Stochastic Evolution with Perturbed Payoffs and Rapid Play*

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Abstract

A population of agents are recurrently matched to play a symmetric normal form game. The strategies' payoffs are subject to i.i.d. shocks, and agents face many rounds of play, and hence many rounds of payoff shocks, during each period. At the end of the period, one agent receives a revision opportunity, and switches to the strategy whose average perturbed payoff during the period was highest. Our stochastic stability analysis concerns long run behavior as the number of rounds per period approaches infinity. We prove that risk dominant equilibria are always selected in 2×2 games, but that more generally equilibrium selection depends on the exact nature of the shock distribution.

To perform this analysis, we define the *unlikelihood* of choosing strategy i given payoff vector π to be the exponential rate at which the probability of choosing strategy i vanishes as the number of rounds grows large. We characterize unlikelihoods using techniques from large deviations theory, and prove that the unlikelihood of choosing strategy i is a convex function of the payoff vector π. As a byproduct of this analysis, we derive the rates of decay of choice probabilities in the multinomial probit model as the shock variance approaches zero.

1. Introduction

Models of stochastic evolution offer a boundedly rational approach to equilibrium selection in games. The pioneers of this approach, Foster and Young (1990), Kandori et al.

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(1993), and Young (1993), examine the long-run behavior of evolutionary game dynamics subject to random mutations. In Kandori et al. (1993), a population of agents recurrently play a symmetric normal form game. When switching strategies, each agent chooses his myopic best response with probability $1-\varepsilon$ and a random action with probability ε, where ε does not depend on the current population state. A population state is called *stochastically stable* if the weight it receives in the stationary distribution of this evolutionary process stays bounded away from zero as ε approaches zero. Since the stochastically stable state is typically unique, this analysis provides a unique equilibrium selection even in games with multiple strict equilibria.

In their critique of this approach to equilibrium selection, Bergin and Lipman (1996) observe that the conclusions drawn from the Kandori et al. (1993) model depend on the assumption that mutation probabilities are independent of the population state. They show that by carefully choosing the rates at which mutation probabilities vanish at different states, any invariant distribution of unperturbed process can be made stochastically stable. Bergin and Lipman (1996) conclude that "the nature of the mutation process must be analyzed more carefully to derive some economically justifiable restrictions".

In this paper, we build upon Bergin and Lipman's (1996) suggestion by introducing a simple and natural model of perturbed optimization for evolutionary game dynamics, a model based on stochastic payoff shocks and rapid play. In each period, each agent in our model is matched m times to play an $n \times n$ symmetric normal form game. During each match, the payoffs to the agents' n strategies are perturbed by an i.i.d. zero-mean shock vector. At the period's end, one agent receives a revision opportunity and switches to the strategy whose average realized payoff during the period was highest. Since agents make optimal choices with respect to perturbed payoffs, "mistake" probabilities exhibit an appealing form of state-dependence: the probability of a "mistake" is decreasing in the consequent payoff loss. Furthermore, the assumption of rapid play introduces vanishing mistake probabilities in a natural fashion: as the number of matches per period grows large, the probability with which the revising agent chooses his unperturbed best response approaches one.

Consider a model of rapid play with a fixed distribution of payoff shocks. As a basis for our analyses of stochastic stability, we define the *unlikelihood of choosing strategy i* given base payoff vector π to be the exponential rate of decay of the probability of choosing strategy i as the number of matches per period approaches infinity. Using techniques from large deviations theory, we obtain a simple characterization of the unlikelihood function and derive some of its basic properties. It is obvious that the unlikelihood of choosing strategy i is nonincreasing in strategy i's payoff and nondecreasing in other strategies' payoffs.

But we also show that the unlikelihood of choosing strategy i is a convex function of the payoff vector π. This restriction is of clear importance for stochastic stability analyses. For instance, convexity ensures that overcoming multiple "small" payoff deficits is less unlikely than overcoming a single large one, influencing which paths transitions between equilibria are likely to take.

The restrictions on unlikelihood functions described above hold force regardless of the distribution of payoff shocks. Are these restrictions themsleves sufficient to determine the stochastically stable state? In 2×2 coordination games, the answer is yes: we prove that for any distribution of payoff shocks, an equilibrium is stochastically stable if and only if it is risk dominant. But beyond the 2×2 setting, the invariance of the equilibrium selection to the shock distribution no longer obtains. Intuitively, different shock distributions generate unlikelihood functions with different degrees of convexity. While with just two strategies this is unimportant, with more than two strategies it can affect the relative likelihoods of different transitions between equilibria, altering equilibrium selection results.

A number of other papers consider the robustness of stochastic stability models to alternative ways of specifying the probabilities of "suboptimal" choices. Most of these papers focus on equilibrium selection in two-strategy games. In this context, Blume (2003) and Maruta (2002) directly specify a parameterized map from payoff differences to mistake probabilities. In Blume (2003), the odds ratio of choosing strategy 1 over strategy 2 is approximately $\exp(\varepsilon h(\delta))$, where δ is strategy 1's payoff advantage and ε is a noise parameter. Blume (2003) shows that if the noise technology treats strategies anonymously, in that $h(\delta) = -h(-\delta)$, then the risk-dominant equilibrium is stochastically stable. van Damme and Weibull (2002) endogenize error probabilities by subjecting agents' choices of mixed strategies to convex control costs, which make placing too little probability on any given pure strategy prohibitively expensive. They provide mild conditions on control cost functions that ensure selection of the risk dominant equilibrium in 2×2 games.

In unpublished work, Ui (1998) presents an example of a 3×3 game in which stochastic evolutionary models based on logit and probit choice lead to different equilibrium selections. These two models are based on one-shot payoff shocks: extreme-value distributed shocks for the logit model, and normally distributed shocks for the probit model. Building on Ui's (1998) example, we show that differences in equilibrium selection results can persist even after averaging of payoff shocks through rapid play.

Myatt and Wallace (2003) consider a model of stochastic evolution in which payoffs are subjected to i.i.d. normal shocks. As in Ui (1998), the stochastically stable equilibria are those that retain weight in the limiting stationary distribution as the shock variance is taken to zero. Myatt and Wallace (2003) argue that their multinomial probit model can generate different equilibrium selections than the mutation model of Kandori et al. (1993) in 3×3 games.

In the course of their analysis, Myatt and Wallace (2003) derive a general formula for the rate of decay of choice probabilities in the multinomial probit model as the shock variance approaches zero. Since the average of i.i.d. normal random variables is itself normally distributed, Myatt and Wallace's (2003) model can be obtained as a special case of ours. Our analysis reveals that Myatt and Wallace's (2003) formula for the rate of decay of multinomial probit choice probabilities is incorrect. We derive the correct formula for the rate of decay, and we offer an intuitive explanation for the difference between the formulas using the language of large deviations theory.

2. The Model

A population of N agents is recurrently matched to play a symmetric normal form game with strategy set $S = \{1, \cdots, n\}$ and payoff matrix $A \in \mathbf{R}^{n \times n}$. Entry A_{ij} is the base payoff an agent receives when he plays strategy i and his opponent plays strategy j. Since agents play pure strategies, each population state x is an element of the simplex $X = \{x \in \mathbf{R}^n_+ : \sum_{i=1}^n x_i = 1\}$; here x_i represents the proportion of agents playing strategy $i \in S$.

During each period of play, every agent is involved in m matches, being matched with each of his opponents exactly $\frac{m}{N-1} \in \mathbf{Z}$ times. Thus, before payoff perturbations are introduced, the per-match payoff to playing strategy i is deterministic.[1] If we let $\chi \in X$ represent the distribution of strategies of an agent's opponents, then the agent's payoff to playing strategy i is given by $F_i(\chi) = \sum_{j \in S} A_{ij} \chi_j$. For future reference, observe that if the current population state is x and an agent's current strategy is k, then the strategy distribution of the agent's opponents is $\chi = \xi^k(x) \equiv \frac{1}{N-1}(Nx - e_k)$, where e_k denotes the kth standard basis vector in $\mathbf{R}^n$.[2]

Payoffs in each match are subject to random shocks. If during his lth match an agent choosing strategy i faces an opponent playing strategy j, the agent's payoff is not just the base payoff A_{ij}, but the perturbed payoff $A_{ij} + \varepsilon_i^l$. As is usual in evolutionary models with payoff perturbations, we assume that after being matched with an opponent who chooses strategy j, an agent observes the perturbed payoffs $A_{ij} + \varepsilon_i^l$ to all strategies $i \in S$; the base

[1] One could also consider a model in which the matching is random. In this case, one would need to account not only for large deviations in payoff shocks, but also for large deviations in the matching process itself.

[2] It is important to incorporate this change in the state into agents' decision rules when the population size is small—see Sandholm (1998).

payoffs A_{ij} are not directly observed.

