Richard Rado (1906 - 1989)

1. **Quote.** There are almost as many types of mathematicians as there are types of human being. Among them are technicians, there are artists, there are poets, there are dreamers, men of affairs, and many more.

   Richard Rado

2. **Who was Richard Rado?** A mathematician who earned two Ph.D.s: in 1933 from the University of Berlin under Isai Schur, and in 1935 from the University of Cambridge under Godfrey Harold Hardy.

   Mathematical work of Richard Rado:

   - Convergence of sequences and series
   - Inequalities
   - Geometry and measure theory
   - Graph theory
   - Number theory
   - Ramsey theory

3. **Birth and Death.** Richard Rado was born on 28 April 1906 in Berlin, Germany, and died on 23 December 1989 in Henley-on-Thames, Oxfordshire, England.

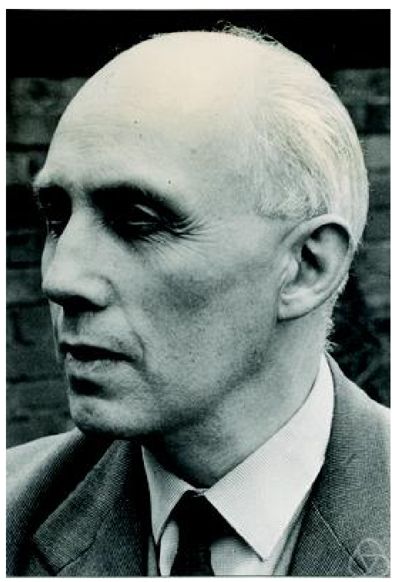


Figure 1: Richard Rado

4. **World in 1906.**

Jan 13th 1st radio set advertised (Telimco for $7.50 in Scientific American) claimed to receive signals up to one mile

March 6th Nora Blatch is 1st woman elected to American Society of Civil Engineers

April 6th 1st animated cartoon copyrighted

April 11th Albert Einstein introduces his Theory of Relativity

May 22nd Wright Brothers patent an airplane

Sept 1st Alberta adopts Mountain Standard Time

Sept 11th Mahatma Gandhi coins the term "Satyagraha" to characterize the Non-Violence movement in South Africa.

Oct 3rd SOS adopted as warning signal by 1st conference on wireless telegraphy

Oct 25th US inventor Lee de Forest patents "Audion," a 3-diode amplification valve which proved a pioneering development in radio & broadcasting

5. **World in 1989.**

6. **Richard's Family.** Richard was born in Berlin. He was the second son of Leopold Rado, who was a Hungarian from Budapest. As a young man he had to choose between being a concert pianist or a mathematician. He chose to become a mathematician in the belief that he could continue with music as a hobby, but that he could never treat mathematics in that way.

   In 1933 he married Luise Zadek, whom he had met when he needed a partner to play piano duets. They had one son, Peter Rado, born in 1943.

Figure 2: Luise and Richard Rado

> "Rado and his wife had a double partnership: she went with him to mathematical conferences and meetings and kept contact with his mathematical friends, he was an accomplished pianist and she was a singer of professional standard. They gave many recitals both public and private, often having musical evenings in their home in Reading. Rado was the kindest and gentlest of men."

7. **Paul Erdős about Richard Rado:**

> I first became aware of Richard Rado's existence in 1933 when his important paper *Studien zur Kombinatorik* appeared. I thought a great deal about the many fascinating and deep unsolved problems stated in this paper but I never succeeded to obtain any significant results here (...)
>
> Our joint work extends to more than 50 years; we wrote 18 joint papers (...)
>
> Our most important work in undoubtedly in set theory and, in particular, the creation of the partition calculus. The term partition calculus is, of course, due to Rado. Without him, I often would have been content in stating only special cases.

8. Timeline.

9. **Canadian Connection.**

   - Canadian Commonwealth Fellow, University of Waterloo, 1971-1972
   - Visiting Professor, University of Calgary, 1973-1974
   - Hon.D. Mathematics, University of Waterloo, 1986

10. **Rado's Work - Three Examples.**

    (a) **Partition Calculus:** In Erdős' words:

    > The investigation centres round what we call partition relations connecting given cardinal numbers or order types and in each given case the problem arises of deciding whether a particular partition relation is true or false. It appears that a large number of seemingly unrelated arguments in set theory are, in fact, concerned with just such a problem. It might therefore be of interest to study such relations for their own sake and to build up a partition calculus which might serve as a new and unifying principle in set theory.

    Ramsey's Theorem: For any $n, m < \omega$, one has $\omega \to (\omega)^n_m$.

    In the early 1950s, Erdős and Rado introduced, what they called, the *decomposition relation* $a \to (b_0, b_1)^2$ between cardinals a, b_0, and b_1. Here, $a \to (b_0, b_1)^2$ holds if, for any set S such that $|S| = a$ and for any colouring $c : S^{(2)} \to \{0, 1\}$, there are $i \in \{0, 1\}$ and a set $S_i \subseteq S$, $|S_i| = b_i$, such that $c\left(S_i^{(2)}\right) = \{i\}$, i.e. for any $T \in S_i^{(2)}$, $c(T) = i$.

