

# Evolutionary Game Theory: Overview and Recent Results

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## Oversviews:

nontechnical survey: “Evolutionary Game Theory”  
(in *Encyclopedia of Complexity and System Science*, Springer, 2009)

book: *Population Games and Evolutionary Dynamics*  
(MIT Press, 2010 (December))

## New work presented here:

“Large Deviations, Reversibility, and Equilibrium Selection  
under Evolutionary Game Dynamics” (with Michel Benaïm)

# Evolutionary Game Theory

Population games model strategic interactions in which:

1. The number of agents is large.
2. Individual agents are small.
3. Agents interact anonymously.

Applications can be found in many fields:

economics: externalities, macroeconomic spillovers

biology: animal conflict, genetic natural selection; prebiotic evolution

sociology: demographic dynamics

transportation science: highway congestion, travel mode choice

computer science/engineering: network congestion, routing

## Evolutionary dynamics in game theory

Traditionally, predictions in game theory are based on **equilibrium**: each agent's choice is optimal given the choices of the others.

These predictions rely on the **assumption of equilibrium knowledge**: that agents correctly anticipate how other agents will behave.

In contexts with large numbers of agents, this assumption seems untenable.

We therefore consider explicitly dynamic models of individual choice:

We describe how myopic individual agents decide whether and when to switch strategies using a **revision protocol**.

Given a revision protocol and a population game, we can describe aggregate behavior by a Markov chain.

## Deterministic and stochastic evolutionary dynamics: overview

A population game, a revision protocol, and a population size define a Markov process  $\{X_t^N\}$  on the set of population states.

There have been two main approaches to studying this process:

1. If the population is large, then a suitable law of large numbers shows that over finite time spans, the evolution of aggregate behavior is well approximated by solutions to an **ODE**, the **mean dynamic**.

The behavior of these differential equations is the focus of deterministic evolutionary game theory.

## Examples:

- replicator dynamic (Taylor and Jonker (1978))
  - best response dynamic (Gilboa and Matsui (1991))
  - many others (to be described soon)

## Deterministic and stochastic evolutionary dynamics: overview

3. To study infinite horizon behavior, one instead looks at the **stationary distributions** of the original stochastic process.

The behavior of these stationary distributions is the focus of **stochastic evolutionary game theory**.

When certain limits are taken (in the noise level  $\varepsilon$ , in the population size  $N$ ), this approach can provide a unique prediction of very long run behavior: the **stochastically stable state**.

Examples:

- best response w. mutations (Kandori et al. (1993), Young (1993))
- logit choice (Blume (1993))
- noisy imitation (Binmore and Samuelson (1997))
- others

## Deterministic and stochastic evolutionary dynamics: overview

2. For a middle way between the finite and infinite horizon analyses, one can consider **large deviations analysis**.

This describes the occasional excursions of the stochastic process against the flow of the mean dynamic.

Large deviations analysis provides a general foundation for stochastic stability analysis, and is of interest in its own right.

Examples: Benaïm and Weibull (2001), Benaïm and Sandholm (new).

## Some specific objectives of evolutionary game theory

1. Define interesting classes of games and revision protocols
  - easy to describe
  - relevant to applications
  - have attractive theoretical properties
2. Study the resulting dynamic processes
  - 2.1 Finite horizon, large population  $\Rightarrow$  approximation by ODE
    - local stability
    - global convergence
    - cycling or chaos
  - 2.2 Intermediate horizon, large population  $\Rightarrow$  large deviations analysis
  - 2.3 Infinite horizon  $\Rightarrow$  description using stationary distribution
    - equilibrium selection
3. Provide models for applications, especially ones with
  - large populations
  - dynamics in a central role

# The Model

## I. Population Games

For simplicity, we consider games played by a single unit-mass population (e.g., in traffic networks, one origin/destination pair).

$S = \{1, \dots, n\}$  strategies

$X = \{x \in \mathbf{R}_+^n : \sum_{i \in S} x_i = 1\}$  population states/mixed strategies

$F_i: X \rightarrow \mathbf{R}$  payoffs to strategy  $i$

$F: X \rightarrow \mathbf{R}^n$  payoffs to all strategies

State  $x \in X$  is a Nash equilibrium if

$$x_i > 0 \Rightarrow [F_i(x) \geq F_j(x) \text{ for all } j \in S]$$

ex. 1. Matching in (symmetric two-player) normal form games

$A \in \mathbf{R}^{n \times n}$  payoff matrix

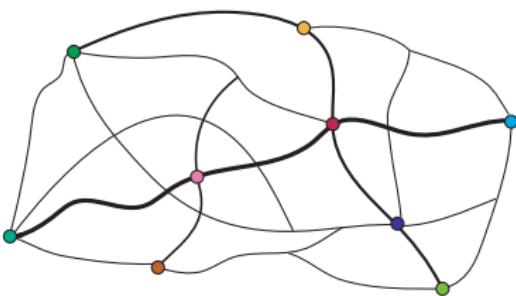
$A_{ij} = e'_i A e_j$  payoff for playing  $i \in S$  against  $j \in S$

$$F_i(x) = e'_i A x \quad \text{total payoff for playing } i \text{ against } x \in X$$

$F(x) = Ax$  payoffs to all strategies

When  $A$  is symmetric ( $A_{ij} = A_{ji}$ ), it is called a **common interest game**.

ex. 2. Congestion games (Beckmann, McGuire, and Winsten (1956))



Home and Work are connected by paths  $i \in S$  consisting of links  $\lambda \in \Lambda$ .

