Best Experienced Payoff Dynamics and Cooperation in the Centipede Game

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Introduction

Backward induction is a pillar of game theory.

But its epistemic foundations require very demanding assumptions.
(Binmore (1987), Reny (1992), Stalnaker(1996), Ben Porath (1997), Halpern (2001), Perea (2014))

And at least in some games, it has questionable descriptive or normative merit.

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Games of Perfect Information, Predatory Pricing and the Chain-Store Paradox

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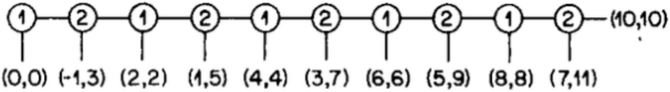


the Centipede game

Attempts to obtain more realistic predictions in Centipede and related games have been few and not entirely satisfying.

The best-known is Kreps et al. (1982), who alter the game by introducing incomplete information about opponents' preferences.

The prediction of cooperative play in this model requires

- augmenting the original game
- assuming this augmentation is common knowledge
- invoking the equilibrium knowledge assumption

Best experienced payoff dynamics

We consider population game dynamics under which agents occasionally receive opportunities to switch strategies.

A revising agent:

- chooses a set of candidate strategies according to a test-set rule τ;
- plays each strategy against κ opponents drawn at random from the opposing population, with each play of each strategy being against a newly drawn opponent;
- switches to the strategy that achieved the highest total payoff, breaking ties according to a tie-breaking rule β.

We look at the differential equations obtained in the large population limit.

We call these best experienced payoff dynamics (or BEP(τ, κ, β) dynamics).

Key precursors: Osborne and Rubinstein (1998), Sethi (2000).

Main results

We analyze the behavior of BEP dynamics in Centipede games.

Most results focus on cases where strategies are tested once ($\kappa = 1$), so that choices only depend on ordinal properties of payoffs.

1. The backward induction state $x^\dagger$ is a rest point under some tie-breaking rules (e.g., those that do not abandon optimal strategies).

   But this state is repelling: solutions from all nearby initial states move away from $x^\dagger$.

Basic intuition:

Near the backward induction state, the most cooperative agents would obtain higher expected payoffs by stopping earlier.

However, a revising agent tests each strategy in his test set against independently drawn opponents.

He may thus test a cooperative strategy against a cooperative opponent, and less cooperative strategies against less cooperative opponents.

In this case, his best experienced payoff will come from the cooperative strategy.

2. The dynamics have exactly one other rest point, x^*.

   The form of x^* depends on the specification of the dynamics (τ, κ, β), but is essentially independent of the length of the game.

   In all cases, x^* has virtually all players choosing to continue until the last few nodes of the game.

   Moreover, x^* is dynamically stable, attracting solutions from all initial conditions other than $x^\dagger$.

   Local stability is proved analytically; global stability is verified by numerically evaluating Lyapunov functions.

Robustness

The results above are robust to

- Replacing substantial fractions of each population with "backward induction agents".
- Allowing the number of trials κ of each tested strategy to be fairly large (say $\kappa \approx 25$ or 50 or 100).

Analytical methods: computational algebra et al.

Because they are based on sampling, BEP dynamics are represented by systems of polynomial differential equations with rational coefficients.

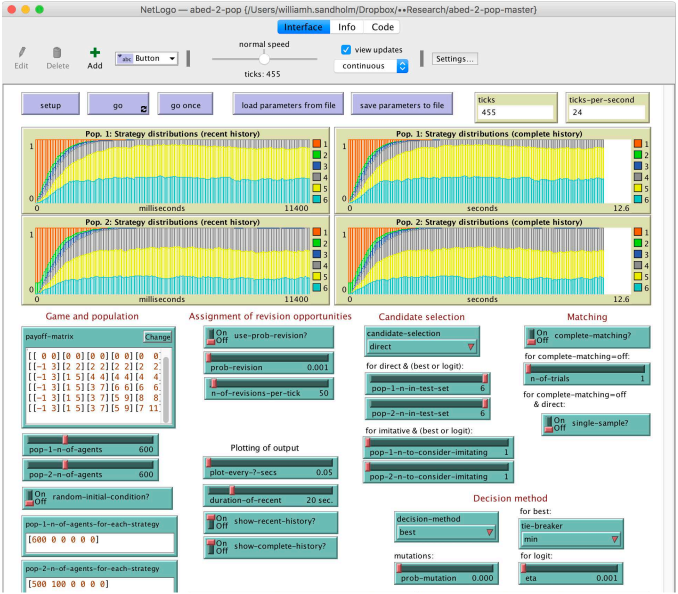
The zeroes (= rest points) of such a system can be determined by computing a Gröbner basis for the set of polynomials.

This new set of polynomials has the same zeroes as the original set. But its zeroes can be computed by finding the roots of a single (possibly high-degree) univariate polynomial.

Exact representations of these roots using algebraic numbers can be obtained using algorithms based on classical theorems.

From there, local stability (of x^*) can be assessed via linearization plus a variety of tricks (eigenvalue perturbation theorems, matrix condition number bounds. . .)

All computations are posted online.



