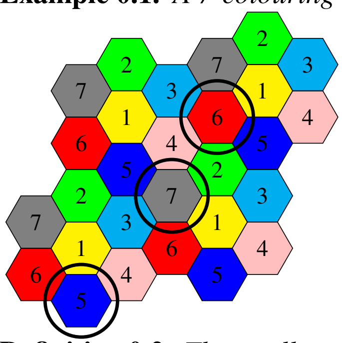
Polychromatic Number of the Plane

Things forbidden have a secret charm. – Publius Cornelius Tacitus, a senator and a historian of the Roman Empire, c. 56 – 117

Problem. What is the smallest number of colours needed for colouring the plane in such a way that no colour realizes all distances? (Paul Erdős, 1958)

Example 0.1. *A 7-colouring that avoids the distance 1 in each colour:*



Hugo Hadwiger in 1961: A 7-colouring of a tessellation of the plane by regular hexagons, with diameter slightly less than one. Observe that each hexagon is surrounded by hexagons of a different colour.

Definition 0.2. *The smallest number of colours sufficient for colouring the plane in such a way that no colour realizes all distances is called the polychromatic number of the plane and it is denoted by* χ_p.

Observation 0.3. $\chi_p \leq \chi$

The Lower Bound: $4 \leq \chi_p$. (Established by Dmitry E. Raiskii in 1970. This proof is by Alexei Merkov from 1997.)

Proof. 1. Assume that there is a 3-colouring of the plane

$$c : \mathbb{E}^2 \longrightarrow \{\color{red}{\bullet}, \color{blue}{\bullet}, \color{green}{\bullet}\}$$

such that

- There are no two points coloured red at the distance r;
- There are no two points coloured blue at the distance b;
- There are no two points coloured green at the distance g.

2. Let a Cartesian coordinate system in $\mathbb{E}^2$ be given.

3. We construct three Moser spindles like on Figure 1:

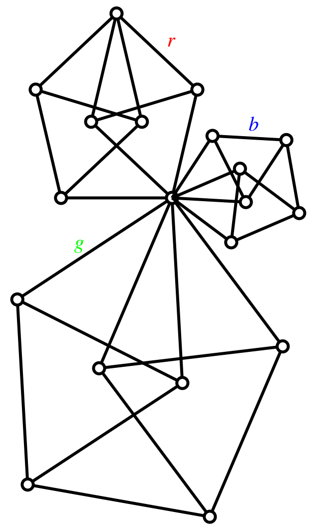


Figure 1: Three Moser spindles share the origin O as a common point and with the edges of lengths r, b, and g.

4. Consider 18 vectors, each of them with its initial point at the origin and the terminal point being a vertex in one of the three Moser spindles. Call those vectors

$$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_6, \vec{v}_7, \vec{v}_8, \dots, \vec{v}_{12}, \vec{v}_{13}, \vec{v}_{14}, \dots, \vec{v}_{18}.$$

   Here the terminal points of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_6$ belong to the Moser spindle with all edges of length r, the terminal points of the vectors $\vec{v}_7, \vec{v}_8, \dots, \vec{v}_{12}$ belong to the Mosers spindle with all edges of length b, and the terminal points of the vectors $\vec{v}_{13}, \vec{v}_{14}, \dots, \vec{v}_{18}$ belong to the Moser spindle with all edges of length g. See Figure 2.

5. Next we define a 3-colouring c' of the vector space

$$\mathbb{E}^{18} = \{(a_1, a_2, \dots, a_{18}) : a_1, a_2, \dots, a_{18} \in \mathbb{R}\}$$

   by

$$c'(a_1, a_2, \dots, a_{18}) = c(P)$$

   where P is the terminal point of the vector

$$a_1 \cdot \vec{v}_1 + \cdots + a_6 \cdot \vec{v}_6 + a_7 \cdot \vec{v}_7 + \cdots + a_{12} \cdot \vec{v}_{12} + a_{13} \cdot \vec{v}_{13} + \cdots + a_{18} \cdot \vec{v}_{18}.$$

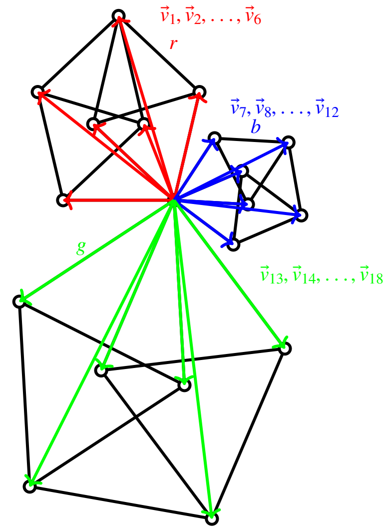


Figure 2: Eighteen vectors with the same initial point.