The payoff to choosing path  $i$  is

-(the delay on path  $i$ ) = -(the sum of the delays on links in path  $i$ )

$$F_i(x) = -\sum_{\lambda \in \Lambda_i} c_\lambda(u_\lambda(x)) \quad \text{payoff to path } i$$

$x_i$  mass of players choosing path  $i$

$$u_\lambda(x) = \sum_{i: \lambda \in \Lambda_i} x_i \quad \text{utilization of link } \lambda$$

$c_\lambda(u_\lambda)$  (increasing) cost of delay on link  $\lambda$

## II. Revision protocols

Agents make decisions by applying a revision protocol.

$$\rho: \mathbf{R}^n \times X \rightarrow \mathbf{R}_+^{n \times n}$$

$\rho_{ij}(\pi, x)$  is the conditional switch rate from strategy  $i$  to strategy  $j$

Interpretation:

Each agent is equipped with a rate  $R$  Poisson alarm clock, where

$$R \geq \max_{x,i} \sum_{j \neq i} \rho_{ij}(F(x), x).$$

When an  $i$  player's clock rings, he receives a revision opportunity.

He switches to strategy  $j$  with probability  $\rho_{ij}(F(x), x)/R$ .

ex. Pairwise proportional imitation (Helbing (1992), Schlag (1998))

protocol:  $\rho_{ij} = x_j [\pi_j - \pi_i]_+$

interpretation: Pick an opponent at random. Only imitate him if his payoff is higher than yours, doing so with probability proportional to the payoff difference.

(to be continued...)

### III. The Markov process

A population game  $F$ , a revision protocol  $\rho$ , and a population size  $N$  define a Markov process  $\{X_t^N\}$  on the finite state space

$$\mathcal{X}^N = X \cap \frac{1}{N}\mathbf{Z}^n = \{x \in X : Nx \in \mathbf{Z}^n\}.$$

This process is described by an initial state  $X_0^N = x_0^N$ , the jump rates  $\lambda_x^N = NR$ , and the transition probabilities

$$P_{xy}^N = \begin{cases} \frac{x_i \rho_{ij}(F(x), x)}{R} & \text{if } y = x + \frac{1}{N}(e_j - e_i), i, j \in S, i \neq j, \\ 1 - \sum_{i \in S} \sum_{j \neq i} \frac{x_i \rho_{ij}(F(x), x)}{R} & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

## Finite horizon deterministic approximation

Q: How does the process  $\{X_t^N\}$  behave over time interval  $[0, T]$  when  $N$  is large?

### The Mean Dynamic

The **mean dynamic** for  $\rho$  and  $F$  is defined by  
the **expected increments** of the Markov process.

$$\begin{aligned} (\text{M}) \quad \dot{x}_i &= V_i^F(x) \\ &= \sum_{j \in S} x_j \rho_{ji}(F(x), x) - x_i \sum_{j \in S} \rho_{ij}(F(x), x) \\ &= \text{inflow into } i - \text{outflow from } i \end{aligned}$$

(The map that assigns to each population game  $F$  a differential equation  $\dot{x} = V^F(x)$  is called an **evolutionary dynamic**.)

Over finite time spans, the process is well-approximated by solutions trajectories of the mean dynamic.

**Theorem (Benaïm and Weibull (2003)):**

Let  $\{X_t^N\}_{N=N_0}^\infty$  be a sequence of stochastic evolutionary processes.

Let  $\{x_t\}_{t \geq 0}$  be the solution to the mean dynamic  $\dot{x} = V(x)$  from  $x_0 = \lim_{N \rightarrow \infty} x_0^N$ .

Then for all  $T < \infty$  and  $\varepsilon > 0$ ,

$$\lim_{N \rightarrow \infty} \mathbb{P} \left( \sup_{t \in [0, T]} |X_t^N - x_t| < \varepsilon \right) = 1.$$

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Deterministic 2  
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Stochastic stability  
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Large deviations  
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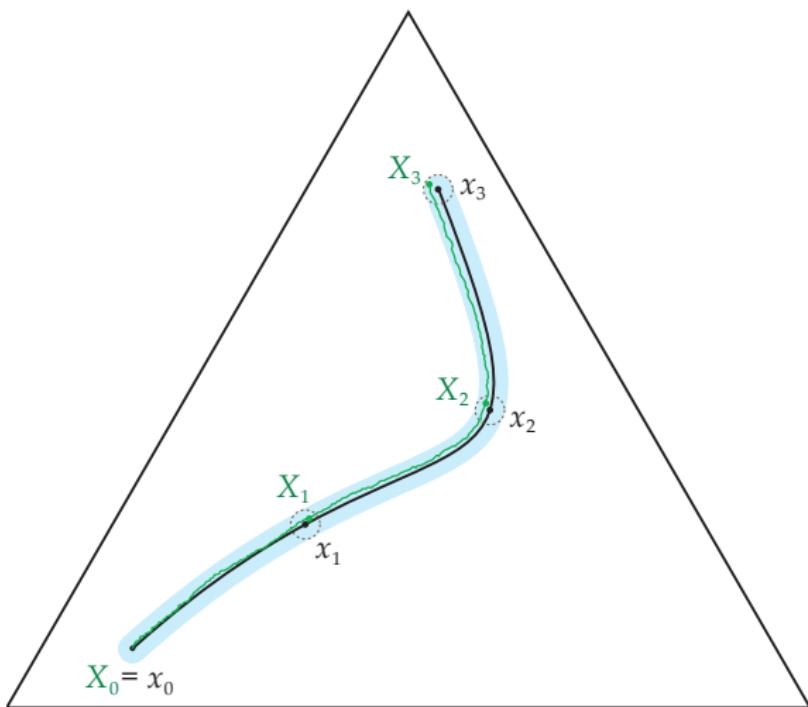


Figure: Finite horizon deterministic approximation.

## ex. Derivation of the replicator dynamic

protocol:  $\rho_{ij} = x_j[\pi_j - \pi_i]_+$

interpretation: Pick an opponent at random. Only imitate him if his payoff is higher than yours, doing so with probability proportional to the payoff difference.

mean dynamic:  $\dot{x}_i = x_i (F_i(x) - \bar{F}(x))$ , where  $\bar{F}(x) = \sum_{k \in S} x_k F_k(x)$

This is the **replicator dynamic**, introduced in the mathematical biology literature by Taylor and Jonker (1978).

