Erdős-Szekeres Problem of Convex Polygons – Part 2

All generalizations are false, including this one. – Mark Twain, American author, 1835 – 1910

Reminder. For any integer $n \geq 3$, determine the smallest positive integer $N(n)$ such that any set of at least $N(n)$ points in general position in the plane (i.e. no three of the points are on a line) contains n points that are the vertices of a convex n-gon.

1. $N(3) =$
2. $N(4) =$
3. $N(5) =$

Question 0.1. *1. Does $N(n)$ exist for any $n \geq 3$?*

2. Which values of $N(n)$ are known?

3. Are there any bounds for the size of $N(n)$?

Pattern?

1. $N(3) = 3 = 2^1 + 1 = 2^{3-2} + 1$
2. $N(4) = 5 = 2^2 + 1 = 2^{4-2} + 1$
3. $N(5) = 9 = 2^3 + 1 = 2^{5-2} + 1$

Well... Szekeres and Peters, 2006:

$$N(6) = 17 = 2^4 + 1 = 2^{6-2} + 1.$$

Conjecture 0.2. *For any $n \geq 3$*

$$N(n) = 2^{n-2} + 1.$$

Not long before his death in 1996, Erdős wrote that he would pay $500 for a proof of this conjecture.
Still... How do we prove that $N(n)$ exists for all $n \geq 3$?

Two Theorems:
Recall:
Ramsey's Theorem. For any natural numbers $k, r, l_1, \ldots, l_r$ there exists the least natural number $m_0 = R(k; l_1, l_2, \ldots, l_r)$ such that for any $m \geq m_0$, if the set of all k-element subset of the set S_m, where $|S_m| = m$, is r-coloured then there exists $i \in [1, r]$ and the l_i-element subset $\Delta_{l_i} \subseteq S_m$ such that all its k-element subsets have the colour i.
Lemma 0.3. *Let $n \geq 4$ be an integer. Then n points in the plane form a convex polygon if and only if every four of them form a convex quadrilateral.*
(Proof by induction.)

Theorem 0.4. ***Erdős-Szekeres' Theorem*** *$N(n)$ exists for any $n \geq 3$.*

Proof. We already know that $N(3) = 3$, $N(4) = 5$, and $N(5) = 9$.
Let $n \geq 4$. Let $m > R(4; n, 5)$ and let S_m be a set of m points in the plane in general position. Let

$$S_m^{(4)} = \{\{A, B, C, D\} : A, B, C, D \in S_m\},$$

i.e., let $S_m^{(4)}$ be the set of all four-element subsets of S_m.
We define a 2-colouring

$$c : S_m^{(4)} \longrightarrow \{\bullet, \bullet\}$$

in the following way. For $T \in S_m^{(4)}$

$$c(T) = \begin{cases} \bullet & \text{if } T \text{ forms a concave quadrilateral} \\ \bullet & \text{if } T \text{ forms a convex quadrilateral.} \end{cases}$$

See Figure 1.

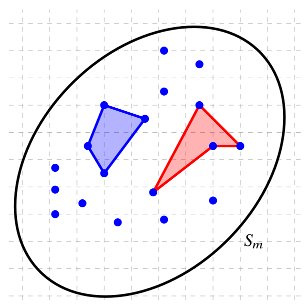

Figure 1: A convex quadrilateral and a non-convex quadrilateral

By Ramsey's theorem (and our choice of $m > R(4; n, 5)$) there is an n-element set $\Delta_n \subset S_m$ such that all of its four-element subsets are coloured blue or a 5-element set $\Delta_5 \subset S_m$ such that all of its four-element subsets are coloured red.
Since $N(4) = 5$, any set of five points in the plane in general position contains a convex quadrilateral. This implies that it is impossible to find a 5-element set $\Delta_5 \subset S_m$ such that all of its four-element subsets are coloured red.
Hence there must be an n-element set $\Delta_n \subset S_m$ such that all of its four-element subsets are coloured blue. But then, by Lemma, the set Δ_n forms a convex n-gon. $\square$

Therefore... For any $n \geq 4$, $N(n) \leq R(4; n, 5)$.

