Recursive Rigid Body Dynamics

a Geometric Approach based on Lagrange Equations and Screw Theory

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Contents

Chapter 1

Full Recursive Dynamics

The derivations are done using methods from [5] and [3]. The notation from [5] is adopted. The description of the connectivity is used as done in [1][2].

1.1 Kinematics

1.1.1 Position Kinematics

The generalized coordinates are

$$\boldsymbol{q} = [q_1 \ \cdots \ q_n]^T \in \mathbb{R}^n. \tag{1.1}$$

We then define the connectivity vector $\boldsymbol{\lambda}$ such that $\boldsymbol{\lambda}(i)$ returns the parent body of body i. Transformations between bodies are defined as homogeneous transformation matrices as some function of te generalized coordinates as shown below.

$$\mathbf{H}_i^{\lambda(i)}(\boldsymbol{q}) \in SE(3). \tag{1.2}$$

The global transformations are found as

$$\mathbf{H}_i^0(\boldsymbol{q}) = \mathbf{H}_{\lambda(i)}^0(\boldsymbol{q})\mathbf{H}_i^{\lambda(i)}(\boldsymbol{q}). \tag{1.3}$$

The arguments of $\mathbf{H}$ will be omitted from here on.

1.1.2 Velocity Kinematics

The local twists from body i with respect to body $i-1$ is expressed in body (i) coordinates as

$$\tilde{\mathbf{T}}_i^{i\,\lambda(i)} = \mathbf{H}_i^{\lambda(i)^{-1}}\dot{\mathbf{H}}_i^{\lambda(i)} \in \mathfrak{se}(3). \tag{1.4}$$

Twists have the form of

$$\tilde{\mathbf{T}} = \begin{bmatrix} \tilde{\boldsymbol{\omega}} & \boldsymbol{v} \\ \mathbf{0} & 0 \end{bmatrix}, \text{ or } \boldsymbol{T} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} \tag{1.5}$$

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Geometric Jacobian

Twists can be transformed by using an Adjoint transformation matrix as

$$\tilde{\mathbf{T}}_i^{k\,\lambda(i)} = \mathbf{H}_i^k \tilde{\mathbf{T}}_i^{i\,\lambda(i)} \mathbf{H}_i^{k^{-1}}, \text{ or } \boldsymbol{T}_i^{k\,\lambda(i)} = \mathrm{Ad}_{\mathbf{H}_i^k} \boldsymbol{T}_i^{i\,\lambda(i)}. \tag{1.6}$$

This fact can be used to recursively express the global twists, expressed in body coordinates, as shown below.

$$\begin{aligned}
\tilde{\mathbf{T}}_i^{i\,0} &= \mathbf{H}_i^{0^{-1}} \dot{\mathbf{H}}_i^0 \\
&= \mathbf{H}_i^{0^{-1}} \left(\dot{\overline{\mathbf{H}_{\lambda(i)}^0 \mathbf{H}_i^{\lambda(i)}}} \right) \\
&= \mathbf{H}_i^{0^{-1}} \left(\dot{\overline{\mathbf{H}_{\lambda(i)}^0 \mathbf{H}_i^{\lambda(i)}}} \right) \\
&= \mathbf{H}_i^{0^{-1}} \dot{\mathbf{H}}_{\lambda(i)}^0 \mathbf{H}_i^{\lambda(i)} + \mathbf{H}_i^{0^{-1}} \mathbf{H}_{\lambda(i)}^0 \dot{\mathbf{H}}_i^{\lambda(i)} \\
&= \mathbf{H}_i^{\lambda(i)^{-1}} \mathbf{H}_{\lambda(i)}^{0\ \ -1} \dot{\mathbf{H}}_{\lambda(i)}^0 \mathbf{H}_i^{\lambda(i)} + \mathbf{H}_i^{\lambda(i)^{-1}} \mathbf{H}_{\lambda(i)}^{0\ \ -1} \mathbf{H}_{\lambda(i)}^0 \dot{\mathbf{H}}_i^{\lambda(i)} \\
\tilde{\mathbf{T}}_i^{i\,0} &= \mathbf{H}_i^{\lambda(i)^{-1}} \tilde{\mathbf{T}}_{\lambda(i)}^{\lambda(i)\,0} \mathbf{H}_i^{\lambda(i)} + \tilde{\mathbf{T}}_i^{i\,\lambda(i)} && (1.7) \\
\boldsymbol{T}_i^{i\,0} &= \mathrm{Ad}^{-1}_{\mathbf{H}_{\lambda(i)}^i} \boldsymbol{T}_{\lambda(i)}^{\lambda(i)\,0} + \boldsymbol{T}_i^{i\,\lambda(i)} && (1.8)
\end{aligned}$$