Results discovered using simulations:

`www.ssc.wisc.edu/~whs/research`

The Model

Two-player (reduced) normal form games $G = \{(S^1, S^2), (U^1, U^2)\}$:

$S^p = \{1, \ldots, s^p\}$	strategy set
$U^p \in \mathbf{R}^{s^p \times s^q}$	payoff matrix
U^p_{ij}	payoff to $i \in S^p$ vs. $j \in S^q$

We define a two-population game via random matching in G:

Ξ^p	simplex in $\mathbf{R}^{s^p}$
$\xi^p \in \Xi^p$	population state (for population p)
$\xi^p_i \in [0, 1]$	mass of agents playing $i \in S^p$ in population p
$\Xi = \Xi^1 \times \Xi^2$	set of population states (for both populations)

Revision protocols and evolutionary dynamics

At all times $t \in [0, \infty)$, each agent has a strategy he uses when matched to play game G.

$\xi(t) = (\xi^1(t), \xi^2(t)) \in \Xi$ is the time t population state.

Agents occasionally receive opportunities to switch strategies according to independent rate 1 Poisson processes.

An agent who receives an opportunity considers switching to a new strategy, applying a *revision protocol* σ^p.

$\sigma^p_{ij}(U^p, \xi^q)$ is the probability that a revising agent playing strategy $i \in S^p$ switches to strategy $j \in S^p$.

In the deterministic limit:

$$\dot{\xi}^p_i = \sum_{j \in S^p} \xi^p_j \sigma^p_{ji}(U^p, \xi^q) - \xi^p_i \qquad i \in S^p, p \in \{1, 2\}.$$

Best experienced payoff protocols and dynamics

A *best experienced payoff protocol* is defined by a triple (τ, κ, β):

$\tau = (\tau_i^p)_{i\in S^p}^{p\in\{1,2\}}$ a test set rule

κ a number of trials

$\beta = (\beta_i^p)_{i\in S^p}^{p\in\{1,2\}}$ a tie-breaking rule

A revising agent playing strategy $i \in S^p$:

- draws a set of strategies $R^p \subseteq S^p$ according to distribution τ_i^p
- plays each strategy in R^p in κ random matches
  (and so has $\#R^p \times \kappa$ random matches in total)
- selects the strategy in R^p that earned him the highest total payoff, breaking ties according to rule β^p

Test set rules

Always test current strategy and at least one other strategy.

ex.: test-all

$$\tau_i^p(S^p) = 1.$$

ex.: test-two

$$\tau_i^p(\{i, j\}) = \frac{1}{s^p - 1} \text{ for all } j \in S^p \setminus \{i\}.$$

ex.: test-adjacent

$$\tau_i^p(\{i, i-1\}) = \tau_i^p(\{i, i+1\}) = \tfrac{1}{2} \text{ for } i \in S^p \setminus \{1, s^p\},$$
$$\tau_1^p(\{1, 2\}) = 1, \ \tau_{s^p}^p(\{s^p, s^p - 1\}) = 1.$$

notation: $\tau^{\text{all}}, \tau^{\text{two}}, \tau^{\text{adj}}$

Tie-breaking rules

Important in extensive form games

ex.: min-if-tie (a conservative rule in Centipede)

$$\beta^p_{ij}(\pi^p, R^p) = 1, \quad \text{where } j = \min\left[\operatorname*{argmax}_{k \in R^p} \pi^p_k\right].$$

ex.: stick-if-tie rules

$$\beta^p_{ii}(\pi^p, R^p) = 1 \;\; \text{whenever } i \in \operatorname*{argmax}_{k \in R^p} \pi^p_k.$$

ex.: uniform-if-tie

$$\beta^p_{ij}(\pi^p, R^p) = \begin{cases} \frac{1}{\#(\operatorname{argmax}_k \pi^p_k)} & \text{if } j \in \operatorname{argmax}_{k \in R^p} \pi^p_k, \\ 0 & \text{otherwise.} \end{cases}$$

notation: β^{min}, β^{stick} (for stick-min), β^{unif}

Best experienced payoff dynamic (BEP(τ, κ, β) dynamic for short)

$$\text{(B)} \qquad \dot{\xi}_i^p = \sum_{j \in S^p} \xi_j^p \left(\sum_{R^p \subseteq S^p} \tau_j^p(R^p) \left[\sum_r \left(\prod_{k \in R^p} \prod_{m=1}^{\kappa} \xi_{r_{km}}^q \right) \beta_{ji}^p(\pi^p(r), R^p) \right] \right) - \xi_i^p,$$

$$\text{where } \pi_k^p(r) = \sum_{m=1}^{\kappa} U_{k r_{km}}^p \text{ for all } k \in R^p,$$

Remarks:

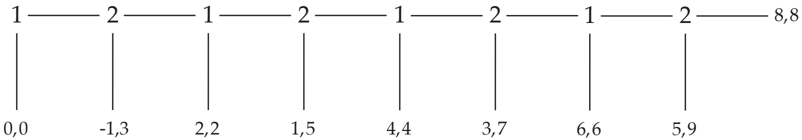
1. Rest points of the BEP($\tau^{\text{all}}, k, \beta^{\text{unif}}$) dynamic are the $S(k)$ equilibria of Osborne and Rubinstein (1998), and the dynamic itself is that of Sethi (2000).
2. Under any BEP(τ, κ, β) dynamic for which β is a stick-if-tie rule, all pure Nash equilibria are rest points.

