Exact Algorithms via Monotone Local Search

Paper.

The main result of the paper is that for a large class of problems, an algorithm with running time $c^k N^{\mathcal{O}(1)}$ for any $c > 1$ immediately implies an exact algorithm with running time $(2 - \frac{1}{c})^{n+o(n)}$.

Subset problem: Let U be a universe on n elements, and let $\mathcal{F}$ be a collection of subsets of U. The task is to determine whether $|\mathcal{F}| > 0$.

Examples:

1. Weighted d-SAT

   Input: A propositional formula F in conjunctive normal form where each clause has at most d literals, a weight function $w : var(F) \to \mathbb{Z}$, and integers k and W.
   Question: Is there a set $S \subseteq var(F)$ of size at most k and weight at most W such that F is satisfied by the assignment that sets the variables in S to 1 and all other variables to 0?

2. Feedback Vertex Set

   Input: A graph G and a positive integer k.
   Question: Does there exist a subset $S \subseteq V(G)$ of size at most k such that graph $G - S$ is acyclic?

In the area of exact exponential-time algorithms, the objective is to design algorithms that outperform brute-force for computationally intractable problems. For subset problems, the brute-force algorithm that tries all possible solutions has running time $2^n n^{\mathcal{O}(1)}$. Thus our goal is typically to design an algorithm with running time $c^n n^{\mathcal{O}(1)}$ for $c < 2$, and we try to minimize the constant c.

Def. *Implicit set system* is a function $\Phi : I \to (U_I, \mathcal{F}_I)$, where $I \in \{0,1\}^*$ is referred to as an *instance*, and we denote by $|U_I| = n$ the size of the universe and by $|I| = N$ the size of the instance. We assume that $N \geq n$.

The implicit set system Φ is said to be polynomial time computable if (a) there exists a polynomial time algorithm that given I produces U_I , and (b) there exists a polynomial time algorithm that given I, U_I and a subset S of

U_I determines whether $S \in F_I$.

Φ-Subset Problem.
Input: An instance I
Question: A set $S \in F_I$ if one exists.

Φ-Extension.
Input: An instance I, a set $X \subseteq U_I$, and an integer k
Question: Does there exists a subset $S \subseteq (U_I \setminus X)$ such that $S \cup X \in F_I$ and $|S| \leq k$?

Theorem 1. If there exists an algorithm for Φ-Extension with running time $c^k N^{\mathcal{O}(1)}$ then there exists a randomized algorithm for Φ-Subset with running time $(2 - \frac{1}{c})^n N^{\mathcal{O}(1)}$.

Approximate proof. To prove it, we use *monotone* local search: we randomly sample a set X, which is supposed to be a subset of the answer S, and then run the parameterized algorithm for Φ-Extension in $c^{|S|-|X|} N^{\mathcal{O}(1)}$ time. We run this algorithm for each possible $k' = |S|$ and repeat it $\frac{\binom{n}{|X|}}{\binom{k'}{|X|}}$ times. We choose the size of $|X|$ in such way that the running time is minimized.

Theorem 2. If there exists an algorithm for Φ-Extension with running time $c^k N^{\mathcal{O}(1)}$ then there exists an algorithm for Φ-Subset with running time $(2 - \frac{1}{c})^{n+o(n)} N^{\mathcal{O}(1)}$.

Approximate proof. To prove it, we need to derandomize our algorithm.

Let U be a universe of size n and let $0 \leq q \leq p \leq n$. A family $\mathcal{C} \subseteq \binom{U}{q}$ is an (n,p,q)-set-inclusion-family, if for every set $S \in \binom{U}{p}$, there exists a set $Y \in \mathcal{C}$ such that $Y \subseteq S$.

Let $\kappa(n,p,q) = \frac{\binom{n}{q}}{\binom{p}{q}}$. There is a deterministic construction of an (n,p,q)-set-inclusion-family, $\mathcal{C}$, of size at most $\kappa(n,p,q) \cdot 2^{o(n)}$. The running time of the algorithm constructing $\mathcal{C}$ is also upper bounded by $\kappa(n,p,q) \cdot 2^{o(n)}$. Now, instead of $\kappa(n,k',|X|)$ independent repetitions of sampling X, the new algorithm loops over all $Y \in \mathcal{C}$.