Each component ε_i^l of the shock vector ε^l is mean zero and continuous with convex support, with a moment generating function that is finite in a neighborhood of 0. We are most interested in cases where the components of the vector ε^l are i.i.d., but our analysis will allow these components to be independent but not identically distributed. Finally, we assume that the sequence of shock vectors $\{\varepsilon^l\}_{l=1}^m$ affecting the m matches that occur during a period is i.i.d.[3]

At the end of each period, a randomly chosen agent receives a revision opportunity, switching to the strategy whose average perturbed payoff during the period was highest. If the distribution of his opponents' strategy is χ, the agent will choose the strategy i that satisfies

$$i \in \operatorname*{argmax}_{j \in S} F_j(\chi) + \bar{\varepsilon}_j^m,$$

where $\bar{\varepsilon}_j^m = \frac{1}{m} \sum_{l=1}^m \varepsilon_j^l$ is strategy j's average payoff shock during the previous period. It follows that probabilities with which the agent chooses each strategy are given by the *perturbed best response function* $\tilde{B}^m$, defined by

$$\tilde{B}_i^m(\chi) = P\left(\bigcap_{j \neq i}^{n} \left\{ F_i(\chi) + \bar{\varepsilon}_i^m \geq F_j(\chi) + \bar{\varepsilon}_j^m \right\} \right). \tag{1}$$

The revision process we have described defines a Markov chain $\{X_t^m\}_{t=0}^\infty$ on the finite state space $\mathcal{X}^N = \{x \in X | Nx \in \mathbf{Z}^n\}$. Since the probability that the next revision opportunity goes to a current strategy i player is x_i, the transition probabilities for $\{X_t^m\}_{t=0}^\infty$ are

$$p_{xy}^m = \begin{cases} x_i \tilde{B}_j^m(\xi^i(x)) & \text{if } y = x - \frac{1}{N}(e_j - e_i), \\ \sum_{k=1}^n x_k \tilde{B}_k^m(\xi^k(x)) & \text{if } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

We henceforth assume that the distributions of the payoff perturbations have large enough supports that each strategy always has a positive probability of being chosen by the revising agent. Under this assumption, the Markov chain $\{X_t^m\}_{t=0}^\infty$ is irreducible and aperiodic, and so admits a unique stationary distribution, μ^m. This distribution describes

[3]Payoff shocks are also assumed to be independent across periods; given the updating procedure below, it is irrelevant whether payoff shocks are identical, correlated, or independent across agents. Our notation for payoff shocks, ε_i^l, suppresses both the period in which the shock occurs and the identity of the agent facing the shock.

the long run behavior of $\{X_t^m\}_{t=0}^{\infty}$ in two distinct ways: it approximates the distribution of X_t^m for all sufficiently large t, and it describes the long run empirical distribution of behavior with probability one.

In general, a population state in a model of stochastic evolution is said to be *stochastically stable* if it receives positive mass in the limit of the model's stationary distributions as the noise in the agents' decision rule vanishes. In the present model, the level of noise is lowered by increasing the number of matches played by each agent during each period. We thus call state $x \in \mathcal{X}^N$ stochastically stable if $\lim_{m\to\infty} \mu_x^m > 0$.

The limiting stationary distribution can be characterized using the well-known Freidlin and Wentzell (1998) technique. For any state $x \in \mathcal{X}^N$, define an *x-tree* to be a tree with node set $\mathcal{X}^N$ and root x. Let T_x denote the set of all x-trees. For any pair of states $x, y \in \mathcal{X}^N$, define the cost of a direct transition from x to y by

$$c(x, y) = -\lim_{m\to\infty} \frac{1}{m} \log p_{xy}^m, \tag{2}$$

and define the cost of x-tree τ_x by $c(\tau_x) = \sum_{(x,y)\in\tau_x} c(x, y)$. Kandori et al. (1993) and Young (1993, 1998) show that state x is stochastically stable if and only if there is a cost-minimizing x-tree: that is, a $\tau_x \in T_x$ satisfying

$$c(\tau_x) = \min_{y\in\mathcal{X}^N} \min_{\tau_y\in T_y} c(\tau_y).$$

3. Rates of Decay of Choice Probabilities

To determine which states are stochastically stable, we need to evaluate rate of decay of the transition probabilities p_{xy}^m as the number of matches m grows large, as expressed in equation (2). Since the state x is fixed as m increases, this amounts to finding the rates of decay of the perturbed best response probabilities $\tilde{B}_i^m(\chi)$. In this section, we show how to determine these probabilities using techniques from the theory of large deviations.

3.1 Background

Let $\{Y^l\}_{l=1}^{\infty}$ be an i.i.d. sequence of random vectors taking values in $\mathbf{R}^d$. Each random vector Y^l is continuous with convex support, with a moment generating function that exists in a neighborhood of the origin. Let $\bar{Y}^m = \frac{1}{m}\sum_{l=1}^m Y^l$ denote the sequence's mth sample mean. We say that the sequence of sample means $\{\bar{Y}^m\}_{m=1}^{\infty}$ satisfies the *large* *deviation principle* with *rate function* $R : \mathbf{R}^d \to \mathbf{R}$ if R is lower semicontinuous and if

$$(3) \qquad -\lim_{m\to\infty} \frac{1}{m} \log P(\bar{Y}^m \in U) = \inf_{y\in U} R(y)$$

whenever $U \subseteq \mathbf{R}^d$ is a *continuity set* of R (that is, whenever $\inf_{y\in \mathrm{int}(U)} R(y) = \inf_{y\in \mathrm{cl}(U)} R(y)$).

Roughly speaking, equation (3) says that $P(\bar{Y}^m \in U)$ is of order $\exp(-m\, R(y^*))$ (that is, that the exponential rate of decay of $P(\bar{Y}^m \in U)$ is $R(y^*)$), where y^* minimizes the rate function R on the set U. If after a large number of trials the realization of $\bar{Y}^m$ is in U, it is overwhelmingly likely that this realization is one that achieves as small a value of R as possible given this constraint; thus, the rate of decay is determined by this smallest value.

Cramér's Theorem provides the means of identifying the rate function for $\{\bar{Y}^m\}_{m=1}^\infty$. Define the *Cramér transform* of Y^l by

$$\Lambda^*(y) = \sup_{\lambda\in\mathbf{R}^d} (\lambda' y - \Lambda(\lambda)), \quad \text{where} \ \ \Lambda(\lambda) = \log E \exp(\lambda' Y^l).$$

Put differently, Λ^* is the convex conjugate of the logarithmic moment generating function of Y^l. It can be shown that Λ^* is a convex, lower semicontinuous, nonnegative function that satisfies $\Lambda^*(E(Y^l)) = 0$. Moreover, Λ^* is finite, strictly convex, and continuously differentiable on the interior of the support of Y^l, and is infinite outside the support of Y^l.[4]

Theorem 3.1 (Cramér's Theorem)**.** *Let Λ^* be the Cramér transform of Y^1. Then the sequence $\{\bar{Y}^m\}_{m=1}^\infty$ satisfies the large deviation principle with convex rate function Λ^*.*

We now record three easily computed Cramér transforms of random variables ($d = 1$) for later use.

Example 3.2. (i) If $Z \sim N(0, \sigma^2)$, then $\Lambda^*(z) = \frac{z^2}{2\sigma^2}$.

(ii) Let Z have a demeaned exponential distribution (i.e., $Z = T - 1$, where $T \sim$ *exponential*(1)). Then $\Lambda^*(z) = z - \log(z + 1)$ for $z > -1$, and $\Lambda^*(z) = \infty$ otherwise.

(iii) Let Z have a Laplace distribution (i.e., $Z = T' - T$, where T, $T' \sim$ *exponential*(1) are independent). Then $\Lambda^*(z) = \sqrt{1 + z^2} - 1 - \log(\frac{1}{2}(\sqrt{1 + z^2} + 1))$. §

In addition, we make a simple observation about Cramér transforms of random vectors with independent components.

Observation 3.3. *Let $Z_1, \ldots, Z_d$ be independent random variables with Cramér transforms $\Lambda_1^*, \ldots, \Lambda_d^*$, and let $Z = (Z_1, \ldots, Z_d)'$. Then the Cramér transform of Z is $\Lambda^*(z) = \sum_{k=1}^d \Lambda_k^*(z_k)$.*

[4] These properties of the Cramér transform and Cramér's Theorem can be found in Section 2.2 of Dembo and Zeitouni (1998). In particular, the finiteness, strict convexity, and smoothness of Λ^* on the interior of its domain follow from the assumptions that Y^l has convex support and that its moment generating function exists—see Exercises 2.2.24 and 2.2.39 in Dembo and Zeitouni (1998).