Centipede

Change in notation:

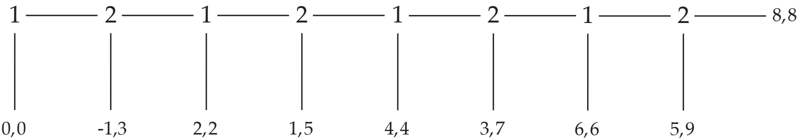
$\xi = (x, y)$	population states
$\Xi = X \times Y$	set of population states
(A, B)	payoff bimatrix

Centipede



d	length (number of decision nodes)
$d^1 = \lfloor \frac{d+1}{2} \rfloor, d^2 = \lfloor \frac{d}{2} \rfloor$	numbers of decision nodes for 1 and 2
$s^p = d^p + 1$	number of strategies for player p
$i \in \{1, \ldots d^p\}$	continue until ith decision node; then stop
$i = s^p$	continue at all decision nodes
$[k] \equiv s^p - k$	strategies numbered backward
$[0]$	continue at all decision nodes
$[k], \; k \geq 1$	stop at one's kth-to-last decision node

Centipede



Payoffs:

$$(A_{ij}, B_{ij}) = \begin{cases} (2i-2, 2i-2) & \text{if } i \leq j, \\ (2j-3, 2j+1) & \text{if } j < i. \end{cases}$$

A player is better off continuing at a given decision node if and only if his opponent will continue at the next decision node.

Only ordinal payoffs matter for the dynamics when $\kappa = 1$.

Best experienced payoff dynamics for the Centipede game

The BEP($\tau^{\text{all}}, 1, \beta^{\min}$) dynamic for Centipede:

$$\dot{x}_i = \left(\sum_{k=i}^{s^2} y_k\right)\left(\sum_{m=1}^{i} y_m\right)^{s^1-i} + \sum_{k=2}^{i-1} y_k \left(\sum_{\ell=1}^{k-1} y_\ell\right)^{i-k} \left(\sum_{m=1}^{k} y_m\right)^{s^1-i} - x_i,$$

$$\dot{y}_j = \begin{cases} \left(\sum_{k=2}^{s^1} x_k\right)(x_1 + x_2)^{s^2-1} + (x_1)^{s^2} - y_1 & j = 1, \\ \left(\sum_{k=j+1}^{s^1} x_k\right)\left(\sum_{m=1}^{j+1} x_m\right)^{s^2-j} + \sum_{k=2}^{j} x_k \left(\sum_{\ell=1}^{k-1} x_\ell\right)^{j-k+1} \left(\sum_{m=1}^{k} x_m\right)^{s^2-j} - y_j & j > 1. \end{cases}$$

This is a system of polynomial equations with rational coefficients.

The BEP($\tau^{\text{all}}, 1, \beta^{\min}$) dynamic for Centipede:

$$\dot{x}_i = \left(\sum_{k=i}^{s^2} y_k\right)\left(\sum_{m=1}^{i} y_m\right)^{s^1-i} + \sum_{k=2}^{i-1} y_k \left(\sum_{\ell=1}^{k-1} y_\ell\right)^{i-k} \left(\sum_{m=1}^{k} y_m\right)^{s^1-i} - x_i$$

When the agent tests i, his opponent plays i or higher (so that the agent is the one who stops the game);

when the agent tests strategies above i, his opponents play i or lower.

$\Rightarrow$ Only strategy i yields the agent his highest payoff.

The BEP($\tau^{\text{all}}, 1, \beta^{\min}$) dynamic for Centipede:

$$\dot{x}_i = \left(\sum_{k=i}^{s^2} y_k\right)\left(\sum_{m=1}^{i} y_m\right)^{s^1-i} + \sum_{k=2}^{i-1} y_k \left(\sum_{\ell=1}^{k-1} y_\ell\right)^{i-k} \left(\sum_{m=1}^{k} y_m\right)^{s^1-i} - x_i$$

When the agent tests i, his opponent plays $k < i$, stopping the game;

when he tests strategies in $\{k, \ldots, i-1\}$,
his opponents play strategies less than k;

when he tests strategies above k,
his opponents play strategies k or lower.

$\Rightarrow$ Strategy i is the lowest one that achieves the optimal payoff.

Let $\xi^\dagger \in \Xi$ denote the backward induction state
(at which all agents in both populations stop immediately).

Observation: *Under any BEP(τ, κ, β) dynamic for which β is a stick-if-tie rule or the min-if-tie rule, the backward induction state $\xi^\dagger$ is a rest point.*

Results for test-all, min-if-tie, $\kappa = 1$

Proposition: *In Centipede games of lengths $d \geq 3$, the backward induction state $\xi^\dagger$ is repelling under the BEP$(\tau^{\text{all}}, 1, \beta^{\min})$ dynamic.*

Proposition: *In Centipede games of lengths $d \geq 3$, the backward induction state $\xi^{\dagger}$ is repelling under the $BEP(\tau^{\text{all}}, 1, \beta^{\text{min}})$ dynamic.*

Proof by a nonstandard linearization argument:

Find general expression for derivative matrix $DV(\xi^{\dagger})$.

Derive general formulas for eigenvectors of $DV(\xi^{\dagger})$ in subspace $T\Xi$ and their corresponding eigenvalues.

$d - 1$ of the eigenvalues are negative and one is positive, so $\xi^{\dagger}$ is unstable.