6. Let $M \subset \mathbb{E}^{18}$ be the set of all 18-tuples such that $(a_1, a_2, \ldots, a_{18}) \in M$ if and only if all of the following conditions are satisfied:

   (a) $a_i \in \{0, 1\}$ for all $i \in \{1, 2, \ldots, 18\}$;

   (b) $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \in \{0, 1\}$

   (c) $a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} \in \{0, 1\}$

   (d) $a_{13} + a_{14} + a_{15} + a_{16} + a_{17} + a_{18} \in \{0, 1\}$

   For example
$$(\underbrace{1, 0, 0, 0, 0, 0,}_{1 \le i \le 6} \underbrace{0, 0, 0, 0, 0, 1,}_{7 \le i \le 12} \underbrace{1, 0, 0, 0, 0, 0}_{13 \le i \le 18}) \in M$$
   but
$$(\underbrace{1, 0, 0, 0, 0, 0,}_{1 \le i \le 6} \underbrace{0, 0, 0, 0, 0, 0,}_{7 \le i \le 12} \underbrace{1, 1, 0, 0, 0, 0}_{13 \le i \le 18}) \notin M.$$

7. Note that
$$|M| = 7^3.$$

8. Consider the set
$$M_r = \{(a_1, a_2, a_3, a_4, a_5, a_6, \underbrace{0, 0, \cdots, 0}_{\text{All 0's}}) \in M : a_1, \ldots, a_6 \in \{0, 1\}\}$$
   and note that $|M_r| = 7$.

9. **Two observations and a conclusion:**

(a) If $(a_1, a_2, a_3, a_4, a_5, a_6, \underbrace{0, 0, \cdots, 0}_{\text{All 0's}}) \in M_r$ and $a_i \neq 0$ for some $i \in \{1, \ldots, 6\}$, then

$$\vec{OP} = a_1 \cdot \vec{v}_1 + \cdots + a_6 \cdot \vec{v}_6 + 0 \cdot \vec{v}_7 + \cdots + 0 \cdot \vec{v}_{12} + 0 \cdot \vec{v}_{13} + \cdots + 0 \cdot \vec{v}_{18} = \vec{v}_i$$

and P is one of the points in the Moser spindle that has all edges of length r.

(b) The Moser spindle that has all edges of length r cannot have three red vertices:

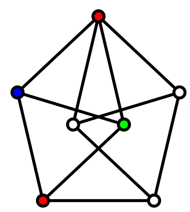


Figure 3: If there are three red vertices then two of them are r units apart.

(c) The set M_r can have at most two elements coloured red by the colouring c'.

Another observation:

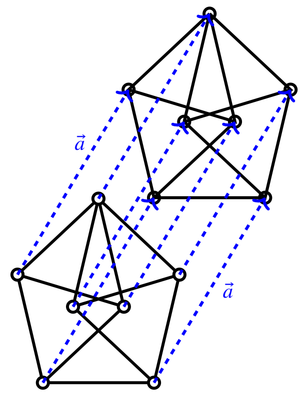


Figure 4: A translate of the Moser spindle is the Moser spindle.

For each of the 49 elements of the set

$$M_{bg} = \{(0,0,0,0,0,0,a_7,a_8,\ldots,a_{18}) \in M : a_7,\ldots,a_{18} \in \{0,1\}\}$$

we make a translate of M_r in $\mathbb{E}^{18}$:

$$M_r^a = a + M_r, \; a \in M_{bg}.$$

Clearly

$$M = \cup_{a \in M_{bg}} M_r^a$$

and, for all $a, b \in M_{bg}$,

$$a \neq b \Longrightarrow M_r^a \cap M_r^b = \emptyset.$$

In other words we have divided the set M into $7^2 = 49$ mutually disjunct copies of M_r.