## Revision protocols and ev. dynamics: five fundamental examples

Revision protocol	Evolutionary dynamic	Name
$\rho_{ij} = x_j[\pi_j - \pi_i]_+$	$\dot{x}_i = x_i(F_i(x) - \bar{F}(x))$	replicator
$\rho_{i\cdot} \in \operatorname{argmax}_{y \in X} y' \pi(x)$	$\dot{x} \in \operatorname{argmax}_{y \in X} y' F(x) - x$	best response
$\rho_{ij} = \frac{\exp(\eta^{-1}\pi_j)}{\sum_{k \in S} \exp(\eta^{-1}\pi_k)}$	$\dot{x}_i = \frac{\exp(\eta^{-1}F_i(x))}{\sum_{k \in S} \exp(\eta^{-1}F_k(x))} - x_i$	logit
$\rho_{ij} = [\hat{\pi}_j]_+$	$\dot{x}_i = m [\hat{F}_i(x)]_+ - x_i \sum_{j \in S} [\hat{F}_j(x)]_+$	BNN
$\rho_{ij} = [\pi_j - \pi_i]_+$	$\dot{x}_i = \sum_{j \in S} x_j [F_i(x) - F_j(x)]_+$ $- x_i \sum_{j \in S} [F_j(x) - F_i(x)]_+$	Smith

# Analysis of deterministic evolutionary dynamics

Some questions:

1. Are there classes of games for which many dynamics converge to equilibrium?
2. How common is nonconvergence?
3. To what extent does the long run behavior of the dynamics agree with traditional game-theoretic solution concepts (e.g., Nash equilibrium, elimination of dominated strategies)?

## I. General Convergence Results

There are three classes of games under which many evolutionary dynamics are known to converge to equilibrium from most initial conditions.

These classes can be defined in terms of the derivative of the payoff function  $F$ :

1. **Potential games** (Monderer and Shapley (1996), Sandholm (2001))  
defined by **externality symmetry**:  $DF(x)$  is symmetric.
2. **Stable games** (Hofbauer and Sandholm (2009))  
defined by **self-defeating externalities**:  $DF(x)$  is negative definite.
3. **Supermodular games** (Topkis (1979))

defined by **strategic complementarities**:  $\frac{\partial(F_{i+1} - F_i)}{\partial(e_{j+1} - e_j)}(x) \geq 0$ .

## Potential games (Monderer and Shapley (1996), Sandholm (2001))

$F$  is a potential game if it admits a potential function  $f: \mathbf{R}^n \rightarrow \mathbf{R}$ :

$$\nabla f(x) = F(x) \text{ for all } x \in X \quad (\text{i.e., } \frac{\partial f}{\partial x_i}(x) = F_i(x)).$$

Potential games are characterized by externality symmetry:

(ES)  $DF(x)$  is symmetric for all  $x \in X$ .

**Theorem: Global convergence in potential games (Sandholm (2001))**

Let  $F$  be a potential game, and let  $V^F$  be a dynamic satisfying

(PC) Positive correlation: If  $V^F(x) \neq \mathbf{0}$ , then  $V^F(x)'F(x) > 0$ , and

(NS) Nash stationarity: If  $V^F(x) = \mathbf{0}$ , then  $x \in NE(F)$ .

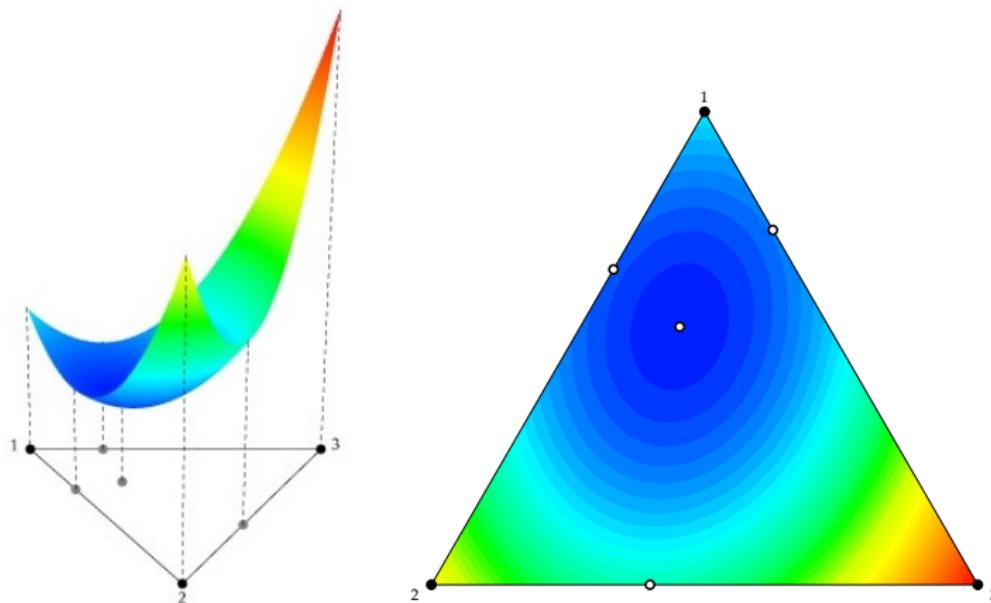
Then nonstationary solutions of  $V^F$  ascend the potential function  $f$  and converge to sets of Nash equilibria.