To prove that $\xi^{\dagger}$ is repelling: Show that the hyperplane through $\xi^{\dagger}$ defined by the span of the eigenvectors with negative eigenvalues supports the convex state space Ξ at state $\xi^{\dagger}$.

Then apply the Hartman-Grobman and stable manifold theorems.

Example: the BEP($\tau^{\text{all}}, 1, \beta^{\min}$) dynamic in Centipede of length $d = 4$.

$$\begin{aligned} \dot{x}_1 &= (y_1)^2 - x_1, & \dot{y}_1 &= (x_2 + x_3)(x_1 + x_2)^2 + (x_1)^3 - y_1, \\ \dot{x}_2 &= (y_2 + y_3)(y_1 + y_2) - x_2, & \dot{y}_2 &= x_3 + x_2 x_1 (x_1 + x_2) - y_2, \\ \dot{x}_3 &= y_3 + y_2 y_1 - x_3, & \dot{y}_3 &= x_2 (x_1)^2 + x_3 (x_1 + x_2) - y_3. \end{aligned}$$

Linearization at $(x^\dagger, y^\dagger)$ has the positive eigenvalue 1 corresponding to eigenvector $(z^1, z^2) = ((-2, 1, 1), (-2, 1, 1))$.

$\therefore$ At $(x, y) = ((1 - 2\varepsilon, \varepsilon, \varepsilon), (1 - 2\varepsilon, \varepsilon, \varepsilon))$,
we have $(\dot{x}, \dot{y}) \approx ((-2\varepsilon, \varepsilon, \varepsilon), (-2\varepsilon, \varepsilon, \varepsilon))$.

Example: the BEP$(\tau^{\text{all}}, 1, \beta^{\min})$ dynamic in Centipede of length $d = 4$.

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\dot{x}_1 &= (y_1)^2 - x_1, & \dot{y}_1 &= (x_2 + x_3)(x_1 + x_2)^2 + (x_1)^3 - y_1, \\
\dot{x}_2 &= (y_2 + y_3)(y_1 + y_2) - x_2, & \dot{y}_2 &= x_3 + x_2 x_1 (x_1 + x_2) - y_2, \\
\dot{x}_3 &= y_3 + y_2 y_1 - x_3, & \dot{y}_3 &= x_2 (x_1)^2 + x_3 (x_1 + x_2) - y_3.
\end{aligned}$$

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Example: the BEP($\tau^{\text{all}}, 1, \beta^{\text{min}}$) dynamic in Centipede of length $d = 4$.

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\dot{x}_1 &= (y_1)^2 - x_1, & \dot{y}_1 &= (x_2 + x_3)(x_1 + x_2)^2 + (x_1)^3 - y_1,\\
\dot{x}_2 &= (y_2 + y_3)(y_1 + y_2) - x_2, & \dot{y}_2 &= x_3 + x_2 x_1 (x_1 + x_2) - y_2,\\
\dot{x}_3 &= y_3 + y_2 y_1 - x_3, & \dot{y}_3 &= x_2(x_1)^2 + x_3(x_1 + x_2) - y_3.
\end{aligned}$$

At $(x, y) = ((1 - 2\varepsilon, \varepsilon, \varepsilon), (1 - 2\varepsilon, \varepsilon, \varepsilon))$, a revising population 2 agent switches to strategy 3 (always continue) if:

i. when testing strategy 3 she meets an opponent playing 2, and when testing strategies 1 and 2 she meets opponents playing 1, or

ii. when testing strategy 3 she meets an opponent playing 3, and when testing strategy 2 she meets an opponent playing 1 or 2.

These events have total probability $\varepsilon(1 - 2\varepsilon)^2 + \varepsilon(1 - \varepsilon) \approx 2\varepsilon$.

Agents switch away from strategy 3 at rate $y_3 = \varepsilon$.

Combining the inflow and outflow terms shows that $\dot{y}_3 \approx 2\varepsilon - \varepsilon = \varepsilon$.

State $\xi^\dagger$ is unstable. What other rest points are there?

Proposition: *For Centipede games of lengths $3 \leq d \leq 6$, the $BEP(\tau^{\text{all}}, 1, \beta^{\min})$ dynamic has exactly two rest points, $\xi^\dagger$ and $\xi^* \in \text{int}(\Xi)$. The rest point ξ^*, whose exact components are known, is asymptotically stable.*

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Set of rest points determined using Gröbner bases.

Limitation on length of the game due to computational demands:

	test-all	test-two	test-adjacent
min	6 (221)	8 (97)	7 (202)
stick	5 (65)	8 (128)	6 (47)
uniform	6 (168)	8 (128)	7 (230)

