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HYPOTHESIS TESTING OF GEOGRAPHICALLY WEIGHTED MULTIVARIATE POISSON REGRESSION

Triyanto*, Purhadi, Bambang Widjanarko Otok and Santi Wulan Purnami

*Department of Mathematics Education
Sebelas Maret University
Surakarta, Indonesia

Department of Statistics
Sepuluh Nopember Institute of Technology
Surabaya, Indonesia

Abstract

Geographically weighted multivariate Poisson regression (GWMPR) model is a local form of the multivariate Poisson regression (MPR) that allows the modelling of spatially heterogeneous processes. This model results in a set of local parameter estimates which depends on the geographical location where the data are observed. In this paper, hypothesis testing in GWMPR model is studied which includes a goodness of fit test, an overall test and test of individual parameters. The test statistic of all the three hypotheses testing of the GWMPR model is done by using likelihood ratio test (LRT) method. The test statistic of the goodness of fit test follows the asymptotic distribution of chi-square with the degree of freedom is the difference between the effective number of parameters in the MPR and GWMPR models. The

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second test statistic of the overall test follows the asymptotic distribution of chi-square with the degree of freedom is the difference between the effective number of parameters in the GWMPR without the covariate in the model and GWMPR model whereas the test statistic of the test of individual parameters follows the asymptotic distribution of the standard normal.

1. Introduction

Poisson regression is one of the nonlinear regression models used to analyze the relationship between the response variable that consists of "count data" with one or more independent variables. The model is often used when the Poisson-distributed response variable is supposed to represent the number of occurrences of some random event in an interval of time, space, or some volume of matter [7], for examples, the number of telephone calls arriving at a central telephone exchange per minute, the number of airline accidents per year, the number of maternal mortality in the region per year and others.

In a recent study, Poisson regression for spatial data has developed. Cressie [3] stated that spatial data is a type of data which showed dependence on the location where the data are observed. For example, the number of malnutrition in children in a country has a high correlation with the socioeconomy of the society, but for other regions which have a policy of health subsidies for the poor society, the possibility of correlation is weak. Consequently, as stated by Anselin [1], if we use the model of global Poisson regression on spatial data, then it produces an invalid model. The estimator of the parameters becomes inefficient, which in turn means that the standard errors of the parameters are too large.

Geographically weighted regression (GWR) is an effective method to estimate the regression parameters from data with spatial heterogeneity. The underlying idea of GWR model is that each estimator of regression parameters depends on the location where the data are observed. In this case, the location is expressed as vector coordinate in two-dimensional geographic space (latitude and longitude) [4]. Based on the idea that is adapted from

GWR models, Nakaya et al. [8] have developed geographically weighted Poisson regression (GWPR) model for spatial count data and applied to a dataset of working-age deaths in the Tokyo metropolitan area. The results indicate that there are significant spatial variations in the relationships between working-age mortality and occupational segregation and unemployment throughout the area. In the next paper, Nakaya et al. [9] proposed a generalized framework for semiparametric geographically weighted regression (S-GWR) by combining several theoretical aspects of GWR. In their paper, GWR 4.0 has been developed as a platform for the practical implementation of S-GWR modelling including Poisson (GWPR) and logistic (GWLR). Hadayeghi et al. [5] showed that the GWPR models are useful for capturing spatially dependent relationships and generally performing better than the conventional generalized linear model. Yang et al. [12] applying GWPR to investigate the relationship of marriage postponement, cohabitation rates and divorce rates with infant mortality in the US.

In the multivariate case, Triyanto et al. [10] introduced geographically weighted multivariate Poisson regression (GWMPR) as a development from GWPR model when there are two or more spatial count data of response variables and they are correlated to each other. In their study, an estimate of the parameters is done by using the maximum likelihood estimation (MLE) method and applied iterative procedure by the Newton-Raphson algorithm. As a follow-up, in this study, hypothesis testing of GWMPR is developed which includes a goodness of fit test that is used to determine if there is a difference between models of GWMPR and MPR, an overall test and the significance test of individual parameters.