How many elements in M_r^a, $a \in M_{bg}$, are coloured red by c'?

10. Let $(0,0,0,0,0,0,a_7,a_8,\ldots,a_{18}) \in M_{bg}$ and let

$$\vec{a} = 0 \cdot \vec{v}_1 + \cdots + 0 \cdot \vec{v}_6 + a_7 \cdot \vec{v}_7 + \cdots + a_{12} \cdot \vec{v}_{12} + a_{13} \cdot \vec{v}_{13} + \cdots + a_{18} \cdot \vec{v}_{18}.$$

Then the elements of M_r^a are coloured by c' in the same way that c colours the vertices of the Moser spindle that is obtained as the translate of the original Moser spindle by $\vec{a}$!

Therefore, for each $a \in M_{bg}$, the set M_r^a can have at most TWO red elements.

11.

$$\begin{aligned} \text{\# of red elements of } M &= \sum_{a \in M_{bg}} \text{\# of red elements of } M_r^a \\ &\leq \sum_{a \in M_{bg}} 2 = 2 \cdot 49 = 98. \end{aligned}$$

12. Similarly

$$\text{\# of blue elements of } M \leq 98$$

and

$$\text{\# of green elements of } M \leq 98.$$

Therefore

$$\begin{aligned} 7^3 &= (\text{\# of red elements of } M) + (\text{\# of blue elements of } M) \\ &+ (\text{\# of green elements of } M) \leq 3 \cdot 98 = 3 \cdot (2 \cdot 7^2) = 6 \cdot 7^2. \end{aligned}$$

Contradiction!

13. Therefore, our assumption that there is a 3-colouring of the plane

$$c : \mathbb{E}^2 \longrightarrow \{\bullet, \bullet, \bullet\}$$

such that

- There are no two points coloured red at the distance r;
- There are no two points coloured blue at the distance b;
- There are no two points coloured green at the distance g

led to a contradiction!

14. Each colour of every 3-colouring of the plane realizes all distances. This implies

$$4 \leq \chi_p.$$

$\square$

The Upper Bound. $\chi_p \leq 6$. (S.B. Stechkin, 1970)

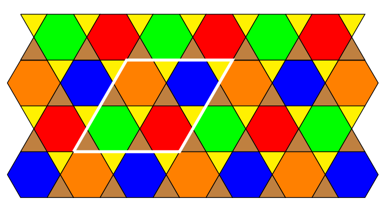


Figure 5: Steichkin's 6-coloring of the plane.

Take a Closer Look.
Note:

- All sides of all triangles and hexagons are of length 0.5.
- Every hexagon includes its boundary except its rightmost and two lowest vertices.

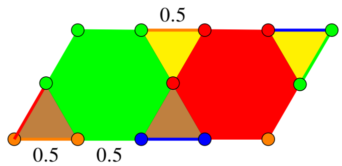


Figure 6: Steichkin's 6-coloring of the plane - a closer look.

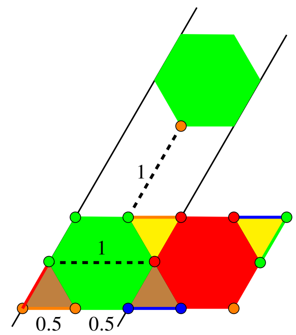


Figure 7: No two green points that are 1 unit apart.

- Triangles do not include their boundaries.

Which Distances are Avoided?
Note:

- Four colours used to colour hexagons do not realize the distance 1.
- Two colours used to colour triangles do not realize the distance 0.5.

Notation. Steichkin's colouring is of the type $(1, 1, 1, 1, \frac{1}{2}, \frac{1}{2})$.
Theorem 0.4. $4 \leq \chi_p \leq 6$.
Resources.

1. Soifer, A. The Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of its Creators, Springer, New York, 2008, pp 32-44.
2. The Erdos-Szekeres problem on points in convex position - a survey by W. Morris and V. Soltan