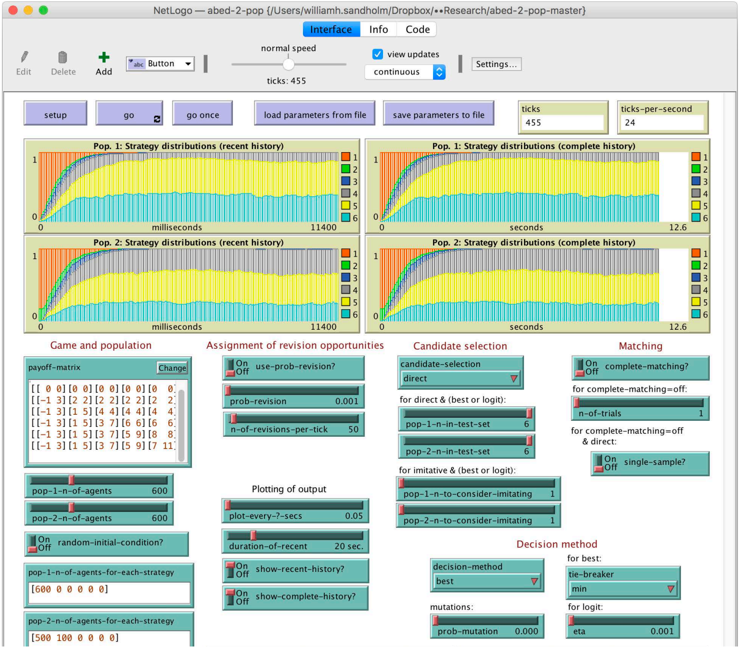
maximal lengths d of Centipede (degree of leading polynomial of Gröbner basis)

Local stability proved using linearization plus a variety of tricks.

Proposition: *For Centipede games of lengths $3 \leq d \leq 6$, the $BEP(\tau^{\text{all}}, 1, \beta^{\text{min}})$ dynamic has exactly two rest points, $\xi^\dagger$ and $\xi^* \in \text{int}(\Xi)$. The rest point ξ^*, whose exact components are known, is asymptotically stable.*

	population p				population q			
	[3]	[2]	[1]	[0]	[3]	[2]	[1]	[0]
$d = 3$			.6180	.3820		.3820	.3820	.2361
$d = 4$		.1136	.5017	.3847		.3371	.4197	.2432
$d = 5$		.1135	.5018	.3847	.0015	.3357	.4197	.2432
$d = 6$	$\approx 10^{-9}$	.1135	.5018	.3847	.0015	.3357	.4197	.2432

"Exact" interior rest points ξ^* of the BEP($\tau^{\text{all}}, 1, \beta^{\text{min}}$) dynamic. p denotes the owner of the penultimate decision node, q the owner of the last decision node.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Interface
Info
Code
Edit
Delete
Add
Button
normal speed
ticks: 455
view updates
continuous
Settings...
setup
go
go once
load parameters from file
save parameters to file
ticks
455
ticks-per-second
24
Pop. 1: Strategy distributions (recent history)
milliseconds
11400
Pop. 1: Strategy distributions (complete history)
seconds
12.6
Pop. 2: Strategy distributions (recent history)
milliseconds
11400
Pop. 2: Strategy distributions (complete history)
seconds
12.6
Game and population
payoff-matrix
Change
[[0 0] [0 0] [0 0] [0 0] [0 0]
[[-1 3] [2 2] [2 2] [2 2] [2 2]
[[-1 3] [1 5] [4 4] [4 4] [4 4]
[[-1 3] [1 5] [3 7] [6 6] [6 6]
[[-1 3] [1 5] [3 7] [5 9] [8 8]
[[-1 3] [1 5] [3 7] [5 9] [7 11]
pop-1-n-of-agents 600
pop-2-n-of-agents 600
On Off random-initial-condition?
pop-1-n-of-agents-for-each-strategy
[600 0 0 0 0 0]
pop-2-n-of-agents-for-each-strategy
[500 100 0 0 0 0]
Assignment of revision opportunities
On Off use-prob-revision?
prob-revision 0.001
n-of-revisions-per-tick 50
Plotting of output
plot-every-?-secs 0.05
duration-of-recent 20 sec.
On Off show-recent-history?
On Off show-complete-history?
Candidate selection
candidate-selection
direct
for direct & (best or logit):
pop-1-n-in-test-set 6
pop-2-n-in-test-set 6
for imitative & (best or logit):
pop-1-n-to-consider-imitating 1
pop-2-n-to-consider-imitating 1
Decision method
decision-method
best
mutations:
prob-mutation 0.000
for best:
tie-breaker
min
for logit:
eta 0.001
Matching
On Off complete-matching?
for complete-matching=off:
n-of-trials 1
for complete-matching=off & direct:
On Off single-sample?

We compute rest points for longer games numerically.

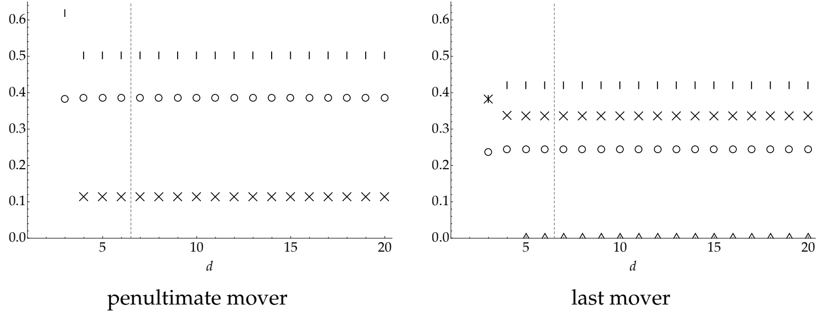


Figure: The stable rest point under the BEP($\tau^{\text{all}}, 1, \beta^{\text{min}}$) dynamic for game lengths $d = 3, \ldots, 20$. Markers $\circ$, $|$, $\times$, and $\triangle$, represent weights on strategies [0], [1], [2], and [3]. Other weights are less than 10^{-8}. The dashed line separates exact ($d \leq 6$) and numerical ($d \geq 7$) results.