2. Multivariate Poisson Distribution

Multivariate Poisson distribution is the joint distribution from two or more random variables, where each of the random variables is Poisson distribution and correlated each other. Multivariate Poisson distribution was constructed by Johnson et al. [6], the so-called $(p+1)$ variate reduction method. Let the random variables $Z_0, Z_1, Z_2, ..., Z_p$ be independently Poisson distribution with means $\lambda_0, \lambda_1, \lambda_2, ..., \lambda_p$, respectively. New random variables $Y_1, Y_2, ..., Y_p$ can be constructed by:

$$Y_h = Z_h + Z_0; \quad h = 1, 2, ..., p. \tag{1}$$

Y_h $(h = 1, 2, ..., p)$ is a convolution of two Poisson random variables with moment generating function as:

$$M_{Y_h}(t) = e^{(\lambda_h + \lambda_0)(e^t - 1)}. \tag{2}$$

So, marginally each Y_h follows a Poisson distribution with mean $\lambda_h + \lambda_0$.

The joint distribution of random variables $Y_1, Y_2, ..., Y_p$ follows a multivariate Poisson distribution with joint probability function given by

$$f(y_1, y_2, ..., y_p) = \begin{cases} e^{-\lambda_0 - \sum_{h=1}^{p} \lambda_h} \sum_{v=0}^{s} \frac{\lambda_0^v}{v!} \prod_{h=1}^{p} \frac{\lambda_h^{y_h - v}}{(y_h - v)!}; & y_h = 0, 1, 2, ..., \\ 0; & \text{otherwise}, \end{cases} \tag{3}$$

where $s = \min(y_1, y_2, ..., y_p)$.

3. Multivariate Poisson Regression

Multivariate Poisson regression (MPR) is a development of univariate Poisson regression when there are two or more count data of response variables and they are correlated to each other. Let the random samples $Y_{hi} \sim P(\lambda_{hi} + \lambda_0)$; $i = 1, 2, ..., n$ and $h = 1, 2, ..., p$ with $E[Y_{hi}] = \lambda_{hi} + \lambda_0$. Multivariate Poisson regression models can be written as the marginal expectation of Y_{hi} which is an exponential function of independent variables, as follows:

$$(Y_{1i}, Y_{2i}, \ldots, Y_{pi}) \sim MP(\lambda_0, \lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{pi}),$$

$$\lambda_{hi} + \lambda_0 = e^{\mathbf{x}_i^T \boldsymbol{\beta}_h}; \quad i = 1, 2, \ldots, n; \quad h = 1, 2, \ldots, p, \tag{4}$$

where

$$\mathbf{x}_i = [1 \ x_{1i} \ x_{2i} \ldots x_{ki}]^T,$$

$$\boldsymbol{\beta}_h = [\beta_{h0} \ \beta_{h1} \ \beta_{h2} \ldots \beta_{hk}]^T.$$

4. Geographically Weighted Multivariate Poisson Regression

Geographically weighted multivariate Poisson regression (GWMPR) is a development from GWPR model which was proposed by Nakaya et al. [8]. This model is used in situations when there are two or more spatial count data of response variables and they are correlated to each other. Based on the idea, which is adapted from GWPR models, the GWMPR model uses the point spatial approach by considering the factor of location that is expressed as vector coordinate in two-dimensional geographic space.

In GWMPR model, response variables $Y_{1i}, Y_{2i}, \ldots, Y_{pi}$ will be predicted by independent variables $\mathbf{x}_i$ where each of the regression coefficients depends on the location where the data are observed. Let $\mathbf{u}_i = (u_{1i} \ u_{2i})$ be a vector of point coordinate in two-dimensional geographic space (latitude and longitude) at location *i*. Then GWMPR model can be written as follows:

$$(Y_{1i}, Y_{2i}, \ldots, Y_{pi}) = MP(\lambda_0(\mathbf{u}_i), \lambda_{1i}(\mathbf{u}_i), \lambda_{2i}(\mathbf{u}_i), \ldots, \lambda_{pi}(\mathbf{u}_i)),$$

$$\lambda_{hi}(\mathbf{u}_i) + \lambda_0(\mathbf{u}_i) = e^{\mathbf{x}_i^T \boldsymbol{\beta}_h(\mathbf{u}_i)}; \quad i = 1, 2, \ldots, n; \quad h = 1, 2, \ldots, p, \tag{5}$$

where

$$\boldsymbol{\beta}_h(\mathbf{u}_i) = [\beta_{h0}(\mathbf{u}_i) \ \beta_{h1}(\mathbf{u}_i) \ \beta_{h2}(\mathbf{u}_i) \ \cdots \ \beta_{hk}(\mathbf{u}_i)]^T.$$

