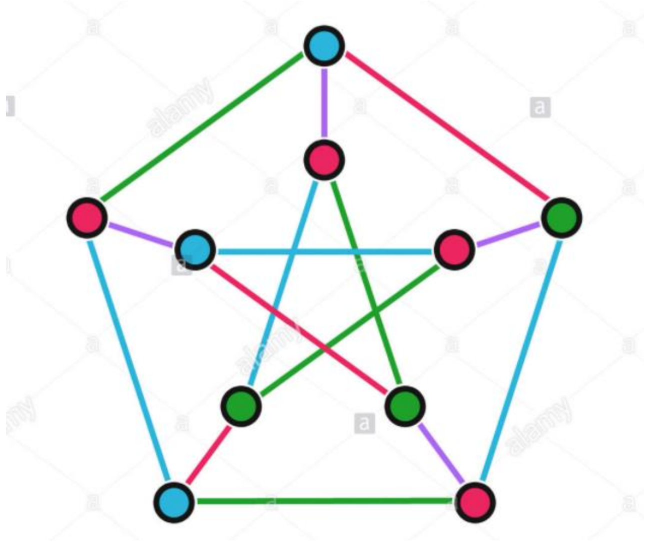
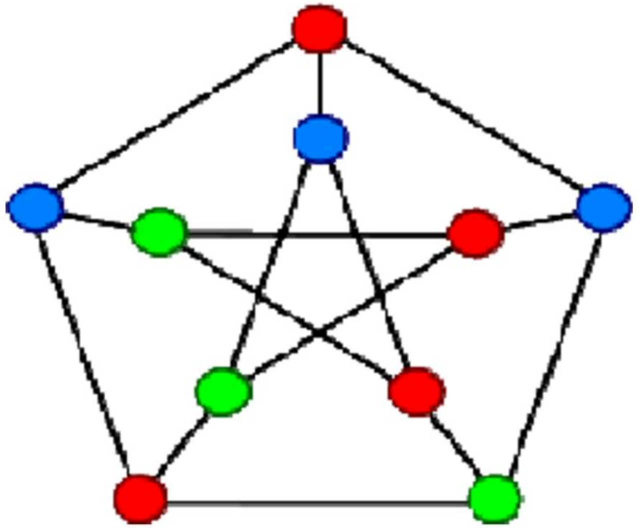
Coloring

- Coloring (or proper coloring) of a graph is done by assigning each vertex a color such that no two adjacent vertices have the same color. A graph in which every vertex has been assigned a color according to coloring definition is called a properly colored graph.
- A graph can be colored in many ways. Example shown below:





- The minimum number of colors required to color a graph G is called the chromatic number of G. It is usually denoted by *k*. A graph with chromatic number *k* is said to be a *k*-chromatic graph. The chromatic number of the graph given in example is 3.
- Note: For coloring problems, we will consider mostly simple and connected graphs only.

Some observations that follow directly from the definitions just introduced are

1. A graph consisting of only isolated vertices is 1-chromatic.
2. A graph with one or more edges (not a self-loop, of course) is at least *2-chromatic* (also called *bichromatic*).
3. A complete graph of *n* vertices is *n*-chromatic, as all its vertices are adjacent. Hence a graph containing a complete graph of *r* vertices is at least *r*-chromatic. For instance, every graph having a triangle is at least 3-chromatic.

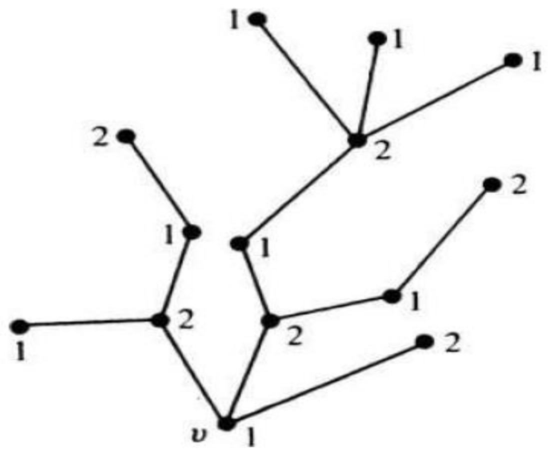
4. A graph consisting of simply one circuit with $n \geq 3$ vertices is 2-chromatic if n is even and 3-chromatic if n is odd. (This can be seen by numbering vertices 1, 2, . . ., n in sequence and assigning one color to odd vertices and another to even. If n is even, no adjacent vertices will have the same color. If n is odd, the nth and first vertex will be adjacent and will have the same color, thus requiring a third color for proper coloring.)

Note – Normally we use numbers to denote different colors.

Theorem 1: Every tree with 2 or more vertices is 2-chromatic.

