Van der Waerden's Theorem - Part 1

1. **Quote.** Say what you know, do what you must, come what may.
   Sofia Vasilyevna Kovalevskaya, Russian mathematician,1850 -1891

2. **Three Reminders.**

   (a) An l-term arithmetic progression is any set of the form

   $$a, a+d, a+2d, \ldots, a+(l-1)d$$

   where $a, d \in \mathbb{R}$, $d \neq 0$.

   (b) A k-colouring of a set A is any function

   $$c : A \to \{1, 2, \ldots, k\} = [1, k].$$

   (c) If c is a k-colouring of the set A and if $B \subseteq A$ is such that for any $x, y \in B$

   $$c(x) = c(y)$$

   then we say that the set B is monochromatic.

3. **Challenge.** Colour with 2-colours avoiding monochromatic 3-term arithmetic progressions:

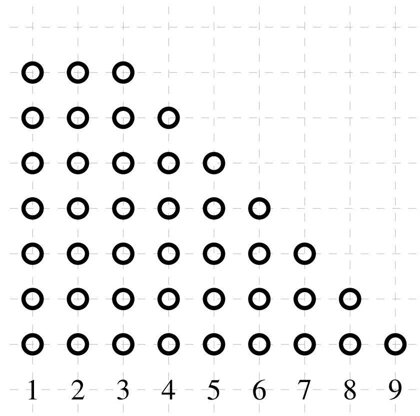

Figure 1: Colour each set with 2-colours avoiding monochromatic 3-term arithmetic progressions

4. **Check and Extend:** Check if the following 3-colouring of the the set $\{1, 2, \ldots, 17\}$ avoids monochromatic 4-term arithmetic progressions:

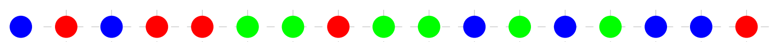

Figure 2: Can you find a monochromatic 4-term arithmetic progression?

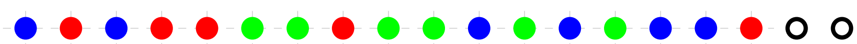

Figure 3: Can you colour numbers 18 and 19 to avoid monochromatic 4-term arithmetic progression?

5. **Question:** Do you think that it is possible to extend the colouring above (and keep it with no monochromatic 4-term arithmetic progression) to the interval $[1, 25]$? $[1, 50]$? $[1, 100]$? Forever?

6. **Baudet's Conjecture:** If the sequence of integers $1, 2, 3, \ldots$ is divided into two classes, at least one of the classes contains an arithmetic progression of l terms, no matter how large the length l is.

7. **Van der Waerden's Theorem:** If the sequence of integers $1, 2, 3, \ldots$ is divided into two classes, at

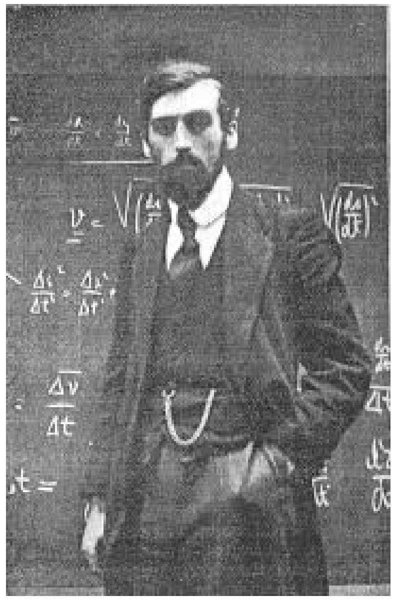

Figure 4: Pierre Joseph Henry Baudet, 1891-1921

least one of the classes contains an arithmetic progression of l terms, no matter how large the length l is.

Reference: van der Waerden, B. L. (1927). “Beweis einer Baudetschen Vermutung”. Nieuw. Arch. Wisk. 15: 212-216

“Beweis einer Baudetschen Vermutung” = “Proof of a Baudet Conjecture”

8. **Van der Waerden's theorem - any number of colours, length 3:** Let $k \in \mathbb{N}$. Any k-colouring of positive integers contains a monochromatic 3-term arithmetic progression. Moreover, there is a natural number N such that any k colouring of the segment of positive integers $[1, N]$ contains a monochromatic 3-term arithmetic progression.

