7.9* Show that the asymptotic variance when a Markov chain sample from P of equation (7.42) is used to estimate π_1 is $\nu = \pi_1(1-\pi_1)(2\pi_1 - 1)$. [Hint: One way is to use equation (7.37) with the eigenvalues and eigenvectors given in Problem 7.8.]

Solution.

Referring to Eq. (7.37) in (Yang 2014a), the asymptotic variance v is defined as

$$v = \sum_{k\geq 2}^{K} \frac{1+\lambda_k}{1-\lambda_k}(E^T Bh)_k^2.$$

According to Eqs. (7.35-7.36) of (Yang 2014a), we have

$$B = \mathrm{diag}\{\pi_1, \dots, \pi_K\} = \begin{bmatrix} \pi_1 & 0 & 0 & \cdots & 0 \\ 0 & \pi_2 & 0 & \cdots & 0 \\ 0 & 0 & \pi_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \pi_K \end{bmatrix},$$

$$h = (1,0,0, \dots, 0)^T.$$

Noting that multiplying a matrix by the column vector h on the right retrieves its first column, we have

$$Bh = \begin{bmatrix} \pi_1 & 0 & 0 & \cdots & 0 \\ 0 & \pi_2 & 0 & \cdots & 0 \\ 0 & 0 & \pi_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \pi_K \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \pi_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Likewise, multiplying E^T by Bh is equivalent to retrieving the first column of E^T and by π_1 times. Hence, it is not difficult to see the following

$$E^T Bh = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & a_2 & -\frac{\pi_2}{\pi_3}a_2 & 0 & \cdots & 0 \\ 0 & a_3 & 0 & -\frac{\pi_2}{\pi_4}a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{K-1} & 0 & 0 & 0 & -\frac{\pi_2}{\pi_K}a_{K-1} \\ \left(1-\frac{1}{\pi_1}\right)a_K & a_K & a_K & a_K & \cdots & a_K \end{bmatrix} \times \begin{bmatrix} \pi_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \pi_1 \\ 0 \\ 0 \\ \vdots \\ \left(1-\frac{1}{\pi_1}\right)a_K \cdot \pi_1 \end{bmatrix}.$$

Thus, we have

$$(E^T Bh)_k^2 = \begin{cases} \pi_1^2, k = 1, \\ 0, 2 \leq k \leq K-1, \\ \left(\left(1-\frac{1}{\pi_1}\right)a_K \cdot \pi_1\right)^2, k = K. \end{cases}$$

Because $a_K = \left[\frac{\pi_1}{1-\pi_1}\right]^{\frac{1}{2}}$, by plugging the above into Eq. (7.37) of (Yang 2014a), the asymptotic variance v can be written as

$$
\begin{aligned}
v &= \sum_{k\geq 2}^{K} \frac{1+\lambda_k}{1-\lambda_k}(E^T Bh)_k^2 \\
&= \frac{1+\lambda_2}{1-\lambda_2}(E^T Bh)_2^2 + \frac{1+\lambda_3}{1-\lambda_3}(E^T Bh)_3^2 + \cdots + \frac{1+\lambda_K}{1-\lambda_K}(E^T Bh)_K^2 \\
&= \frac{1+0}{1-0}\times 0 + \cdots + \frac{1+0}{1-0}\times 0 + \frac{1+1-\frac{1}{\pi_1}}{1-\left(1-\frac{1}{\pi_1}\right)} \times \left(\left(1-\frac{1}{\pi_1}\right) a_K \cdot \pi_1\right)^2 \\
&= \frac{2-\frac{1}{\pi_1}}{\frac{1}{\pi_1}} \times \pi_1(1-\pi_1) \\
&= \pi_1(1-\pi_1)(2\pi_1 - 1).
\end{aligned}
$$