Van der Waerden's Theorem - Part 4

1. **Quote.** Never measure the height of a mountain until you have reached the top. Then you will see how low it was.

   Dag Hjalmar Agne Carl Hammarskjöld, Swedish diplomat, economist, and author, 1905 -1961

2. **Van der Waerden's Theorem:** Let $l, k \in \mathbb{N}$. Any k-colouring of positive integers contains a monochromatic l-term arithmetic progression. Moreover, there is a natural number N such that any k-colouring of the segment of positive integers $[1, N]$ contains a monochromatic l-term arithmetic progression.

3. **Question:** Is it true that any 2-colouring of positive integers contains an infinite monochromatic arithmetic progression?

4. **Question:** Is it true that any infinite colouring of positive integers contains a monochromatic l-term arithmetic progression, for $l \in \mathbb{N}$?

5. **Canonical form of van der Waerden's theorem:** If f is an arbitrary function from the positive integers to the positive integers, then there are arbitrarily large arithmetic progressions P such that the restriction of f to P is either constant or one-to-one.

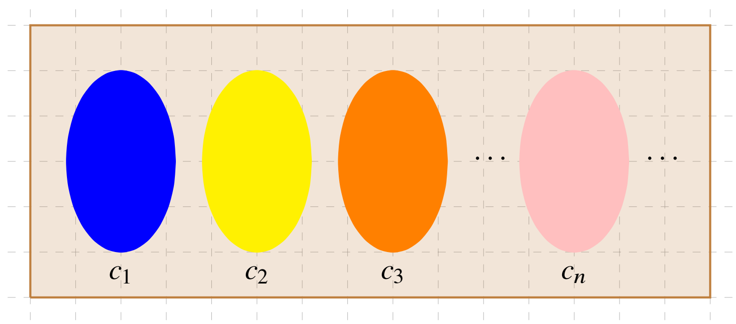
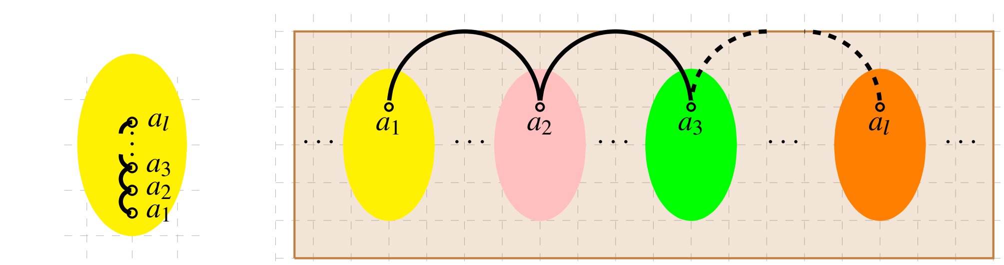


Figure 1: Divide positive integers in as many parts as you wish. Possibly infinite. . .



A monochromic l-AP

A rainbow l-AP

Figure 2: Canonical form: monochromatic or rainbow

6. **Two Questions.**

(a) Is it true that any finite colouring of positive integers contains a monochromatic l-term arithmetic progression with an **odd common difference**?

(b) Is it true that any finite colouring of positive integers contains a monochromatic l-term arithmetic progression with an **even common difference**?

7. **Polynomial van der Waerden Theorem.** Let $l, r \in \mathbb{N}$ and let p be a polynomial with integer coefficients such that $p(0) = 0$, i.e., let

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x,\ \ a_n, a_{n-1}, \cdots, a_1 \in \mathbb{Z}.$$

Then for any r-colouring of $\mathbb{Z}$ there are $a, d \in \mathbb{Z}$ such that the l-term arithmetic progression

$$a, a + p(d), a + 2p(d), \ldots, a + (l-1)p(d)$$

is monochromatic.

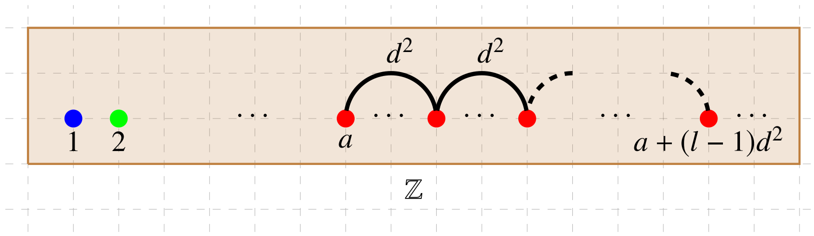


Figure 3: A monochromatic l-term arithmetic progression with step d^2.

