Best Experienced Payoff Dynamics and Cooperation in the Centipede Game

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Introduction

Backward induction is a pillar of game theory.

But its epistemic foundations require very demanding assumptions.
(Binmore (1987), Reny (1992), Stalnaker(1996), Ben Porath (1997), Halpern (2001), Perea (2014))

And at least in some games, it has questionable descriptive or normative merit.

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Games of Perfect Information, Predatory Pricing and the Chain-Store Paradox

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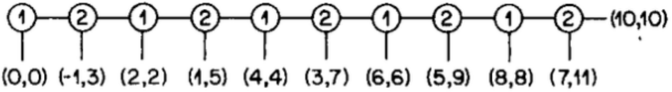


the Centipede game

Attempts to obtain more realistic predictions in Centipede and related games have been few and not entirely satisfying.

The best-known is Kreps et al. (1982), who alter the game by introducing incomplete information about opponents' preferences.

The prediction of cooperative play in this model requires

- augmenting the original game
- assuming this augmentation is common knowledge
- invoking the equilibrium knowledge assumption

We place the Centipede game in a large population setting.

We do not assume equilibrium play, or even that agents know the current population state.

Instead we assume that agents respond optimally to direct experience playing the game.

We show that if this experience is not exhaustive, cooperative behavior is sustained.

Best experienced payoff dynamics

We consider population game dynamics under which agents occasionally receive opportunities to switch strategies.

A revising agent:

- chooses a set of candidate strategies according to a test-set rule τ;
- plays each strategy against κ opponents drawn at random from the opposing population, with each play of each strategy being against a newly drawn opponent;
- switches to the strategy that achieved the highest total payoff, breaking ties according to a tie-breaking rule β.

We look at the differential equations obtained in the large population limit.

We call these best experienced payoff dynamics (or BEP(τ, κ, β) dynamics).

Key precursors: Osborne and Rubinstein (1998), Sethi (2000).

Best experienced payoff dynamics for the Centipede game

The BEP($\tau^{\text{all}}, 1, \beta^{\min}$) dynamic for Centipede:

$$\dot{x}_i = \left(\sum_{k=i}^{s^2} y_k\right)\left(\sum_{m=1}^{i} y_m\right)^{s^1-i} + \sum_{k=2}^{i-1} y_k \left(\sum_{\ell=1}^{k-1} y_\ell\right)^{i-k} \left(\sum_{m=1}^{k} y_m\right)^{s^1-i} - x_i,$$

$$\dot{y}_j = \begin{cases} \left(\sum_{k=2}^{s^1} x_k\right)(x_1 + x_2)^{s^2-1} + (x_1)^{s^2} - y_1 & j = 1, \\ \left(\sum_{k=j+1}^{s^1} x_k\right)\left(\sum_{m=1}^{j+1} x_m\right)^{s^2-j} + \sum_{k=2}^{j} x_k \left(\sum_{\ell=1}^{k-1} x_\ell\right)^{j-k+1} \left(\sum_{m=1}^{k} x_m\right)^{s^2-j} - y_j & j > 1. \end{cases}$$

This is a system of polynomial equations with rational coefficients.

The BEP($\tau^{\text{all}}, 1, \beta^{\text{min}}$) dynamic for Centipede:

$$\dot{x}_i = \left(\sum_{k=i}^{s^2} y_k\right)\left(\sum_{m=1}^{i} y_m\right)^{s^1-i} + \sum_{k=2}^{i-1} y_k \left(\sum_{\ell=1}^{k-1} y_\ell\right)^{i-k} \left(\sum_{m=1}^{k} y_m\right)^{s^1-i} - x_i$$

When the agent tests i, his opponent plays i or higher (so that the agent is the one who stops the game);

when the agent tests strategies above i, his opponents play i or lower.

$\Rightarrow$ Only strategy i yields the agent his highest payoff.