To express the Jacobians of the spatial velocities of body i expressed in body coordinates, we derive with respect to $\dot{\boldsymbol{q}}$.

$$\mathbf{J}_i^b = \boldsymbol{\nabla}_{\dot{\boldsymbol{q}}} \boldsymbol{T}_i^{i\,0} = \left[\frac{\partial}{\partial \dot{q}_1} \boldsymbol{T}_i^{i\,0} \ \cdots \ \frac{\partial}{\partial \dot{q}_n} \boldsymbol{T}_i^{i\,0} \right]. \tag{1.9}$$

Here, we can write

$$\begin{aligned}
\frac{\partial}{\partial \dot{q}_j} \boldsymbol{T}_i^{i\,0} &= \frac{\partial}{\partial \dot{q}_j} \left(\mathrm{Ad}^{-1}_{\mathbf{H}_{\lambda(i)}^i} \boldsymbol{T}_{\lambda(i)}^{\lambda(i)\,0} + \boldsymbol{T}_i^{i\,\lambda(i)} \right) \\
&= \mathrm{Ad}^{-1}_{\mathbf{H}_{\lambda(i)}^i} \frac{\partial}{\partial \dot{q}_j} \boldsymbol{T}_{\lambda(i)}^{\lambda(i)\,0} + \frac{\partial}{\partial \dot{q}_j} \boldsymbol{T}_i^{i\,\lambda(i)}, && (1.10)
\end{aligned}$$

such that we can found a recursive formulation for the Jacobians expressed in body coordinates as

$$\begin{aligned}
\mathbf{J}_i^b &= \left[\mathrm{Ad}^{-1}_{\mathbf{H}_{\lambda(i)}^i} \frac{\partial}{\partial \dot{q}_1} \boldsymbol{T}_{\lambda(i)}^{\lambda(i)\,0} + \frac{\partial}{\partial \dot{q}_1} \boldsymbol{T}_i^{i\,\lambda(i)} \ \cdots \ \mathrm{Ad}^{-1}_{\mathbf{H}_{\lambda(i)}^i} \frac{\partial}{\partial \dot{q}_n} \boldsymbol{T}_{\lambda(i)}^{\lambda(i)\,0} + \frac{\partial}{\partial \dot{q}_n} \boldsymbol{T}_i^{i\,\lambda(i)} \right] \\
&= \mathrm{Ad}^{-1}_{\mathbf{H}_{\lambda(i)}^i} \left[\frac{\partial}{\partial \dot{q}_1} \boldsymbol{T}_{\lambda(i)}^{\lambda(i)\,0} \ \cdots \ \frac{\partial}{\partial \dot{q}_n} \boldsymbol{T}_{\lambda(i)}^{\lambda(i)\,0} \right] + \left[\frac{\partial}{\partial \dot{q}_1} \boldsymbol{T}_i^{i\,\lambda(i)} \ \cdots \ \frac{\partial}{\partial \dot{q}_n} \boldsymbol{T}_i^{i\,\lambda(i)} \right] \\
&= \mathrm{Ad}^{-1}_{\mathbf{H}_{\lambda(i)}^i} \mathbf{J}_{\lambda(i)}^b + \left[\frac{\partial}{\partial \dot{q}_1} \boldsymbol{T}_i^{i\,\lambda(i)} \ \cdots \ \frac{\partial}{\partial \dot{q}_n} \boldsymbol{T}_i^{i\,\lambda(i)} \right]. && (1.11)
\end{aligned}$$