## Example: Common interest games

$$F(x) = Ax, \text{ } A \text{ symmetric}$$

$$\Rightarrow f(x) = \frac{1}{2}x'Ax = \frac{1}{2} \times \text{average payoffs}$$

## Example: A common interest game and its potential function



$$F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \quad f(x) = \frac{1}{2} \left( (x_1)^2 + 2(x_2)^2 + 3(x_3)^2 \right).$$

## II. Nonconvergence of Evolutionary Dynamics

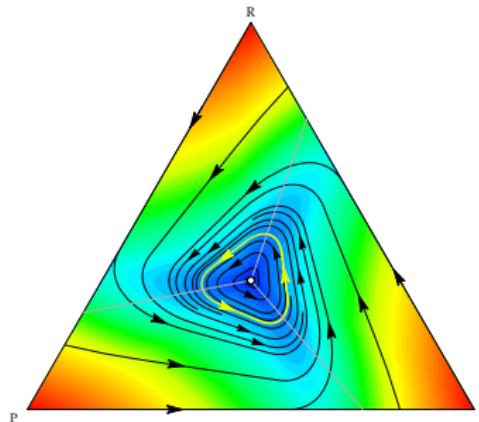
Outside of the classes of games noted above, convergence to equilibrium need not occur.

Indeed, simple games and decision rules can generate complex dynamics.

## Example 1: Cycling in Bad Rock-Paper-Scissors

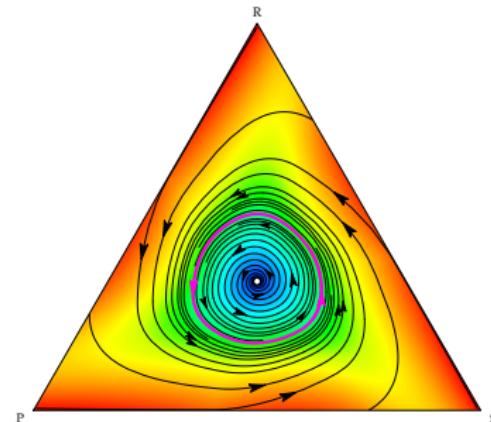
$$F(x) = Ax, \text{ where } A = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 0 & -2 \\ -2 & 1 & 0 \end{pmatrix}$$

All of the dynamics introduced earlier approach limit cycles in this game.



the BNN dynamic in bad RPS

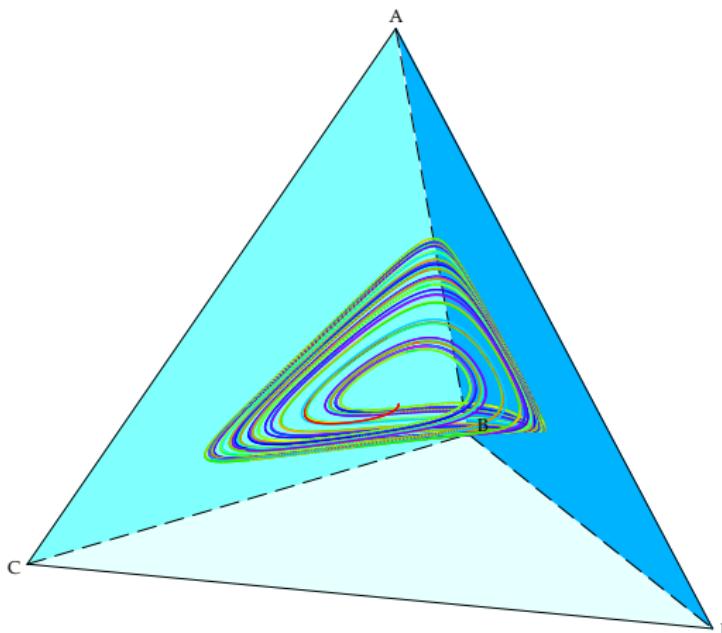
(colors represent speed: red is fast, blue is slow)



the Smith dynamic in bad RPS

## Example 2: Chaos under the replicator dynamic (Arneodo et al. (1980))

$$F(x) = Ax, \text{ where } A = \begin{pmatrix} 0 & -12 & 0 & 22 \\ 20 & 0 & 0 & 10 \\ -21 & -4 & 0 & 35 \\ 10 & -2 & 2 & 0 \end{pmatrix}.$$



A solution trajectory of the replicator dynamic in the ACT game.

## Example 3: Survival of dominated strategies

(Berger and Hofbauer (2006), Hofbauer and Sandholm (2010).)

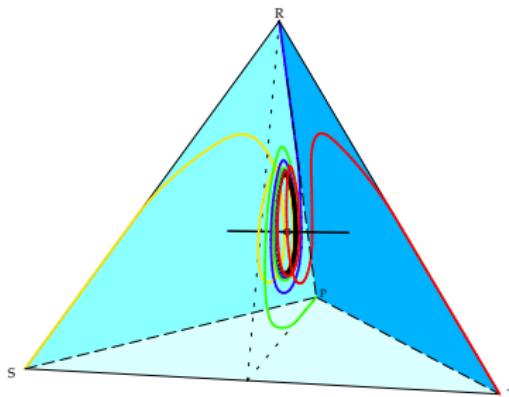
Strategy  $i$  is **strictly dominated** by strategy  $j$  if  $F_i(x) < F_j(x)$  for all  $x \in X$ .

Dominance is the mildest requirement employed in standard decision-theoretic analysis, so it is natural to expect evolutionary dynamics to accord with it.

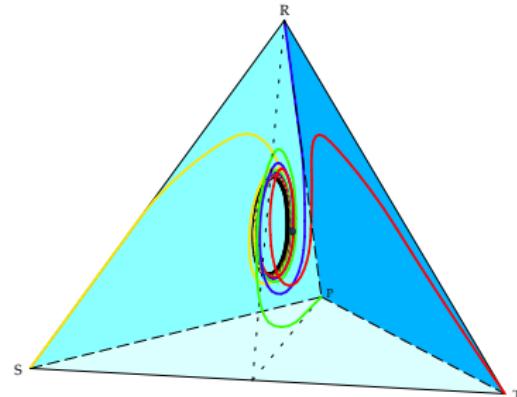
## Example: Survival of a dominated strategy

bad RPS with a feeble twin:

$$A = \begin{pmatrix} 0 & -2 & 1 & 1 \\ 1 & 0 & -2 & -2 \\ -2 & 1 & 0 & 0 \\ -2-d & 1-d & -d & -d \end{pmatrix}$$



Smith dynamic for  $d = 0$



Smith dynamic for  $d = .1$

More generally, Hofbauer and Sandholm (2010) prove that any evolutionary dynamic satisfying four natural conditions fails to eliminate strictly dominated strategies in some games.

