Erdős-Szekeres Problem of Convex Polygons

Where there is love there is life. – Mahatma Gandhi, Indian leader, 1869 – 1948

Warm Up. Consider a finite set of points S in the plane, and ask, for example, this question: Is it true that there will always be a set of three points in S that are the vertices of a triangle?

Points in General Position in Plane. We say that the set of points A in the plane is in general position if there is no line that contains three points from A. See Figure 1.

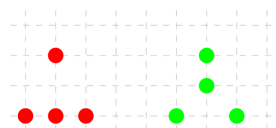


Figure 1: Which of the two sets is a set of points in general position?

Problem. For any integer $n \geq 3$, determine the smallest positive integer $N(n)$ such that any set of at least $N(n)$ points in general position in the plane (i.e., no three of the points are on a line) contains n points that are the vertices of a convex n-gon.

Convex n-gon. A convex n-gon is an n-gon with the property that if two points A and B are inside of the n-gon then the whole segment $\overline{AB}$ is inside of the n-gon. See Figure 3.

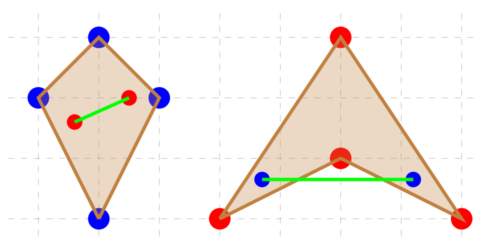


Figure 2: A convex quadrilateral and a non-convex quadrilateral

Example 0.1. $N(3) = 3$.

Example 0.2. $n = 4$. *In 1932 Esther Klein made the following observation: Among any five points in general position in the Euclidean plane, it is always possible to select four points that form the vertices of a convex quadrilateral.*
Proof: The **convex hull** of a set of points S in the Euclidean plane is the smallest convex polygon that encloses all points from S.

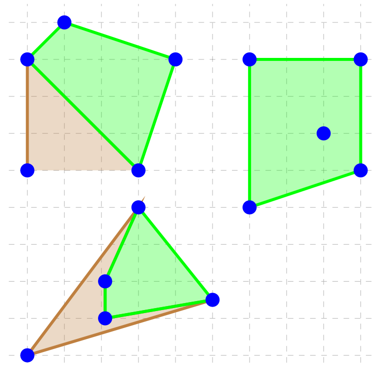


Figure 3: Five points and three cases: Types $(5, 0)$, $(4, 1)$, and $(3, 2)$. Here "Type (n, m)" means that the convex hull is an n-gon.

Question 0.3. *Is it possible to find four points in the plane that do not form a convex quadrilateral?*

Therefore... $N(4) =$

$n = 5$**.** See Figures 4 an d 5.

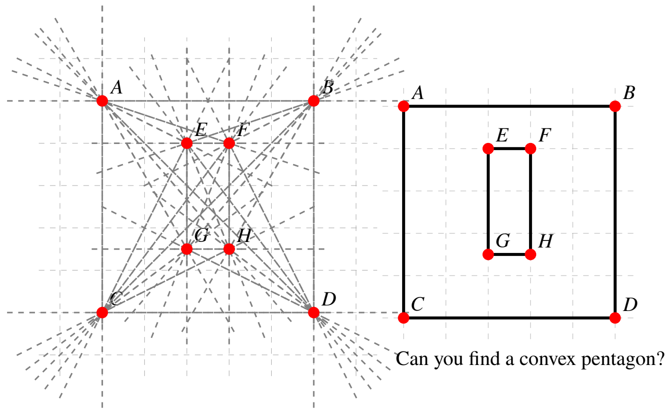


Figure 4: Eight points in general position.