Cups and Caps. See Figure 2.

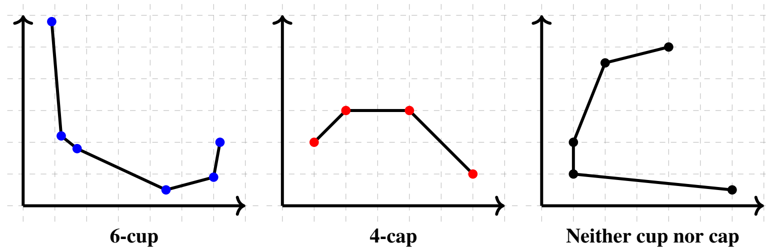

Figure 2: Cups and caps

Observation 0.5. *Note that in a k-cup, the sequence of slopes is increasing and that in an l-cap, the sequence of slopes is decreasing.*

Observation 0.6. *It is clear that if we find a cup or a cap in a set S in some system of coordinates, then we will also find a convex polygon. See Figure 3.*

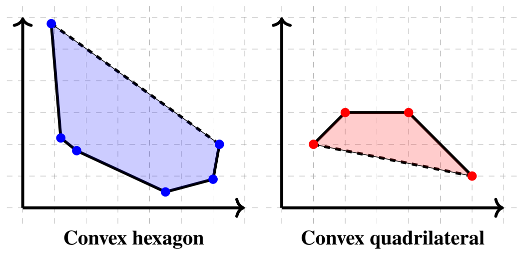

Figure 3: Convex polygons from cups and caps

Observation 0.7. *The expression"in some system of coordinates" can be substituted for the expression "in any system of coordinates, in which there are no two points in S that belong to the vertical line". Let us call any system of coordinates with such property right for S. In what follows we will always assume that, for a given set S of points in general position, we have chosen a coordinate system that is right for S. See Figure 4.*

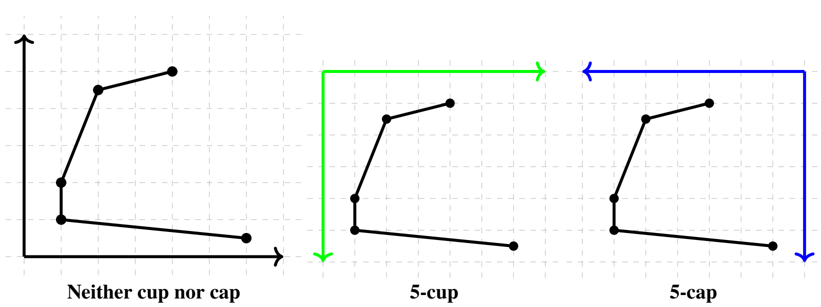

Figure 4: Making a *right* coordinate system

Definition 0.8. *For $k, l \geq 3$ we define $f(k,l)$ to be the least positive integers such that any set S of points in the plane (with a given coordinate system that is 'right' for S) in general position such that*

$$|S| \geq f(k,l)$$

contains either a k-cup or an l-cap.

Theorem 0.9. ***Theorem about cups and caps:*** *For $k, l \geq 3$ the number $f(k,l)$ exists and, for $k, l \geq 4$ we have that*

$$f(k,l) \leq f(k-1,l) + f(k,l-1) - 1.$$

Proof. We prove the theorem by induction on $m = k + l$ if $m \geq 6$.

1. Base Case: For any $k \geq 3$,

$$f(k,3) = f(3,k) = k.$$

Take a set S with k points in the plane in general position. Let

$$S = \{A_1, A_2, \dots, A_k\}$$

where for $i < j$, the x-coordinate of A_i is less than the the x-coordinate of A_j.

Let s_i be the slope of the line segment $\overline{A_i A_{i+1}}$, for $i \in [1, k-1]$.

