Ramsey's Theorem - Part 2

1. **Quote.** To see things in the seed, that is genius.

   Laozi 老子, Chinese philosopher, 6th century BC

2. **Reminder.**

   (a) $R(3,3) = 6$

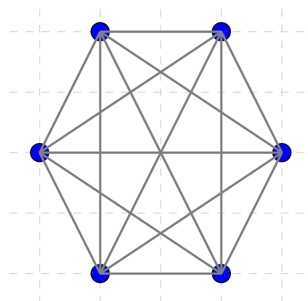


Figure 1: K_6 - a complete graph on six vertices

   (b) $R(4,3) = R(3,4) = 9$

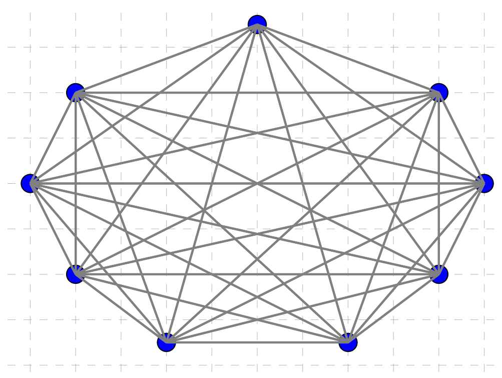


Figure 2: K_9 - a complete graph on nine vertices

(c) $R(4,4) = 18$: Consider a blue-red edge colouring of a K_{18}.

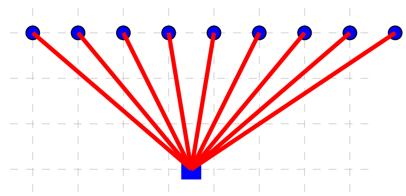
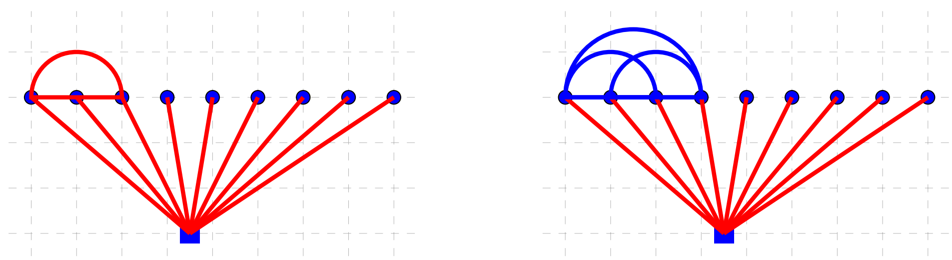


Figure 3: Step 1: Observe that each vertex ■ is adjacent to 9 edges of the same colour - say red



Case 1- There is a red K_3 n the induced K_9 Case 2: There is a blue K_4 in the induced K_9

Figure 4: Step 2 – Two cases

3. **Jeopardy!**

(a) There are only **two** 2-colourings of K_{16} without a monochromatic K_4.

(b) There is only **one** 2-colouring of K_{17} without a monochromatic K_4.

4. **Definition:** The Ramsey number $R(s,t)$ is the minimum number n for which any edge 2-coloring of K_n, a complete graph on n vertices, in red and blue contains a red K_s or a blue K_t.

5. **Three BIG Questions:**

   (a) Does the Ramsey number $R(s,t)$ exist for any choice of natural numbers $s \geq 2$ and $t \geq 2$?

   (b) If $R(s,t)$ exists, can we find the exact value of $R(s,t)$?

   (c) If $R(s,t)$ exists and if we cannot find the exact value of $R(s,t)$, what are the best known bounds for $R(s,t)$?

6. **What About...**

   (a) $R(s,2)$**?**

   (b) $R(2,t)$**?**

7. **Ramsey's Theorem - Two Colours.** For any $s, t \in \mathbb{N}\backslash\{1\}$ the Ramsey number $R(s,t)$ exists and, for $s, t \geq 3$,
$$R(s,t) \leq R(s-1,t) + R(s,t-1).$$

8. **Observation 1:**
$$R(2,2) = 2,\ R(3,2) = R(2,3) = 3,\ R(3,3) = 6,\ R(4,2) = R(2,4) = 4.$$

9. **Observation 2:** If $s, t \in \mathbb{N}\backslash\{1\}$ are such that
$$s + t = 4 \text{ or } s + t = 5 \text{ or } s + t = 6$$
then $R(s,t)$ exists!