5. Hypothesis Testing of GWMPR

The hypothesis tests of the GWMPR model are considered for the goodness of fit test, the overall test and test of individual parameters. The first test is a goodness of fit test of GWMPR model that is used to determine if there is a difference between models of GWMPR and MPR. The form of hypothesis can be expressed as follows:

$$H_0 : \beta_{hl}(\mathbf{u}_i) = \beta_{hl}; \quad h = 1, 2, ..., p; \quad l = 1, 2, ..., k; \quad i = 1, 2, ..., n,$$

$$H_1 : \text{at least one of the } \beta_{hl}(\mathbf{u}_i) \neq \beta_{hl}. \tag{6}$$

The test statistic of a goodness of fit test of GWMPR model is done by using likelihood ratio test (LRT) method as follows:

Let $\omega_1 = \{\lambda_0, \boldsymbol{\beta}_h\}$ and $\Omega_1 = \{\lambda_0(\mathbf{u}_i), \boldsymbol{\beta}_h(\mathbf{u}_i)\}$ for $h = 1, 2, ..., p$; $i = 1, 2, ..., n$ denote the parameter space under the null hypothesis and alternative hypothesis, respectively. The LRT formulated as follows:

$$G_1 = -2 \ln \frac{L(\hat{\boldsymbol{\theta}}_{10})}{L(\hat{\boldsymbol{\theta}}_1)}$$

$$= 2[\ln L(\hat{\boldsymbol{\theta}}_1) - \ln L(\hat{\boldsymbol{\theta}}_{10})],$$

where

$$L(\hat{\boldsymbol{\theta}}_{10}) = \max L(\omega_1)$$

$$= \prod_{i=1}^{n} \left(\exp\left((p-1)\hat{\lambda}_0 - \sum_{h=1}^{p} e^{\mathbf{x}_i^T \hat{\boldsymbol{\beta}}_h} \right) \hat{A}_i \right), \tag{7}$$

$$L(\hat{\boldsymbol{\theta}}_1) = \max L(\Omega_1)$$

$$= \prod_{i=1}^{n} \left(\exp\left((p-1)\hat{\lambda}_0(\mathbf{u}_i) - \sum_{h=1}^{p} e^{\mathbf{x}_i^T \hat{\boldsymbol{\beta}}_h(\mathbf{u}_i)} \right) \hat{C}_i \right). \tag{8}$$

Considering equations (7) and (8), the test statistic to the hypothesis testing equation (6) can be written as follows:

$$G_1 = 2\Bigg((p-1)\left(\sum_{i=1}^{n}\hat{\lambda}_0(\mathbf{u}_i) - n\hat{\lambda}_0\right) - \sum_{i=1}^{n}\sum_{h=1}^{p}\left(e^{\mathbf{x}_i^T\hat{\boldsymbol{\beta}}_h(\mathbf{u}_i)} - e^{\mathbf{x}_i^T\hat{\boldsymbol{\beta}}_h}\right) + \sum_{i=1}^{n}(\ln \hat{C}_i - \ln \hat{A}_i)\Bigg), \tag{9}$$

where

$$\hat{A}_i = \sum_{v=0}^{s}\frac{\hat{\lambda}_0^v \prod_{h=1}^{p}\left(e^{\mathbf{x}_i^T\hat{\boldsymbol{\beta}}_h} - \hat{\lambda}_0\right)^{y_{hi}-v}}{v!\prod_{h=1}^{p}(y_{hi}-v)!}$$

and

$$\hat{C}_i = \sum_{v=0}^{s}\frac{\hat{\lambda}_0^v(\mathbf{u}_i)\prod_{h=1}^{p}\left(e^{\mathbf{x}_i^T\hat{\boldsymbol{\beta}}_h(\mathbf{u}_i)} - \hat{\lambda}_0(\mathbf{u}_i)\right)^{y_{hi}-v}}{v!\prod_{h=1}^{p}(y_{hi}-v)!},$$

$\hat{\lambda}_0$ and $\hat{\boldsymbol{\beta}}_h$ are parameter estimators of MPR model, whereas $\hat{\lambda}_0(\mathbf{u}_i)$ and $\hat{\boldsymbol{\beta}}_h(\mathbf{u}_i)$ are parameter estimators of GWMPR model (see [10]).

The test statistic G_1 follows the asymptotic distribution of chi-square with the degree of freedom is the difference between the effective number of parameters in the MPR (m_1) and GWMPR models (m_2). So, at the significance level α, reject the null hypothesis, when G_1 value falls into the rejection region, i.e., $G_1 > \chi^2_{(\alpha,\, m_2 - m_1)}$.