If

$$s_1 < s_2 < \cdots < s_{k-1}$$

then S contains a k-cup.

If, for some $i \in [1, k-2]$,

$$s_i > s_{i+1}$$

then the set S contains a 3-cap $\{A_i, A_{i+1}, A_{i+2}\}$.

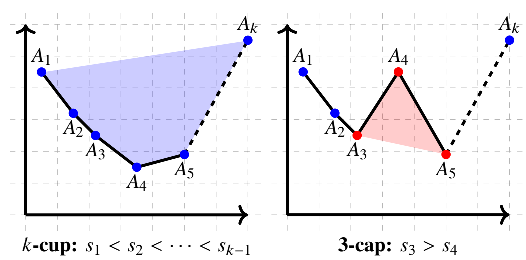

Figure 5: $f(k,3) = f(3,k) = k$

2.

Observation 0.10. *Note that we have proved that* $f(3,3) = 3$ *and* $f(4,3) = f(3,4) = 4$*. In particular this means that if* $k, l \geq 3$ *are such that* $k + l \leq 7$ *then* $f(k,l)$ *exists.*

3. Inductive Step: Let $m \geq 7$ and $k, l \geq 3$ be such whenever $k + l = m$ then $f(k,l)$ exists.

   We choose $k, l \geq 3$ such that
   $$k + l = m + 1.$$
   This implies that
   $$(k-1) + l = k + (l-1) = m$$
   and, hence $f(k-1,l)$ and $f(k,l-1)$ exist.

   Let
   $$n = f(k-1,l) + f(k,l-1) - 1.$$
   Let us fix a set S of cardinality n and any right for S system of coordinates. We have to prove that S contains either a k-cup or an l-cap.

   Let L be the set of all points that are the left ends of $(k-1)$-cups in S.

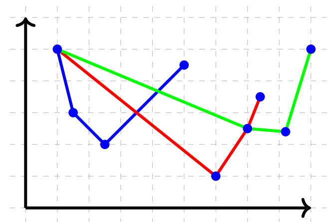

Figure 6: For $k - 1 = 4$, the top left point belongs to L.

   (a) Let us assume first that the set $S \backslash L$ has at least $f(k-1,l)$ points. Then it contains either a $(k-1)$-cup or an l-cap. But taking the set L out of S *destroys* all $(k-1)$-cups in S. Hence $S \backslash L$ does not contain any $(k-1)$-cups and therefore must contain an l-cap.

   (b) Suppose then that $|S \backslash L| \leq f(k-1,l) - 1$. It follows
   $$|L| = |S| - |S \backslash L| \geq n - f(k-1,l) + 1 = f(k,l-1).$$
   Therefore, there exists a k-cup in L (and everything is alright) or there exists a $(l-1)$-cap in L.

   Let us consider the point Y, the right end of that cap. Let X be the point that is immediately to the left of Y in the $(l-1)$-cap in L.

   Since $Y \in L$, the point Y is a left end of some $(k-1)$-cup in S. Let Z be the point that is immediately to the right of Y in the $(k-1)$-cup in S.

   If
   $$\text{Slope of } \overline{XY} > \text{Slope of } \overline{YZ}$$

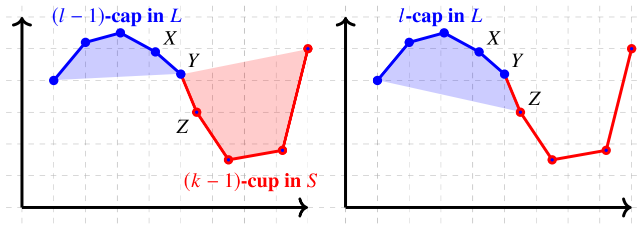

Figure 7: Getting an l-cap in S from an $(l-1)$-cap in L.

then adding the point Z to the $(l-1)$-cap in L makes an l-cap in S. (See Figure 7.) If

$$\text{Slope of } \overline{XY} < \text{Slope of } \overline{YZ}$$

then adding the point X to the $(k-1)$-cup in S makes an k-cup in S. (See Figure 8.)