**Intuition:** Evolutionary dynamics describe the aggregate behavior of agents who switch to strategies whose current payoffs are good, though not necessarily optimal.

If a solution trajectory of a dynamic converges, payoffs converge as well. In this case, even simple rules ensure that strategies in use are optimal.

But in many games, solutions do not converge, and payoffs remain in flux. In this situation, simple choice rules need not eliminate strategies whose payoffs are often good but never optimal.

**Moral:** Evolutionary dynamics provide limited support for a basic rationality postulate.

# Large Deviations, Stochastic Stability, and Equilibrium Selection

(joint work with Michel Benaïm)

In games with multiple equilibria, predictions generally depend on the initial behavior of the system.

But suppose that

- (i) if we are interested in very long time spans, and
- (ii) there is noise in agents' decisions.

Then we can base predictions on the **unique stationary distribution**  $\mu^N$  of the Markov process  $\{X_t^N\}$ , which describes the infinite horizon behavior of the process.

When the noise level is small or the population size large, this distribution often concentrates its mass around a single state, which is said to be **stochastically stable**.

Intuition: Rare but unequally unlikely transitions between equilibria.

## The standard stochastic stability model: best responses and rare mutations (Kandori, Mailath, and Rob (1993), Young (1993))

Each agent occasionally receives opportunities to switch strategies.

At such moments, the agent plays a best response with probability  $1 - \varepsilon$ , chooses a strategy at random with probability  $\varepsilon$ .

Key assumption: the probability of mutation is independent of the payoff consequences.

⇒ Analysis of equilibrium selection via mutation counting:

To compute the probability of a transition between equilibria, count the number of mutations required.

Then use graph-theoretic methods to determine the asymptotics of the stationary distribution.

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Deterministic 1  
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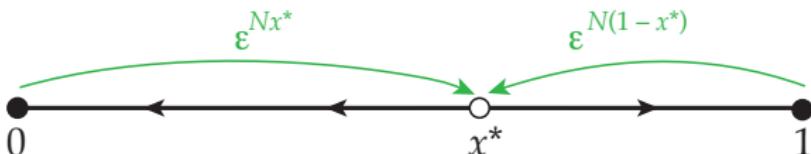
Deterministic 2  
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Stochastic stability  
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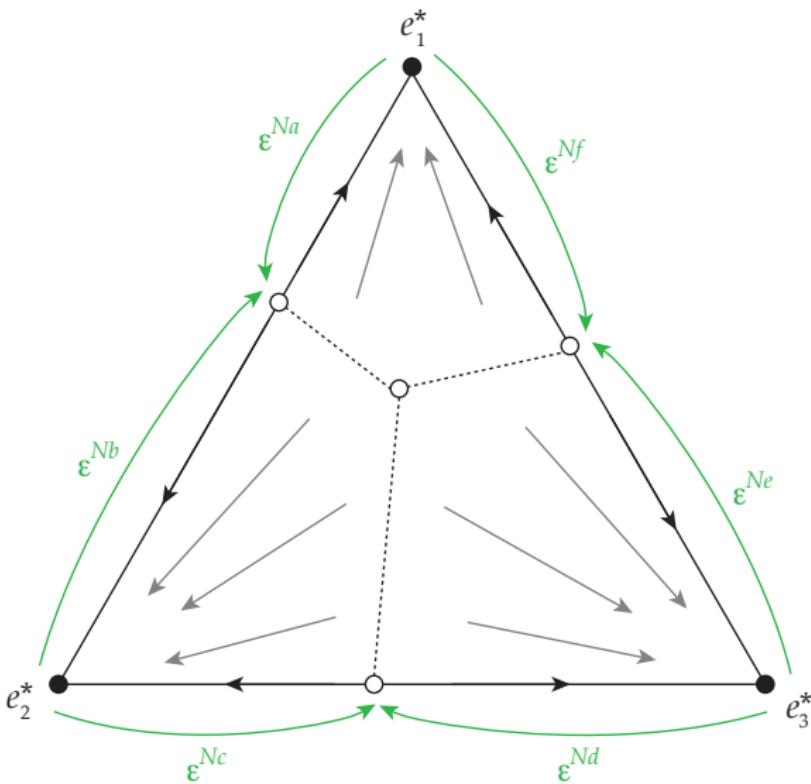
Large deviations  
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## Best response/mutations in a two-strategy coordination game



## Best response/mutations in a three-strategy coordination game



Mutation counting works because of a special property of the BRM model: **the probability of any given suboptimal choice is independent of the payoff consequences.**

It seems more realistic to expect the reverse.

ex.: **logit choice** (Blume (1993))

$$\rho_j^\eta(\pi) = \frac{\exp(\eta^{-1}\pi_j)}{\sum_{k \in S} \exp(\eta^{-1}\pi_k)}.$$

Once mistake probabilities depend on payoffs, evaluating the probabilities and paths of transitions between basins of attraction becomes much more difficult.