The second test is an overall test that is used to determine the simultaneous significance of the regression parameters predicting any of the response variables. The form of hypothesis can be expressed as:

$$H_0 : \beta_{h1}(\mathbf{u}_i) = \beta_{h2}(\mathbf{u}_i) = \cdots = \beta_{hk}(\mathbf{u}_i) = 0; \quad h = 1, 2, ..., p; \quad i = 1, 2, ..., n,$$

$$H_1 : \text{at least one of the } \beta_{hl}(\mathbf{u}_i) \neq 0; \quad l = 1, 2, ..., k. \tag{10}$$

The test statistic of an overall test of GWMPR model is done by using likelihood ratio test (LRT) method as follows:

Let $\omega_2 = \{\lambda_{00}(\mathbf{u}_i), \beta_{h0}(\mathbf{u}_i)\}$ and $\Omega_2 = \{\lambda_0(\mathbf{u}_i), \boldsymbol{\beta}_h(\mathbf{u}_i)\}$ for $h = 1, 2, ..., p$; $i = 1, 2, ..., n$ denote the parameter spaces under the null hypothesis and alternative hypothesis, respectively. Analogously, the goodness of fit test of GWMPR model, the LRT is formulated as follows:

$$G_2 = 2[\ln L(\hat{\boldsymbol{\theta}}_2) - \ln L(\hat{\boldsymbol{\theta}}_{20})],$$

where

$$L(\hat{\boldsymbol{\theta}}_{20}) = \max L(\omega_2)$$

$$= \prod_{i=1}^{n} \left(\exp\left((p-1)\hat{\lambda}_{00}(\mathbf{u}_i) - \sum_{h=1}^{p} e^{\hat{\beta}_{h0}(\mathbf{u}_i)} \right) \hat{C}_{i0} \right), \tag{11}$$

$$L(\hat{\boldsymbol{\theta}}_2) = \max L(\Omega_2)$$

$$= \prod_{i=1}^{n} \left(\exp\left((p-1)\hat{\lambda}_0(\mathbf{u}_i) - \sum_{h=1}^{p} e^{\mathbf{x}_i^T \hat{\boldsymbol{\beta}}_h(\mathbf{u}_i)} \right) \hat{C}_i \right). \tag{12}$$

Considering equations (11) and (12), the test statistic to the hypothesis testing equation (10) can be written as follows:

$$G_2 = 2\left((p-1)\sum_{i=1}^{n}(\hat{\lambda}_0(\mathbf{u}_i) - \hat{\lambda}_{00}(\mathbf{u}_i)) \right.$$

$$\left. - \sum_{i=1}^{n}\sum_{h=1}^{p}(e^{\mathbf{x}_i^T\hat{\boldsymbol{\beta}}_h(\mathbf{u}_i)} - e^{\hat{\beta}_{h0}(\mathbf{u}_i)}) + \sum_{i=1}^{n}(\ln \hat{C}_i - \ln \hat{C}_{i0}) \right), \tag{13}$$

where

$$\hat{C}_{i0} = \sum_{v=0}^{s} \frac{\hat{\lambda}_{00}^{v}(\mathbf{u}_i)\prod_{h=1}^{p}(e^{\hat{\beta}_{h0}(\mathbf{u}_i)} - \hat{\lambda}_{00}(\mathbf{u}_i))^{y_{hi}-v}}{v!\prod_{h=1}^{p}(y_{hi}-v)!},$$

$\hat{\lambda}_0(\mathbf{u}_i)$ and $\hat{\boldsymbol{\beta}}_h(\mathbf{u}_i)$ are parameter estimators of GWMPR model (see [10]), whereas $\hat{\lambda}_{00}(\mathbf{u}_i)$ and $\hat{\beta}_{h0}(\mathbf{u}_i)$ are parameter estimators of GWMPR without the covariate in the model.

The MLEs of $\lambda_{00}(\mathbf{u}_i)$ and $\beta_{h0}(\mathbf{u}_i)$ are found with maximizing the log-likelihood function as follows:

Considering joint probability function as expressed by equation (3) and GWMPR model by equation (5) without the covariate in the model, the log-likelihood function can be written as:

$$\ln L(\omega_2) = \sum_{i=1}^{n}(p-1)\lambda_{00}(\mathbf{u}_i) - \sum_{j=1}^{n}\sum_{h=1}^{p}e^{\beta_{h0}(\mathbf{u}_i)} + \sum_{j=1}^{n}\ln C_{j0}. \tag{14}$$

To estimate the parameters in GWMPR model, it is required a weighting function based on the proximity of observations at location i with other locations without an explicit relationship being stated. Brunsdon et al. [2] proposed some types of weighting functions that can be used to describe the relationship between the observations on the location i with other locations (for example location j), one of which is Gaussian function that formulated as:

$$w_{ij} = \exp\left[-\frac{1}{2}\left(\frac{d_{ij}}{r}\right)^2\right], \tag{15}$$

where

w_{ij} : the spatial weight of location j on location i,

r : the kernel bandwidth,

d_{ij} : Euclidean distance between location i and location j.