1.2 Dynamics

1.2.1 Mass Matrix

The mass matrix in configuration space can be found by summing the individual mass matrices. The final matrix $\mathbf{M}_n$ shall be denoted as $\mathbf{M}$.

$$\mathbf{M}_i = \mathbf{M}_{\lambda(i)} + \mathbf{J}_i^{b\,T} \mathcal{I}_i \mathbf{J}_i^b. \tag{1.12}$$

1.2.2 Coriolis Matrix

First, the partial derivatives of the Jacobian can be calculated recursively by taking partial derivatives of (1.11).

$$\begin{aligned} \frac{\partial}{\partial q_k}\mathbf{J}_i^b &= \frac{\partial}{\partial q_k}\left(\mathrm{Ad}^{-1}_{\mathbf{H}^i_{\lambda(i)}}\mathbf{J}^b_{\lambda(i)} + \left[\frac{\partial}{\partial \dot{q}_1}\boldsymbol{T}_i^{i\,\lambda(i)} \;\cdots\; \frac{\partial}{\partial \dot{q}_n}\boldsymbol{T}_i^{i\,\lambda(i)}\right]\right) \\ &= \frac{\partial}{\partial q_k}\left(\mathrm{Ad}^{-1}_{\mathbf{H}^i_{\lambda(i)}}\right)\mathbf{J}^b_{\lambda(i)} + \mathrm{Ad}^{-1}_{\mathbf{H}^i_{\lambda(i)}}\frac{\partial}{\partial q_k}\mathbf{J}^b_{\lambda(i)} \\ &\quad + \frac{\partial}{\partial q_k}\left(\left[\frac{\partial}{\partial \dot{q}_1}\boldsymbol{T}_i^{i\,\lambda(i)} \;\cdots\; \frac{\partial}{\partial \dot{q}_n}\boldsymbol{T}_i^{i\,\lambda(i)}\right]\right). \end{aligned} \tag{1.13}$$

The partial derivatives of the mass matrix can now be expressed recursively as well by deriving (1.12).

$$\frac{\partial}{\partial q_k}\mathbf{M}_i = \frac{\partial}{\partial q_k}\mathbf{M}_{\lambda(i)} + \left(\frac{\partial}{\partial q_k}\mathbf{J}_i^b\right)^T \mathcal{I}_i \mathbf{J}_i^b + \mathbf{J}_i^{b\,T}\mathcal{I}_i\frac{\partial}{\partial q_k}\mathbf{J}_i^b. \tag{1.14}$$

With $\frac{\partial}{\partial q_k}\mathbf{M}_n$ shall be refered to as $\frac{\partial}{\partial q_k}\mathbf{M}$. The Christoffel symbols are found by summing elements of mass matrix derivatives as

$$\Gamma(i,j,k) = \frac{1}{2}\left(\frac{\partial}{\partial q_k}M_{i,j} + \frac{\partial}{\partial q_j}M_{i,k} - \frac{\partial}{\partial q_i}M_{k,j}\right). \tag{1.15}$$

Finally, the Coriolis matrix elements can be found as shown below.

$$C_{i,j} = \sum_{k=1}^{n}\Gamma(i,j,k)\dot{q}_k \tag{1.16}$$

1.2.3 Gravity Wrench

The gravity component can be found by calculating the joint torques due to the gravity wrench on each body.

$$\begin{gathered} \boldsymbol{G}_i = \boldsymbol{G}_{\lambda(i)} + \mathbf{J}_i^{b\,T}\begin{bmatrix}\boldsymbol{F}_{g,i} \\ \mathbf{0}\end{bmatrix}, \\ \text{with } \boldsymbol{F}_{g,i} = \mathbf{R}_i^{0\,-1}[0\;0\;\; -m_i g]^T \end{gathered} \tag{1.17}$$

Here, the gravity wrench is expressed in body coordinates such that it can be mapped to joint torques by using the body Jacobian. For now, it will be assumed that the homogeneous transformations $\mathbf{H}$ and therefore the Jacobians $\mathbf{J}^b$ are expressed at the center of mass of each body.