3.2 Unlikelihood Functions

Using these ideas, we can determine the rate of decay the perturbed best response probabilities $\tilde{B}^m_i(\chi)$ as the number of matches grows large.

To state our results in a simple and general way, we decompose the perturbed best response function as $\tilde{B}^m = C^m \circ F$, where $C^m : \mathbf{R}^n \to X$ is the *choice probability function*

$$C^m_i(\pi) = P\left(\bigcap_{j\neq i}^{n} \left\{\pi_i + \bar{\varepsilon}^m_i \geq \pi_j + \bar{\varepsilon}^m_j\right\}\right) = P\left(D^i(\pi + \bar{\varepsilon}^m) \geq \mathbf{0}\right).$$

In the last expression, $D^i \in \mathbf{R}^{n\times n}$ is the matrix $\mathbf{1}e_i' - I$, where I is the identity matrix and $\mathbf{1}$ the vector of ones, so that $(D^i\pi)_j = \pi_i - \pi_j$.

Define the *unlikelihood function* $\Upsilon : \mathbf{R}^n \to \mathbf{R}^n_+$ by

$$\Upsilon_i(\pi) = -\lim_{m\to\infty} \frac{1}{m} \log C^m_i(\pi).$$

$\Upsilon_i(\pi)$ is the exponential rate of decay of the choice probability $C^m_i(\pi)$ as the number of matches m grows large. We sometimes refer to $\Upsilon_i(\pi)$ as the *unlikelihood of choosing strategy i given payoff vector π*.

Suppose that the components of the random vector $\varepsilon^l = (\varepsilon^l_1, \dots, \varepsilon^l_n)$ have Cramér transforms $R_1(\eta_1), \dots, R_n(\eta_n)$. Cramér's Theorem tells us that the unlikelihood $\Upsilon_i(\pi)$ can be computed by minimizing the rate function for the sequence $\{\bar{\varepsilon}^m\}_{m=1}^{\infty}$ on an appropriate domain. In particular, Theorem 3.1 and Observation 3.3 imply that

$$\Upsilon_i(\pi) = \min \sum_{k=1}^{n} R_k(\eta_k) \quad \text{subject to} \quad D^i(\pi + \eta) \geq \mathbf{0}. \tag{4}$$

Since the unlikelihood function Υ is the ultimate determinant of the transition costs of the evolutionary process, it is worth exploring its qualitative properties. Some of these are easy to deduce from the function's definition.

Observation 3.4. (*i*) $\Upsilon_i(\pi) = 0$ *if and only if* $\pi_i \geq \pi_j$ *for all* $j \neq i$.

(*ii*) $\Upsilon_i(\pi)$ *is nonincreasing in* π_i *and is nondecreasing in* π_j *for* $j \neq i$.

(*iii*) *Suppose that* $\varepsilon^l_1, \dots, \varepsilon^l_n$ *are i.i.d., and let* $\Xi \in \mathbf{R}^{n\times n}$ *be a permutation matrix. Then* $\Upsilon(\Xi\pi) = \Xi\Upsilon(\pi)$. *If in addition* $(\Xi\pi)_i = \pi_i$, *then* $\Upsilon_i(\pi) = \Upsilon_i(\Xi\pi)$.

Observation 3.4 tells us that the unlikelihood of choosing strategy i is zero precisely when strategy i is optimal, nonincreasing in π_i, and nondecreasing in other strategies' payoffs. Furthermore, if the payoff shocks to different strategies are not only independent, but also identically distributed, then the unlikelihood function Υ is symmetric in its arguments, and each component Υ_i treats the payoffs of strategies other than i symmetrically as well.

While these properties are easy to verify, a less obvious property of the components of Υ can be deduced from the form of the minimization problem (4).

Proposition 3.5. *Each component Υ_i of the unlikelihood function Υ is convex.*

Proof. The objective function of the program that defines Υ_i is convex, and the function defining the program's constraints is linear in the vector $(\eta, \pi) \in \mathbf{R}^{2n}$. The convexity of Υ_i thus follows from Lemma 1 of Mangasarian and Rosen (1964). ■

Proposition 3.5 tells us that the unlikelihood of choosing strategy i depends on the payoff vector π in a convex fashion. To understand the relevance of this result to equilibrium selection, suppose that $\varepsilon_1^l, \ldots, \varepsilon_n^l$ are i.i.d., and imagine two x-trees: τ_1, in which the sole costly transition involves an agent switching to a strategy whose payoff is c units less than that of every alternative, and τ_2, in which the two costly transitions each involve an agent switching to a strategy whose payoff is $\frac{c}{2}$ units less than that of every alternative. Since Υ_i is convex and satisfies $\Upsilon_i(\mathbf{0}) = 0$, it follows that $\Upsilon_i(0, c, \ldots, c) \geq 2\Upsilon_i(0, \frac{c}{2}, \ldots, \frac{c}{2})$, and so that tree τ_1 is weakly more costly than tree τ_2. As we will see in Section 4.2, considerations of this sort are central in stochastic stability analyses, particularly for games with more than two strategies per player.

Observation 3.4 and Proposition 3.5 describe properties of the unlikelihoods $\Upsilon_i(\pi)$ that must hold regardless of the distributions of the shocks ε_j^l. In Section 4, we will see that these restrictions tightly constrain equilibrium selection results in 2×2 games, but not in more general games.

To proceed further with our analysis, we require a tractable characterization of the unlikelihood function Υ. Theorem 3.6 provides this. In this theorem, the assumption that $C_i^1(\pi) > 0$ requires the distribution of the shock vector ε^l to provide strategy i with a strictly positive probability of being chosen at base payoff vector π.

Theorem 3.6. *Suppose $C_i^1(\pi) > 0$. Then the unlikelihood function Υ satisfies*

$$\Upsilon_i(\pi) = \sum_{k=1}^{n} R_k(\eta_k^*), \tag{5}$$

where $\eta^* \in \mathbf{R}^n$ is the unique vector satisfying

$$\sum_{k \in S} R_k'(\eta_k^*) = 0, \tag{M1}$$

(M2) $\qquad \pi_i + \eta_i^* - (\pi_j + \eta_j^*) \geq 0 \quad$ *for all* $j \neq i$,

(M3) $\qquad \eta_j^* \leq 0 \quad$ *for all* $j \neq i$, *and*

(M4) $\qquad \eta_j^* \left(\pi_i + \eta_i^* - (\pi_j + \eta_j^*)\right) = 0 \quad$ *for all* $j \neq i$.

Proof. In the Appendix.

In Theorem 3.6, the vector η^* represents the realization of the (average) shock vector $\bar{\varepsilon}^m$ that is "least unlikely" among those that make strategy i optimal. The theorem implies that the payoff shock to strategy i is nonnegative, the payoff shocks to worse-performing strategies are zero, and the payoff shocks to better-performing strategies are nonpositive.

It is worth drawing attention to one possibility that Theorem 3.6 does not exclude: it may be that the least unlikely way to ensure the optimality of strategy i involves imposing a payoff shock of zero on some better-performing strategy. We show next that this possibility is actually a common occurrence.

3.3 The Unlikelihood Function of the Multinomial Probit Model

We now derive the unlikelihood function generated by vectors ε^l of i.i.d multivariate normal payoff shocks. We are interested in this setup not only because it is an important baseline case, but also because it captures the limiting behavior of the multinomial probit model. If each payoff shock ε_i^l has a $N(0, \sigma^2)$ distribution, then each average payoff shock $\bar{\varepsilon}_i^m$ has a $N(0, \frac{\sigma^2}{m})$ distribution. Therefore, by describing the rates of decay of choice probabilities in our model as the number of observations m goes to infinity, we also describe the rates of decay for the i.i.d. multinomial probit model as the noise variance goes to zero.[5]

We need one more definition to state our result. Fix a strategy i; then for any set of strategies J not containing i, let

$$\bar{\pi}^{J+} = \frac{1}{n^J + 1} \sum_{j \in J \cup \{i\}} \pi_j.$$

denote the average payoff of strategies in $J \cup \{i\}$.

Theorem 3.7. *Suppose that each random vector $\varepsilon^l \sim N(\mathbf{0}, \sigma^2 I)$ is multivariate normal. Then the*

[5]In fact, since any multivariate normal random vector can be converted into one with i.i.d. components by applying a linear transformation, the analysis below is easily extended to cover multivariate probit models with correlated shocks.

unlikelihood function Υ *is given by*

$$(6) \qquad \Upsilon_i(\pi) = \sum_{k=1}^{n} \frac{(\eta_k^*)^2}{2\sigma^2}.$$

The vector $\eta^* \in \mathbf{R}^n$ *is given by*

$$(7) \qquad \eta_j^* = \begin{cases} \bar{\pi}^{J+} - \pi_j & \text{if } j \in J \cup \{i\}, \\ 0 & \text{otherwise,} \end{cases}$$

where the set $J \subset S$ *contains the* n^J *strategies whose payoffs are highest; the number* n^J *is uniquely determined by the requirement that*

$$(8) \qquad \pi_j - \bar{\pi}^{J+} > 0 \text{ if and only if } j \in J.$$

Proof. In the Appendix.

