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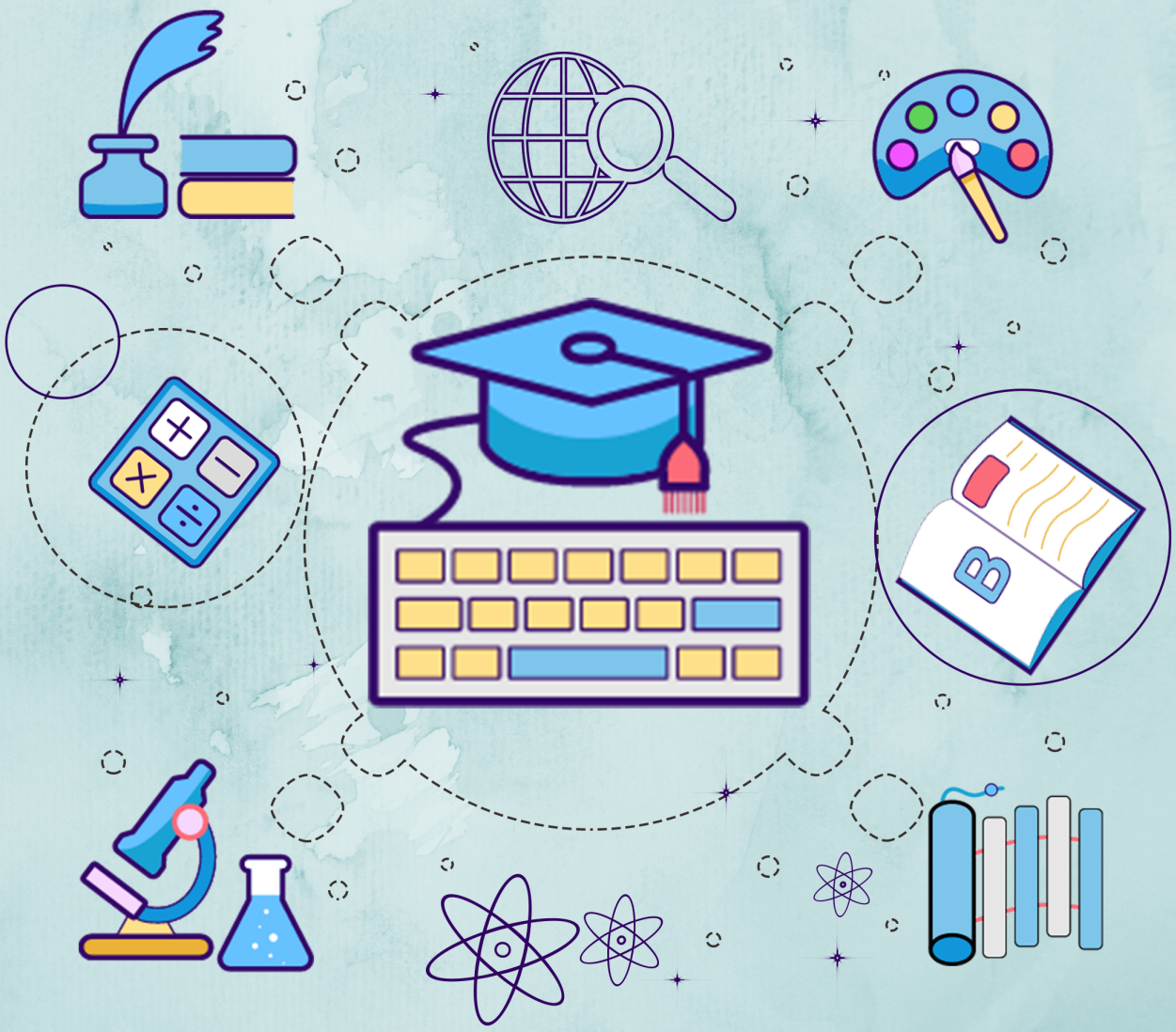

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GRAPH THEORY (MAT206)

Module 5

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Module 5

1. **Show that if *A(G)* is an incidence matrix of a connected graph *G* with *n* vertices, then rank of *A(G)* is *n−1*.**

Soln :

- ✓ Each row in an incidence matrix A(G) may be regarded as a vector in the vector space of graph G.
- ✓ Let the vector in the first row be called as A1, in the second row as A2, and so on.

$$A(G) = \begin{bmatrix} A_1 \\ A_2 \\ . \\ . \\ A_n \end{bmatrix}$$

- ✓ Since there are only two 1s in every column of A, the sum of all these vectors (mod 2) is 0.
- ✓ Thus the vectors A1, A2, ….. An are not linearly independent.
- ✓ Therefore the rank of A is less than "n".
- ✓ We can say that rank of $A \le n-1$
- ✓ Now, consider the sum of any "m" vectors (m rows) such that $m \le n-1$
- ✓ We cannot find any such m rows, whose modulo 2 sum turns to be zero.
- ✓ We cannot partition A(G) into A(g1) with "m" rows and A(g2) with "n-m" rows such that modulo 2 sum of those "m" rows is 0.
- ✓ Since there can be only two values o or 1, the addition of all vectors taken "m" at a time for m=1,2,…(n-1) exhausts all possible linear combinations of (n-1) row vectors.
- ✓ Thus no linear combination of m row vectors is zero. This means that rank of A(G) must be at least (n-1).
- ✓ Since rank of A(G), $A \le n-1$
- ✓ And rank of A(G) $A(G) \ge n-1$
- ✓ We have rank of A(G) is exactly equal to (n-1).
- ✓ Hence the theorem.