Chapter 2

Recursive Dynamics

In the following chapter, the transformations will be explicitly defined such that it results in simplified recursive expressions.

2.1 Kinematics

2.1.1 Position Kinematics

Now let us define the homogeneous transformation matrices. Let both bodies be connected by joint with one single degree of freedom. The transformation matrix that describes the position and orientation of body i relative to its parent $\lambda(i)$ for revolute and prismatice joints respectively take the form of

$$\mathbf{H}_i^{\lambda(i)}(q_i) = \begin{bmatrix} \mathbf{R}_i^{\lambda(i)}(q_i) & \boldsymbol{d}_i \\ \mathbf{0} & 0 \end{bmatrix}, \tag{2.1}$$

$$\mathbf{H}_i^{\lambda(i)}(q_i) = \begin{bmatrix} \mathbf{I} & \boldsymbol{d}_i(q_i) \\ \mathbf{0} & 0 \end{bmatrix}, \tag{2.2}$$

where $\mathbf{R}_i^{\lambda(i)}(q_i) \in SO(3)$ is a rotation matrix, $\mathbf{I} \in \mathbb{R}^{3\times 3}$ the identity matrix and $\boldsymbol{d}_i, \boldsymbol{d}_i(q_i) \in \mathbb{R}^3$ represent translations. The transformation consists of a successive translation and rotation as depicted in figure 2.1. For a revolute joint, for example, the vector d_i is the constant offset of a joint from its parent body, expressed in parent body coordinates. The rotation matrix then rotates body i with an angle q_i from its parent joint.

Equation (1.3) becomes

$$\mathbf{H}_i^0(\boldsymbol{q}) = \mathbf{H}_{\lambda(i)}^0(\boldsymbol{q})\mathbf{H}_i^{\lambda(i)}(q_i). \tag{2.3}$$

Note that the CoM location is not yet included. It shall be used later. For now, let us define the position of the CoM as a homogeneous transformation matrix

$$\mathbf{H}_{mi}^i = \begin{bmatrix} \mathbf{I} & \boldsymbol{r}_i \\ \mathbf{0} & 0 \end{bmatrix}, \tag{2.4}$$

where $\boldsymbol{r}_i$ the offset of the CoM from the body frame. The transformation $\mathbf{H}_{mi}^i$ then transforms from body frame i to the CoM frame mi.

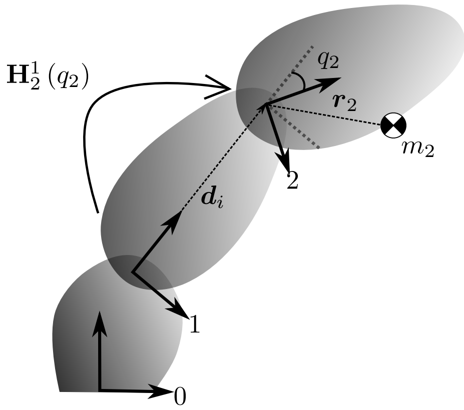

Figure 2.1: A representation of a kinematic chain with revolute joints.