The BEP($\tau^{\text{all}}, 1, \beta^{\min}$) dynamic for Centipede:

$$\dot{x}_i = \left(\sum_{k=i}^{s^2} y_k\right)\left(\sum_{m=1}^{i} y_m\right)^{s^1-i} + \sum_{k=2}^{i-1} y_k \left(\sum_{\ell=1}^{k-1} y_\ell\right)^{i-k} \left(\sum_{m=1}^{k} y_m\right)^{s^1-i} - x_i$$

When the agent tests i, his opponent plays $k < i$, stopping the game;

when he tests strategies in $\{k, \ldots, i-1\}$,
his opponents play strategies less than k;

when he tests strategies above k,
his opponents play strategies k or lower.

$\Rightarrow$ Strategy i is the lowest one that achieves the optimal payoff.

Main results

We analyze the behavior of BEP dynamics in Centipede games.

Most results focus on cases where strategies are tested once ($\kappa = 1$), so that choices only depend on ordinal properties of payoffs.

1. The backward induction state $x^\dagger$ is a rest point under some tie-breaking rules (e.g., those that do not abandon optimal strategies).

   But this state is repelling: solutions from all nearby initial states move away from $x^\dagger$.

Basic intuition:

Near the backward induction state, the most cooperative agents would obtain higher expected payoffs by stopping earlier.

However, a revising agent tests each strategy in his test set against independently drawn opponents.

He may thus test a cooperative strategy against a cooperative opponent, and less cooperative strategies against less cooperative opponents.

In this case, his best experienced payoff will come from the cooperative strategy.

2. The dynamics have exactly one other rest point, x^*.

   The form of x^* depends on the specification of the dynamics (τ, κ, β), but is essentially independent of the length of the game.

   In all cases, x^* has virtually all players choosing to continue until the last few nodes of the game.

   Moreover, x^* is dynamically stable, attracting solutions from all initial conditions other than x^*.

   Local stability is proved analytically; global stability is verified by numerically evaluating Lyapunov functions.

The stable rest point of the BEP($\tau^{\text{all}}, 1, \beta^{\min}$) dynamic

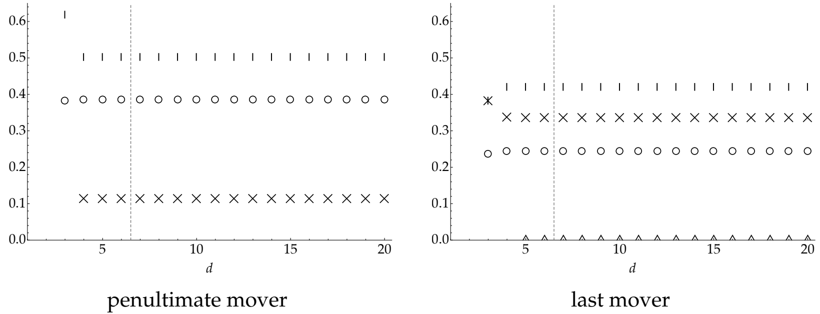


Figure: The stable rest point under the BEP($\tau^{\text{all}}, 1, \beta^{\min}$) dynamic for game lengths $d = 3, \ldots, 20$. Markers $\circ$, $|$, $\times$, and $\triangle$, represent weights on strategies [0], [1], [2], and [3]. Other weights are less than 10^{-8}. The dashed line separates exact ($d \leq 6$) and numerical ($d \geq 7$) results.

Each population's three most cooperative strategies predominate.

Weights are essentially independent of $d \geq 4$.