Myatt and Wallace (2003) study a model of stochastic evolution based on multinomial probit choice. Their Proposition 1 states that the rate of decay of choice probabilities in this model takes the form described in our Theorem 3.7, but with the set J of strategies for which η_j^* is negative containing all strategies earning a higher payoff than strategy i.[6] In contrast, Theorem 3.7 says that while the set J should contain all strategies with high enough payoffs, it does not always contain every strategy whose payoff exceeds that of strategy i. The reason why some strategies better than i may be left out is easy to illustrate through an example.

Example 3.8. Let $n = 3$, suppose that payoff shocks are i.i.d. standard normal, and consider a base payoff vector of $\pi = (\pi_1, \pi_2, \pi_3) = (0, b, c)$ with $b > 0$. If $c \leq 0$, so that only strategy 2's base payoff is higher than strategy 1's, then both Myatt and Wallace's (2003) Proposition 1 and our Theorem 3.7 specify the unlikelihood of choosing strategy 1 as $\Upsilon_1(\pi) = \frac{b^2}{4}$, obtained from shock vector $\eta^* = (\frac{b}{2}, -\frac{b}{2}, 0)$. Our large deviations analysis shows that the least unlikely way to satisfy the inequality $\bar{\varepsilon}_1^m - \bar{\varepsilon}_2^m \geq b$ is to have the shocks to strategies 1 and 2 "share the burden equally".

Now suppose instead that $c > 0$, so that strategies 2 and 3 both have higher base payoffs than strategy 1. In this case, Myatt and Wallace (2003) suggest unlikelihood of choosing strategy 1 of $\Upsilon_1(\pi) = \frac{1}{3}(b^2 - bc + c^2)$, obtained from shock vector $\eta = (\frac{b+c}{3}, \frac{c-2b}{3}, \frac{b-2c}{3})$. But

[6] Since Myatt and Wallace (2003) parameterize the level of noise directly in terms of the overall shock variance, they do not provide the large deviation interpretation of η^* offered here.

if $c < \frac{b}{2}$, then the payoff deficit of strategy 1 to strategy 3 of $\pi_1 - \pi_3 = -c$ is already fully addressed by the positive shock to strategy 1 of $\eta_1 = \frac{b+c}{3} > c$. Indeed, the shock to strategy 3 specified above, $\eta_3 = \frac{b-2c}{3}$, is positive, which can only be counterproductive.

In fact, when $c < \frac{b}{2}$, Theorem 3.7 tells us that the optimal choice of η is still $\eta^* = (\frac{b}{2}, -\frac{b}{2}, 0)$, for an unlikelihood of $\Upsilon_1(\pi) = \frac{b^2}{4}$. More generally, the theorem shows that when $b, c > 0$,

$$\eta^* = \begin{cases} (\frac{b}{2}, -\frac{b}{2}, 0) & \text{if } c < \frac{b}{2}, \\ (\frac{b+c}{3}, \frac{c-2b}{3}, \frac{b-2c}{3}) & \text{if } c \in [\frac{b}{2}, 2b], \\ (\frac{c}{2}, 0, -\frac{c}{2}) & \text{if } c > 2b, \end{cases} \quad \text{and} \quad \Upsilon_1(\pi) = \begin{cases} \frac{b^2}{4} & \text{if } c < \frac{b}{2}, \\ \frac{b^2 - bc + c^2}{3} & \text{if } c \in [\frac{b}{2}, 2b], \\ \frac{c^2}{4} & \text{if } c > 2b. \quad \S \end{cases}$$

4. Equilibrium Selection

4.1 2×2 Games

Consider the 2×2 normal form game defined by payoff matrix A:

	0	1
0	A_{00}, A_{00}	A_{01}, A_{10}
1	A_{10}, A_{01}	A_{11}, A_{11}

(Note the labeling of the two strategies, which will prove convenient below.) We suppose that A defines a coordination game: $A_{00} > A_{10}$, $A_{11} > A_{01}$.

Let $\Delta_1 : [0, 1] \to \mathbf{R}$ denote the expected payoff advantage of strategy 1 as a function of the weight on strategy 1 in the opponent's mixed strategy. Then $\Delta_1(z) = d(z - z^*)$, where $d = (A_{00} - A_{10}) + (A_{11} - A_{01}) > 0$, and where $z^* = (A_{00} - A_{10})/d$ is the weight on strategy 1 in the mixed strategy Nash equilibrium of A. In similar fashion, define $\Delta_0(z) = -d(z - z^*)$ to be the payoff advantage of strategy 0 as a function of the weight the opponent places on strategy 1.

Let R_0 and R_1 be the Cramér transforms of payoff shocks ε_0^l and ε_1^l, respectively. Define

$$\rho(a) = R_1(\eta_1^*(a)) + R_0(\eta_1^*(a) - a),$$

where $\eta_1^*(a)$ is implicitly (and uniquely) defined by the requirement that

$$R_1'(\eta_1^*(a)) + R_0'(\eta_1^*(a) - a) = 0.$$

Our first proposition expresses the unlikelihood function for the binary choice model in terms of the function ρ.[7]

Proposition 4.1. *The function ρ is the rate function for $\{\bar{\varepsilon}_1^m - \bar{\varepsilon}_0^m\}_{m=1}^{\infty}$. Equivalently, the unlikelihood function Υ for the binary choice model is given by*

$$\Upsilon(\pi) = (\Upsilon_0(\pi), \Upsilon_1(\pi)) = \begin{cases} (\rho(\pi_0 - \pi_1), 0) & \text{if } \pi_0 < \pi_1, \\ (0, 0) & \text{if } \pi_0 = \pi_1, \\ (0, \rho(\pi_0 - \pi_1)) & \text{if } \pi_0 > \pi_1. \end{cases} \tag{9}$$

Proof. Immediate from Theorem 3.6. ■

To determine which state $x \in \mathcal{X}^N$ is stochastically stable, we need to determine the lowest cost x-tree. In the present case, there are only two plausible candidates. By letting $x^k = (\frac{N-k}{N}, \frac{k}{N})$ denote the population state at which k agents choose strategy 1, we can describe the two candidates explicitly: they are the e_1-tree $\tau_1 = \{(x^0, x^1), \dots, (x^{N-1}, x^N)\}$, and the e_0-tree $\tau_0 = \{(x^N, x^{N-1}), \dots, (x^1, x^0)\}$.

The next result provides formulas for the costs of these two trees.

Proposition 4.2. *The costs of trees τ_1 and τ_0 are*

$$c(\tau_1) = \sum_{k=1}^{\kappa_1} \rho(\Delta_0(\tfrac{k-1}{N-1})) \quad \textit{and} \quad c(\tau_0) = \sum_{k=1}^{\kappa_0} \rho(\Delta_0(\tfrac{N-k}{N-1})),$$

where $\kappa_1 = \lceil (N-1)z^* \rceil$ and $\kappa_0 = \lceil (N-1)(1-z^*) \rceil$.

Proof. Let $\chi^k = (\frac{N-1-k}{N-1}, \frac{k}{N-1})$ represent the behavior of $N-1$ agents, of whom k choose strategy 1. Then $F_0(\chi^k) - F_1(\chi^k) = \Delta_0(\frac{k}{N-1}) = -d\,(\frac{k}{N-1} - z^*)$ is decreasing in k, and first becomes nonpositive when $k = \lceil (N-1)z^* \rceil = \kappa_1$. Therefore,

$$\begin{aligned} c(\tau_1) &= \sum_{k=1}^{N} c(x^{k-1}, x^k) \\ &= -\sum_{k=1}^{N} \lim_{m\to\infty} \tfrac{1}{m} \log p^m_{x^{k-1}x^k} \\ &= -\sum_{k=1}^{N} \lim_{m\to\infty} \tfrac{1}{m} \log \left(\tfrac{N-(k-1)}{N} \cdot C_1^m(F(\chi^{k-1})) \right) \end{aligned}$$

[7] Let us draw attention to a possible source of confusion: while ρ is the rate function for the differences $\{\bar{\varepsilon}_1^m - \bar{\varepsilon}_0^m\}_{m=1}^{\infty}$, in expression (9) for the unlikelihood function Υ, the argument of ρ is the difference $\pi_0 - \pi_1$. To see why, remember that strategy 1 is chosen when $\pi_1 + \bar{\varepsilon}_1^m > \pi_0 + \bar{\varepsilon}_0^m$, or equivalently, when $\bar{\varepsilon}_1^m - \bar{\varepsilon}_0^m > \pi_0 - \pi_1$.

$$= \sum_{k=1}^{N} \Upsilon_1(F(\chi^{k-1}))$$

$$= \sum_{k=1}^{\kappa_1} \rho(\Delta_0(\tfrac{k-1}{N-1})).$$

The derivation of $c(\tau_0)$ is similar. ∎

By applying Proposition 4.2, we obtain our equilibrium selection result. Recall that in a 2×2 game, strategy i is *risk dominant* if it is optimal against an opponent who randomizes uniformly over his two strategies. Thus, strategy 1 is risk dominant if $z^* \leq \frac{1}{2}$, while strategy 0 is risk dominant if $z^* \geq \frac{1}{2}$.