   Note: The smallest N guaranteed by the theorem is annotated by $W(3, k)$. We have seen that $W(3, 2) = 9$.

   Proof:

   (a) **Colour-focused arithmetic progressions and spikes:** Let c be a finite colouring of an interval of positive integers $[1, m]$ and $l, r \in \mathbb{N}$. We say that the set of l-term arithmetic progressions $A_1, A_2, \ldots, A_r$, i.e., for all $i \in [1, r]$ we have, for some $a_i, d_i \in \mathbb{N}$,

   $$A_i = \{a_i + jd_i : j \in [0, l - 1]\}$$

   is *colour-focused* at $f \in \mathbb{N}$ if

   i. $A_i \subseteq [1, m]$ for each $i \in [1, r]$.
   ii. Each A_i is monochromatic.
   iii. If $i \neq j$ the A_i and A_j are not of the same colour.
   iv.
   $$a_1 + ld_1 = a_2 + ld_2 = \cdots = a_r + id_r = f.$$

   We call elements of a colour-focussed set **spikes**.

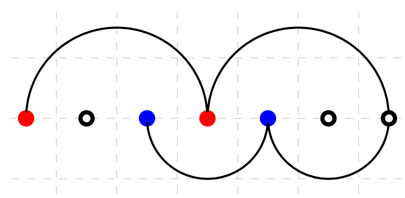

Figure 5: $\{1, 4\}$ and $\{3, 5\}$ are *colour-focussed* at 7.

   (b) **Warm up - $k = 2$:**

   i. Consider a two colouring of $[1, 3]$.

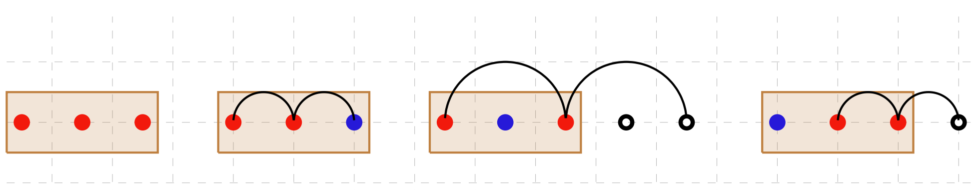

Figure 6: Any 2-colouring of the set $\{1, 2, 3\} = [1, 3]$ produces or a monochromatic 3-term arithmetic progression or **one** coloured-focused 2-term arithmetic progression.

   ii. Consider the interval of positive integers $[1, (2 \cdot 3) \cdot (2^6 + 1)] = [1, 390]$. Divide this interval into 65 consecutive blocks of length 6.

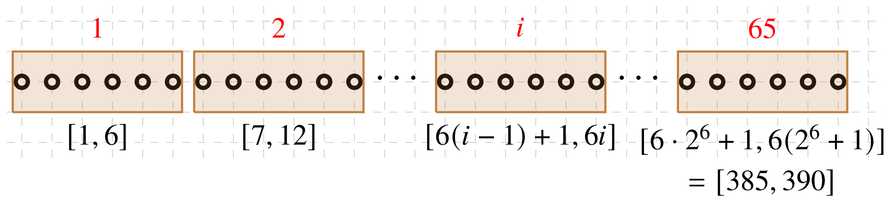

Figure 7: $(2 \cdot 3) \cdot (2^{2 \cdot 3} + 1) = 65$ consecutive blocks of length $2 \cdot 3 = 6$.

iii. In how many ways can we 2-colour six consecutive integers?

iv. Let c be a 2-colouring of the interval $[1, 2 \cdot 390]$.

v. Every 2-colouring of $[1, 780]$ contains a monochromatic 3-term arithmetic progression!

