Restricted 3-body Problem - Ethan Vogelsang

Introduction

The n-body simulation problem is a well known problem in both the physics and high performance computing fields. In this problem, many physical bodies are given some set of initial conditions and acted upon based off an agreed set of rules. In this implementation of the n-body problem, 3 masses are created and acted upon by the law of gravity, and the future state of each mass is determined by its equations of motion.

Motivation

The n-body problem with 3 or more masses has no general closed form solution. As such, in order to simulate and compute these systems, numerical methods must be used [1]. This project explores a highly constrained version of the n-body problem, known as the restricted 3-body problem. In the restricted 3-body problem, 3 masses m_1, m_2, and m_3 are present, but $m_1 << m_2$ and $m_1 << m_3$. This version has many real world corollaries in astronomy, such as a satellite orbiting between two massive bodies like the Earth and Moon.

One advantage of the restricted 3-body problem is that the satellite mass is much smaller than the two other bodies in the system, and therefore it exerts a nearly negligible force on them. As such, the problem can be treated similarly to a two body problem (with a general closed form solution) where the two bodies do interact with each other, and the third body orbits around their center of mass. However, this project seeks to remove those assumptions and explore simulations where all three bodies interact with each other to see the resulting dynamics and if they behave accordingly to the removed assumptions.

Implementation

Classical mechanics state that the force of gravity is proportional to the product of two bodies' masses over the square of the distance between them. The exact equation is shown:

$$F_g = G\frac{m_1 m_2}{r^2}$$

Using numerical integration, the acceleration due to this force can be calculated, and the subsequent equations of motion can be expressed and their results calculated. This n-body program uses the 4th order Runge-Kutta (RK4) method of numerical integration [2]. Given an initial value problem established by the following:

$$\ddot{x} = f(t, x)$$
$$x(t_0) = x_0$$

The next state of x can be calculated using

$$x_{n+1} = x_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)$$

where h is the time step, and k_1, k_2, k_3, and k_4 are defined as follows:

$$k_1 = f(t_n, x_n)$$
$$k_2 = f(t_n + \frac{h}{2}, x_n + h\frac{k_1}{2})$$
$$k_3 = f(t_n + \frac{h}{2}, x_n + h\frac{k_2}{2})$$
$$k_4 = f(t_n + h, x_n + hk_3)$$

This RK4 computation is performed for each time step of the simulation, and in conjunction with the law of gravity, the next state of the body can be derived from the computed acceleration. Velocity is calculated according to $\dot{x} = \ddot{x}t$ and the position from $x = \dot{x}t$.

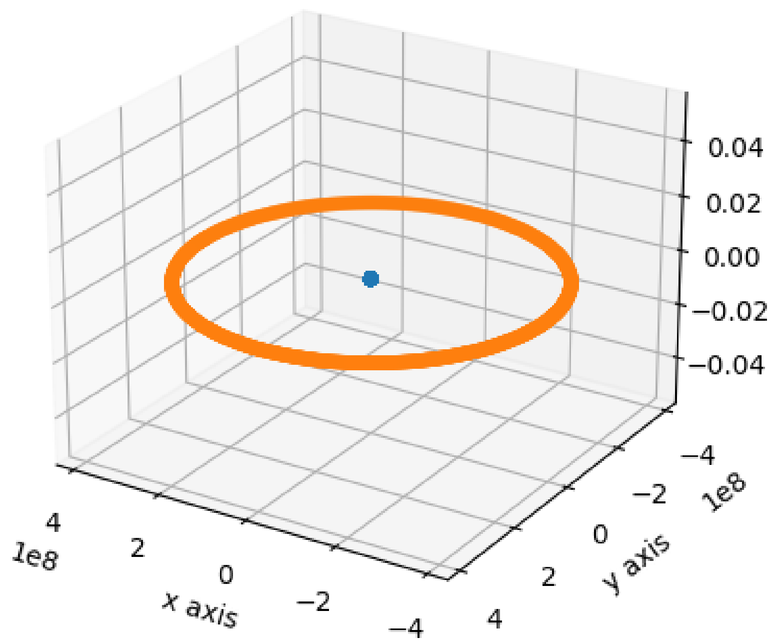
These RK4 computations are handled in the `integration.cpp` file as part of the `RK4` class. For each step of RK4, an approximation of the next state is made, and then the results are combined to create a more accurate final approximation. The code snippet below comes from `RK4::single_body_accel` and shows how one step of the acceleration of is calculated.

```
double grav = G * other->mass / (r * r * r);

k1 = (other->position - target_body.position) * grav;

vel_update = partial_step(target_body.velocity, k1, 0.5);
pos_update = partial_step(target_body.position, vel_update, 0.5);
k2 = (other->position - pos_update) * grav;

vel_update = partial_step(target_body.velocity, k2, 0.5);
pos_update = partial_step(target_body.position, vel_update, 0.5);
k3 = (other->position - pos_update) * grav;

vel_update = partial_step(target_body.velocity, k3, 1);
pos_update = partial_step(target_body.position, vel_update, 1);
k4 = (other->position - pos_update) * grav;

accel += (k1 + k2 * 2 + k3 * 2 + k4) / 6;
```