Theorem 4.3. *Suppose that ε_0^l and ε_1^l are i.i.d. Then state e_i is stochastically stable if and only if strategy i is risk dominant.*

Proof. Suppose that $z^* < \frac{1}{2}$, so that strategy 1 is strictly risk dominant. Then $\kappa_1 = \lceil (N-1)z^* \rceil \leq \lceil (N-1)(1-z^*) \rceil = \kappa_0$. Moreover, since $1 - z^* > z^*$, we have that $\frac{N-k}{N-1} - z^* > z^* - \frac{k-1}{N-1}$. Since Δ_0 is decreasing and linear with $\Delta_0(z^*) = 0$, it follows that $\Delta_0(\frac{N-k}{N-1}) < -\Delta_0(\frac{k-1}{N-1}) < 0$ when $\frac{k-1}{N-1} < z^*$, and hence when $k \leq \kappa_1$.

Now the rate function ρ is convex, and since ε_0^l and ε_1^l are i.i.d., ρ is also symmetric about its minimizer at 0. We can therefore conclude that

$$c(\tau_1) = \sum_{k=1}^{\kappa_1} \rho(\Delta_0(\tfrac{k-1}{N-1})) = \sum_{k=1}^{\kappa_1} \rho(-\Delta_0(\tfrac{k-1}{N-1})) < \sum_{k=1}^{\kappa_1} \rho(\Delta_0(\tfrac{N-k}{N-1})) \leq \sum_{k=1}^{\kappa_0} \rho(\Delta_0(\tfrac{N-k}{N-1})) = c(\tau_0). \tag{10}$$

When $z^* > \frac{1}{2}$, a similar analysis shows that $c(\tau_1) > c(\tau_0)$. Finally, if $z^* = \frac{1}{2}$, then all of the inequalities in (10) become equalities, so $c(\tau_1) = c(\tau_0)$. ∎

The next example presents some of the ideas from this section in pictures, focusing for simplicity on the case where the population size N is large. The intuitions provided here will prove useful in the next section, where we consider equilibrium selection beyond 2×2 games.

Example 4.4. Proposition 4.2 shows that costs $c(\tau_1)$ and $c(\tau_2)$ are obtained by summing up values of the composition $\rho \circ \Delta_0$ over appropriate ranges. If N is large, we can approximate these sums by integrals:[8]

$$\tfrac{1}{N} c(\tau_1) \approx I_1 \equiv \int_0^{z^*} \rho(\Delta_0(z))\,\mathrm{d}z \quad \text{and} \quad \tfrac{1}{N} c(\tau_0) \approx I_0 \equiv \int_{z^*}^{1} \rho(\Delta_0(z))\,\mathrm{d}z.$$

[8] Binmore and Samuelson (1997), Blume (2003), and Sandholm (2007) use this integral approximation in related contexts.

Thus, for large N, state e_1 is stochastically stable if $I_1 < I_0$: that is, if the area under the graph of $\rho \circ \Delta_0$ on $[0, z^*]$ is less than the area under the graph of $\rho \circ \Delta_0$ on $[z^*, 1]$.

To make our discussion more concrete, let us focus on the coordination game

	0	1
0	6, 6	0, 0
1	0, 0	9, 9

whose mixed equilibrium places weight $z^* = \frac{2}{5}$ on strategy 1, implying that strategy 1 is risk dominant.

We consider evolution under three assumptions about the distributions of payoff shocks: (i) ε_0^l and ε_1^l are both $N(0, \sigma^2)$; (ii) ε_0^l and ε_1^l are both (demeaned) exponential(1);[9] and (iii) ε_0^l is demeaned *exponential*(1) and ε_1^l is $N(0, \sigma^2)$. Using Example 3.2 and Proposition 4.1, we can compute the rate function for $\{\bar{\varepsilon}_1^m - \bar{\varepsilon}_0^m\}_{m=1}^\infty$ under each distributional assumption:

(i) $\rho(a) = \frac{a^2}{4\sigma^2}$,

(ii) $\rho(a) = \sqrt{1 + a^2} - 1 - \log\left(\frac{1}{2}(\sqrt{1 + a^2} + 1)\right)$, and

(iii) $\rho(a) = \frac{1}{2\sigma^2}(\eta_1^*(a))^2 + (\eta_1^*(a) - a) - \log(\eta_1^*(a) - a + 1)$, where
$\eta_1^*(a) = \frac{1}{2}(a - \sigma^2 - 1 + \sqrt{(a - \sigma^2 - 1)^2 + 4a\sigma^2})$.

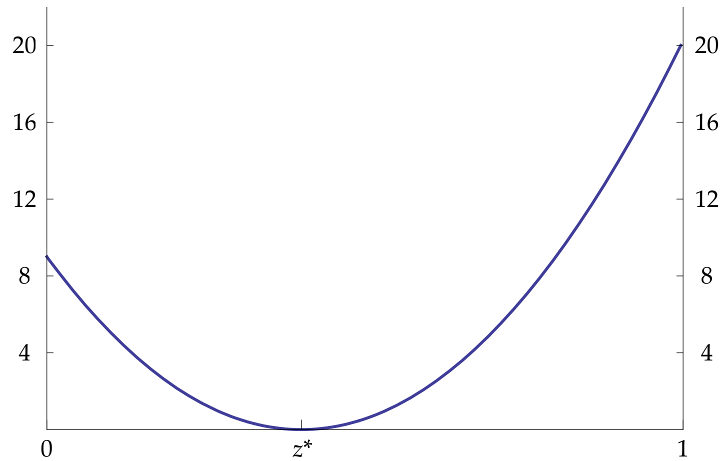
Notice that function (i) is the Cramér transform of the $N(0, 2\sigma^2)$ distribution, while function (ii) is the Cramér transform of the Laplace distribution (cf Example 3.2).

Figure 1 presents graphs the composition $\rho(\Delta_0(z)) = \rho(-15(z - z^*))$ under each of the distributional assumptions above when $\sigma^2 = 1$. In cases (i) and (ii), where the shocks are i.i.d., the rate function ρ is symmetric about 0, so the composition $\rho \circ \Delta_0$ is symmetric about the mixed equilibrium z^*. Since interval $[0, z^*]$ is shorter than interval $[z^*, 1]$, it follows immediately that $I_1 < I_0$, and hence that state e_1 is stochastically stable for large N. Of course, Theorem 4.3 tells us that this selection result obtains regardless of the value of N.

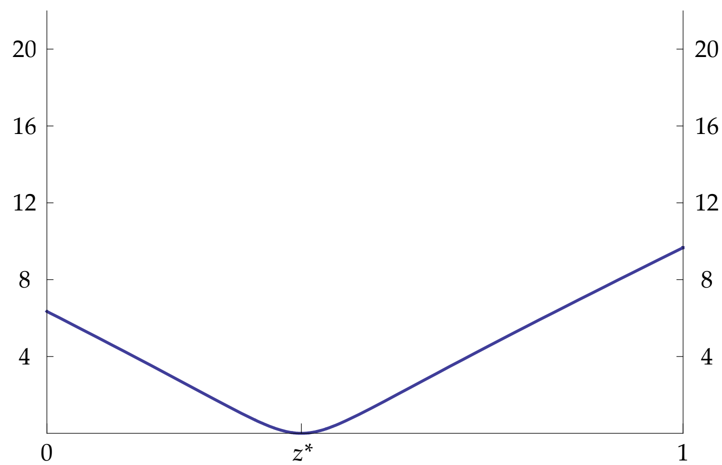
Still, Figures 1(i) and 1(ii) reveal an important difference between the effects of normal and exponential payoff shocks: the former lead unlikelihoods to grow quadratically, while the latter cause them to grow approximately linearly. This difference has no bearing on equilibrium selection in 2×2 games, but as we will see in Section 4.2, it takes on a central role when we consider larger games.