Note : Why is the rank of A(G), "n-1"?

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- **Let "G" be a connected graph and A(G) be its incidence matrix.**
- **Then if we remove any one row from A(G), then the remaining (n-1) rows are linearly independent.**
- **Thus we need only (n-1) rows of A(G) to reconstruct the graph completely.**
- **Because, (n-1) rows contain the same information as "n" rows. The missing row can be easily reconstituted, as each column of A(G) need to contain exactly two 1s. That is why the rank of A(G) is (n-1).**

2. **Show that if *B* is a cycle matrix of a connected graph *G* with *n* vertices and *e* edges, then rank *B = e−n+1*.**

   Soln :

   - ✓ Since A is the incidence matrix, we know that $A.B^T = 0 (\text{mod } 2)$.
   - ✓ According to Slyvester's theorem,

     rank of A + rank of B ≤ e
   - ✓ rank of B ≤ e – rank of A ……..(1)
   - ✓ We know rank of A = n-1
   - ✓ rank of B = e-n+1…….(2)
   - ✓ We have the result rank of B_f = e-n+1
   - ✓ Since B_f is a sub matrix of B, rank of B ≥ e-n+1…..(3)
   - ✓ From (2) and (3) rank of B = e-n+1.

3. Derive the relations between the reduced incidence matrix, the fundamental cycle matrix, and the fundamental cut-set matrix of a graph ***G***.

   Soln : Fundamental circuit matrix, Bf= [Iμ:Bt] μ x e

   - ✓ Fundamental cut-set matrix, Cf= [Cc :Ir]r x e
   - ✓ Reduced incidence matrix, Af= [Ac :At]r x e

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- $A_f . B_f^T = 0 \pmod 2$ (coz $A.B^T=0$)
- $[A_c : A_t]_{r \times e} \begin{bmatrix} I_\mu \\ \hline B_t^T \end{bmatrix}_{e \times \mu} = 0$
- $A_c + A_t . B_t^T = 0$
- $A_c A_t^{-1} + A_t . B_t^T A_t^{-1} = 0 A_t^{-1}$
- $A_t^{-1} A_c = - B_t^T$
- $A_t^{-1} A_c = B_t^T$ --------(1)
- $C_f . B_f^T = 0 \pmod 2$ (coz $C.B^T=0$)
- $[C_c : I_{n-1}]_{r \times e} \begin{bmatrix} I_\mu \\ \hline B_t^T \end{bmatrix}_{e \times \mu} = 0$
- $C_c I_\mu + I_{n-1} . B_t^T = 0$
- $C_c + B_t^T = 0$
- $C_c = - B_t^T$
- $C_c = B_t^T$ --------(2)
- Equating (1) & (2)
- $C_c = A_t^{-1} A_c$ --------(3)

- Summarise:
  1. Given A_f, we can find B_f & C_f
  2. Give B_f, we can find C_f; but not A_f
  3. Given C_f, we can find B_f, but not A_f

4. Characterize simple, self-dual graphs in terms of their cycle and cut-set matrices.
5. Show that an n vertex graph is a tree iff its chromatic polynomial is $Pn(\lambda) = \lambda(\lambda - 1)^{n-1}$
6. Prove the statement: "A covering g of a graph is minimal if g contains no path of length three or more."
7. Find the chromatic polynomial of the graph

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8. Consider the circuit matrix ***B*** and incidence matrix ***A*** of a simple connected graph whose columns are arranged using the same order of edges. Prove that every row of ***B*** is orthogonal to every row of ***A***?

   Soln : Consider a vertex 'v' and a circuit 'Γ' in G.

   - There can be two cases :
     - Case 1: 'V' is in 'Γ' .
       - Then the number of edges in 'Γ' that is incident on 'V' is exactly 2.
     - Case 2 : 'V' is not in 'Γ' .
       - Then no edge of 'Γ' is incident on 'V'.
   - Consider ith row of A (non-zero entries indicates, edge incident on ith vertices)
   - Consider jth row of B (non-zero entries indicated edges included in jth circuit)
   - If ith vertex is not in jth circuit then there is no 1 in their common positions. So their dot product is zero.
   - If ith vertex is in jth circuit, there would be exactly two 1s in common positions. Sum of their dot product turns to be 2, which is 0 (since mod 2).
   - Since dot product of two arbitrary rows is zero. (one from A & one from B).
   - $A.B^T = B.A^T = 0$ (Mod 2)

9. A graph is *critical* if the removal of any one of its vertices (and the edges adjacent to that vertex) results in a graph with a lower chromatic number. Show that ***Kn*** is critical for all ***n > 1***.