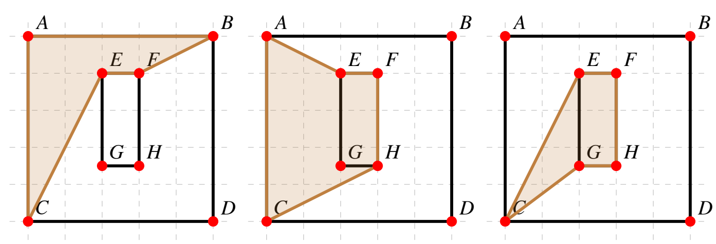


Figure 5: No! - A few cases.

Therefore... $N(5) \geq 9$.
$n = 5$ **... Part II** Let S be a set of nine points in the plane in general position. Let $\overline{S}$ be the convex hull of S.

1. If $\overline{S}$ has **five or more** vertices, we are done. See Figure 6.

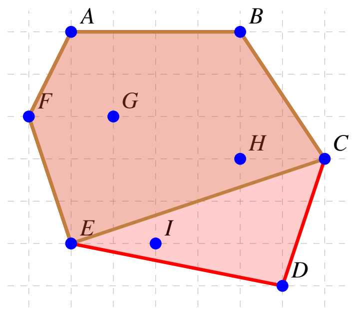


Figure 6: The convex hull of S has six points.

2. Let the convex hull $\overline{S}$, the convex hull of S, has three or four vertices. Then the set $T = S \backslash \overline{S}$ contains six or five (remaining) points of S and they are all inside of $\overline{S}$.

   Let $\overline{T}$ be the convex hull of T.

3. If $|\overline{T}| = 5$ or $|\overline{T}| = 6$... Done! See Figure 7.

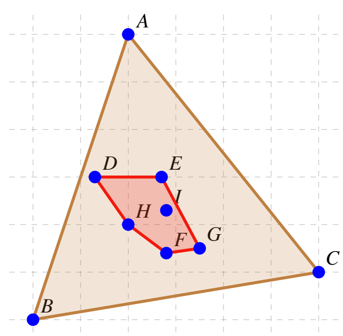
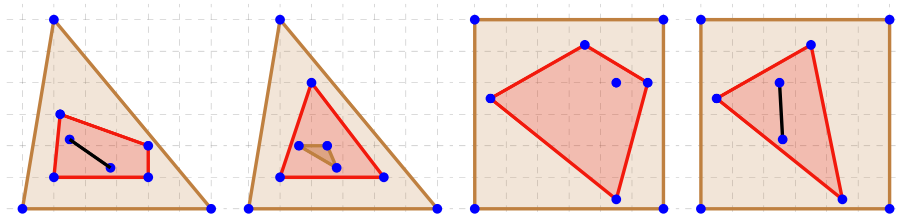


Figure 7: Example: $|S| = |\{A, B, \ldots, I\}| = 9$, $|\overline{S}| = |\{A, B, C\}| = 3$, $|T| = |\{D, E, \ldots, I\}| = 6$, and $|\overline{T}| = |\{D, E, F, G, H\}| = 5$.

4. For the remaining cases see Figure 8.



Type (3, 4, 2) **Type** (3, 3, 3) **Type** (4, 4, 1) **Type** (4, 3, 2)

Figure 8: Four remaining cases.

Configuration of the type (3, 3, 2)**.**

1. **Consider the inside triangle and the line segment.**
   - The line that contains the line segment intersects two sides of the triangle.
   - Notice the vertex where those two sides of the triangle intersect.
   - Draw rays starting at the end points of the line segment as on Figure 9

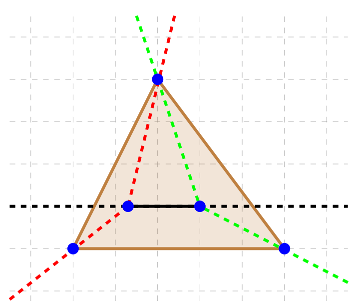


Figure 9: A triangle, a line segment, and four rays.