In Figure 1(iii), $\rho \circ \Delta_0$ is asymmetric: it is approximately quadratic to the left of z^*, and approximately linear to the right. This asymmetry is sufficient to make $I_1 \approx 1.8001 > 1.5965 \approx I_0$, implying that state e_0, in which all agents play the risk dominated strategy, is selected when N is large.

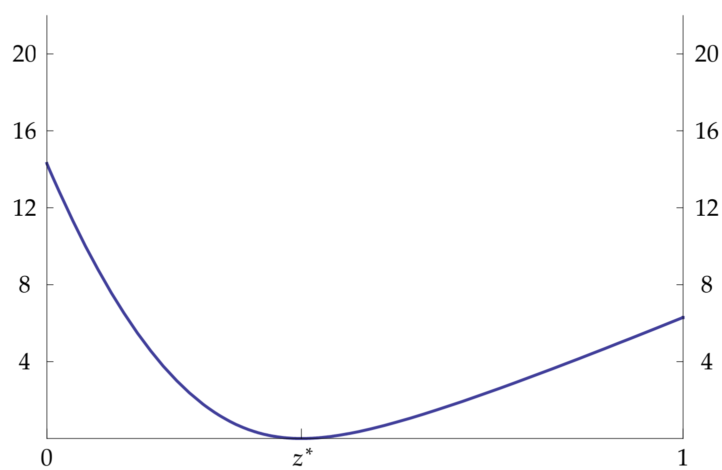
[9] When the trials are i.i.d., demeaning is unnecessary, since subtracting the same quantity from all of the payoff shocks has no effect on agents' choices.



(i) $\varepsilon_0^l, \varepsilon_1^l \sim N(0, 1)$



(ii) $\varepsilon_0^l, \varepsilon_1^l \sim \textit{exponential}(1)$



(iii) $\varepsilon_0^l \sim \textit{exponential}(1) - 1,\ \varepsilon_1^l \sim N(0, 1)$

Figure 1: The composition $\rho \circ \Delta_0$.

For intuition, remember that escaping from the interval $[0, z^*]$ where strategy 0 is optimal (and hence I_1 is determined) requires positive values of the difference $\bar{\varepsilon}_1^m - \bar{\varepsilon}_0^m$. Since the payoff shocks to strategy 0 are demeaned exponentials, negative values of $\bar{\varepsilon}_0^m$ quickly take on very high unlikelihoods—actually, infinite unlikelihoods for values below -1. Therefore, escape from $[0, z^*]$ requires positive values of payoff shocks to strategy 1. Since these shocks are normally distributed, $\rho \circ \Delta_0$ is approximately quadratic on $[0, z^*]$.

On the other hand, escaping from the interval $[z^*, 1]$ where strategy 1 is optimal (and hence I_0 is determined) requires positive values of $\bar{\varepsilon}_0^m - \bar{\varepsilon}_1^m$. This time, the fact that the payoff shocks to strategy 0 are demeaned exponentials implies that positive values of $\bar{\varepsilon}_0^m$ have relatively low unlikelihoods: as we saw in Example 3.2, the Cramér transform of this distribution is $R_0(\eta_0) = \eta_0 - \log(\eta_0 + 1)$, which is approximately linear on $\mathbf{R}_+$. Therefore, it is these payoff shocks to strategy 0 that drive the escape from $[z^*, 1]$, creating the approximately linear portion of $\rho \circ \Delta_0$ on this interval. Because of this difference in the way unlikelihoods grow, e_0 can be selected despite the fact that strategy 0 is optimal on a smaller interval.

What happens if we vary σ^2, the variance of the normal shock ε_1^l? If we make σ^2 smaller, then large payoff shocks to strategy 1 become even more unlikely. The main effect of this is to make interval $[0, z^*]$ even harder to escape, further favoring the selection of strategy 0. In contrast, making σ^2 larger makes large payoff shocks to strategy 1 more likely; thus, interval $[0, z^*]$ becomes easier to escape, tipping the balance in favor of strategy 1. In fact, this risk dominant strategy begins being selected once σ^2 reaches 1.1656, at which point $I_1 \approx 1.57002 < 1.57009 \approx I_0$. §

4.2 $n \times n$ Games

In Section 3.2 we established a number of restrictions on the unlikelihood function for the rapid play model: in particular, Υ_i is nonincreasing in π_i, symmetric and nondecreasing in π_j, and convex in the full payoff vector π. Theorem 4.3 showed that these restrictions alone are enough to completely determine the nature equilibrium selection in 2×2 games.

In this section, we demonstrate that beyond the 2×2 case, these restrictions on the unlikelihood function are not enough to uniquely determine the stochastically stable state. In particular, unlikelihood functions exhibiting different degrees of convexity can generate different equilibrium selection results. Our examples build on one introduced by Ui (1998) to show that the logit and probit choice rules can generate different stochastically stable states.

Consider the following symmetric 3×3 game with $a, b, c > 0$.

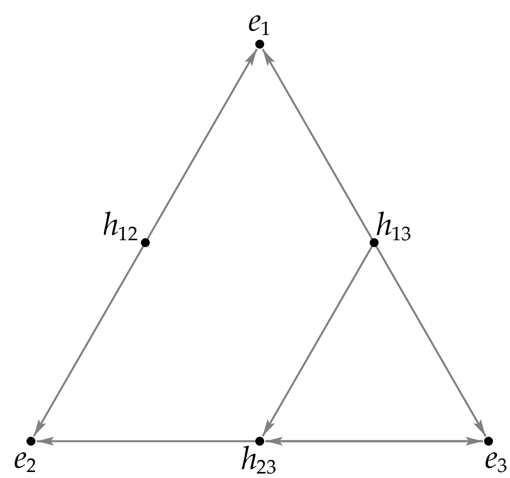


Figure 2: The state space $\mathcal{X}^2$. Arrows represent zero-cost edges.

	1	2	3
1	a, a	$0, 0$	$0, 0$
2	$0, 0$	$b + c, b + c$	b, b
3	$0, 0$	b, b	b, b

Since strategy 3 is weakly dominated by strategy 2, the symmetric pure equilibria in undominated strategies are $(1, 1)$ and $(2, 2)$.

To keep the analysis as simple as possible, we suppose in our model of evolution that the game above is played by a population of just $N = 2$ agents. The state space $\mathcal{X}^2$ of our evolutionary process has just 6 states: the three "pure" states, e_1, e_2, and e_3, and the three "mixed" states $h_{12} = \frac{1}{2}e_1 + \frac{1}{2}e_2$, $h_{13} = \frac{1}{2}e_1 + \frac{1}{2}e_3$, and $h_{23} = \frac{1}{2}e_2 + \frac{1}{2}e_3$.

The discussion in the rest of this section revolves around the following proposition.

Proposition 4.5. (i) *For some positive values of a, b, and c, state e_1 is stochastically stable under i.i.d. $N(0, 1)$ shocks, while state e_2 is stochastically stable under i.i.d. demeaned exponential(1) shocks.*

(ii) *For other positive values of a, b, and c, state e_2 is stochastically stable under i.i.d. $N(0, 1)$ shocks, while state e_1 is stochastically stable under i.i.d. demeaned exponential(1) shocks.*

Once we fix the distribution of shocks, we can determine the stochastically stable state by finding the minimum cost x-tree. In the present case, finding this tree is simplified by the fact that of the 18 directed edges between adjacent states in $\mathcal{X}^2$, 8 have zero cost. In Figure 2, we draw these 8 zero-cost edges on a diagram of the state space $\mathcal{X}^2$.

Taking note of these zero-cost edges, it is easy to verify that the minimal cost can always be generated by one of three trees. The first, τ_1^d, is an e_1-tree whose sole positive cost edge is (e_2, h_{12}); we call this the *direct e_1-tree*. The second, τ_1^i, is an e_1-tree whose sole positive cost edges are (e_2, h_{23}) and (e_3, h_{13}); we call this the *indirect e_1-tree*. The third, τ_2^d, is an e_2-tree whose sole positive cost edge is (e_1, h_{12}); we call this the *direct e_2-tree*. The costs of these three trees are:

$$
\begin{aligned}
&c(\tau_1^d) = \Upsilon_1(0, b + c, b),\\
&c(\tau_1^i) = \Upsilon_3(0, b + c, b) + \Upsilon_1(0, b, b), \text{ and}\\
&c(\tau_2^d) = \Upsilon_2(a, 0, 0).
\end{aligned}
$$

To prove Proposition 4.5, it is enough to provide appropriate values of a, b, and c, compute the costs of trees τ_1^i, τ_1^d, and τ_2^d under normal and exponential noise, and verify that trees with the desired roots are the ones with the lowest costs. Rather than doing this immediately, we offer an argument that provides deeper intuition for our results.