Each population's three most cooperative strategies predominate.

Weights are essentially independent of $d \geq 4$.

Why does each population concentrate on three strategies?

Easy to explain by examining the law of motion:
when most agents behave cooperatively, choice probabilities are almost entirely determined by a small number of terms.

$$
\begin{aligned}
\Pr(j = [0]) &\approx x_{[0]}(x_{[d^1]} + \cdots + x_{[1]}) + x_{[1]}(x_{[d^1]} + \cdots + x_{[2]})^2 \\
\Pr(j = [1]) &\approx x_{[0]} + x_{[1]}(x_{[d^1]} + \cdots + x_{[2]})(x_{[d^1]} + \cdots + x_{[1]}) \\
\Pr(j = [2]) &\approx (x_{[1]} + x_{[0]})(x_{[d^1]} + \cdots + x_{[1]})^2 \\
\Pr(j = [3]) &\approx (x_{[2]} + x_{[1]} + x_{[0]})(x_{[d^1]} + \cdots + x_{[2]})^3
\end{aligned}
$$

Almost global convergence

We cannot prove analytically that x^* is almost globally stable
(i.e., that it attracts all solutions besides the one from $x^\dagger$.)

To provide strong numerical evidence of global stability, we consider the candidate Lyapunov function

$$L(x, y) = \sum_{i=2}^{s_1} (x_i - x_i^*)^2 + \sum_{j=2}^{s_2} (y_j - y_j^*)^2$$

by numerically evaluating its time derivative $\dot{L}$ at 10^9 points in Ξ.

For game lengths $d \leq 20$, $\dot{L}(x)$ always evaluates to a negative number.

Other specifications of the dynamics

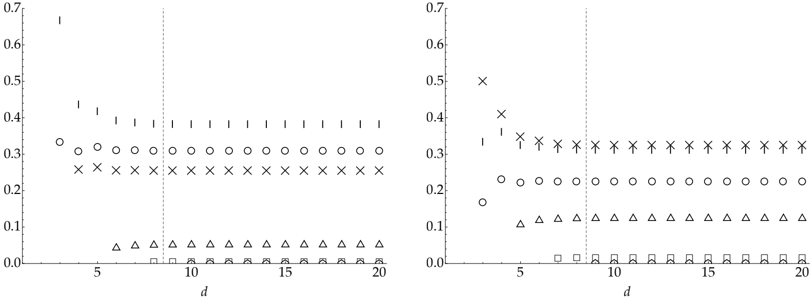
Essentially the same results hold for all BEP($\tau, 1, \beta$) dynamics
with $\tau \in \{\tau^{\text{all}}, \tau^{\text{two}}, \tau^{\text{adj}}\}$ and $\beta \in \{\beta^{\min}, \beta^{\text{stick}}, \beta^{\text{unif}}\}$.

Novelty: the backward induction state $x^\dagger$ is not a rest point under β^{unif}.

At the stable rest point x^*, both populations are concentrated
on the most cooperative strategies.

Weights in x^* depend on τ, and to a lesser extent on β.

The stable rest point of the BEP($\tau^{\text{two}}, 1, \beta^{\min}$) dynamic



penultimate mover — last mover

Figure: The stable rest point under the BEP($\tau^{\text{two}}, 1, \beta^{\min}$) dynamic for game lengths $d = 3, \ldots, 20$. Markers ○, |, ×, △, □, and ⬠ represent weights on strategies [0], [1], [2], [3], [4], and [5]. Other weights are less than 10^{-4}. The dashed line separates exact ($d \leq 8$) and numerical ($d \geq 9$) results.

The populations' four and five most cooperative strategies predominate.

Weights are essentially independent of $d \geq 6$.

The stable rest point of the BEP($\tau^{\text{adj}}, 1, \beta^{\min}$) dynamic

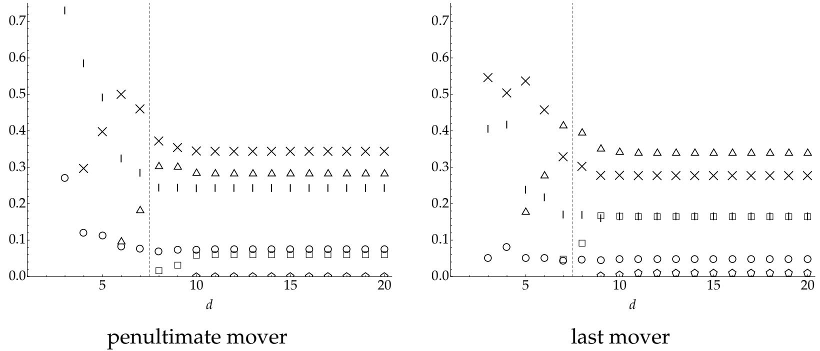


Figure: The stable rest point of Centipede under the BEP($\tau^{\text{adj}}, 1, \beta^{\min}$) dynamic for game lengths $d = 3, \ldots, 20$. Markers ○, |, ×, △, □, and ⬠ represent weights on strategies [0], [1], [2], [3], [4], and [5]. Other weights are less than 10^{-5}. The dashed line separates exact ($d \leq 7$) and numerical ($d \geq 8$) results.

The population's five and six most cooperative strategies predominate.

Weights are essentially independent of $d \geq 10$.

Robustness to including backward induction agents

So far we have assumed that the population is homogeneous.