Natural logarithm of likelihood function in equation (14) can be written as:

$$\begin{aligned} Q_0 &= w_{ij} \ln L(\omega_2) \\ &= \sum_{j=1}^{n} w_{ij}(p-1)\lambda_{00}(\mathbf{u}_i) - \sum_{j=1}^{n}\sum_{h=1}^{p} w_{ij} e^{\beta_{h0}(\mathbf{u}_i)} + \sum_{j=1}^{n} w_{ij} \ln C_{j0}. \end{aligned} \tag{16}$$

Let $\boldsymbol{\theta}_{20}(\mathbf{u}_i) = (\lambda_{00}(\mathbf{u}_i)\ \ \beta_{10}^T(\mathbf{u}_i)\ \ \beta_{20}^T(\mathbf{u}_i)\ \cdots\ \beta_{p0}^T(\mathbf{u}_i))^T$ be a vector of regression parameters. Estimator of the regression parameters is a solution of the following simultaneous equations, which are obtained by taking the partial derivative of equation (16) with respect to each element of the vector $\boldsymbol{\theta}_{20}(\mathbf{u}_i)$ and setting the result to zero, as follows:

$$\frac{\partial Q_0}{\partial \boldsymbol{\theta}_{20}(\mathbf{u}_i)} = \mathbf{0}.$$

Let

$$C_{j0} = \sum_{v=0}^{s} C_{v0j0} \prod_{h=1}^{p} C_{vhj0},$$

where

$$C_{v0j0} = \frac{\lambda_{00}^{v}(\mathbf{u}_i)}{v!} \quad \text{and} \quad C_{vhj0} = \frac{(e^{\beta_{h0}(\mathbf{u}_i)} - \lambda_{00}(\mathbf{u}_i))^{y_{hj}-v}}{(y_{hj}-v)!}; \quad h = 1, 2, \ldots, p.$$

Hence, by differentiating Q_0 with respect to $\lambda_{00}(\mathbf{u}_i)$ and $\beta_{h0}(\mathbf{u}_i)$, we have:

$$\frac{\partial Q_0}{\partial \lambda_{00}(\mathbf{u}_i)} = \sum_{j=1}^{n} w_{ij}(p-1) + \sum_{j=1}^{n} w_{ij} \frac{1}{C_{j0}} \frac{\partial C_{j0}}{\partial \lambda_{00}(\mathbf{u}_i)}, \tag{17}$$

$$\frac{\partial Q_0}{\partial \beta_{h0}(\mathbf{u}_i)} = -\sum_{j=1}^{n} w_{ij} e^{\beta_{h0}(\mathbf{u}_i)} + \sum_{j=1}^{n} w_{ij} \frac{1}{C_{j0}} \frac{\partial C_{j0}}{\partial \beta_{h0}(\mathbf{u}_i)} \tag{18}$$

and differentiating C_{j0} with respect to $\lambda_{00}(\mathbf{u}_i)$ and $\beta_{h0}(\mathbf{u}_i)$, we have:

$$\frac{\partial C_{j0}}{\partial \lambda_{00}(\mathbf{u}_i)} = \sum_{v=0}^{s} \left[\frac{\lambda_{00}^{v-1}(\mathbf{u}_i)}{(v-1)!} \prod_{h=1}^{p} C_{vhj0} - \frac{\lambda_{00}^{v}(\mathbf{u}_i)}{v!} \sum_{h=1}^{p} \left(C_{vhj0}^{*} \prod_{\substack{g=1 \\ g \neq h}}^{p} C_{vgj0} \right) \right], \tag{19}$$

$$\frac{\partial C_{j0}}{\partial \beta_{h0}(\mathbf{u}_i)} = \sum_{v=0}^{s} \left[\frac{\lambda_{00}^{v}(\mathbf{u}_i)}{v!} C_{vhj0}^{*} e^{\beta_{h0}(\mathbf{u}_i)} \prod_{\substack{g=1 \\ g \neq h}}^{p} C_{vgj0} \right], \tag{20}$$

where

$$C_{vhj0}^{*} = \frac{(e^{\beta_{h0}(\mathbf{u}_i)} - \lambda_{00}(\mathbf{u}_i))^{y_{hj}-v-1}}{(y_{hj}-v-1)!}.$$

