Van der Waerden's Theorem - Part 3

1. **Quote.** Do not, however, confuse elementary with simple.

Figure 1: Aleksandr Yakovlevich Khinchin, Soviet mathematician,1894 -1959

2. **Van der Waerden's Theorem:** Let $l, k \in \mathbb{N}$. Any k-colouring of positive integers contains a monochromatic l-term arithmetic progression. Moreover, there is a natural number N such that any k-colouring of the segment of positive integers $[1, N]$ contains a monochromatic l-term arithmetic progression.

3. **Note:** The smallest N guaranteed by the theorem is annotated by $W(l, k)$. We have seen that $W(3, 2) = 9$ and that $W(3, k)$ exists for any $k \in \mathbb{N}$.

4. **Two Questions.**

   (a) How big is $W(l, k)$?

   (b) If $\mathbb{N}$ is k-coloured can we be sure that a certain colour contains an l-term arithmetic progression?

5. **Van der Waerden's Numbers.** In 1951, Paul Erdős and Richard Rado introduced van der Waerden's function:

$$W : (l, k) \to W(l, k).$$

   The values of van der Waerden's function are called *van der Waerden's numbers*.

6. **Best known lower bounds to Van der Waerden numbers.**

k # of colours\ l length	3	4	5	6	7	8	9
2	9	35	178	1132	> 3703	> 7484	> 27113
3	27	> 292	> 1209	> 8886	> 43855	> 238400	
4	76	> 1048	> 10437	> 90306	> 387967		
5	> 125	> 2254	> 24045	> 246956			
6	> 207	> 9778	> 56693	> 600486			

7. **Big Question:**

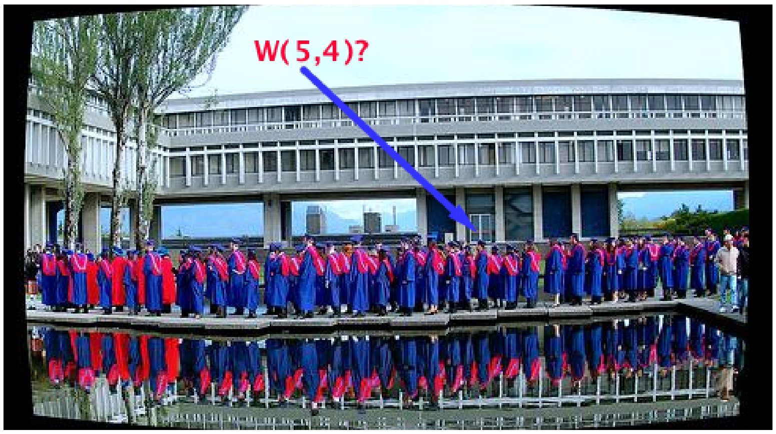

Figure 2: $W(l,k)$: Can you find me?

8. **Two Lower Bounds:** It is a convention to write $W(l)$ instead of $W(l, 2)$. Hence,

   - $W(3) = 9$ and $W(4) = 35$ (Chvatal, 1970)
   - $W(5) = 178$ (Stevens and Shantaram, 1978)
   - $W(6) = 1132$ (Kouril and Paul, 2008)

   (a) (Berlekamp, 1969) If l is a prime then

$$W(l + 1) > l \cdot 2^l.$$

Figure 3: Elwyn Ralph Berlekamp, American mathematician, 1940-

   (b) (Szabó, 1990) For any $\varepsilon > 0$,

$$W(l) \geq \frac{2^l}{l^\varepsilon}$$

   for large enough l.

Figure 4: Zoltán Szabó, Hungarian mathematician, 1965-

9. **Upper Bounds.**

(a) **Prelude**

- $f_1(x) = \text{DOUBLE}(x) = 2x$
- $f_2(x) = \text{EXPONENT}(x) = 2^x$
  Note that

$$f_1^{(2)}(1) = f_1(f_1(1))) = f_1(2 \cdot 1) = 2 \cdot 2 = 2^2 = f_2(2),\ f_1^{(3)}(1) = f_1(2^2) = 2 \cdot 2^2 = f_2(3)$$

  and in general

$$f_2(x) = f_1^{(x)}(1).$$

- $f_3(x) = \text{TOWER}(x) = 2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}} \Big\} x = f_2^{(x)}(1)$
- $f_4(x) = \text{WOW}(x) = f_3^{(x)}(1)$
- $f_{i+1}(x) = f_i^{(x)}(1)$
- $f_\omega(x) = \text{ACKERMANN}(x) = f_x(x)$