10. **Observation 3:** Since, for any $s \geq 2$
$$R(s,2) = R(2,s) = s$$
we are interested only in the question if $R(s,t)$ exists for $s, t \geq 3$.

11. **Observation 4:**
$$(s-1) + t = s + (t-1) = (s+t) - 1$$

12. **Observation 5:** To prove that $R(s,t)$, $s, t \geq 3$, exists it is enough to prove that any 2-colouring of a complete graph K_M where
$$M = R(s-1,t) + R(s,t-1)$$
yields a monochromatic K_s or a monochromatic K_t. Why?

13. **Strategy:** We prove that any 2-colouring of a complete graph K_M where
$$M = R(s-1,t) + R(s,t-1)$$
yields a monochromatic K_s or a monochromatic K_t via induction on the sum $s + t$.

14. **Proof of Ramsey's Theorem - Two Colours:** Let $s, t \geq 3$. We use mathematical induction on the sum $s + t$ to prove that $R(s, t)$ exists.

The base case of induction, $s + t = 6$, follows from the fact that $R(3, 3) = 6$.

Suppose that $n \geq 6$ is such that for any $u, v \geq 3$ such that $u + v = n$ the Ramsey number $R(u, v)$ exists.

Let $s, t \geq 3$ by such that

$$s + t = n + 1.$$

Then, since

$$(s - 1) + t = s + (t - 1) = n,$$

by the induction hypothesis $R(s - 1, t)$ and $R(s, t - 1)$ exist. Let

$$M = R(s - 1, t) + R(s, t - 1)$$

and we consider a 2-colouring of K_M.

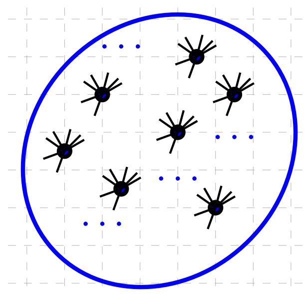
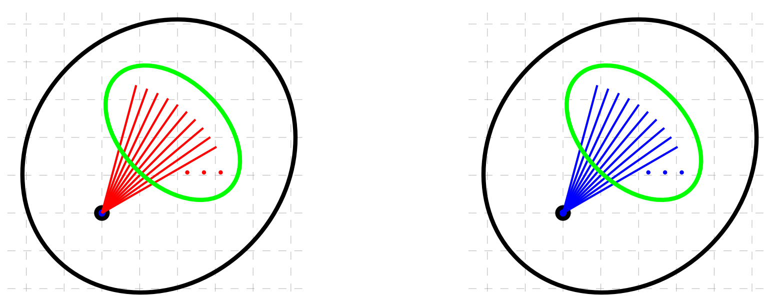


Figure 5: K_M: Each vertex is incident to $M - 1 = R(s - 1, t) + R(s - 1, t) - 1$ edges

Fix a vertex. There are two possibilities:



Case 1: At least $R(s - 1, t)$ red edges Case 2: At least $R(s, t - 1)$ blue edges