Three regions. Notice the three open regions in the plane on the Figure 10:

- None of the three regions intersects the interior of the triangle
- Region 1 and Region 2 intersect (part of the plane 'above' the top vertex.)
- Region 3 does not intersect either Region 1 or Region 2.

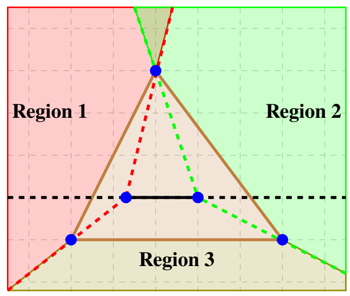
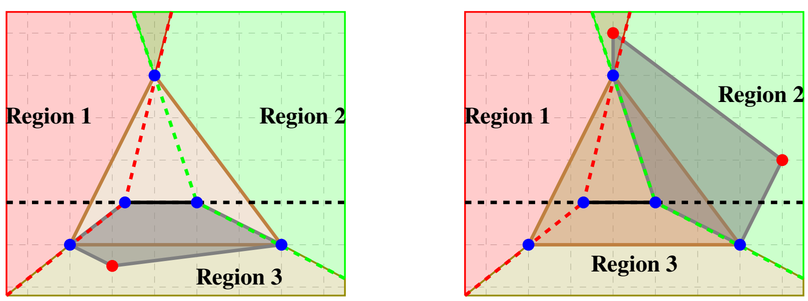


Figure 10: Three regions.

Three points outside of the triangle. Note that the remaining three points in the configuration $3-3-2$ cannot be on the boundary of any of Regions 1-3. (Why?) See Figure 11.



One of the *outside points* belongs to Region 3. None of the *outside points* belongs to Region 3.

Figure 11: There is a convex pentagon!

Configuration of the type $(3, 3, 3)$**.** Note that the configuration the 8-point configuration $(3, 3, 2)$ is contained in the configuration $(3, 3, 3)$. See Figure 12.

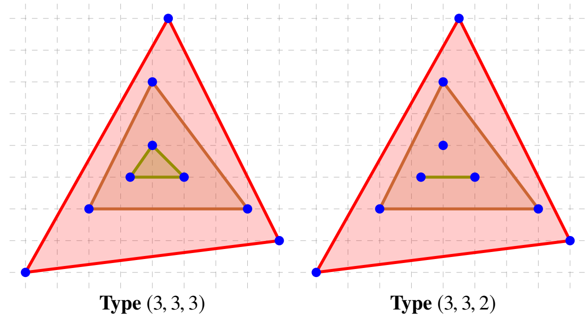


Figure 12: Type $(3, 3, 3)$ contains Type $(3, 3, 2)$.

Therefore the configuration of the type $(3, 3, 3)$ contains a convex pentagon.

Configuration of the type $(4, 3, 1)$**.**

1. Consider a triangle and a single point inside of it, and note three regions, Figure 13.

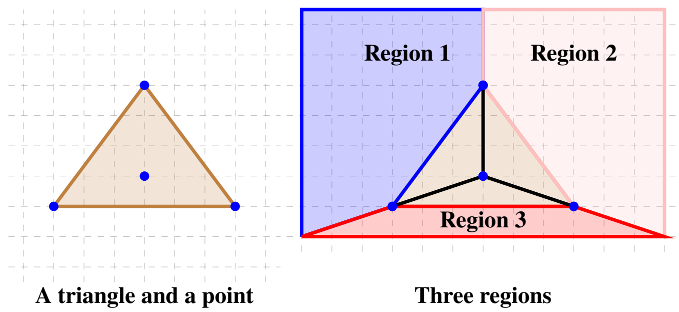


Figure 13: Type $(*, 3, 1)$.

2. By the Pigeonhole Principle, at least two of the remaining four points must belong to the same region, say Region 2. See Figure 14.