The above computations are performed on each body for every time step of the simulation. Afterwards, the results are then propagated and used to update the velocity and the position of each body in `RK4::update_velocity` and `RK4::update_position` respectively..

Demonstration

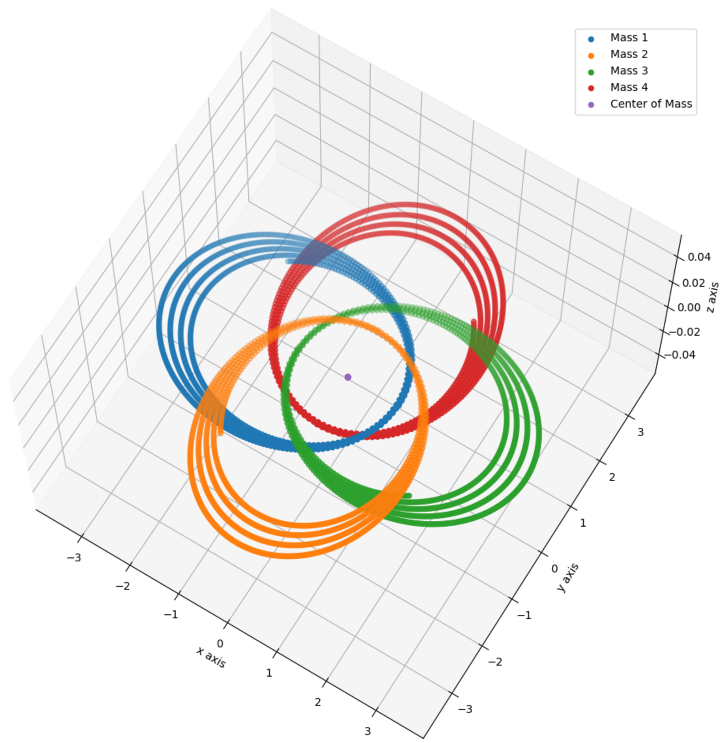
To first demonstrate that the laws governing the simulation are implemented correctly, a confirmation can be performed using systems that have known solutions. An example of one such system would be the Earth-Moon system. Because this problem only involves two bodies, not only can the expected results be computed analytically, the resulting dynamics should closely model that of the real world. Using the Python library Matplotlib, we can look at the simulated dynamics of this system.