Estimate of the parameters by MLE cannot be available in closed form since equations (17) and (18) are a system of interdependent nonlinear equations. Hence, we can apply iterative procedure that works well for concave objective functions, that is the Newton-Raphson algorithm as follows:

$$\hat{\boldsymbol{\theta}}_{20(m+1)}(\mathbf{u}_i) = \hat{\boldsymbol{\theta}}_{20(m)}(\mathbf{u}_i) - \mathbf{H}^{-1}(\hat{\boldsymbol{\theta}}_{20(m)}(\mathbf{u}_i))\mathbf{g}(\hat{\boldsymbol{\theta}}_{20(m)}(\mathbf{u}_i)), \tag{21}$$

where $\hat{\boldsymbol{\theta}}_{20(m)}(\mathbf{u}_i)$ is an estimator of the parameters at mth iteration.

$$\mathbf{g}^{T}(\hat{\boldsymbol{\theta}}_{20(m)}(\mathbf{u}_i)) = \left(\frac{\partial Q_0}{\partial \lambda_{00}(\mathbf{u}_i)} \;\; \frac{\partial Q_0}{\partial \beta_{10}(\mathbf{u}_i)} \;\; \cdots \;\; \frac{\partial Q_0}{\partial \beta_{p0}(\mathbf{u}_i)} \right)_{\boldsymbol{\theta}_{20}(\mathbf{u}_i) = \hat{\boldsymbol{\theta}}_{20(m)}(\mathbf{u}_i)}. \tag{22}$$

Equation (22) is a gradient vector. It was obtained from equation (17) up to (20):

$$\mathbf{H}(\hat{\boldsymbol{\theta}}_{20(m)}(\mathbf{u}_i)) = \begin{pmatrix} \frac{\partial^2 Q_0}{\partial \lambda_{00}^2(\mathbf{u}_i)} & \frac{\partial^2 Q_0}{\partial \lambda_{00}(\mathbf{u}_i)\partial \beta_{10}(\mathbf{u}_i)} & \cdots & \frac{\partial^2 Q_0}{\partial \lambda_{00}(\mathbf{u}_i)\partial \beta_{p0}(\mathbf{u}_i)} \\ \frac{\partial^2 Q_0}{\partial \lambda_{00}(\mathbf{u}_i)\partial \beta_{10}(\mathbf{u}_i)} & \frac{\partial^2 Q_0}{\partial \beta_{00}^2(\mathbf{u}_i)} & \cdots & \frac{\partial^2 Q_0}{\partial \beta_{10}(\mathbf{u}_i)\partial \beta_{p0}(\mathbf{u}_i)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 Q_0}{\partial \lambda_{00}(\mathbf{u}_i)\partial \beta_{p0}(\mathbf{u}_i)} & \frac{\partial^2 Q_0}{\partial \beta_{10}(\mathbf{u}_i)\partial \beta_{p0}(\mathbf{u}_i)} & \cdots & \frac{\partial^2 Q_0}{\partial \beta_{p0}^2(\mathbf{u}_i)} \end{pmatrix}_{\boldsymbol{\theta}_{20}(\mathbf{u}_i)=\hat{\boldsymbol{\theta}}_{20(m)}(\mathbf{u}_i)}. \tag{23}$$

Equation (23) is a Hessian matrix, where each element of the matrix was obtained from second order partial derivatives of Q_0 with respect to combination of $\lambda_{00}(\mathbf{u}_i)$ and $\beta_{h0}(\mathbf{u}_i)$ as follows:

$$\frac{\partial^2 Q_0}{\partial \lambda_{00}^2(\mathbf{u}_i)} = \sum_{j=1}^{n} w_{ij}\left[\left(\frac{1}{C_{j0}}\frac{\partial^2 C_{j0}}{\partial \lambda_{00}^2(\mathbf{u}_i)}\right) - \frac{1}{C_{j0}^2}\left(\frac{\partial C_{j0}}{\partial \lambda_{00}(\mathbf{u}_i)}\right)^2\right], \tag{24}$$