The stable rest point of the BEP($\tau^{\text{two}}, 1, \beta^{\text{min}}$) dynamic

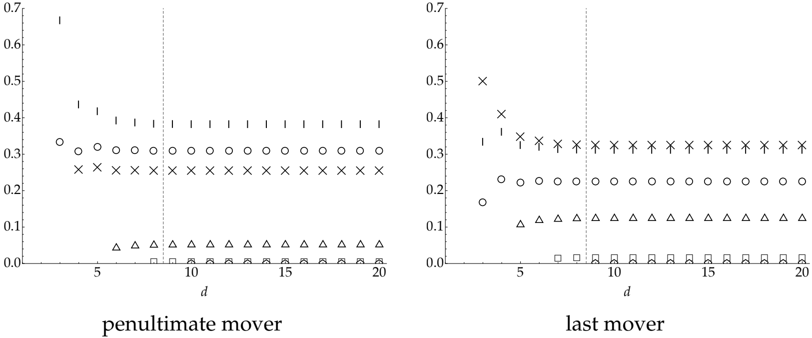


Figure: The stable rest point under the BEP($\tau^{\text{two}}, 1, \beta^{\text{min}}$) dynamic for game lengths $d = 3, \ldots, 20$. Markers ○, |, ×, △, □, and ⬠ represent weights on strategies [0], [1], [2], [3], [4], and [5]. Other weights are less than 10^{-4}. The dashed line separates exact ($d \leq 8$) and numerical ($d \geq 9$) results.

The populations' four and five most cooperative strategies predominate.

Weights are essentially independent of $d \geq 6$.

The stable rest point of the BEP($\tau^{\text{adj}}, 1, \beta^{\text{min}}$) dynamic

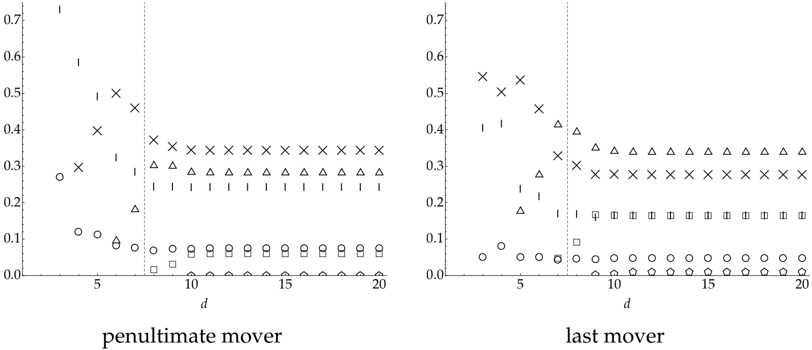


Figure: The stable rest point of Centipede under the BEP($\tau^{\text{adj}}, 1, \beta^{\text{min}}$) dynamic for game lengths $d = 3, \ldots, 20$. Markers ○, |, ×, △, □, and ⬠ represent weights on strategies [0], [1], [2], [3], [4], and [5]. Other weights are less than 10^{-5}. The dashed line separates exact ($d \leq 7$) and numerical ($d \geq 8$) results.

The population's five and six most cooperative strategies predominate.

Weights are essentially independent of $d \geq 10$.

Analytical methods: computational algebra et al.

Because they are based on sampling, BEP dynamics are represented by systems of polynomial differential equations with rational coefficients.

The zeroes (= rest points) of such a system can be determined by computing a Gröbner basis for the set of polynomials.

This new set of polynomials has the same zeroes as the original set. But its zeroes can be computed by finding the roots of a single (possibly high-degree) univariate polynomial.

Exact representations of these roots using algebraic numbers can be obtained using algorithms based on classical theorems.

From there, local stability (of x^*) can be assessed via linearization plus a variety of tricks (eigenvalue perturbation theorems, matrix condition number bounds. . .)

All computations are posted online.

	test-all	test-two	test-adjacent
min	6 (221)	8 (97)	7 (202)
stick	5 (65)	8 (128)	6 (47)
uniform	6 (168)	8 (128)	7 (230)

maximal lengths d of Centipede (degree of leading polynomial of Gröbner basis)

Robustness to including backward induction agents

What happens if we introduce heterogeneity, with some agents who always stop immediately?