2.1.2 Velocity Kinematics

Because all transformations only contain a single degree of freedom, the constant unit twist can be defined as

$$\boldsymbol{T} = \hat{\boldsymbol{T}}\,\dot{q}. \tag{2.5}$$

For the homogeneous transformations of the form (2.1) and (2.2), these unit twists have simple, intuitive forms. A joint rotating around the y-axis, for example, has an associated unit twist of the form $[0\,0\,0\,0\,1\,0\,]^T$. Using the twist exponent, elements of $\mathfrak{se}(3)$ can be related with elements of $SE(3)$. In other words, the exponent of a twist $\tilde{\mathbf{T}}$ returns a homogeneous transformation $\mathbf{H}$. By taking the exponent of $\hat{\boldsymbol{T}}\,\dot{q}$, it can indeed be shown that the simple unit twist results in a transformation matrix as given in (2.1). See [5] and [3] for details. Now, we can use this fact to find

$$\begin{aligned}
\frac{\partial}{\partial \dot{q}_i}\tilde{\mathbf{T}}_i^{i\,\lambda(i)} &= \mathbf{H}_i^{\lambda(i)\,-1}\frac{\partial}{\partial \dot{q}_i}\mathbf{H}_i^{\lambda(i)} \\
&= \exp(-\hat{\tilde{\mathbf{T}}}_i^{i\,\lambda(i)}\dot{q}_i)\exp(\hat{\tilde{\mathbf{T}}}_i^{i\,\lambda(i)}\dot{q}_i)\hat{\tilde{\mathbf{T}}}_i^{i\,\lambda(i)} \\
&= \hat{\tilde{\mathbf{T}}}_i^{i\,\lambda(i)}.
\end{aligned}$$

such that

$$\frac{\partial}{\partial \dot{q}_l}\tilde{\mathbf{T}}_i^{i\,\lambda(i)} = \begin{cases} l = i & \hat{\tilde{\mathbf{T}}}_i^{i\,\lambda(i)} \\ l \neq i & \mathbf{0} \end{cases}, \tag{2.6}$$

and

$$\frac{\partial}{\partial q_k}\left(\frac{\partial}{\partial \dot{q}_l}\tilde{\mathbf{T}}_i^{i\,\lambda(i)}\right) = 0,\ \forall k, l. \tag{2.7}$$

These expressions will be used in the following sections to simply the formulas from the previous chapter.

Geometric Jacobian

Using (2.6) we can simplify (1.11) to

$$\mathbf{J}_i^b = \mathrm{Ad}^{-1}_{\mathbf{H}^i_{\lambda(i)}} \mathbf{J}^b_{\lambda(i)} + \begin{bmatrix} \mathbf{0}_{6\times(i-1)} & \hat{\hat{\mathbf{T}}}_i^{i\,\lambda(i)} & \mathbf{0}_{6\times(n-i)} \end{bmatrix} \tag{2.8}$$

where the unit twist is located at the ith column. The Jacobian above gives the velocities expressed in a coordinate frame located at the origin of the parent joint. To obtain the velocities expressed in a coordinate frame located at the CoM position of the body, we can do an Adjoint transformation as

$$\mathbf{J}^b_{mi} = \mathrm{Ad}^{-1}_{\mathbf{H}^i_{mi}} \mathbf{J}^b_i. \tag{2.9}$$

Analytic Jacobian

The analytic Jacobian can be found by the direct derivative of the CoM position as follows

$$\mathbf{J}^a_i = \boldsymbol{\nabla}_{\dot{\boldsymbol{q}}} \mathbf{S}_4 \mathbf{H}^0_{mi} \boldsymbol{v}_4 \tag{2.10}$$

with

$$\mathbf{S}_i = \begin{bmatrix} \mathbf{I}_{3\times 3} & \mathbf{0}_{3\times(i-3)} \end{bmatrix} \tag{2.11}$$

$$\boldsymbol{v}_i = \begin{bmatrix} \mathbf{0}_{1\times(i-1)} & 1 & \mathbf{0}_{1\times(4-i)} \end{bmatrix}^T. \tag{2.12}$$