Figure 6: Pigeonhole principle: Two cases

Suppose

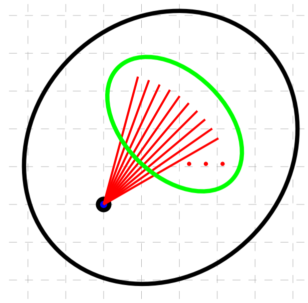
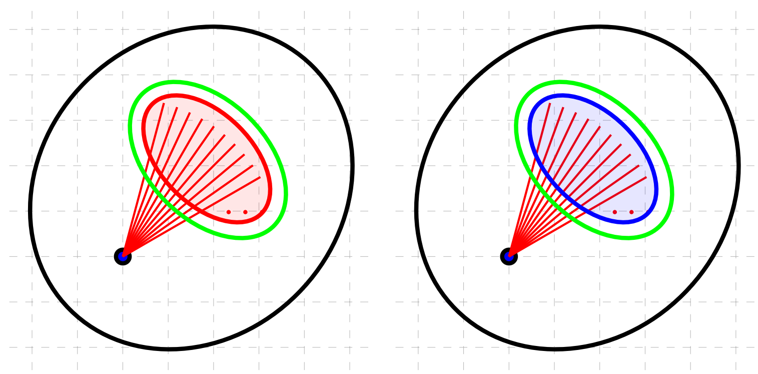


Figure 7: Case 1: At least $R(s-1,t)$ red edges

Then



Case 1.1: There is a red K_{s-1} Case 1.2: There is a blue K_t

Figure 8: Recall the definition of $R(s-1,t)$

Hence, any red/blue 2-colouring of K_M yields a red K_s or a red K_t. Therefore $R(s,t)$ exists and

$$R(s,t) \leq R(s-1,t) + R(s,t-1).$$

15. **Known Ramsey Numbers.**

s	t	$R(s,t)$	Who & When
3	3	6	Greenwood & Gleason 1955
3	4	9	Greenwood & Gleason 1955
3	5	14	Greenwood & Gleason 1955
3	6	18	Graver & Yackel 1968
3	7	23	Kalbfleisch 1966
3	8	28	McKay & Min 1992
3	9	36	Grinstead & Roberts 1982
4	4	18	Greenwood & Gleason 1955
4	5	25	McKay & Radziszowski 1995

16. **More Known Facts.**

s	t	$R(s,t)$	Who and When
3	10	[40, 42]	Exoo 1989, Radziszowski & Kreher 1988
3	11	[46, 51]	Radziszowski & Kreher 1988
4	6	[35, 41]	Exoo, McKay & Radziszowski 1995
4	7	[49, 61]	Exoo 1989, Mackey 1994
5	5	[43, 48]	Exoo 1989, McKay & Radziszowski 1995
5	6	[58, 87]	Exoo 1993, Walker 1971
6	6	[102, 165]	Kalbfleisch 1965, Mackey 1994
6	7	[113, 298]	Exoo & Tatarevic 2015, Xu & Xie 2002

17. **In Erdős' Words.** Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.

18. **How Big - Upper Bound.**

$$R(s,t) \leq \binom{s+t-2}{t-1}.$$

19. **Therefore:** For $s \geq 3$

$$2^{s/2} < R(s,s) \leq \binom{2s-2}{s-1}.$$

20. **Question:** Suppose that we decide to use three colours, say blue, red, and green. Is there a something like $R(s, t, u)$, for $s, t, u \in \mathbb{N}$? In other words, is it possible to find a number n so that if each edge of K_n is coloured by one of the three colours then there will be always possible to find a blue K_s or a red K_t or a green K_u?

One can go all the way and consider a situation in which, for $m \in \mathbb{N}\backslash\{1\}$, the edges of a complete graph K_n are coloured with m colours.

21. **Definition: Ramsey number**]Let $k \in \mathbb{N}\backslash\{1\}$ and $m_1, m_2, \ldots, m_k \in \mathbb{N}\backslash\{1\}$ be given. The Ramsey number $R(m_1, m_2, \ldots, m_k)$ is the minimum number n for which any edge k–colouring of K_n, with colours $c_1, c_2, \ldots, c_k$, contains a c_i– monochromatic K_{m_i}, for some $i \in [1, k]$.

    Question now becomes: Does the Ramsey number $R(m_1, m_2, \ldots, m_k)$ exist for any choice of natural numbers $k, m_1, m_2, \ldots, m_k \geq 2$?

22. **Ramsey's Theorem For Graphs**. For any $k, m_1, m_2, \ldots, m_k \in \mathbb{N}\backslash\{1\}$ the Ramsey number $R(m_1, m_2, \ldots, m_k)$ exists.