$d = 4, \ \tau = \tau^{\text{all}}$

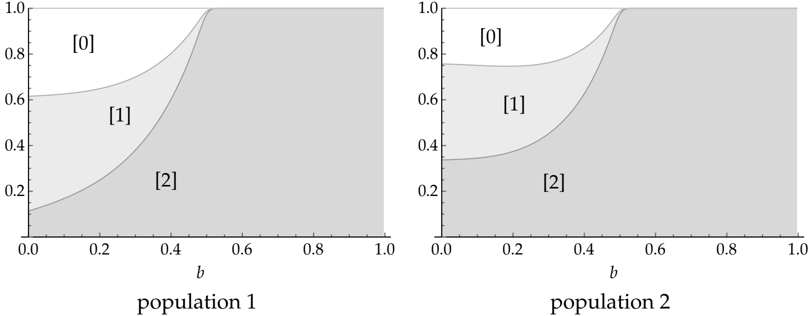


Figure: Behavior of BEP(τ^{all}, 1, β^{min}) agents at the stable rest point when proportion $b \in [0, 1]$ always stops immediately ($d = 4$).

$d = 20, \;\; \tau = \tau^{\text{all}}$

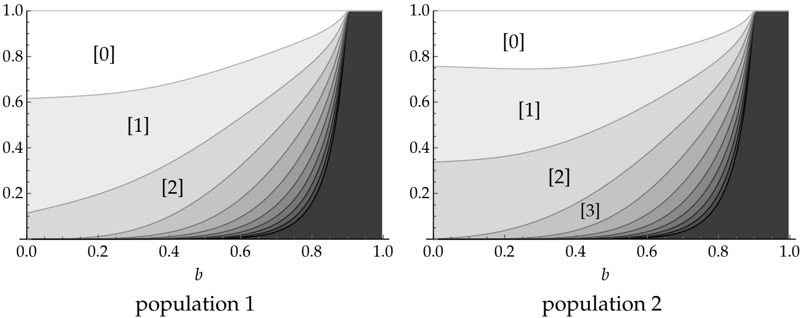


Figure: Behavior of BEP($\tau^{\text{all}}, 1, \beta^{\min}$) agents at the stable rest point when proportion $b \in [0, 1]$ always stops immediately ($d = 20$).

Larger numbers of trials

So far we have assumed that agents test the strategies in their test sets exactly once.

What happens if we increase the number of trials κ of each tested strategy?

The case $\kappa = \infty$ corresponds to the best response dynamic.

Results of Xu (2016) imply that every solution converges to the set of Nash equilibria $\Rightarrow$ all population 1 agents stop immediately.

What happens in intermediate cases?

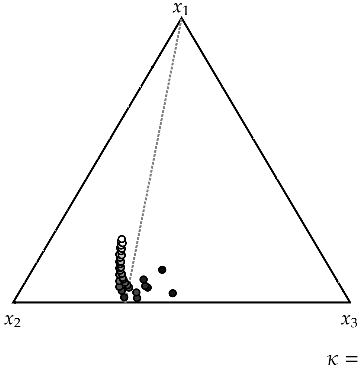


Figure: The stable rest point in Centipede of length $d = 4$ under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{min}}$) dynamics for $\kappa = 1, \ldots, 34$ trials of each tested strategy. Lighter shading corresponds to larger numbers of trials.

Intuition from central limit theorem.

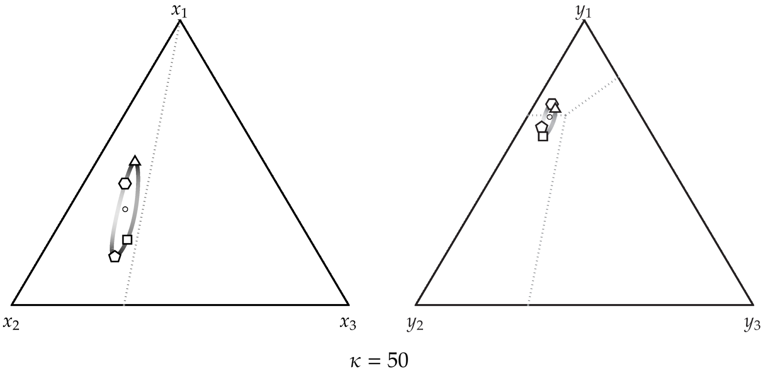


Figure: Stable cycle in Centipede of length $d = 4$ under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{min}}$) dynamics for $\kappa = 50$. Lighter shading represents faster motion.