    For example, the fact that $R(3, 4) = 9$ means that $a \to (3, 4)^2$ holds for any integer $a \geq 9$ and that $8 \to (3, 4)^2$ does not hold.

    In a follow up paper, Erdős and Rado extended this notation (and introduced the term *partition relation*) to

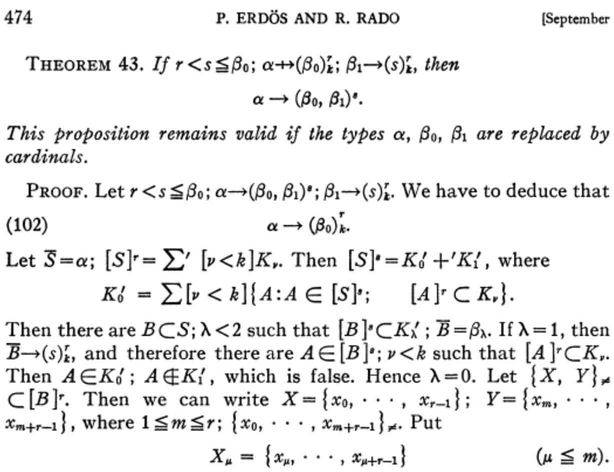
    $$a \to (b_0, b_1, \ldots)^r_k,$$

    with a, b_i, r, k being cardinals such that $b_i < k$.

    Here, $a \to (b_0, b_1, \ldots)^r_k$ holds if for any set S such that $|S| = a$ and for any k–colouring c of $S^{(r)}$ there is $i < k$ and a set $S_i \subseteq S$, $|S_i| = b_i$, such that $c\left(S_i^{(r)}\right) = \{i\}$.

    If $b_0 = b_1 = \ldots = b$, the convention is to write $a \to (b)^r_k$. If k is a finite cardinal, we write $a \to (b_0, b_1, \ldots, b_{k-1})^r$.

    For example, Ramsey's theorem may be stated as:



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THEOREM 43. *If* $r<s\leqq\beta_0$; $\alpha\nrightarrow(\beta_0)^r_k$; $\beta_1\rightarrow(s)^r_k$, *then*

$$\alpha \rightarrow (\beta_0, \beta_1)^s.$$

This proposition remains valid if the types α, β_0, β_1 *are replaced by cardinals.*

PROOF. Let $r<s\leqq\beta_0$; $\alpha\rightarrow(\beta_0, \beta_1)^s$; $\beta_1\rightarrow(s)^r_k$. We have to deduce that

(102) $$\alpha \rightarrow (\beta_0)^r_k.$$

Let $\overline{S}=\alpha$; $[S]^r=\sum'[\nu<k]K_\nu$. Then $[S]^s=K_0'+'K_1'$, where

$$K_0' = \sum[\nu < k]\{A:A \in [S]^s; \quad [A]^r \subset K_\nu\}.$$

Then there are $B\subset S$; $\lambda<2$ such that $[B]^s\subset K_\lambda'$; $\overline{B}=\beta_\lambda$. If $\lambda=1$, then $\overline{B}\rightarrow(s)^r_k$, and therefore there are $A\in[B]^s$; $\nu<k$ such that $[A]^r\subset K_\nu$. Then $A\in K_0'$; $A\notin K_1'$, which is false. Hence $\lambda=0$. Let $\{X, Y\}_{\neq}\subset[B]^r$. Then we can write $X=\{x_0, \cdots, x_{r-1}\}$; $Y=\{x_m, \cdots, x_{m+r-1}\}$, where $1\leqq m\leqq r$; $\{x_0, \cdots, x_{m+r-1}\}_{\neq}$. Put

$$X_\mu = \{x_\mu, \cdots, x_{\mu+r-1}\} \qquad (\mu \leqq m).$$

Figure 3: From Erdős-Rado's paper

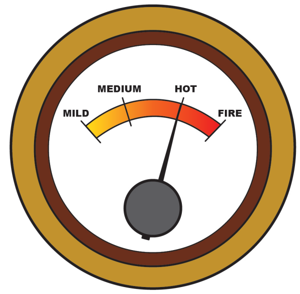

Figure 4: How complicated?

> For every choice of positive integers r, n, μ, there is an integer m_0 such that, if $m \geq m_0$ then $m \rightarrow (n)^r_\mu$.

Similarly, it is common to state Theorem 2.4.2, Ramsey's Theorem for $\mathbb{N}^{(2)}$ – Two Colours, as: $\omega \rightarrow (\omega)^2_2$. Here, ω represents the smallest infinite ordinal.