10. Find a maximum matching in the graph below and use Hall's theorem to show that it is indeed maximum.

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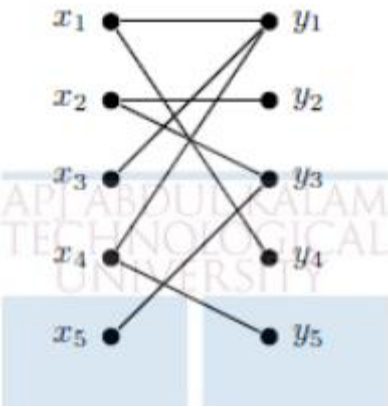

Soln:

- **Halls Theorem states about matching in bipartite graphs.**

Matching is independent edge set in a graph. This means no two edges share a common vertex.

Let G be a bipartite graph with partite sets U and W, where $r = |U| \leq |W|$. Then G satisfies Hall's condition if, for every $S \leq U$, $|S| \leq |N(S)|$. Furthermore, G contains a matching of cardinality "r" if and only if G satisfies Hall's condition.

- **In a bipartite graph a complete matching of V1 into V2 exist if there is a positive integer "m" for which the following condition is satisfied.**

$$degree\ of\ every\ vertex\ in\ V1 \geq m \geq degree\ of\ every\ vertex\ in\ V2$$

- **Deficiency $\delta(G)$, of a bipartite graph G is the maximum value of the number "*r-q*". r = 1,2,.....$|V1|$.**

  r = set of vertices of V1

  q = set of vertices of V2

- **A complete matching in a bipartite graph G exists if and only if**

$$\delta(G) \leq 0$$

- **The maximal number of vertices in set V1 that can be matched into V2 is equal to**

$$number\ of\ vertices\ in\ V1 - \delta(G)$$

- **<u>MATCHING</u>**

❖ Matching in a graph is a subset of edges in which no two edges are adjacent.

❖ Single edge in a graph is a matching.

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- **Maximal Matching** is a matching to which no edge in the graph can be added.
- A graph can have different maximal matchings, and of different sizes.
- The maximal matchings with the largest number of edges are called **largest maximal matchings.**
- The number of edges in a largest maximal matching is called the **matching number of the graph.**
- **In a bipartite graph** having vertex partition V1 and V2, a **complete matching** of vertices in set V1 into those in V2 is a matching in which there is one edge incident with every vertex in V1. In other words **every vertex in V1 is matched against some vertex in V2.**
- **Complete matching is a largest maximal matching, whereas the converse is not necessarily true.**
  - **For the existence of a complete matching of set V1 into set V2, we must have at least as many vertices in V2 as there are in V1.**
  - **For a complete matching : Every subset of "r" vertices in V1 must collectively be adjacent to at least "r" vertices in V2, for all values of "r". r = 1,2,.....$|V1|$.**
  - **A complete matching of V1 into V2 in a bipartite graph exists if and only if every subset of "r" vertices in V1 is collectively adjacent to "r" or more vertices in V2 for all values of "r".**

1. Suppose that four applicants a1, a2, a3 and a4 are available to fill six vacant positions p1, p2, p3 ,p4, p5 and p6. Applicant a1 is qualified to fill positions p2 or p5. Applicant a2 can fill p2 or p5. Applicant a3 is qualified for p1, p2, p3, p4, or p6. Applicant a4 can fill jobs p2 and p5. Is it possible to hire all the applicants and assign each a position for which he is suitable? If the answer is no, what is the maximum number of positions that can be filled from the given set of applicants?
   Soln: There must be at least as many vacant positions as the number of applicants if all the applicants are to be hired. This condition is not sufficient.

2.

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11. A graph ***G*** has chromatic polynomial ***PG(k) = k⁴-4k³+5k²-2k***. How many vertices and edges does ***G*** have? Is ***G*** bipartite? Justify your answers.

Soln :

The **degree of the chromatic polynomial is the number of vertices**, and so G has 4 vertices.

If **G has e edges and n vertices, the coefficient of k^{n-1} in PG(k) is −e**, and so G has 4 edges.

Being bipartite is equivalent to G having a coloring with only two colors. By definition, the PG(2) is the number of colorings of G with 2 colors. We have PG(2) $= 24 - 4 \cdot 2^3 + 5 \cdot 2^2 - 2 \cdot 2 = 16 - 32 + 20 - 4 = 0$, and so G is not bipartite.

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