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Deterministic 1  
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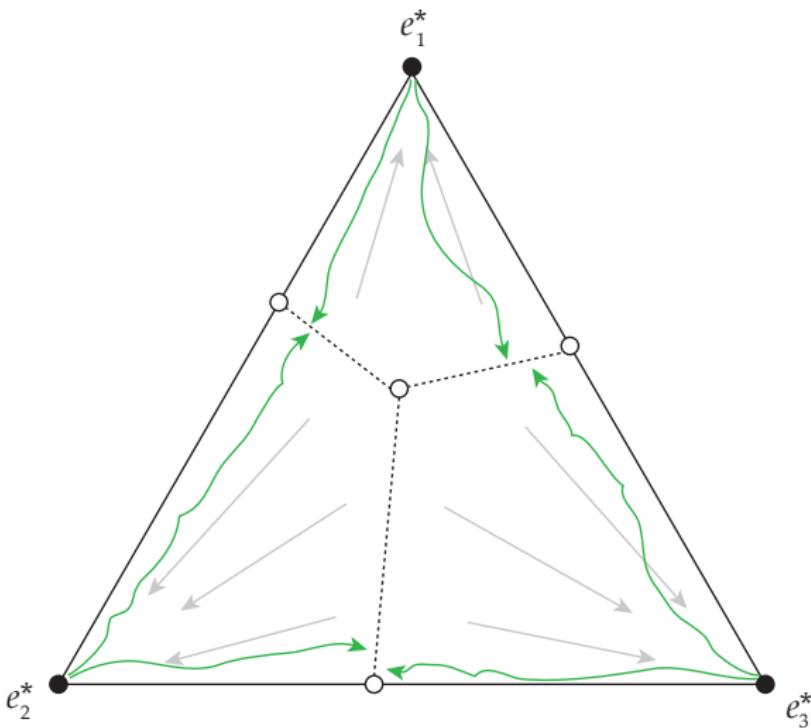
Deterministic 2  
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Stochastic stability  
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## General escape dynamics in a three-strategy coordination game



## Large deviations analysis of the stochastic evolutionary process

To describe transitions between basins of attraction, we use techniques from **large deviations theory** to describe the behavior of the process in the large population limit.

The analysis proceeds in three steps:

1. Obtain deterministic approximation via the mean dynamic ( $M$ ).
2. Evaluate probabilities of excursions away from stable rest points of ( $M$ ).
3. Combine (1) and (2) to understand global behavior:
  - probabilities of transitions between rest points of ( $M$ )
  - expected times until exit from general domains
  - asymptotics of the stationary distribution & stochastic stability

Key mathematics references: Freidlin and Wentzell (1998), Dupuis (1986).

## Evolution in potential games under the logit choice protocol

Let  $F: X \rightarrow \mathbf{R}^n$  be a potential game with potential function  $f: X \rightarrow \mathbf{R}$ :

$$F(x) = \nabla f(x).$$

Let  $\rho$  be the  $\text{logit}(\eta)$  choice protocol:

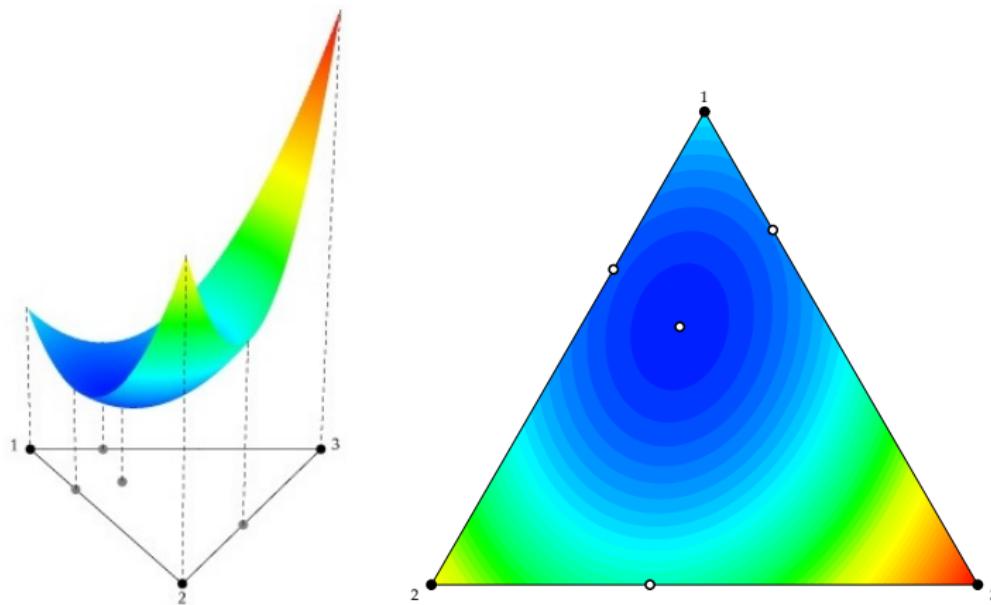
$$\rho_j^\eta(\pi) = \frac{\exp(\eta^{-1}\pi_j)}{\sum_{k \in S} \exp(\eta^{-1}\pi_k)}.$$

Define the  $\text{logit}(\eta)$  potential function:

$$f^\eta(x) = \eta^{-1} f(x) + h(x),$$

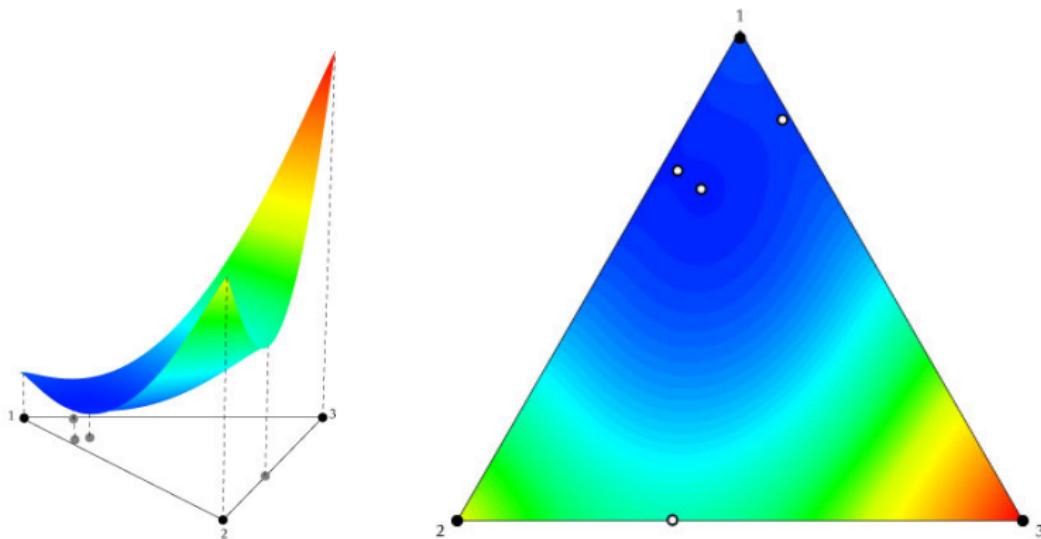
where  $h(x) = -\sum_{i \in S} x_i \log x_i$  is the entropy function.