Proof: Identify a vertex *v* and let it be the root of the tree. Give color 1 to vertex *v*. Give color 2 to all the vertices adjacent to *v*. Then give color 1 to all the vertices adjacent to these vertices (the ones which were given color 2). Continue the process till every vertex of tree gets colored. Hence we can see that all vertices that are at odd distance from vertex *v* have color 2 and all vertices at even distance from *v* and vertex *v* have color 1.

An example has been given to understand the coloring of a tree.



Theorem 2: A graph with at least one edge is 2-chromatic if and only if it has no odd circuits ~~of even length.~~

Proof: Let G be a connected graph with circuits of only even lengths. Consider a spanning tree T in G. From theorem 1, we can color T with 2 colors. Add the chords to T one by one. Since G has no circuits of odd length, the end vertices of every chord being replaced are the end vertices of a path whose length is odd in T (only then we will have a circuit of even length). From previous theorem, the end vertices of an odd length path in T will have different colors and hence the colors of the end vertices of each chord will be also different. Thus, G is colored with 2 colors, with no adjacent vertices having the same color. Therefore, G is 2-chromatic.

Conversely, if G is 2-chromatic, then G cannot have circuits of odd length as then we would need 3 colors for a proper coloring of G.

Theorem 3: A graph G is bipartite if and only if it is bichromatic (2-chromatic).

Proof: Suppose G is bipartite. The vertex set is partitioned into 2 sets and no 2 vertices in same partition are adjacent. Hence we can color each vertex in same partition with the same color. Hence G is bichromatic.

Conversely, if G is bichromatic, all the vertices with same color will form a partition and all the edges in G will have its end vertices in 2 different partitions. Hence G is bipartite.

Note: Every tree is a bipartite graph.

Chromatic Polynomial

We have seen that a given graph G of n vertices can be properly colored in many ways using a sufficient large number of colors. This property of a graph is expressed by means of a polynomial. This polynomial is called the chromatic polynomial of G.

Definition: The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n vertices gives the number of ways of properly coloring the graph, using λ or fewer colors.

- Let c_i be the different ways of properly coloring G using i different colors.
- Since i colors can be chosen out of λ colors in $\binom{\lambda}{i}$ different ways, there are $c_i\binom{\lambda}{i}$ different ways of properly coloring G using exactly i colors out of λ colors.
- Since i can be any positive integer from 1 to n (it is not possible to use more than n colors on n vertices), the chromatic polynomial is a sum of these terms. That is
- $P_n(\lambda) = \sum_{i=1}^{n} c_i\binom{\lambda}{i} = c_1\lambda + c_2\frac{\lambda(\lambda-1)}{2!} + c_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \cdots + c_n\frac{\lambda(\lambda-1)\ldots(\lambda-n+1)}{n!}$

- Note: Each c_i is evaluated individually for a given graph. For example, any graph with one edge requires at least 2 colors for proper coloring and hecnce $c_1 = 0$.

- A graph with n vertices and using n colors can be properly colored in $n!$ Different ways. That is $c_n = n!$

Theorem: A graph of n vertices is a complete graph iff its chromatic polynomial is $P_n(\lambda) = \lambda(\lambda - 1) \ldots (\lambda - n + 1)$.

Proof: With λ colors there are λ different ways of coloring any selected vertex. A second vertex can be properly colored in exactly in $(\lambda - 1)$ different ways, third in $(\lambda - 2)$ different ways,…, and the nth in $(\lambda - n + 1)$ ways iff every vertex is adjacent to every other vertices which is possible if and only if the graph is complete**.**

Theorem : An n −vertex graph is a tree iff its chromatic polynomial $P_n(\lambda) = \lambda(\lambda - 1)^{(n-1)}$

Proof: Proof of this theorem can be done by induction on number of vertices of the graph.

- When $n = 1$, that is an isolated vertex can be colored using λ colors hence the chromatic polynomial $P_n(\lambda) = P_1(\lambda) = \lambda$.

- When $n = 2$, $P_2(\lambda) = \lambda(\lambda - 1)$.
- Assume that the result is true for all trees with number of vertices less than or equal to k vertices.
- Hence the chromatic polynomial $P_n(\lambda) = P_k(\lambda) = \lambda(\lambda - 1)^{(k-1)}$.
- Consider a tree with $k + 1$ vertices. We know that a tree with more than 2 vertices have at least 2 pendant vertices.
- If we remove a pendant vertex along with an edge of the tree with $k + 1$ vertices, we are left with a tree with k vertices. Hence the resulting tree will have $\lambda(\lambda - 1)^{(k-1)}$different ways of coloring.
- After coloring all the vertices, the removed pendent vertex can be colored in $(\lambda - 1)$ different ways.
- Hence the chromatic polynomial of the tree with $k + 1$ vertices is $P_k(\lambda) = \lambda(\lambda - 1)^{(k-1)}$ $(\lambda - 1) = \lambda(\lambda - 1)^k$.