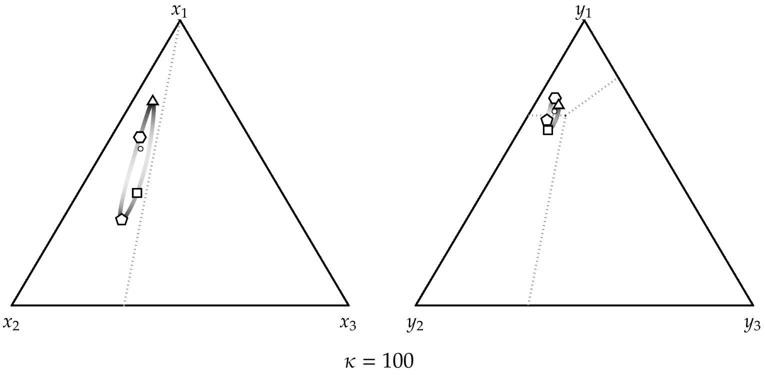


Figure: Stable cycle in Centipede of length $d = 4$ under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{min}}$) dynamics for $\kappa = 100$. Lighter shading represents faster motion.

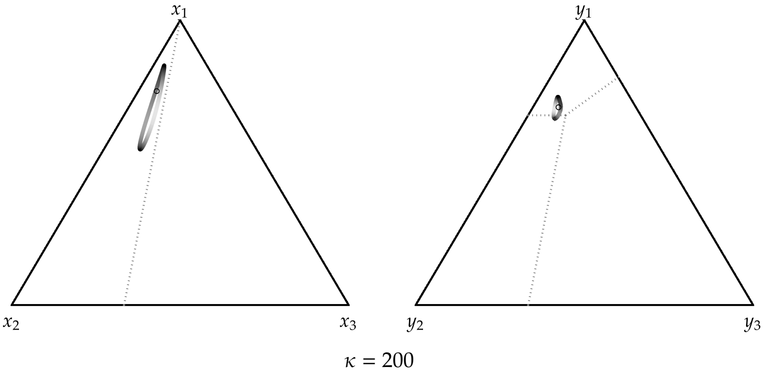
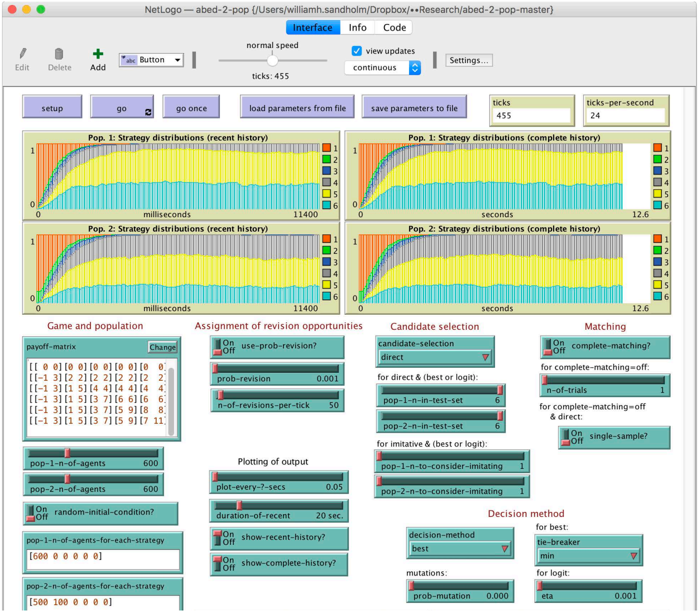


Figure: Stable cycle in Centipede of length $d = 4$ under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{min}}$) dynamics for $\kappa = 200$. Lighter shading represents faster motion.



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