(b) In 1940, Rado characterized so–called *distributive sequences*:

> We say that a sequence $(\kappa_\nu, \lambda_\nu)$, $\nu = 1, 2, \ldots$, of pairs of positive integers is distributive if the convergence of the $\Sigma_{\kappa=1}^\infty X_\kappa$ and $\Sigma_{\lambda=1}^\infty Y_\lambda$ implies the convergence of the series $\Sigma_{i=1}^\infty X_{\kappa_i} Y_{\lambda_i}$ and the validity of the equation
>
> $$(\Sigma_{\kappa=1}^\infty X_\kappa) \cdot (\Sigma_{\lambda=1}^\infty Y_\lambda) = \Sigma_{i=1}^\infty X_{\kappa_i} Y_{\lambda_i}.$$

In order to determine all distributive sequences, Rado used Ramsey's theorem to prove the following statement.

> For any $l \in \mathbb{N}$ there is $N = N(l) \in \mathbb{N}$ such that, for any choice of nonempty sets $M_1, M_2, \ldots, M_N$, it is always possible to select l sets $M_{a_\lambda} = M'_\lambda$, $(\lambda \leq l)$, where $a_1 < a_2 < \ldots < a_l \leq N$, and to find a number $p \leq l + 1$ such that, however we choose $r \leq l$ numbers $e_1 < e_2 < \ldots < e_r$, we have
>
> $$M'_{e_1} \cap M'_{e_2} \cap \ldots \cap M'_{e_r} \begin{cases} \neq \emptyset & \text{if } r < p \\ = \emptyset & \text{if } p \leq r \leq l. \end{cases}$$

Proof: Let $l \in \mathbb{N}$. Let $N = N(l)$ be a positive integer with the property that for any l–colouring of $[1, N]^{(l)}$, the set of all l–subsets of $[1, N]$, there is a subset S of $[1, N]$, $|S| = 2l - 1$, such that $S^{(l)}$ is monochromatic. Recall that, by Ramsey's theorem, we can take $N(l) = R_l(l; 2l - 1) = R(l; \underbrace{2l - 1, \ldots, 2l - 1}_{l})$.

Let $M_1, M_2, \ldots, M_N$ be non–empty sets. We define an l–colouring c of $[1, N]^{(l)}$ in the following way. Let $T = \{x_1, x_2, \ldots, x_l\} \subset [1, N]$, with $x_i < x_{i+1}$, for all $i \in [1, l-1]$. By definition, $c(T) = j \in [1, l]$, if j is the largest number such that $M_{x_1} \cap M_{x_2} \cap \ldots \cap M_{x_j} \neq \emptyset$. By our choice of N, there is a set $S = \{a_1, a_2, \ldots, a_{2l-1}\} \subset [1, N]$, with $a_i < a_{i+1}$ for all $i \in [1, 2l-2]$, such that $S^{(l)}$ is c–monochromatic. Let $c(T) = j \in [1, l]$, for all $T \in S^{(l)}$, and let $p = j + 1$. Let $M_{a_i} = M'_i$, for $i \in [1, l]$.

Let $r \in [1, l]$ and let $1 \leq e_1 < e_2 < \cdots < e_r \leq l$. If $r = l$ then $\{a_{e_1}, a_{e_2}, \ldots, a_{e_r}\} \in S^{(l)}$. If $r < l$, then $\{a_{e_1}, a_{e_2}, \ldots, a_{e_r}, a_{l+1}, \ldots a_{l+r-1}\} \in S^{(l)}$. By the definition of the colouring c, if $r \leq j < p$ then $M'_{e_1} \cap M'_{e_2} \cap \ldots \cap M'_{e_r} \neq \emptyset$, and if $j < p \leq r \leq l$ then $M'_{e_1} \cap M'_{e_2} \cap \ldots \cap M'_{e_r} = \emptyset$.

(c) **Rado Graph.**

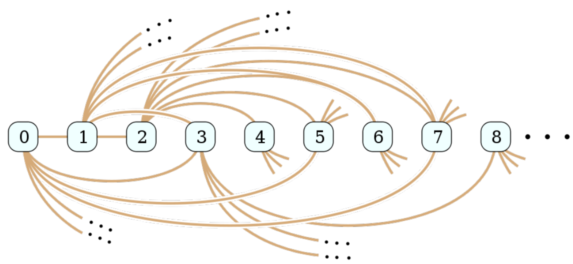

Figure 5: Rado graph

In 1964 Rado constructed the Rado graph by identifying the vertices of the graph with the natural numbers $0, 1, 2, \ldots$. An edge connects vertices x and y in the graph (with $x < y$) whenever the xth bit of the binary representation of y is nonzero. Thus, for instance, the neighbours of vertex 0 consist of all odd-numbered vertices, while the neighbours of vertex 1 consist of vertex 0 (the only vertex whose bit in the binary representation of 1 is nonzero) and all vertices with numbers congruent to 2 or 3 modulo 4.