Note: This is a very special case of the Polynomial van der Waerden Theorem proved by Bergelson and Leibman (Polynomial extensions of van der Waerden's and Szemeredi's theorems, Journal of the American Math Society, Vol. 9, 1996, 725-753.)

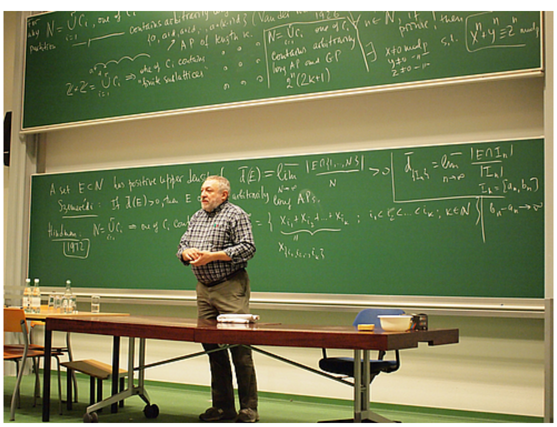


Figure 4: Vitaly Bergelson, American mathematician

Figure 5: Alexander Leibman, Israeli-American mathematician,

8. 2**-Large and Large Sets:**

   We say that a set $L \subseteq \mathbb{N}$ is 2-large if any 2-colouring of $\mathbb{N}$ contains long monochromatic arithmetic progressions with common difference in L.

   We say that a set $L \subseteq \mathbb{N}$ is large if any finite colouring of $\mathbb{N}$ contains long monochromatic arithmetic progressions with common difference in L.

9. **Examples:**

10. **Conjecture:** Every 2-large set is large.

    (T.C. Brown, R.L. Graham, and B.M. Landman, On the set of common differences in van der WaerdenÕs theorem on arithmetic progressions, Canad. Math. Bull. 42 (1999), 25-36.)

Figure 6: Bruce Landman, American mathematician

11. **Problem.** Colour the 12 points below with three colours so that you use each colour four times. Can you avoid 3-term rainbow arithmetic progressions?

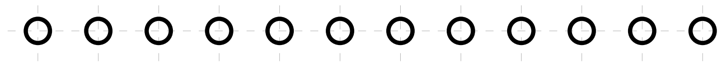


Figure 7: Colour with three colours; use each colour four times; look for a rainbow 3-term arithmetic progression.

12. **What About...** Colour the 15 points below with three colours so that you use each colour five times. Can you avoid 3-term rainbow arithmetic progressions?

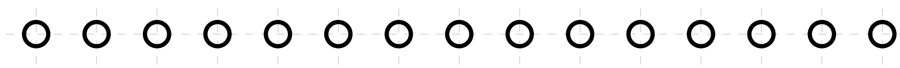


Figure 8: Can you avoid rainbow 3-term arithmetic progressions?

13. **Fact:** Every equinumerous 3-colouring of $[1, 3n]$ contains a rainbow 3-term arithmetic progression.

    (Jungić, V., Radoičić, R., Rainbow Arithmetic Progressions, Integers, Electron. J. Combin. Number Theory 3 (2003) A18)

Figure 9: Radoš Radoičić, Bosnian-born American mathematician

14. **van der Waerden Games:** Two players, Maker and Baker, play the following game on a one-way endless strip of empty 1×1 cells: Maker and Breaker take turns. On their turn, Maker puts one triangle into an empty cell, and they wins if they form a three term arithmetic progression. On their turn, Breaker puts 1000 circles into any 1000 (not necessarily adjacent) distinct empty cells and they wins if they prevents Maker from forming a three-term arithmetic progression of triangles.

    (a) Prove that Maker can win if they goes first.
    (b) What would happen if the players switch the order who starts the game?
    (c) What would happen if "three-term arithmetic progression of triangles" is replaced by "k-term arithmetic progression of triangles", $k > 3$?