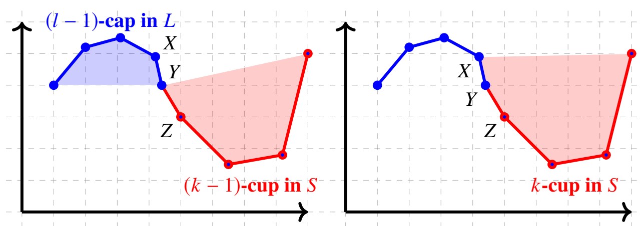

Figure 8: Getting a k-cup in S from an l-cap in L.

Therefore, any set S such that

$$|S| = n = f(k-1, l) + f(k, l-1) - 1$$

contains either a k-cup or an l-cup which implies that $f(k, l)$ exists and that

$$f(k, l) \leq f(k-1, l) + f(k, l-1) - 1.$$

By the Principle of Mathematical Induction, $f(k, l)$ exists for any $k, l \geq 3$.

$\square$

How big is $f(k, l)$?

$$f(k, l) \leq \binom{k+l-4}{k-2} + 1.$$

Proof via induction on $k + l$: If $k + l = 6$ then $k = l = 3$ and

$$f(3, 3) = 3 \text{ and } \binom{3+3-4}{3-2} + 1 = 2 + 1 = 3.$$

If $k + l = 7$ then $k = 4$ and $l = 3$ or $k = 3$ and $l = 4$. From

$$f(4,3) = f(3,4) = 4 \text{ and } \binom{4+3-4}{4-2} + 1 = 3 + 1 = 4 \text{ and } \binom{3+4-4}{3-2} + 1 = 3 + 1 = 4$$

we conclude that in the case that $k + l = 7$ we have

$$f(k,l) \leq \binom{k+l-4}{k-2} + 1.$$

Suppose that $m \geq 7$ is such that whenever $k, l \geq 3$ are such such that $k + l = m$ then

$$f(k,l) \leq \binom{k+l-4}{k-2} + 1.$$

Actually. . .

$$f(k,l) = \binom{k+l-4}{k-2} + 1.$$

Back to $N(n)$:

$$N(n) \leq f(n,n) \leq \binom{2n-4}{n-2} + 1.$$

Also. . .

$$N(n) \geq 2^{n-2} + 1.$$

What is known?

1. The next step is if not to prove the hypothesis, then at least to improve the estimation a little bit.

2. The inequality
$$N(n) \leq \binom{2n-4}{n-2} + 1$$
was proved by Erdős and Szekeres in 1935.

3. And in 1998 there were three improvements at once!

   (a) The first of them was made by F. Chung and R. Graham:
   $$N(n) \leq \binom{2n-4}{n-2}.$$

   (b) D. Kleitman and L. Pachtler showed that it is true that
   $$N(n) \leq \binom{2n-4}{n-2} + 7 - 2n$$

   (c) The third improvement was achieved by G. Tot and P. Vultr:
   $$N(n) \leq \binom{2n-5}{n-2} + 2.$$

   The last result is approximately twice as good as the result of Erdős - Szekeres.

4. The current record also belongs to Tot and Vultr (2005).
$$N(n) \leq \binom{2n-5}{n-2} + 1, \ n \geq 5.$$

5. And that was it, no more improvement. This is a classic problem, and crowds of people are working on it.

6. Chung and Graham offered $100 for the first proof that
$$g(n) \leq c^n,$$
where $c < 4$ is a constant.

Resources.

1. Happy Ending Problem - Wikipedia

2. Happy Ending Problem by D. Harvey

3. Erdős, P.; Szekeres, G. (1935), "A combinatorial problem in geometry", Compositio Math 2: 463Ð470.

4. The Erdos-Szekeres problem on points in convex position - a survey by W. Morris and V. Soltan