		1	2	3	4	5	6
DOUBLE	f_1	2	4	6	8	10	12
EXPONENT	f_2	2	4	8	16	32	64
TOWER	f_3	2	4	16	65536	2^{65536}	$\vdots$
WOW	f_4	2	4	65536	WOW!	$\vdots$	$\vdots$
	f_5	2	4	WOW!	$\vdots$	$\vdots$	$\vdots$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ACKERMANN	f_ω	2	4	16	WOW!	$\vdots$	$\vdots$

Figure 5: Wilhelm Friedrich Ackermann, German mathematician, 1896 -1962

(b) Van der Waerden's proof implies, for $k \geq 10$

$$W(l) \leq \text{ACKERMANN}(l).$$

(c) Shelah, 1988:

$$W(l) < \text{WOW}(l + 2).$$

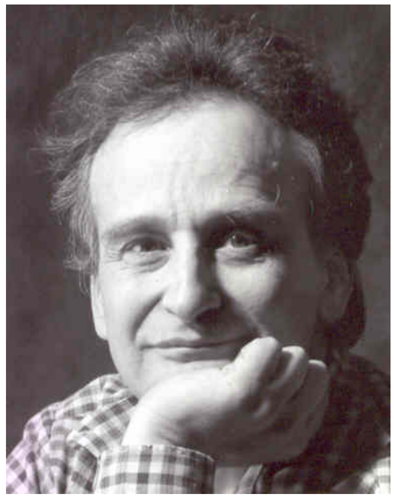

Figure 6: Saharon Shelah, Israeli mathematician, 1945-

(d) Gowers, 1998:

$$W(l) \leq 2^{2^{2^{2^{2^{l+9}}}}}$$

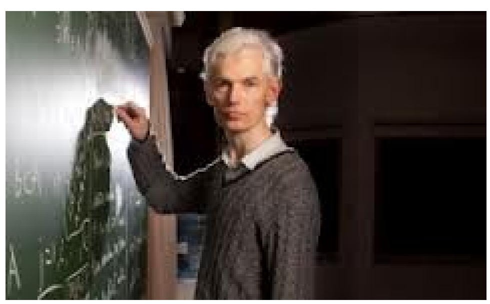

Figure 7: Sir William Timothy Gowers, British mathematician, 1963-

(e) Ron Graham offers \$ 1000 for a proof or disproof of the bound that

$$W(l) \leq 2^{l^2}.$$

Figure 8: Ron Graham, American mathematician, 1935-

10. **Closer to Home:** Given any positive integer r and positive integers $k_1, k_2, \ldots, k_r$, there is an integer m such that given any partition $\{1, 2, \ldots, m\} = P_1 \cup P_2 \cup \ldots \cup P_r$, there is always a class P_j containing an arithmetic progression of length k_j. Let us denote the least m with this property by $w(r; k_1, k_2, \ldots, k_r)$.

    Tom Brown in 1974 found the following:

$w(3; 2, 3, 3) = 14$	$w(3; 2, 4, 4) = 40$	$w(4; 2, 2, 3, 3) = 17$	$w(4; 2, 3, 3, 3) = 40$
$w(3; 2, 3, 4) = 21$	$w(3; 2, 4, 5) = 71$	$w(4; 2, 2, 3, 4) = 25$	
$w(3; 2, 3, 5) = 32$		$w(4; 2, 2, 3, 5) = 43$	
$w(3; 2, 3, 6) = 40$		$w(4; 2, 2, 4, 4) = 53$	

11. **Where to Look for monochromatic arithmetic progressions?**

   (a) **Prelude:** Let A be a subset of the set of natural numbers $\mathbb{N}$. For any $n \in \mathbb{N}$ let

$$A(n) = \{1, 2, \ldots, n\} \cap A \text{ and } a(n) = |A(n)|.$$

We define the upper density $\overline{d}(A)$ of the set A by

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{a(n)}{n}.$$

Similarly, $\underline{d}(A)$, the lower density of A, is defined by

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{a(n)}{n}.$$

We say that A has density $d(A)$ if

$$\underline{d}(A) = \overline{d}(A).$$

Thus

$$d(A) = \lim_{n\to\infty} \frac{a(n)}{n}.$$

Two examples:

- What is the density of the set of all natural numbers divisible by 3?