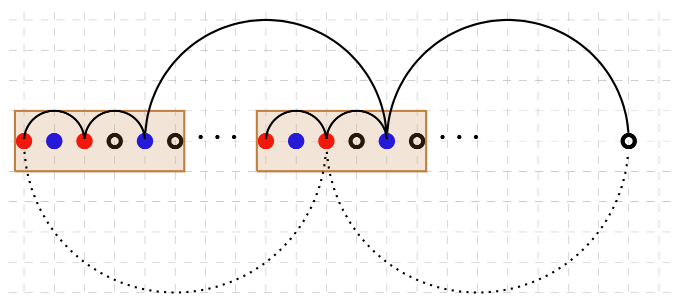

Figure 8: Every 2-colouring of $[1, 780]$ contains a monochromatic 3-term arithmetic progression!

vi. Thus $W(3, 2) \leq 780$.

(c) Next we consider a k-colouring, for any $k \geq 2$.

i. **Strategy:** We use induction on r to prove the following statement:

For all $r \leq k$, there exists a natural number n such that whenever $[1, n]$ is k-coloured, **either** there exists a monochromatic 3-term arithmetic progression **or** there exist r coloured-focussed arithmetic progressions of length 2.

ii. **The base case:** Take $r = 1$ and $n = k + 1$:

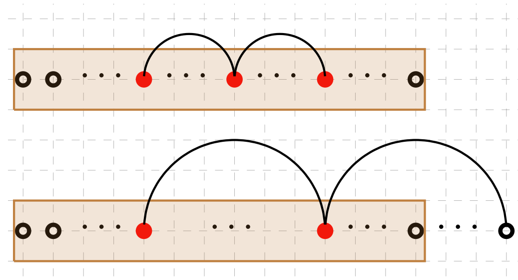

Figure 9: The base step: If the interval $[1, k + 1]$ is k-coloured then there is a monochromatic 3-term arithmetic progression or **one** coloured-focused 2-term arithmetic progression.

iii. **The inductive step:** Suppose that $r \in [2, k]$ is such that there is an n such that any k-colouring of $[1, n]$ contains a monochromatic 3-term arithmetic progression or $r - 1$ 'spikes', i.e., $r - 1$ colour focussed 2-term arithmetic progressions.

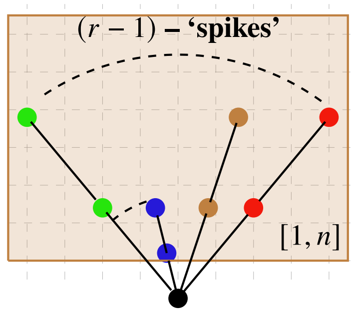

Figure 10: The inductive step: For $1 < r \leq k$ there is an n such that any k-colouring of $[1, n]$ contains a monochromatic 3-term arithmetic progression or $r - 1$ 'spikes', i.e., $r - 1$ colour focussed 2-term arithmetic progressions.

A. How many different k-colourings of the interval $[1, 2n]$ are there?

B. Consider the interval of positive integers $[1, (2 \cdot n) \cdot (k^{2n} + 1)]$. Divide this interval into $k^{2n} + 1$ consecutive blocks of length $2n$. Call those blocks B_i, $1 \leq i \leq k^{2n} + 1$.

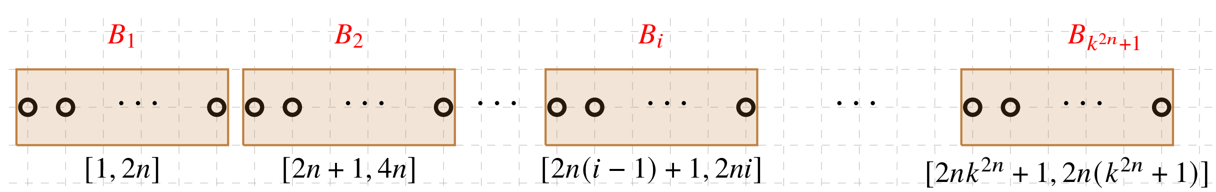

Figure 11: The interval $[1, 2n(k^{2n} + 1)]$ is divided in $k^{2n} + 1$ blocks of length $2n$ and k-coloured.

C. Let c be a k-colouring of the interval $[1, (2 \cdot n) \cdot (k^{2n} + 1)]$. Suppose that c does not contain a monochromatic 3-term arithmetic progression.