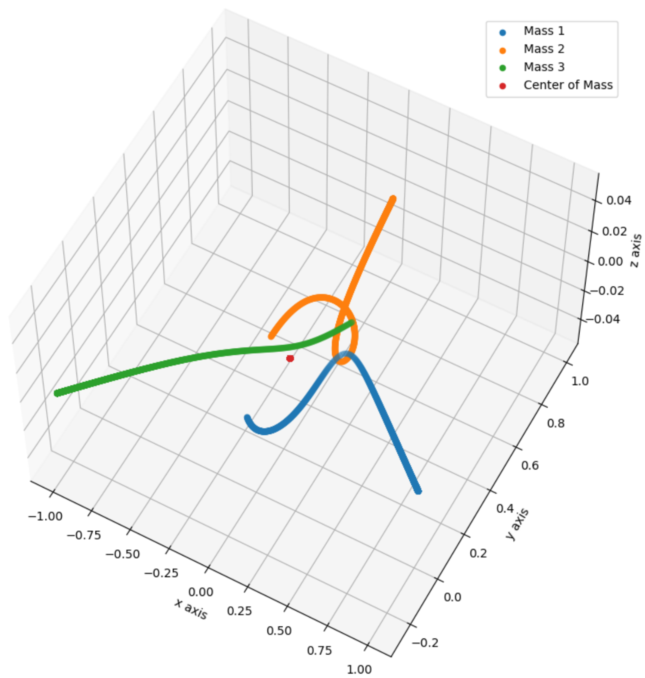


As expected the moon orbits in a circular path around the Earth. While this acts as a demonstration of the physical stability of the simulation, there is still room for numerical instability. Numerical instability would result in bodies drifting apart or experiencing other abnormal behavior even when the physics state that the system should be stable.

Unfortunately it is harder to definitively demonstrate that no numerical instability exists, and in reality there will likely always be some numerical instability due to how computers store floating point numbers. However, one scenario that would absolutely lead to instability is when there are two masses passing very close to each other. In this program all of the bodies exist at a single point, and no collisions are simulated. Because of this, two bodies could pass through the same space or have a distance between them that is nearly zero. This would result in very large forces between them which would then be propagated to the next time step in the simulation. In a perfect world this would not cause issues, but due to the nature of the simulation, and the fact that time passes in a discrete manor, some issues can arise that seemingly violate physical and mathematical laws if not careful.

However in the experimental setup that is primarily discussed in this project, these such scenarios are not a factor, and for the most part the numerical stability of the program can be trusted. This can be confirmed by plotting the center of mass of a stable system (where the total momentum is zero). Two examples of this are shown below. In the first example, four bodies of equal mass are placed equidistant from the origin. Each body is given the same initial speed in the direction that would make it move clockwise with respect to the origin (when looking down from +z). The second figure shows three bodies of equal mass placed to form a scalene triangle. The initial velocities of each mass is zero in this configuration. Both of these scenarios are examples of stable ones in which the masses are expected to orbit a stationary center of mass, and this is shown to be the case.


Mass 1
Mass 2
Mass 3
Mass 4
Center of Mass
x axis
y axis
z axis
−3
−2
−1
0
1
2
3
0.04
0.02
0.00
−0.02
−0.04