$$\frac{\partial^2 Q_0}{\partial \beta_{h0}^2(\mathbf{u}_i)} = -\sum_{j=1}^{n} w_{ij}e^{\beta_{h0}(\mathbf{u}_i)} + \sum_{j=1}^{n} w_{ij}\left[\left(\frac{1}{C_{j0}}\frac{\partial^2 C_{j0}}{\partial \beta_{h0}^2(\mathbf{u}_i)}\right) - \frac{1}{C_{j0}^2}\left(\frac{\partial C_{j0}}{\partial \beta_{h0}(\mathbf{u}_i)}\right)^2\right], \tag{25}$$

$$\frac{\partial^2 Q_0}{\partial \lambda_{00}(\mathbf{u}_i)\partial \beta_{h0}(\mathbf{u}_i)} = \sum_{j=1}^{n} w_{ij}\left[\left(\frac{1}{C_{j0}}\frac{\partial^2 C_{j0}}{\partial \lambda_{00}(\mathbf{u}_i)\partial \beta_{h0}(\mathbf{u}_i)}\right) - \left(\frac{1}{C_{j0}^2}\frac{\partial C_{j0}}{\partial \lambda_{00}(\mathbf{u}_i)}\frac{\partial C_{j0}}{\partial \beta_{h0}(\mathbf{u}_i)}\right)\right]. \tag{26}$$

For $h \neq g$ $(h,\ g = 1,\ 2,\ ...,\ p)$,

$$\frac{\partial^2 Q_0}{\partial\beta_{h0}(\mathbf{u}_i)\partial\beta_{g0}(\mathbf{u}_i)}$$

$$= \sum_{j=1}^{n} w_{ij}\left[\frac{1}{C_{j0}}\frac{\partial^2 C_{j0}}{\partial\beta_{h0}(\mathbf{u}_i)\partial\beta_{g0}(\mathbf{u}_i)} - \frac{1}{C_{j0}^2}\frac{\partial C_{j0}}{\partial\beta_{h0}(\mathbf{u}_i)}\frac{\partial C_{j0}}{\partial\beta_{g0}(\mathbf{u}_i)}\right]. \tag{27}$$

Second order partial derivatives of C_{j0} with respect to the combination of $\lambda_{00}(\mathbf{u}_i)$ and $\beta_{h0}(\mathbf{u}_i)$ are as follows:

$$\frac{\partial^2 C_{j0}}{\partial\lambda_{00}^2(\mathbf{u}_i)}$$

$$= \sum_{v=0}^{s}\left[\frac{\lambda_{00}^{v-2}(\mathbf{u}_i)}{(v-2)!}\prod_{h=1}^{p} C_{vhj0} - \frac{2\lambda_{00}^{v-1}(\mathbf{u}_i)}{(v-1)!}\sum_{h=1}^{p}\left(C_{vhj0}^{*}\prod_{\substack{g=1\\ g\neq h}}^{p} C_{vgj0}\right)\right.$$

$$\left. + \frac{\lambda_{00}^{v}(\mathbf{u}_i)}{v!}\left(\sum_{h=1}^{p}\left(C_{vhj0}^{**}\prod_{\substack{g=1\\ g\neq h}}^{p} C_{vgj0}\right) + \sum_{h=1}^{p}\sum_{\substack{g=1\\ g\neq h}}^{p} C_{vhj0}^{*}\left(C_{vgj0}^{*}\prod_{\substack{t=1\\ t\neq h,\, g}}^{p} C_{vtj0}\right)\right)\right], \tag{28}$$

$$\frac{\partial^2 C_j}{\partial\beta_{h0}^2(\mathbf{u}_i)} = \sum_{v=0}^{s}\left[\frac{\lambda_{00}^{v}(\mathbf{u}_i)}{v!}(C_{vhj0}^{**}e^{2\beta_{h0}(\mathbf{u}_i)} + C_{vhj0}^{*}e^{\beta_{h0}(\mathbf{u}_i)})\prod_{\substack{g=1\\ g\neq h}}^{p} C_{vgj0}\right], \tag{29}$$

$$\frac{\partial^2 C_{j0}}{\partial\lambda_{00}(\mathbf{u}_i)\partial\beta_{h0}(\mathbf{u}_i)}$$

$$= \sum_{v=0}^{s}\left[\frac{\lambda_{00}^{v-1}(\mathbf{u}_i)}{(v-1)!}C_{vhj0}^{*}e^{\beta_{h0}(\mathbf{u}_i)}\prod_{\substack{g=1\\ g\neq h}}^{p} C_{vgj0} - \frac{\lambda_{00}^{v}(\mathbf{u}_i)}{v!}C_{vhj0}^{**}e^{\beta_{h0}(\mathbf{u}_i)}\prod_{\substack{g=1\\ g\neq h}}^{p} C_{vgj0}\right.$$