## Example: A common interest game and its potential function



$$F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \quad f(x) = \frac{1}{2} \left( (x_1)^2 + 2(x_2)^2 + 3(x_3)^2 \right).$$

## Example: A common interest game and its logit(.2) potential function



$$F(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \quad \tilde{f}(x) = 5 \cdot \frac{1}{2} \left( (x_1)^2 + 2(x_2)^2 + 3(x_3)^2 \right) - \sum_i x_i \log x_i.$$

## Step 1: Deterministic approximation

The mean dynamic is the **logit dynamic**:

$$(L) \quad \dot{x}_i = \frac{\exp(\eta^{-1}\pi_i)}{\sum_{k \in S} \exp(\eta^{-1}\pi_k)} - x_i.$$

The rest points of the logit dynamic are called **logit equilibria**.

**Observation:** *The logit equilibria are the critical points of the logit potential function  $f^\eta$ .*

**Theorem** (Hofbauer and Sandholm (2007)):

$f^\eta$  is a strict Lyapunov function for the logit dynamic (L).

Thus, all solutions of (L) converge to logit equilibria.

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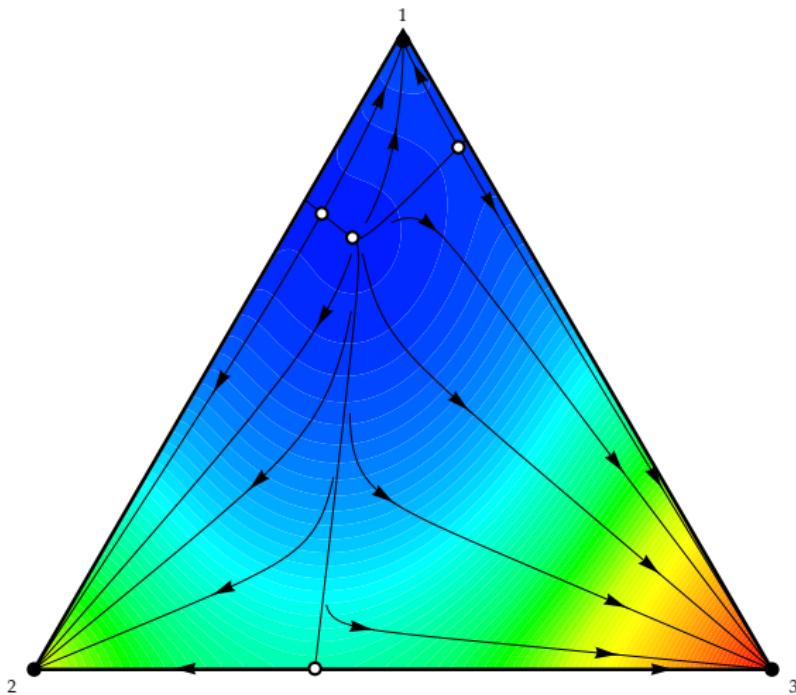
Deterministic 1  
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Deterministic 2  
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Large deviations  
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The logit(.2) dynamic for  $F(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ , plotted atop  $f^\eta(x) = 5f(x) + h(x)$ .

## Step 2: Evaluation of transition costs (in general)

The **transition cost**  $C(x, y)$  describes the unlikelihood of the most likely path from  $x$  to  $y$ .

**Theorem:** Let  $\chi$  be an asymptotically stable rest point of (M).

Let  $y = \underset{z \in \text{bd}(\text{basin}(\chi))}{\operatorname{argmin}} C(\chi, z)$ .

If play begins near  $\chi$ , then for large enough  $N$ :

- (i) Escape from  $\text{basin}(\chi)$  occurs near  $y$  with probability close to 1.
- (ii) The expected escape time is of order  $\exp(N C(\chi, y))$ .
- (iii) The realized escape time is of order  $\exp(N C(\chi, y))$  with probability close to 1.

Results on sample path large deviations show that a function  $C(x, \cdot)$  with these properties can be represented in terms of solutions to suitable calculus of variations problems.

## Step 2: Evaluation of transition costs (in the logit/potential model)

**Proposition:** For any path  $\phi \in C[T, T']$  from  $x$  to  $y$ , we have  $c_{T,T'}(\phi) \geq f^\eta(x) - f^\eta(y)$ . Thus  $C(x, y) \geq f^\eta(x) - f^\eta(y)$ .

**Proposition:** If there is a solution of (L) leading from  $y$  to  $x$ , then its time-reversed version is an extremal yielding  $C(x, y) = f^\eta(x) - f^\eta(y)$ .