To begin, we note some values of the unlikelihood functions generated by normal and exponential shocks. Suppose that $d > 0$. In Example 3.8, we showed that under $N(0, 1)$ shocks,

$$\Upsilon_1^n(0, d, d') = \tfrac{d^2}{4} \text{ when } d' \le \tfrac{d}{2}; \tag{11}$$

$$\Upsilon_1^n(0, d, d) = \tfrac{d^2}{3}. \tag{12}$$

For *exponential*(1) shocks, demeaned or not, one can use Theorem 3.6 to show that

$$\Upsilon_1^e(0, d, 0) = \sqrt{1 + d^2} - 1 - \log\left(\tfrac{1}{2}\left(\sqrt{1 + d^2} + 1\right)\right); \tag{13}$$

$$
\begin{aligned}
\Upsilon_1^e(0, d, d) = {}& \tfrac{1}{2}\left(\sqrt{9 + 6d + 9d^2} - d - 3\right)\\
& - \log\left(\tfrac{1}{6}\left(d^2 - d\sqrt{1 + \tfrac{2}{3}d + d^2} + \sqrt{9 + 6d + 9d^2} + 3\right)\right).
\end{aligned} \tag{14}
$$

While expressions (13) and (14) look daunting, it is easy to check that as d grows large, the dominant term in each of them is linear with coefficient 1. Thus, in our intuitive arguments we will use the approximation

$$\Upsilon_1^e(0, d, d') \approx d \text{ when } d \text{ is large and } d' \le d. \tag{15}$$

The approximate linearity of the unlikelihood function (13) was already noted in Example 4.4; see especially Figure 1(ii).

To obtain the two parts of Proposition 4.5, we introduce two different restrictions on payoffs b and c.

tree	normal	exponential
τ_1^d	25	7.3406
τ_1^i	$14\frac{7}{12}$	6.0568
τ_2^d	16	5.5513

(a) Tree costs when $a = 8$, $b = 5$, and $c = 5$.

tree	normal	exponential
τ_1^d	16.57	4.8208
τ_1^i	16.3358	4.7746
τ_2^d	16	5.5513

(b) Tree costs when $a = 8$, $b = 7$, and $c = \frac{1}{10}$.

Table I: Tree costs

Example 4.6. Suppose that $c = b$. Then using equations (11) and (12), we find that under normal shocks, the costs of our three trees are

$$\begin{aligned}
c^n(\tau_1^d) &= \Upsilon_1^n(0, 2b, b) = b^2;\\
c^n(\tau_1^i) &= \Upsilon_3^n(0, 2b, b) + \Upsilon_1^n(0, b, b) = \tfrac{b^2}{4} + \tfrac{b^2}{3} = \tfrac{7}{12}\, b^2;\\
c^n(\tau_2^d) &= \Upsilon_2^n(a, 0, 0) = \tfrac{a^2}{4}.
\end{aligned}$$

Thus, the minimal cost tree is τ_1^i as long as $a > \sqrt{7/3}\, b \approx 1.53\, b$. Similarly, using equation (15), we find that the costs of the three trees under exponential shocks are approximately

$$\begin{aligned}
c^e(\tau_1^d) &= \Upsilon_1^e(0, 2b, b) \approx 2b;\\
c^e(\tau_1^i) &= \Upsilon_3^e(0, 2b, b) + \Upsilon_1^e(0, b, b) \approx b + b = 2b;\\
c^e(\tau_2^d) &= \Upsilon_2^e(a, 0, 0) \approx a.
\end{aligned}$$

This suggests that the minimal cost tree is τ_2^d if $a < 2b$. Putting this altogether, we find that if $a \in (1.53\, b, 2b)$, $c = b$, and a is large, then normal shocks should lead to the selection of state e_1, while exponential shocks should lead to the selection of state e_2, as described in Proposition 4.5(i).

Since the linear approximation (15) for $\Upsilon_1^e(0, d, d')$ works reasonably well even when d is not especially large, one does not need large entries in the payoff matrix to obtain these selection results. In Table I(a), we report the exact costs of the three trees under both noise assumptions when $a = 8$ and $b = c = 5$. Evidently, normal shocks select state e_1, and exponential shocks state e_2.

The intuition here is straightforward. With normal shocks, unlikelihoods vary quadratically with payoffs. As a consequence, transitions that involve multiple "short" (i.e., small payoff difference) steps are less unlikely than transitions involving a single "large" step; in particular, the cost of the indirect tree τ_1^i is relatively small. With exponential shocks, unlikelihoods vary linearly with payoffs, so indirect transitions offer no particular advantage. By choosing payoffs appropriately, we can convert this difference in costs into different equilibrium selections. §

Example 4.7. Now suppose that $c \approx 0$. In this case, we find that

$$
\begin{aligned}
c^n(\tau_1^d) &\approx \Upsilon_1^n(0,b,b) = \tfrac{b^2}{3}; & c^e(\tau_1^d) &\approx \Upsilon_1^e(0,b,b) \approx b;\\
c^n(\tau_1^i) &\approx \Upsilon_3^n(0,b,b) + \Upsilon_1^n(0,b,b) = \tfrac{b^2}{3}; & c^e(\tau_1^i) &\approx \Upsilon_3^e(0,b,b) + \Upsilon_1^e(0,b,b) \approx b;\\
c^n(\tau_2^d) &\approx \Upsilon_2^n(a,0,0) = \tfrac{a^2}{4}. & c^e(\tau_2^d) &\approx \Upsilon_2^e(a,0,0) \approx a.
\end{aligned}
$$

This suggests that if $a \in (b, \sqrt{4/3}\,b) \approx (b, 1.15\,b)$ is large and $c \approx 0$, then normal shocks will select state e_2, while exponential shocks will select state e_1. In Table I(b), we report the exact costs of the three trees under both noise assumptions when $a = 8$, $b = 7$, and $c = \frac{1}{10}$. Evidently, normal shocks select state e_2 and exponential shocks state e_1, verifying Proposition 4.5(ii).

For intuition, note first that when c is close to 0, the cost of the edge from state e_2 to state h_{23} is itself close to 0. For this reason, the cost of the indirect tree τ_1^i is essentially determined by a single edge, eliminating the forces at work in the previous example.

What matters here instead is the difference between unlikelihood $\Upsilon_1(0,d,d)$ (overcoming a payoff disadvantage of d against both other strategies) and unlikelihood $\Upsilon_1(0,0,d)$ (overcoming a payoff disadvantage of d against just one other strategy). Under normal shocks, we have that $\Upsilon_1^n(0,d,d) = \frac{d^2}{3} > \frac{d^2}{4} = \Upsilon_1^n(0,d,0)$, while under exponential shocks, we have that $\Upsilon_1^e(0,d,d) \approx d \approx \Upsilon_1^n(0,d,0)$. Once again, choosing payoffs to take advantage of this difference leads to our selection results.

Why do the normal unlikelihood functions differ while the exponential unlikelihood functions are almost the same? Recall that standard normal shocks generate the quadratic rate function $R(\eta) = \sum_i \frac{(\eta_i)^2}{2}$. Unlikelihood $\Upsilon_1(0,d,0) = \frac{d^2}{4}$ is generated by the shock realization $\eta^* = (\frac{d}{2}, -\frac{d}{2}, 0)$: as we noted in Example 3.8, the best way to ensure that 1 is preferred to 2, is to "split the burden equally" between the two shocks. In contrast, unlikelihood $\Upsilon_1(0,d,d) = \frac{d^2}{3}$ is generated by shock realization $\eta^* = (\frac{2d}{3}, -\frac{d}{3}, -\frac{d}{3})$: since the shock realization must now contend with the improved payoff to strategy 3, the unlikelihood choosing strategy 1 increases.

Demeaned *exponential*(1) shocks generate the rate function $R(\eta) = \sum_i(\eta_i - \log(\eta_i + 1))$, which increases at an approximately linear rate in the value of each positive shock, but which explodes as the shocks' values approach -1. Therefore, if the payoff vector is $(0,d,0)$ for some d large, the best way of making strategy 1 optimal has $\eta_1^* \approx d$: the payoff advantage of strategy 2 is accounted for almost entirely using a positive shock to strategy 1. Doing this generates an unlikelihood of choosing strategy 1 of $\Upsilon_1^n(0,d,0) \approx d$. If payoffs change to $\hat{\pi} = (0, d, d)$, the payoff shock $\eta_1^* \approx d$ to strategy 1 is already nearly enough to address the improved base payoff to strategy 3; therefore, the unlikelihood $\Upsilon_1^e(0, d, d)$ is approximately d as well. §

While the standard normal distribution is symmetric with full support on the real line, the demeaned *exponential*(1) distribution is asymmetric with support $(-1, \infty)$. Could this difference between the two distributions rather than the difference in the degrees of convexity of the rate functions be driving our selection results? To address this question, we repeat the analysis in this section under the assumption that the payoff shocks ε_i^l are i.i.d. with a Laplace distribution (see Example 3.2). The results of this analysis, which we report in the Appendix, are qualitatively similar to those obtained in the case of exponential shocks. This suggests that the differences in equilibrium selections described above are indeed consequences of the different degrees of convexity of the unlikelihood functions generated by normal and exponential shocks.