$$\left.\left. - \frac{\lambda_{00}^{v}(\mathbf{u}_i)}{v!} C_{vhj0}^{*} e^{\beta_{h0}(\mathbf{u}_i)} \sum_{\substack{g=1 \\ g \neq h}}^{p} \left(C_{vgj0}^{*} \prod_{\substack{t=1 \\ t \neq g, h}}^{p} C_{vtj0} \right) \right] \right], \tag{30}$$

$$\frac{\partial^2 C_{j0}}{\partial \beta_{h0}(\mathbf{u}_i) \partial \beta_{g0}(\mathbf{u}_i)}$$

$$= \sum_{v=0}^{s} \left(\frac{\lambda_{00}^{v}(\mathbf{u}_i)}{v!} C_{vhj0}^{*} e^{\beta_{h0}(\mathbf{u}_i)} C_{vgj0}^{*} e^{\beta_{g0}(\mathbf{u}_i)} \prod_{\substack{t=1 \\ t \neq h, g}}^{p} C_{vtj0} \right), \tag{31}$$

where

$$C_{vhj0}^{**} = \frac{(e^{\beta_{h0}(\mathbf{u}_i)} - \lambda_{00}(\mathbf{u}_i))^{(y_{hj} - v - 2)}}{(y_{hj} - v - 2)!}.$$

The test statistic G_2 follows the asymptotic distribution of chi-square with the degree of freedom is the difference between the effective number of parameters in the GWMPR without the covariate in the model (m_3) and GWMPR model (m_2). So, at the significance level α, reject the null hypothesis when G_2 value falls into the rejection region, i.e., $G_2 > \chi^2_{(\alpha, m_2 - m_3)}$.

Further, the significance test of individual parameters is used to determine the significance of each parameter in the regression model. The form of hypothesis can be expressed as:

$$H_0 : \beta_{hl}(\mathbf{u}_i) = 0,$$

$$H_1 : \beta_{hl}(\mathbf{u}_i) \neq 0. \tag{32}$$

The results of parameters estimation by MLE method and the iterative procedure have a consequence that exact distribution for $\hat{\boldsymbol{\theta}}(\mathbf{u}_i)$ is difficult to be obtained. The inference can be used by the properties of asymptotic distribution of MLE [11] as follows:

$$\hat{\boldsymbol{\theta}}(\mathbf{u}_i) \overset{a}{\sim} \mathbf{N}(\boldsymbol{\theta}(\mathbf{u}_i), [\mathbf{I}(\boldsymbol{\theta}(\mathbf{u}_i))]^{-1}). \tag{33}$$

Considering equation (33), the test statistic to the hypothesis testing equation (32) can be written as follows:

$$Z = \frac{\hat{\beta}_{hl}(\mathbf{u}_i)}{\sqrt{\widehat{\text{var}}(\hat{\beta}_{hl}(\mathbf{u}_i))}} \sim N(0, 1), \tag{34}$$

where $\widehat{\text{var}}(\hat{\beta}_{hl}(\mathbf{u}_i))$ is $[(k+1)(h-1)+(l+2)]$th the diagonal elements of matrix $[\mathbf{I}(\boldsymbol{\theta})(\mathbf{u}_i)]^{-1}$.

The test statistic Z follows the asymptotic distribution of $N(0, 1)$. So, at the significance level α, reject the null hypothesis when Z value falls into the rejection region, i.e., $|Z| > Z_{\alpha/2}$.

6. Conclusion

GWMPR is a local form of MPR that allows the modelling of spatially heterogeneous processes. This model results in a set of local parameter estimates which depends on the geographical location where the data are observed. There are three hypothesis tests of the GWMPR model. The first test is a goodness of fit test that is used to determine if there is a difference between models of GWMPR and MPR. In this test, the test statistic follows the asymptotic distribution of chi-square with the degree of freedom is the difference between the effective number of parameters in the MPR and GWMPR models. The second test is an overall test that is used to determine the simultaneous significance of the regression parameters to predict the response variables. In this test, the test statistic follows the asymptotic distribution of chi-square with the degree of freedom is the difference between the effective number of parameters in the GWMPR without the covariate in the model and GWMPR model. The significance test of individual parameters is used to determine the significance of each parameter in the regression model. The test statistic follows the asymptotic distribution of the standard normal.

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