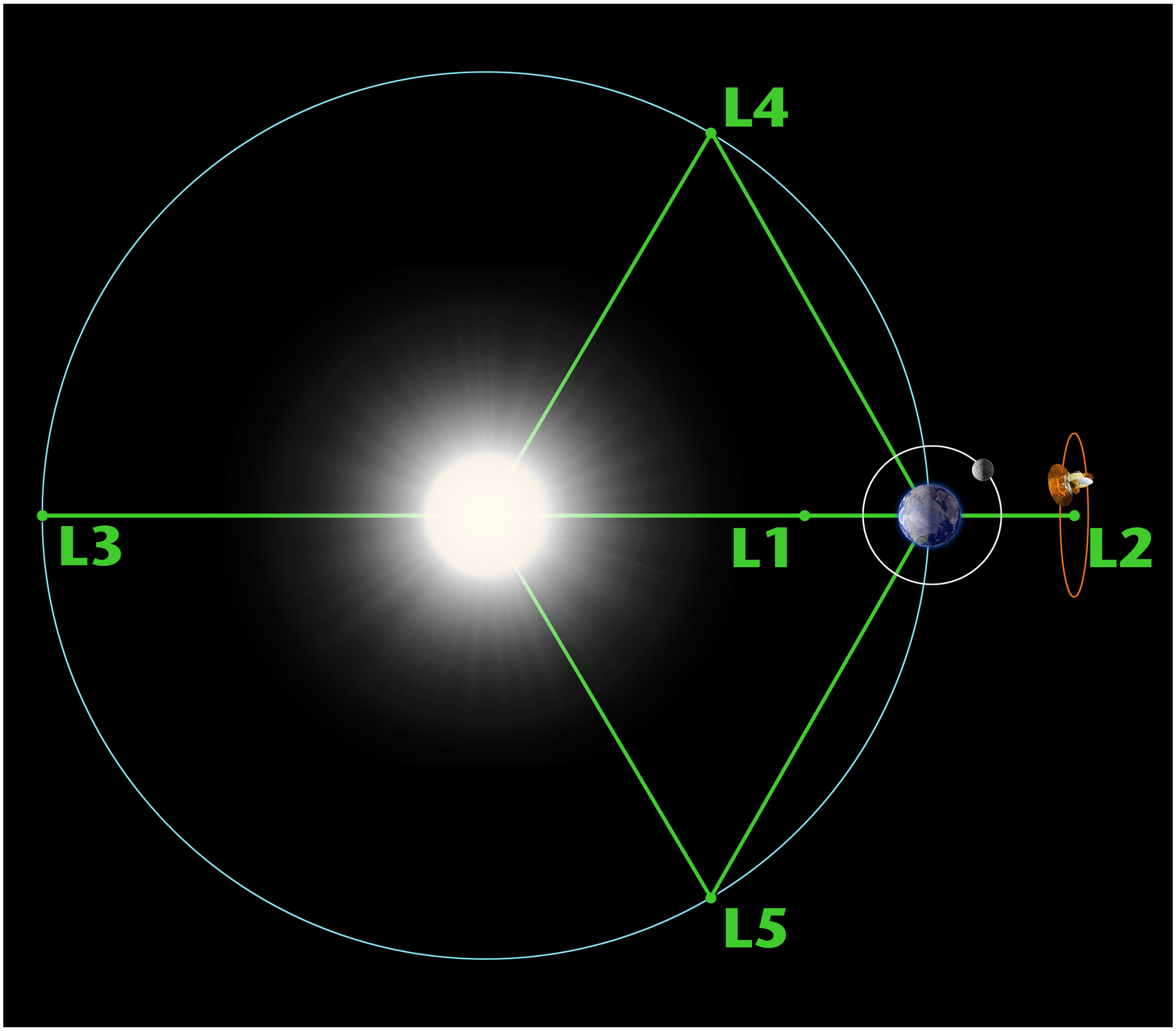


Results

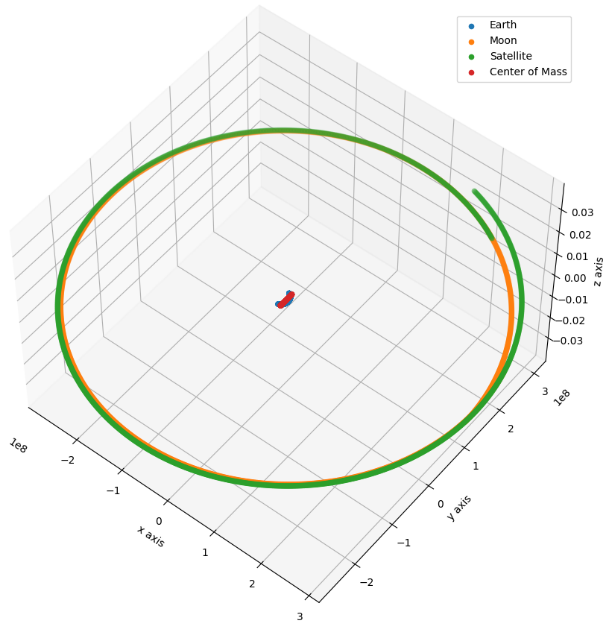
As stated before, this project seeks to primarily explore the restricted 3-body problem. In the experimental setup three bodies are added to the system, Earth with $m_e = 5.97 * 10^{24}$ kg, Moon with $m_m = 7.34 * 10^{22}$ kg, and a satellite with $m_s = 1000$ kg. The Earth is placed at the origin with zero initial velocity. The moon is placed $3.844 * 10^8$ meters away on the x axis and is given an initial velocity of $v_y = 1020$ m/s. The satellite is placed at the L4 Lagrange point (discussed below) and given an initial speed equal to the Moon's, but in the direction of its orbit.

