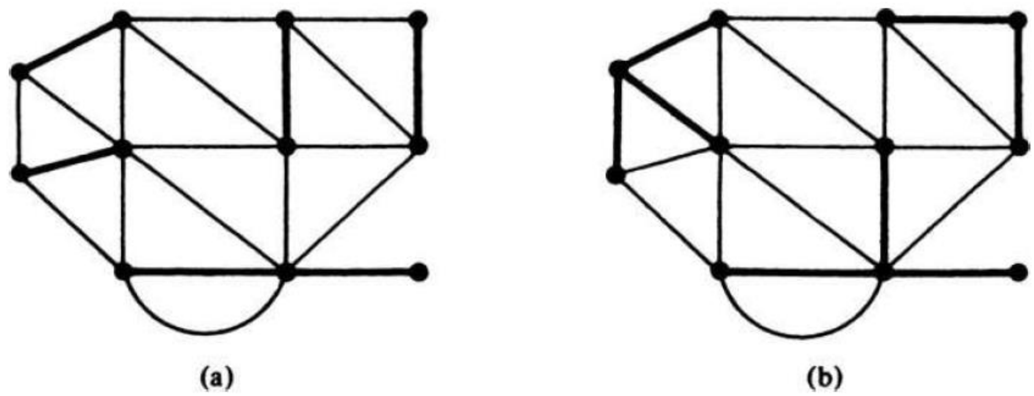
Matching in a bipartite graph has got many applications. Hence the following theorem might prove to be useful in some cases.

Theorem: In a bipartite graph with partition sets V_1 and V_2, a complete matching of V_1 into V_2 exists if there exists a positive integer *m* such that

degree of every vertex in $V_1 \geq m \geq$ degree of every vertex in V_2.

Covering

A set of edges E is said to *cover* graph G if every vertex in graph G is incident to at least one edge in E. The set of edges that covers a graph G is said to be an edge covering or simply a covering of graph G. A covering is said to be *minimal covering* of graph G if removal of any edge from the covering will not cover graph G. The minimum number of edges required to cover a graph G is called the *covering number* of graph G.



The bold lines in the above 2 graphs represent minimal coverings of the same graph and the covering number of the above graph is 6 (see the covering in (a)).

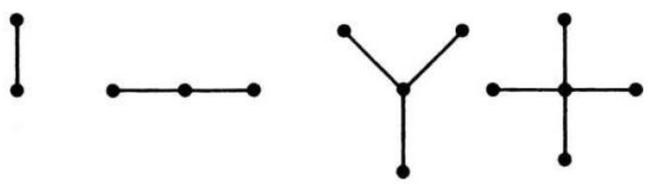
The following are to be noted:

1. A covering exists for a graph if and only if the graph has no isolated vertex.
2. A covering of a graph with n vertices will have at least $\left\lceil \frac{n}{2} \right\rceil$ edges.
3. Every pendant edge (edge incident with pendant vertex) in a graph is included in every covering of the graph.
4. Every covering contains a minimal covering.
5. No minimal covering can contain a circuit, as we can always remove an edge from a circuit and still all the vertices in the circuit will remain covered.

Theorem: A covering E of a graph is minimal if and only if E contains no paths of length 3 or more.

Proof: Let E contain a path of length 3, say a-b-c-d, where a, b, c, d are vertices. Then even if we remove the middle edge (b, c) from this path, still all the 4 vertices a, b, c, d will remain covered. Hence E cannot be a minimal covering, which is a contradiction. Therefore E contains no paths of length 3 or more.

Conversely, if E contains no paths of length 3 or more, all its components must be star graphs as shown below.



From a star graph, if an edge is removed, then the corresponding pendant vertex will remain as uncovered. Hence all the edges in star graph are required to cover the graph. Hence E is a minimal covering.

4 color Theorem

- In graph-theory, the four color theorem states that the vertices of every planar graph can be colored with at most four colors so that no two adjacent vertices receive the same color. i.e. Every planar graph is four-colourable.
- The four – color came into focus during 1852 when attempt was made to color the countries of England with color representation. Regions were considered as vertices and an edge was drawn between vertices if the corresponding regions shared a common boundary. They found that only 4 colors were needed for this work. But it could not be generalized.

Greedy coloring algorithm

In the study of graph coloring problems in mathematics and computer science, a greedy coloring is a coloring of the vertices of a graph formed by a greedy algorithm that considers the vertices of the graph in sequence and assigns each vertex its first available color.

Basic Greedy Coloring Algorithm:

1. Color first vertex with first color.

2. Do following for remaining n-1 vertices.

   a) Consider the currently picked vertex and color it with the lowest numbered color that has not been used on any previously colored vertices adjacent to it.

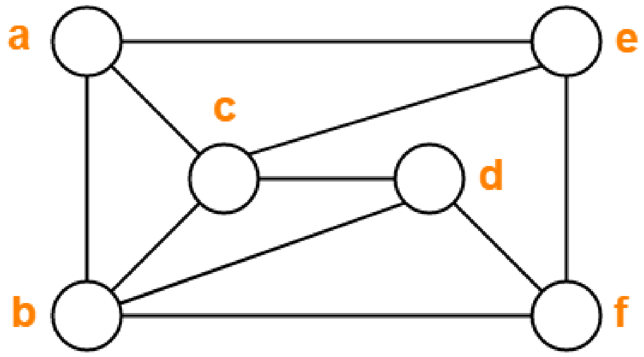
   b)If all previously used colors appear on vertices adjacent to v, assign a new color to it.

Drawbacks of Greedy Algorithm

There are following drawbacks of the above Greedy Algorithm-

- The above algorithm does not always use minimum number of colors.
- The number of colors used sometimes depend on the order in which the vertices are processed

Find chromatic number of the following graph by Greedy Algorithm



- Applying Greedy Algorithm, we have

    1. Color vertex a by C1.
    2. Choose vertex b. The lowest numbered color available is C2 as C1 is already given to vertex a and vertex b is adjacent to vertex a.
    3. Next vertex c. It can be colored only by C3 as C1 and C2 have been given to vertices a and b which are adjacent to vertex c.
    4. Next vertex d. It can be colored by C1 since vertex d is adjacent to vertices b, c and f and only colors C2 and C3 are used to color those (f is still not colored).
    5. To color vertex e, C1 and C3 cannot be used, but C2 can be used.
    6. Using again a similar argument, we can color vertex f by color C3.

Vertex	a	b	c	d	e	f
Color	C1	C2	C3	C1	C2	C3

Therefore, Chromatic Number of the given graph = 3.

The given graph may be properly colored using 3 colors as shown below