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Project topic:
Traffic flow simulation: case studies of traffic in Atlanta and Los Angeles

Goal:
The project objective is to build a mathematical model to simulate the traffic flow in Atlanta and Los Angeles. The ultimate goal is to understand the traffic phenomena in order to optimize traffic networks, alleviate congestion, maximize traffic flow and decrease accident rate. To validify the reliability of our models, we will perform case studies via simulating the real traffic data in Atlanta and Los Angeles with our models. The implemented models would be tweaked based on the validation.

Proposed method:
We intend to simulate the traffic flow with two models (progressively increasing in complexity). The first model would simulate the traffic flow with *linear partial differential equations* (PDE). This model simplifies the discrete traffic flow as continuous vehicle density. We would solve the PDEs both analytically and numerically. The numerical solution would be verified by the analytical solution. The second model (*non-linear*) would target more realistic conditions via taking into account more complex effects, such as the traffic lights, intersections, entrances and exits. etc.

1) *Simple model*
In the simple model, the traffic flow (the density of cars per unit length) is formulated as a function of position and time given the initial density. The ***variables*** that would be included are: traffic flux (J), position (x), time (t), vehicle density (ρ), and vehicle velocity (v).

For the purpose of simulating the traffic flow on a first-order, we make a few ***assumptions***:

1. The vehicle density is continuous. It means that our approach does not use individual car behavior to implement the traffic flow simulation.
2. The vehicle motion is unidirectional from left to right on a one-lane road of infinite length.
3. All vehicles are assumed to have the same length (l) and they are evenly spaced at distance d, which is simplified as a *uniform distribution model* (the vehicle density is homogeneous along the infinite lane).
4. All other factors are not considered in the basic implementation of the model, but they might be tested and discussed given the progress and availability of the schedule.

The *vehicle density* ρ is formulated as a function of position x and time t. The vehicle density at position x0 and time t0 is calculated as follows according to the definition (number of vehicles per unit length).

$$\rho(x0, t0) = 1 / (L + d) \qquad \text{(Eq. 1)}$$

Given the non-negativity of vehicle spacing d, the vehicle density is upper bounded by ($1 / L$).

The *traffic flux J* would be a linear combination of vehicle speed and vehicle density as follows:

$$J(x0, t0) = \rho(x0, t0) * v(x0, t0) \qquad \text{(Eq. 2)}$$

Substitute Eq. 1 in Eq. 2, we get

$$J(x0, t0) = v(x0, t0) / (L + d) \qquad \text{(Eq. 3)}$$

Consider a position interval Δx and time interval Δt around $x0$ and $t0$, according to the *Balance law for density*, the following equation can be derived:

$$\Delta x * [\rho(x0, t0 + \Delta t) - \rho(x0, t0)] = \Delta t * [J(x0 + \Delta x, t0) - J(x0, t0)] \qquad \text{(Eq. 4)}$$

After conducting Taylor expansion on both sides of Eq. 4 and omitting the third-order in terms of Δx and Δt, we get

$$\partial\rho/\partial t = - \partial J/\partial x \qquad \text{(Eq. 5)}$$

According to Eq.2, substitute J with ρv, the linear PDE model for traffic flow is:

$$\partial\rho/\partial t + \partial(\rho v)/\partial x = 0 \qquad \text{(Eq. 6)}$$

with the initial condition (initial vehicle density) of

$$\rho(x, 0) = g(x) \qquad \text{(Eq. 7)}$$

The relationship between velocity and density would be derived from the empirical traffic data, and its correlation would be linearly fitted (according to the *Greenshields model*) as follows:

$$v = v\_max (1 - \rho / \rho\_max) \qquad \text{(Eq. 8)}$$

Then Eq. 6 would be transformed to

$$\partial\rho/\partial t + C(\rho) * \partial\rho/\partial x = 0 \qquad \text{(Eq. 9)}$$

where $C(\rho)$ is expressed as:

$$C(\rho) = v\_max (1 - 2\rho / \rho\_max) \qquad \text{(Eq. 10)}$$

The above PDE would be first solved analytically and then solved numerically. The proposed numerical scheme would be a finite *difference method*.

Denote $\rho(xj, tn) = \rho_j^{n+1}$, *Eq. 9* would be discretized as:

$$(\rho_j^{n+1} - \rho_j^{n}) / \Delta t + C * (\rho_j^{n} - \rho_{j-1}^{n}) / \Delta x = 0 \qquad \text{(Eq. 11)}$$

And the numerical scheme is:

$$\rho_j^{n+1} = \rho_j^{n} + C\Delta t/\Delta x * (\rho_j^{n} - \rho_{j-1}^{n}) \qquad \text{(Eq. 12)}$$

This simple model would be validated to fit the real traffic data (e.g., Atlanta or Los Angeles) via testing with different boundary conditions.

2) *Complex model*

Given many real-world factors in traffic are not considered in the basic model, we expect the performance of the simple model to simulate the traffic data is poor. As a result, our second model would be a non-linear model that incorporates several effects, such as traffic light, intersection, etc. We haven't figured out what type of non-linear model would be best to fit in this section. We will work on the development of the complex model after literature review and further group discussions. Same as the first model, after the implementation, the validity of the model would be tested on the real traffic data to see if the fitting performance is improved. It is expected that parameter tuning would be involved.

Potential data sources [for validation and case studies]:
http://www.dot.ga.gov/DS/Data#tab-1 (Georgia data)
https://gisdata-caltrans.opendata.arcgis.com/datasets/f71f49fb87b3426e9688fe66039170bc_0 (California data)

Reference:

[1] Haberman, Richard. *Mathematical models: mechanical vibrations, population dynamics, and traffic flow.* Vol. 21. Siam, 1998.
[2] http://www.norbertwiener.umd.edu/Education/m3cdocs/Presentation2.pdf
[3] H. Holmes. *Introduction to the Foundations of Applied Mathematics.* Springer (2009)
[4] https://www.math.nyu.edu/faculty/childres/traffic2.pdf
[5] https://www.sciencedirect.com/science/article/pii/S0377042706002159#bib8