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The model  
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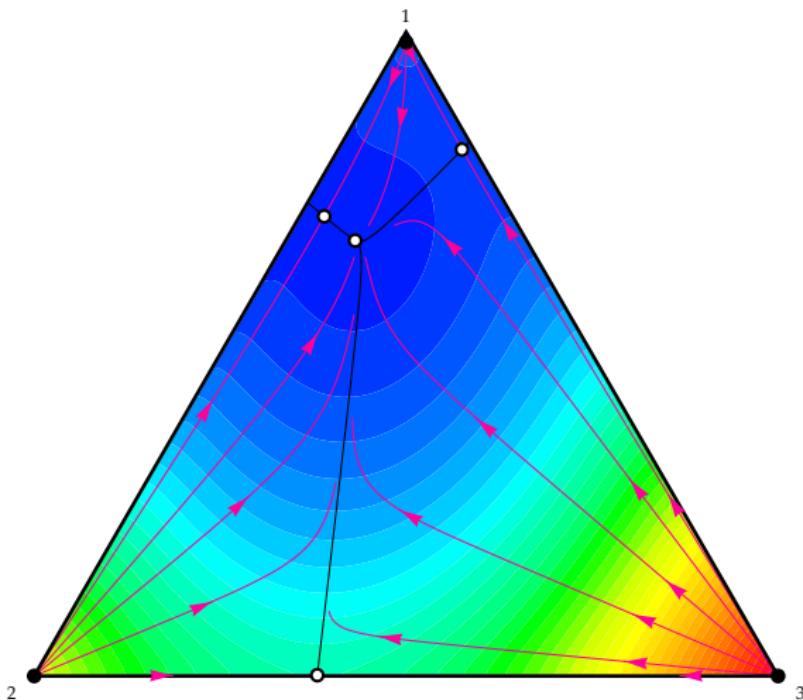
Deterministic 1  
○○○○○

Deterministic 2  
○○○○○○○○○○○○

Stochastic stability  
○○○○○○○

Large deviations  
○○○○○○○●○○

End  
○○



The logit(.2) dynamic for  $F(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ , plotted atop  $f^\eta(x) = 5f(x) + h(x)$ .

## Step 3: Global long-run behavior

By combining the previous results with graph-theoretic arguments we can characterize the asymptotics of the stationary distribution.

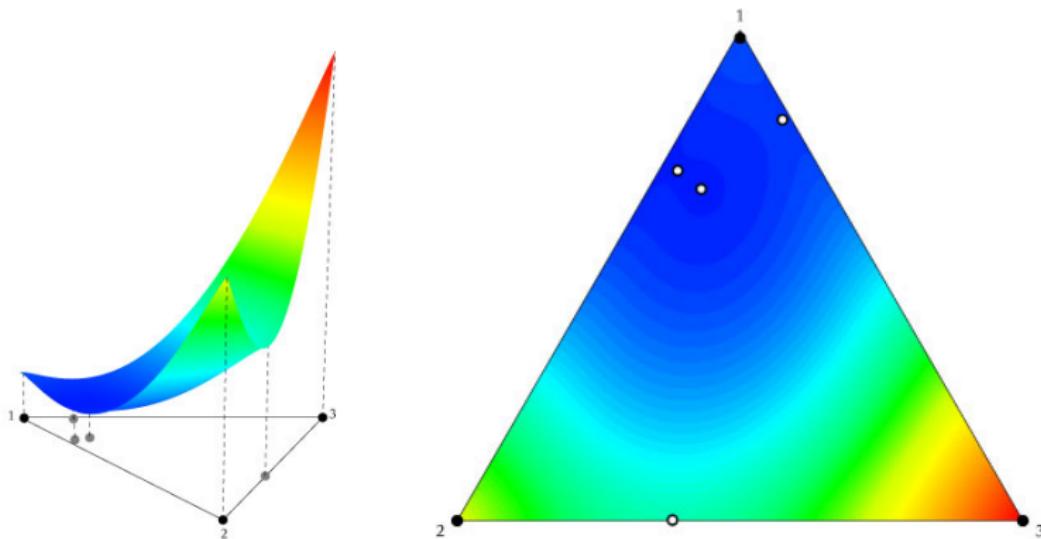
Shift the values of  $f^\eta$  uniformly so that the maximum value is 0:

$$\Delta f^\eta(x) = f^\eta(x) - \max_{y \in X} f^\eta(y)$$

**Theorem:**  $\mu^N(x)$  is of order  $\exp(N \Delta f^\eta(x))$  as  $N \rightarrow \infty$ .

**Corollary:**  $\operatorname{argmax}_{x \in X} f^\eta(x)$  is stochastically stable as  $N \rightarrow \infty$ .

## Example: A common interest game and its logit(.2) potential function



$$F(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \tilde{f}(x) = 5 \cdot \frac{1}{2} \left( (x_1)^2 + 2(x_2)^2 + 3(x_3)^2 \right) - \sum_i x_i \log x_i.$$

## Some specific objectives of evolutionary game theory

1. Define interesting classes of games and revision protocols

- easy to describe
- relevant to applications
- have attractive theoretical properties

2. Study the resulting dynamic processes

2.1 Finite horizon, large population  $\Rightarrow$  approximation by ODE

- local stability
- global convergence
- cycling or chaos

2.2 Intermediate horizon, large population  $\Rightarrow$  large deviations analysis

2.3 Infinite horizon  $\Rightarrow$  description using stationary distribution

- equilibrium selection

3. Provide models for applications, especially ones with

- large populations
- dynamics in a central role

## References

### Overviews:

nontechnical survey: “Evolutionary Game Theory”  
(in *Encyclopedia of Complexity and System Science*, Springer, 2009)

book: *Population Games and Evolutionary Dynamics*  
(MIT Press, 2010 (December))

### New work presented here:

“Large Deviations, Reversibility, and Equilibrium Selection  
under Evolutionary Game Dynamics” (with Michel Benaïm)

### Software for figures:

*Dynamo: Diagrams for Evolutionary Game Dynamics*  
(with Emin Dokumacı and Francisco Franchetti)  
[www.sss.wisc.edu/~whs/dynamo](http://www.sss.wisc.edu/~whs/dynamo)