The expectations of this setup would that the Earth and Moon form a stable orbit, and due to the much smaller mass of the satellite, it is able to be added to the stable system non-disruptively. To demonstrate this, a special case of orbit is used known as a Lissajous orbit.

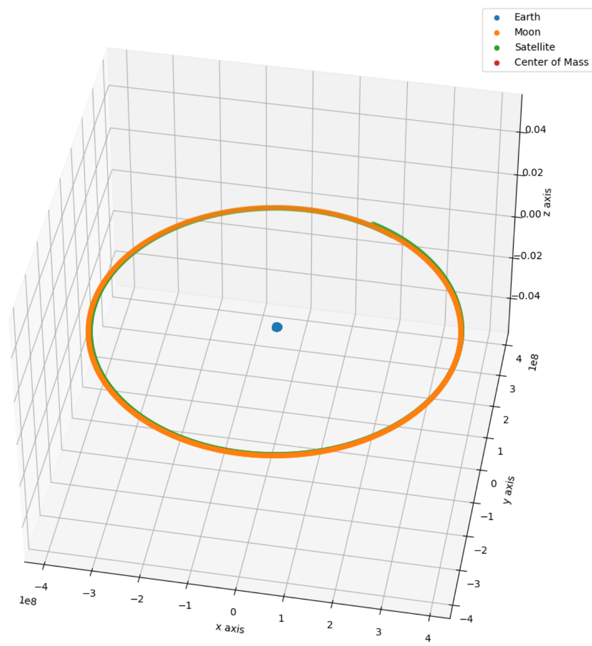
Lissajous orbits are a class of orbits that exist primarily at the L4 and L5 Lagrange points. These points are considered stable areas where the forces between the Earth and Moon balance out. These points exist for all pairs of massive bodies that orbit each other as long as the ratio of their masses roughly exceeds 25 [5]. An example of the Lagrange points between the Sun and Earth is shown below.



Using the priorly discussed initial conditions and running the simulation for 28 days, the following orbits are produced.



Two things may be immediately apparent: the center of mass moves, and the satellite orbit begins to drift outwards. In regards to the center of mass moving, this relates to the total momentum of the system. Because the earth is not given an initial velocity, the total momentum of the system is non-zero. This leads to the Earth “falling” towards the center of mass of the system, and the results is that the entire system begins to slowly drift. This can be solved by giving the Earth a small initial velocity and results in the following figure.



This small change fixes both the center of mass drift and the outwards drift of the satellite. There is still some small drift of the satellite, but this is likely due to numerical errors and slight approximations of values.

Discussion

Based off of the findings above, this project can largely be considered a success. Not only was the restricted 3-body problem successfully simulated, but the results were mostly confirmed through an analysis of the governing physical laws. Additionally, simulation of the L4 Lissajous orbit shows the advanced capabilities of numerical simulations such as this one by producing a stable 3-body orbit without chaos. Further work on this project may seek to reduce sources of numerical error and make improvements to the computational methods such that even greater accuracy can be achieved. Additionally, simulations using more bodies may be good topics of exploration.

Code

Code for this project can be found at https://github.iu.edu/evogelsa/n-body-simulation

References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Background of the n-body problem:
[1] https://en.wikipedia.org/wiki/N-body_problem

Runge-Kutta numerical method and math:
[2] https://en.wikipedia.org/wiki/Runge%E2%80%93Kutta_methods

Lagrange points background:
[3] http://hyperphysics.phy-astr.gsu.edu/hbase/Mechanics/lagpt.html#c1
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Lagrange point calculations of Earth-Moon system:
[4] https://www.researchgate.net/figure/Locations-of-the-five-Earth-Moon-Lagrangian-points-namely-where-

Lagrange point background and image:
[5] https://solarsystem.nasa.gov/resources/754/what-is-a-lagrange-point/

Computation and physical constants:
[6] https://www.wolframalpha.com/