This Jacobian only delivers translational velocities. The complete analytic Jacobian can also be constructed geometrically as done in [4]. This Jacobian shall be referred to as $\mathbf{J}^g_i$. For both cases, though, the Jacobian returns velocities expressed in a frame that has the CoM position as the origin but which is aligned with the global frame, whereas $\mathbf{J}^b_{mi}$ is aligned with the body axes as shown in figure 2.1. So, the analytic Jacobian can be obtained using the geometric body Jacobian by a simple Adjoint transformation. Let $\mathrm{Ad}_{\mathbf{R}^0_i}$ be the Adjoint transformation of $\mathbf{H}^0_i$ with the translation set to zero. Then

$$\mathbf{J}^g_i = \mathrm{Ad}_{\mathbf{R}^0_i} \mathbf{J}^b_{mi} \tag{2.13}$$

$$\mathbf{J}^g_i = \mathrm{Ad}_{\mathbf{R}^0_i} \mathrm{Ad}^{-1}_{\mathbf{H}^i_{mi}} \mathbf{J}^b_i \tag{2.14}$$

Where we have used the fact that $\mathbf{R}^0_{mi} = \mathbf{R}^0_i$. The analytic Jacobian is then

$$\mathbf{J}^a_i = \mathbf{S}_6 \mathbf{J}^g_i \tag{2.15}$$

$$\mathbf{J}^a_i = \mathbf{S}_6 \mathrm{Ad}_{\mathbf{R}^0_i} \mathrm{Ad}^{-1}_{\mathbf{H}^i_{mi}} \mathbf{J}^b_i. \tag{2.16}$$

2.2 Dynamics

2.2.1 Mass Matrix

Now that we have found an expression of the Jacobian expressed at the CoM of the body, equation (1.12) changes to

$$\begin{aligned}\mathbf{M}_i &= \mathbf{M}_{\lambda(i)} + {\mathbf{J}^b_{mi}}^T \mathcal{I}_i \mathbf{J}^b_{mi} \\ &= \mathbf{M}_{\lambda(i)} + {\mathbf{J}^b_i}^T {\mathrm{Ad}^{-1}_{\mathbf{H}^i_{mi}}}^T \mathcal{I}_i \mathrm{Ad}^{-1}_{\mathbf{H}^i_{mi}} \mathbf{J}^b_i \\ &= \mathbf{M}_{\lambda(i)} + {\mathbf{J}^b_i}^T \mathcal{I}_{ii} \mathbf{J}^b_i \end{aligned} \tag{2.17}$$

where we have found an expression for the local mass matrix

$$\mathcal{I}_{ii} = {\mathrm{Ad}^{-1}_{\mathbf{H}^i_{mi}}}^T \mathcal{I}_i \mathrm{Ad}^{-1}_{\mathbf{H}^i_{mi}} \tag{2.18}$$

with $\mathcal{I}_i$ is the local mass matrix at the CoM position, aligned with the principle axes of inertia, which has the form of

$$\mathcal{I}_i = \begin{bmatrix} m_i & 0 & 0 & 0 & 0 & 0 \\ 0 & m_i & 0 & 0 & 0 & 0 \\ 0 & 0 & m_i & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{ix} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{iy} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{iz} \end{bmatrix}. \tag{2.19}$$

Note that when the analytic Jacobian is used to obtain the mass matrix in configuration space, the local mass matrix of equation (2.19) is modified by rotating the inertia. Writing this out confirms the relation between the analytic and geometric Jacobian found in equation (2.14).