What happens if we introduce heterogeneity, with some agents who always stop immediately?

$d = 4, \ \tau = \tau^{\text{all}}$

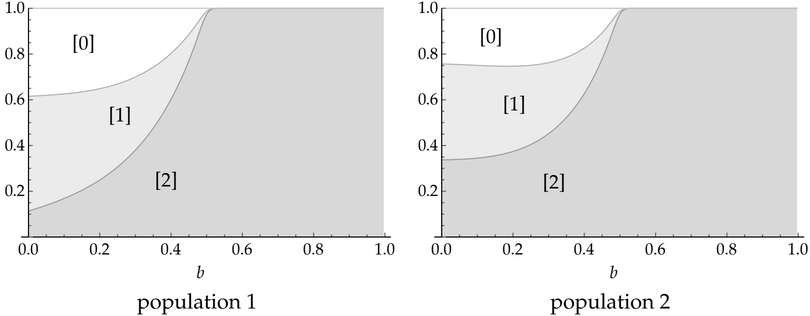


Figure: Behavior of BEP($\tau^{\text{all}}, 1, \beta^{\min}$) agents at the stable rest point when proportion $b \in [0, 1]$ always stops immediately ($d = 4$).

$d = 20,\ \tau = \tau^{\text{all}}$

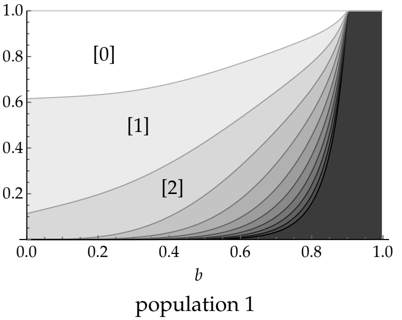


Figure: Behavior of BEP($\tau^{\text{all}}, 1, \beta^{\text{min}}$) agents at the stable rest point when proportion $b \in [0, 1]$ always stops immediately ($d = 20$).

$d = 20, \ \tau = \tau^{\text{two}}$

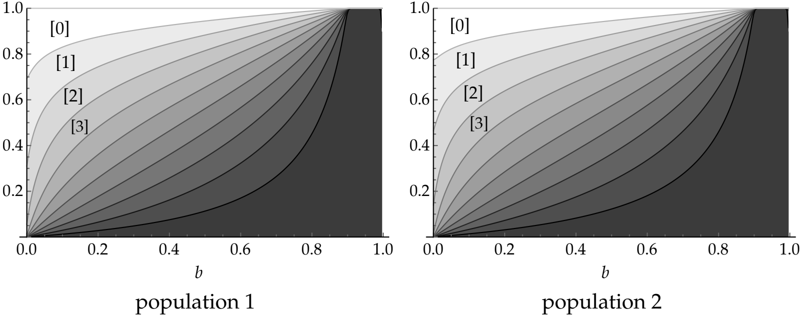


Figure: Behavior of BEP($\tau^{\text{two}}, 1, \beta^{\text{min}}$) agents at the stable rest point when proportion $b \in [0, 1]$ always stops immediately ($d = 20$).

Larger numbers of trials

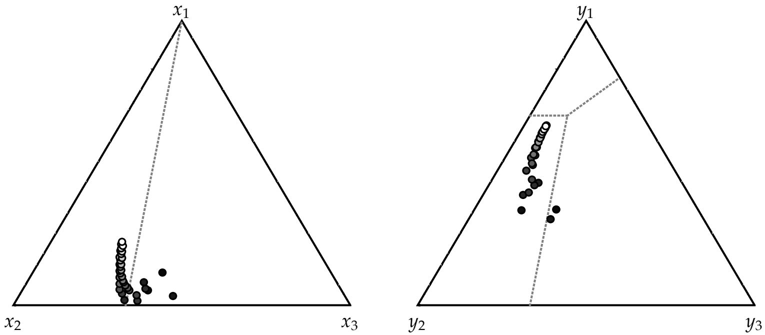
So far we have assumed that agents test the strategies in their test sets exactly once.

What happens if we increase the number of trials κ of each tested strategy?

The case $\kappa = \infty$ corresponds to the best response dynamic.

Results of Xu (2016) imply that every solution converges to the set of Nash equilibria $\Rightarrow$ all population 1 agents stop immediately.

What happens in intermediate cases?



$\kappa = 1, \ldots, 34$

Figure: The stable rest point in Centipede of length $d = 4$ under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{min}}$) dynamics for $\kappa = 1, \ldots, 34$ trials of each tested strategy. Lighter shading corresponds to larger numbers of trials.

Intuition from central limit theorem.

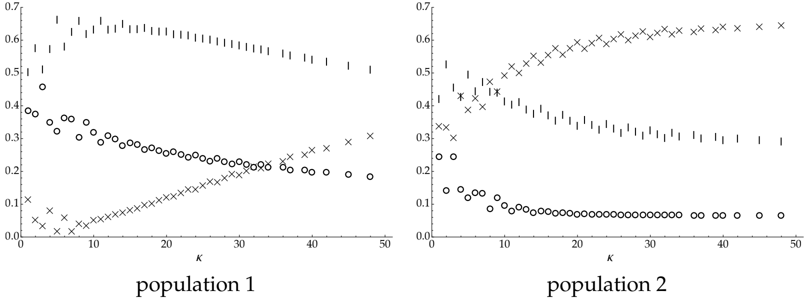


Figure: The stable rest point of the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{min}}$) dynamic in the Centipede game of length $d = 4$, $\kappa = 1, \ldots, 50$. Markers ○, |, and ×, represent weights on strategies 3 (= [0]), 2 (= [1]), and 1 (= [2]).