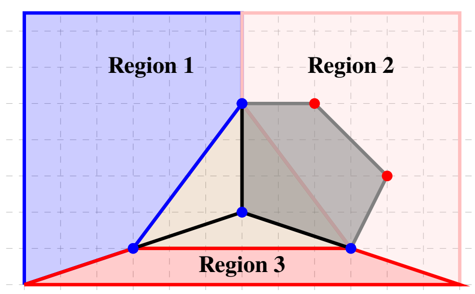
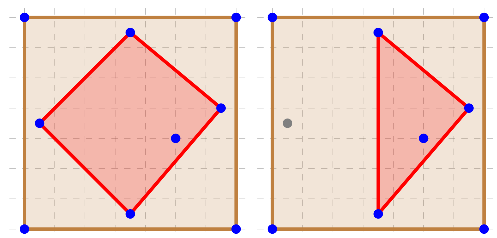


Figure 14: There is a convex pentagon!

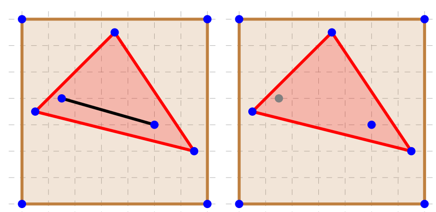
Configuration of the type $(4,4,1)$**.** Note that the configuration $(4,4,1)$ contains the configuration $(4,3,1)$. See Figure 15.



Type $(4,4,1)$ **Type** $(4,3,1)$

Figure 15: The configuration of the type $(4,4,1)$ contains a convex pentagon..

Configuration of the type $(4,3,2)$**.** Note that the configuration $(4,3,2)$ contains the configuration $(4,3,1)$. See Figure 16.



Type $(4,3,2)$ **Type** $(4,3,1)$

Figure 16: The configuration of the type $(4,3,2)$ contains a convex pentagon..

Configuration of the type $(3,4,2)$**.** Consider the inside quadrilateral and the line segment. See Figures 17 – 19.

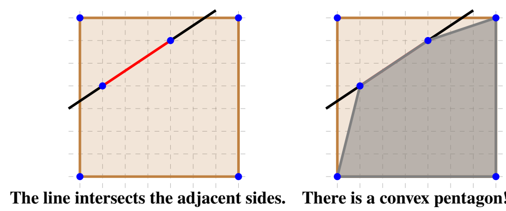


Figure 17: Case 1: The line that contains the line segment intersects the adjacent sides of the quadrilateral.

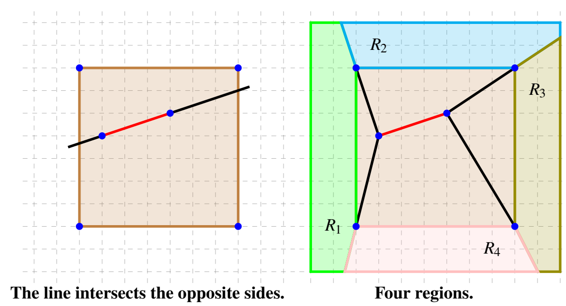


Figure 18: Case 2: The line that contains the line segment intersects the opposite sides of the quadrilateral.

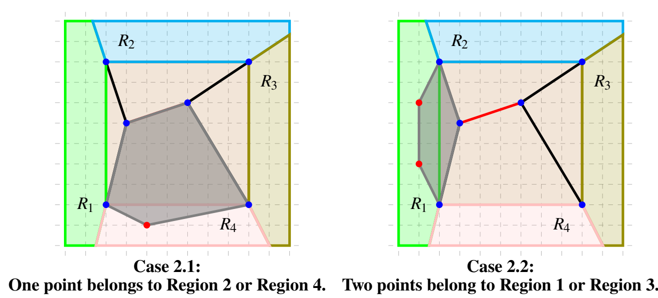


Figure 19: Case 2: There is a convex pentagon!

Therefore

$$N(5) = 9.$$