D. Note that by the inductive hypothesis each block B_i contains $r - 1$ spikes together with their focus.

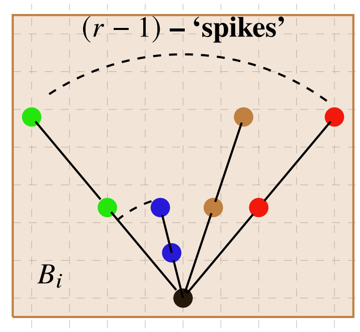

Figure 12: Each block B_i contains $r - 1$ spikes together with their focus.

E. There must be two two blocks coloured in the sam way.

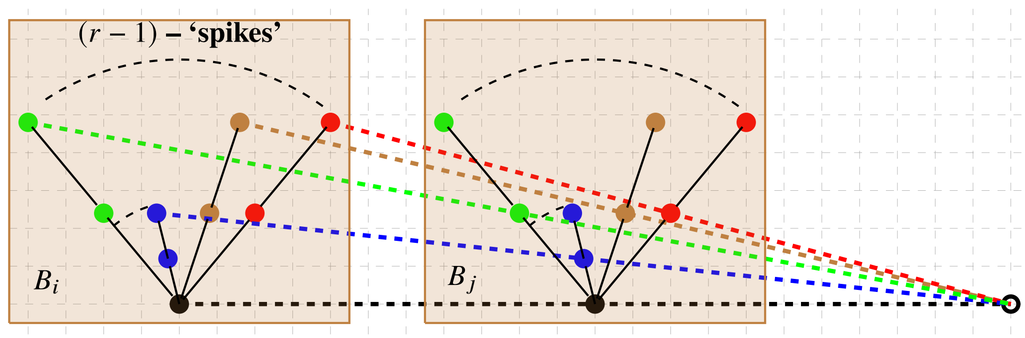

Figure 13: There must be two two blocks coloured in the sam way. Each of them contains $r - 1$ spikes together with their focus. Do you see how r spikes with the same focus emerge?

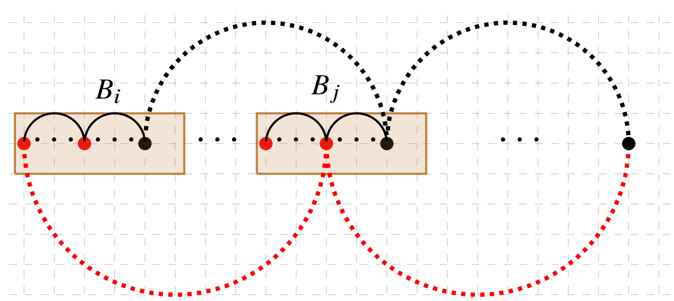

Figure 14: Take a closer look. Two pairs of red spikes (in B_i and B_j) produce a new pair of red spikes.

F. This completes the induction step:

iv. What happens when $r = k$?

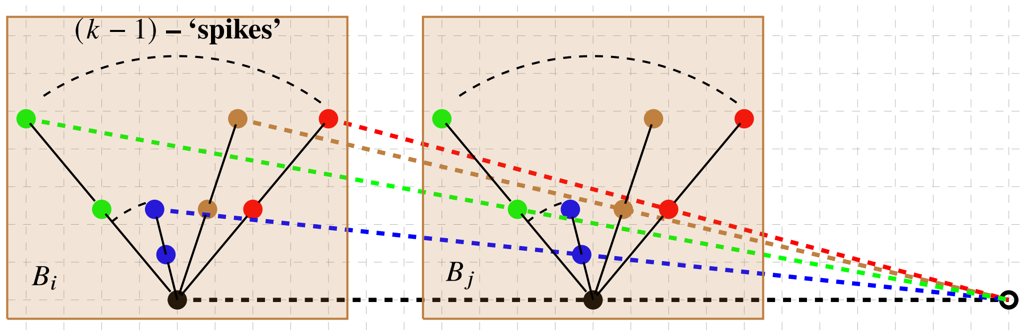

Figure 15: What happens when $r = k$? Do you see how a monochromatic 3-term arithmetic progression emerges?