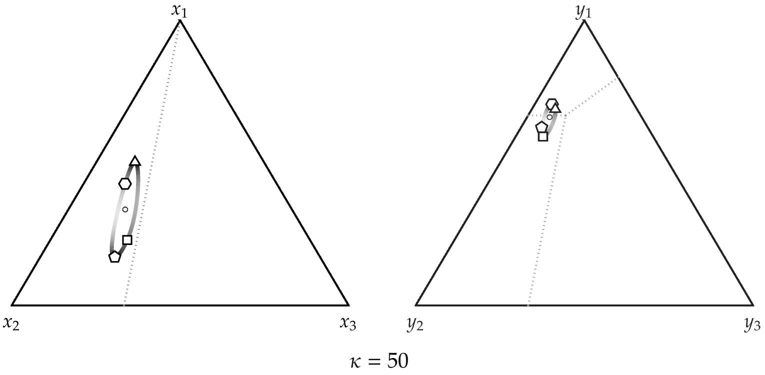
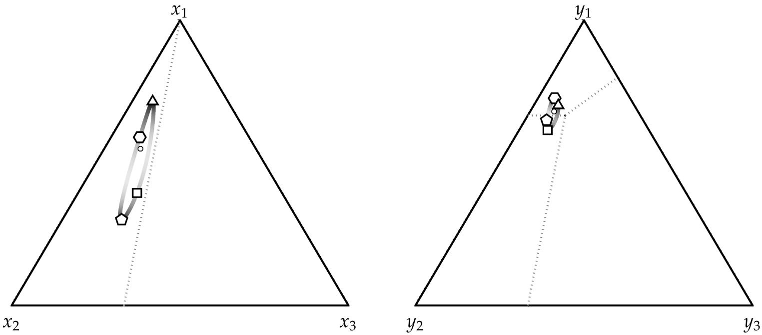


Figure: Stable cycle in Centipede of length $d = 4$ under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{min}}$) dynamics for $\kappa = 50$. Lighter shading represents faster motion.



$\kappa = 100$

Figure: Stable cycle in Centipede of length $d = 4$ under BEP$(\tau^{\text{all}}, \kappa, \beta^{\text{min}})$ dynamics for $\kappa = 100$. Lighter shading represents faster motion.

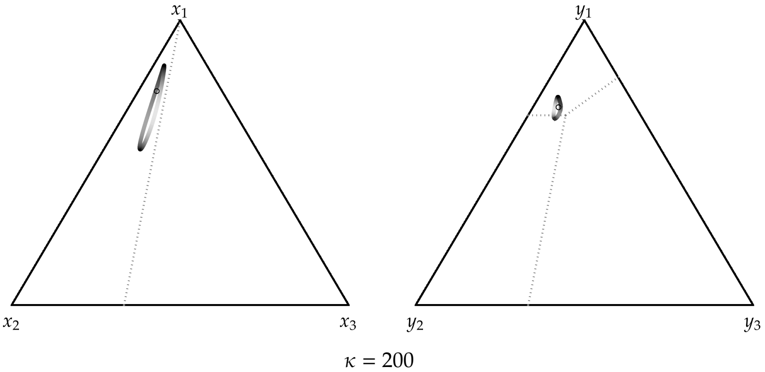


Figure: Stable cycle in Centipede of length $d = 4$ under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{min}}$) dynamics for $\kappa = 200$. Lighter shading represents faster motion.

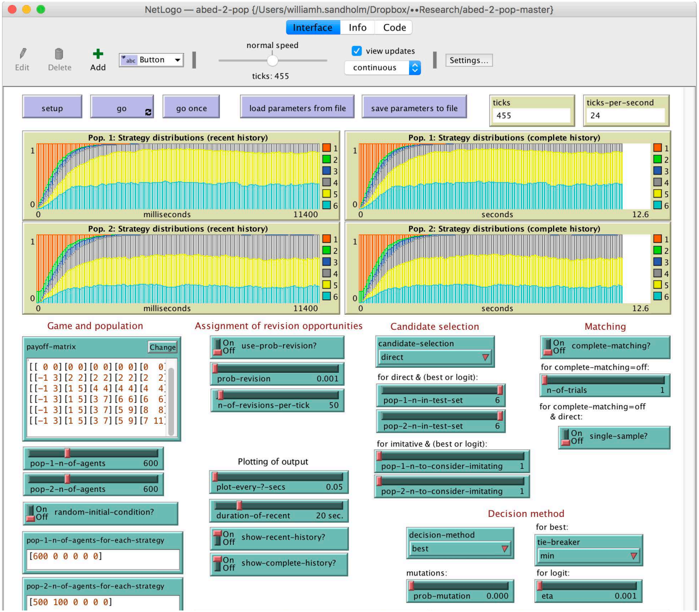
Conclusion

We study best experienced payoff dynamics in Centipede games.

Cooperative behavior is stable and robust.

Future work to include analyses of more general extensive form games.

Methods from computational algebra etc. should be useful
for studying other game dynamics.



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