2.2.2 Coriolis Matrix

Using [5] and [3], the derivative of an Adjoint can be found. Using the equations from the previous chapter, only a few of these derivatives are required. Moreover, for joints with one degree of freedom, many reduce to zero matrices.

$$\frac{\partial}{\partial q_k}\left(\mathrm{Ad}^{-1}_{\mathbf{H}^i_{\lambda(i)}}\right) = \begin{cases} k = i & -\mathrm{ad}_{\hat{\boldsymbol{T}}^{k\ \lambda(k)}_k} \mathrm{Ad}^{-1}_{\mathbf{H}^i_{\lambda(i)}} \\ k \neq i & \mathbf{0} \end{cases} \tag{2.20}$$

Using this fact and the simplification of equation (2.7), the Jacobian derivative of (2.21) reduces to

$$\frac{\partial}{\partial q_k}\mathbf{J}^b_i = \begin{cases} k = i & -\mathrm{ad}_{\hat{\boldsymbol{T}}^{k\ \lambda(k)}_k} \mathrm{Ad}^{-1}_{\mathbf{H}^i_{\lambda(i)}} \mathbf{J}^b_{\lambda(i)} + \mathrm{Ad}^{-1}_{\mathbf{H}^i_{\lambda(i)}} \frac{\partial}{\partial q_k}\mathbf{J}^b_{\lambda(i)} \\ k \neq i & \mathrm{Ad}^{-1}_{\mathbf{H}^i_{\lambda(i)}} \frac{\partial}{\partial q_k}\mathbf{J}^b_{\lambda(i)} \end{cases}. \tag{2.21}$$

The mass matrix derivative is then

$$\frac{\partial}{\partial q_k}\mathbf{M}_i = \frac{\partial}{\partial q_k}\mathbf{M}_{\lambda(i)} + \left(\frac{\partial}{\partial q_k}\mathbf{J}^b_i\right)^T \mathcal{I}_{ii}\mathbf{J}^b_i + {\mathbf{J}^b_i}^T \mathcal{I}_{ii} \frac{\partial}{\partial q_k}\mathbf{J}^b_i, \tag{2.22}$$

where only the new local mass matrix $\mathcal{I}_{ii}$ is used. The Christoffel symbols and the Coriolis matrix can now be found in a the same way as before.

2.2.3 Gravity Wrench

The body Jacobian in (1.17) is replaced to take the position of the CoM into account.

$$\begin{aligned} \boldsymbol{G}_i = \boldsymbol{G}_{\lambda(i)} + \left(\mathrm{Ad}^{-1}_{\mathbf{H}^i_{mi}} \mathbf{J}^b_i\right)^T \begin{bmatrix} \boldsymbol{F}_{g,i} \\ \mathbf{0} \end{bmatrix}, \\ \text{with } \boldsymbol{F}_{g,i} = {\mathbf{R}^0_i}^{-1} [0\ 0\ \ -m_i g]^T \end{aligned} \tag{2.23}$$

Or, now that the analytic Jacobian has been calculated, we can write

$$\boldsymbol{G}_i = \boldsymbol{G}_{\lambda(i)} + \mathbf{J}^a_i [0\ 0\ \ -m_i g\ 0\ 0\ 0]^T. \tag{2.24}$$

When rewriting (2.23), transforming the gravity wrench can be done with $\mathrm{Ad}_{\mathbf{R}^0_i}$ and when making use of the fact that $\mathrm{Ad}^{-1}_{\mathbf{R}^0_i} = {\mathrm{Ad}_{\mathbf{R}^0_i}}^T$ it can indeed be found that (2.23) equals (2.24).

Bibliography

[1] Roy Featherstone. A beginner's guide to 6-d vectors (part 1). *Robotics & Automation Magazine, IEEE*, 17(3):83–94, 2010.

[2] Roy Featherstone. A beginner's guide to 6-d vectors (part 2). *Robotics & Automation Magazine, IEEE*, 17(4):88–99, 2010.

[3] Richard M Murray, Zexiang Li, S Shankar Sastry, and S Shankara Sastry. *A mathematical introduction to robotic manipulation.* CRC press, 1994.

[4] Mark W Spong, Seth Hutchinson, and Mathukumalli Vidyasagar. *Robot modeling and control.* John Wiley & Sons New York, 2006.

[5] Stefano Stramigioli and Herman Bruyninckx. Geometry and screw theory for robotics. *Tutorial during ICRA*, 2001, 2001.