- What is the density of the set of all powers of 2?

Note: For more examples see: Natural density - Wikipedia

(b) Paul Erdős and Paul Turán conjectured in 1936:
Any set of integers with positive density contains a 3-term arithmetic progression.

Figure 9: Paul Turán, Hungarian mathematician, 1910-1976

(c) Roth, 1953:
Any set of integers with positive density contains a 3-term arithmetic progression.

Figure 10: Klaus Friedrich Roth, German-born British mathematician, 1925 -

(d) Szemerédi, 1975:
Any set of integers with positive density contains an arithmetic progression of any length.

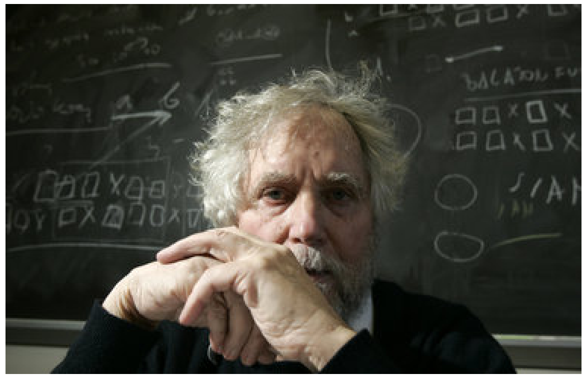

Figure 11: Endre Szemerédi, Hungarian-American mathematician, 1940-

(e) Furstenberg, 1977:
Any set of integers with positive density contains an arithmetic progression of any length.

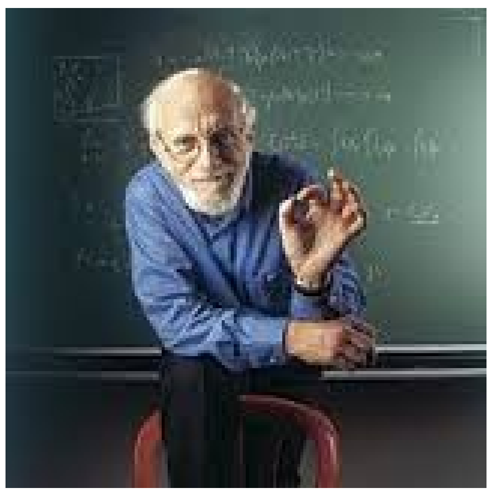

Figure 12: Hillel (Harry) Furstenberg, American-Israeli mathematician, 1935-

12. **Green-Tao Theorem, 2004:** For any $k \in \mathbb{N}$, there is a k-term progression consisting of primes.

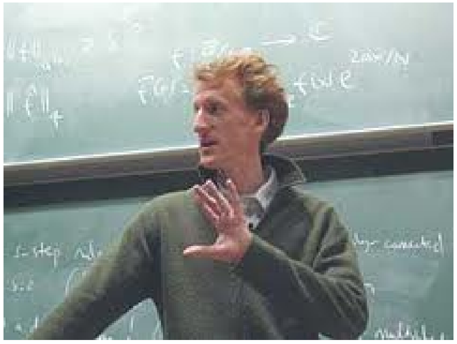

Figure 13: Ben Green, 1977-

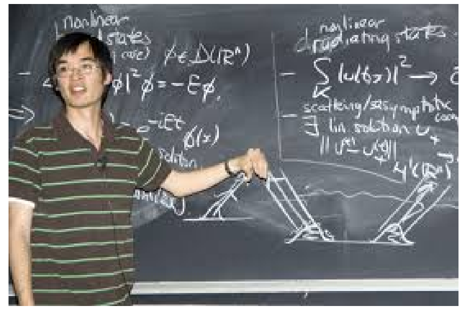

Figure 14: Terence Chi-Shen Tao, 1975-