5. Conclusion

Bergin and Lipman (1996) observe that the results of stochastic stability analyses depend on how the probabilities of suboptimal choices vary with the population state. This paper offers a natural model of this link: during each match, payoffs to each strategy are randomly perturbed; after m matches, an agent is given the opportunity to switch strategies, choosing the one whose average perturbed payoff was highest. We establish some restrictions that this model imposes on the rates of decay of choice probabilities: the unlikelihood of choosing a strategy is nonincreasing in its own payoff, nondecreasing in other payoffs, and convex in the entire vector of payoffs. We find that these restrictions ensure selection of the risk dominant equilibrium in 2×2 games, but that in more general games, different distributions of payoff shocks can generate different stochastically stable states.

A. Appendix

Proof of Theorem 3.6.

Since $C_i^1(\pi) > 0$, program (4) admits a feasible solution on the interior of the support of ε^l. Since the Cramér transform of ε^l is differentiable in this region, we can solve program (4) using the Kuhn-Tucker method.

The Lagrangian for program (4) is

$$\mathcal{L}(\eta, \mu) = \sum_{k=1}^{n} R_k(\eta_k) - \sum_{j \neq i} \mu_j \left(\pi_i + \eta_i - (\pi_j + \eta_j)\right).$$

(Note that the vector of Lagrange multipliers $\mu \in \mathbf{R}^{n-1}$ does not have a component for strategy i.) Since the objective function is convex, η^* is the minimizer if and only it satisfies the constraints (M2) and there exist Lagrange multipliers μ^* such that η^* and μ^* together satisfy

$$R_i'(\eta_i^*) = \sum_{j \neq k} \mu_k^*, \tag{16}$$

$$R_j'(\eta_j^*) = -\mu_j^* \quad \text{for all } j \neq i, \tag{17}$$

$$\mu_j^* \geq 0 \quad \text{for all } j \neq i, \text{ and} \tag{18}$$

$$\mu_j^* \left(\pi_i + \eta_i^* - (\pi_j + \eta_j^*)\right) = 0 \quad \text{for all } j \neq i. \tag{19}$$

Conditions (16) and (17) together imply condition (M1). Since each R_j is strictly convex on its domain and is minimized at 0, R_j' satisfies $\operatorname{sgn}(R_j'(\eta_j)) = \operatorname{sgn}(\eta_j)$. Thus, conditions (17) and (18) imply condition (M3), and conditions (17) and (19) imply condition (M4). ■

Proof of Theorem 3.7.

Part 0: Preliminaries. Example 3.2 tells us that the Cramér transform of component ε_k^l is $R_k(\eta_k) = \frac{(\eta_k)^2}{2\sigma^2}$, thus, Theorem 3.6 shows that the rate function relevant to the present problem takes the form of the sum from equation (6). Also, since $R_k'(\eta_k) = \frac{\eta_k}{\sigma^2}$, condition (M1) from Theorem 3.6 takes the linear form

$$\sum_{k=1}^{n} \eta_k^* = 0. \tag{M1'}$$

Part 1: Necessity. Here we establish that if the vector η^* is an optimal solution to the rate minimization problem—that is, if η^* satisfies conditions (M1')–(M4)—then there exists a set J that satisfies property (8) and that jointly with η^* satisfies property (7).

Define the set $J = \{j \in S : \eta_j^* < 0\}$. Then conditions (M3) and (M4) imply that

$$\eta_k^* = 0 \quad \text{for all } k \notin J \cup \{i\} \text{ and} \tag{20}$$

$$\eta_j^* = \eta_i^* + \pi_i - \pi_j \quad \text{for all } j \in J. \tag{21}$$

Substituting these equalities into equation (M1′) yields

$$\sum_{k \in J \cup \{i\}} (\eta_i^* + \pi_i - \pi_k) = 0.$$

Rearranging this equation shows that

$$\eta_i^* = \bar{\pi}^{J+} - \pi_i, \tag{22}$$

and substituting this expression back into equation (21) shows that

$$\eta_j^* = \bar{\pi}^{J+} - \pi_j \quad \text{for all } j \in J. \tag{23}$$

Together, these two facts and equation (20) imply that equation (7) holds.

Next, equation (23) and the definition of J imply that $\pi_j - \bar{\pi}^{J+} > 0$ for all strategies $j \in J$, verifying condition (8) for these strategies. Similarly, equations (M1′), (20), and (22) and the definition of J imply that $\eta_i^* = \bar{\pi}^{J+} - \pi_i \geq 0$, so (8) holds for strategy i. It follows from this, the definition of J, and equation (M3) that $\eta_k^* = 0$ whenever $k \notin J \cup \{i\}$, so for such strategies, equations (M2) and (22) imply that $\pi_k - \bar{\pi}^{J+} \leq 0$, and so condition (8) holds for k. Thus, J satisfies condition (8).

Part 2: Sufficiency. Here we show that if there is a set $J \subset S$ satisfying condition (8) from the statement of the present theorem, then the vector η^* defined in equation (7) satisfies optimality conditions (M1′)–(M4). Condition (M1′) is easily verified:

$$\sum_{k=1}^{n} \eta_k^* = \sum_{k \in J \cup \{i\}} (\bar{\pi}^{J+} - \pi_k) = (n^J + 1)\bar{\pi}^{J+} - \sum_{k \in J \cup \{i\}} \pi_k = 0.$$

If $j \in J$, then $\eta_j^* = \bar{\pi}^{J+} - \pi_j < 0$ and $\pi_i + \eta_i^* - (\pi_j + \eta_j^*) = \bar{\pi}^{J+} - \bar{\pi}^{J+} = 0$, so conditions (M2)–(M4) hold for this strategy. Similarly, if $k \notin J \cup \{i\}$, then $\eta_k^* = 0$ and $\pi_i + \eta_i^* - (\pi_k + \eta_k^*) = \bar{\pi}^{J+} - \pi_k \geq 0$, so conditions (M2)–(M4) hold for this strategy as well.

Part 3: Uniqueness. It follows easily from Part 2 of the proof that the set J described there is unique: Were there another set $\hat{J} \neq J$ satisfying condition (8), then the vector $\hat{\eta}^* \neq \eta$ defined in equation (7) would also satisfy optimality conditions (M1′)–(M4). But since these are necessary and sufficient conditions, this would imply that both $\hat{\eta}$ and η are optimal solutions to the rate minimization program, contradicting the fact that the minimizer of a strictly convex program is unique.

This completes the proof of the theorem. ■

Analysis of the 3 x 3 games under i.i.d. Laplace shocks.

Suppose that each component ε_i^l of the random vector ε^l follows the Laplace distribution. Applying Example 3.2 and Theorem 3.6, we compute the following values of the unlikelihood function:

$$\Upsilon_1(0,d,0) = \sqrt{4+d^2} - 2 - 2\log\left(\tfrac{1}{2}\left(\sqrt{1+\tfrac{d^2}{4}}+1\right)\right); \tag{24}$$

$$\begin{aligned}\Upsilon_1(0,d,d) &= -3 + 2\sqrt{1+(d-\eta_1^*)^2} + \sqrt{1+(\eta_1^*)^2} \\ &\quad -2\log\left(\tfrac{1}{2}\left(1+\sqrt{1+(d-\eta_1^*)^2}\right)\right) - \log\left(\tfrac{1}{2}\left(1+\sqrt{1+(\eta_1^*)^2}\right)\right), \text{ where} \\ \eta_1^* &= \tfrac{1}{2}d + 2\sqrt{1+\tfrac{1}{16}d^2 - \sqrt{1+\tfrac{1}{9}d^2}}.\end{aligned} \tag{25}$$

As in the demeaned exponential case, both (24) and (25) are dominated by a linear term with coefficient one as d grows large.

When $a = 8$ and $b = c = 5$, the costs for the three trees are $c(\tau_1^d) = 5.9681$, $c(\tau_1^i) = 4.7915$, and $c(\tau_2^d) = 4.3650$. When $a = 8$, $b = 7$, and $c = \frac{1}{10}$, the costs for the three trees are $c(\tau_1^d) = 4.3130$, $c(\tau_1^i) = 4.2712$, and $c(\tau_2^d) = 4.3650$. Thus, the selection results under Laplace shocks are the same as those resulting from exponential shocks.

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