Ramsey's Theorem - Part 1

1. **Quote.** A friend to all is a friend to none.

   (Aristotle, Greek philosopher, 384 BCE - 322 BCE)

2. **Edge 2- Colouring.** Use TWO colours, red and blue, for example, to colour the edges of K_6, a complete graph on six vertices. (Each edge should be coloured by only one colour.)

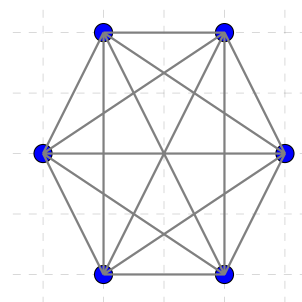


Figure 1: K_6 - a complete graph on six vertices

3. **Two Questions:**

   (a) How many different edge 2-colourings of K_6 are there?

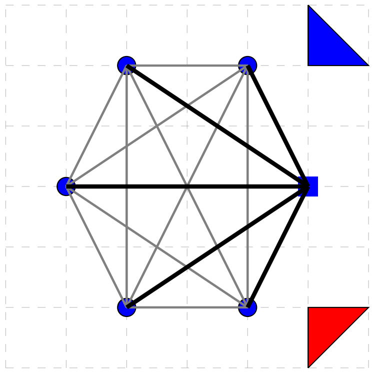
   (b) Can you find a monochromatic triangle in your colouring, i.e., three edges coloured by the same colour that form a triangle?

4. **BIG Question:** Does any edge 2-colouring of K_6 yield a monochromatic triangle?

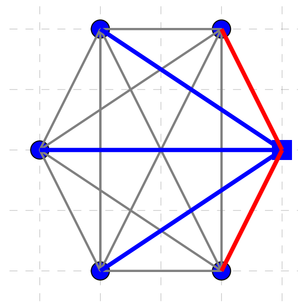
5. **BIG Answer:** Yes, any edge 2-colouring of K_6 yields a monochromatic triangle?

6. **Pigeonhole Principle:** Suppose you have k pigeonholes and n pigeons to be placed in them. If $n > k$ then at least one pigeonhole contains at least two pigeons.

7. **Proof.**

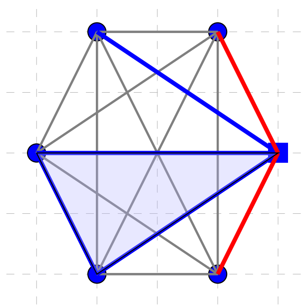


Step 1: Fix one vertex and colour FIVE adjacent edges

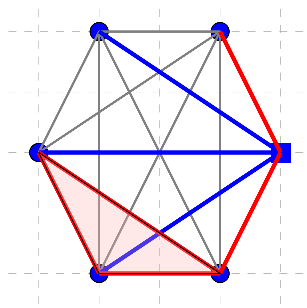


Step 2: At least three edges of the same colour!

Figure 2: Proof: Step 1 and Step 2



Step 3.1: At lease one edge is blue



Step 3.2: All three are red

Figure 3: Proof: Step 3 – Two cases

8. **Ramsey's Theorem - Special Case.** Any edge 2-colouring of K_6 yields a monochromatic K_3.

9. **Question.** Is this true for any edge 2-colouring of K_5?

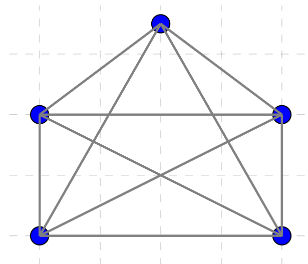


Figure 4: Find an edge 2-colouring of K_5 that avoids monochromatic triangles

10. **A Dinner Party Problem.** Suppose that six people are gathered at a dinner party. Then there is a group of three people at the party who are either all mutual acquaintances or all mutual strangers.

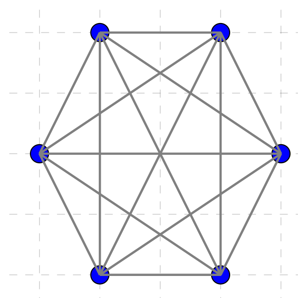


Figure 5: Do we know each other? YES or NO.

11. **Claim 1:** Any edge 2-colouring (blue and red) of K_{10} yields or a red K_4 or a blue K_3.

    Proof:

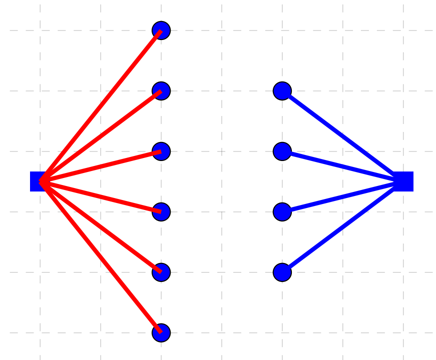
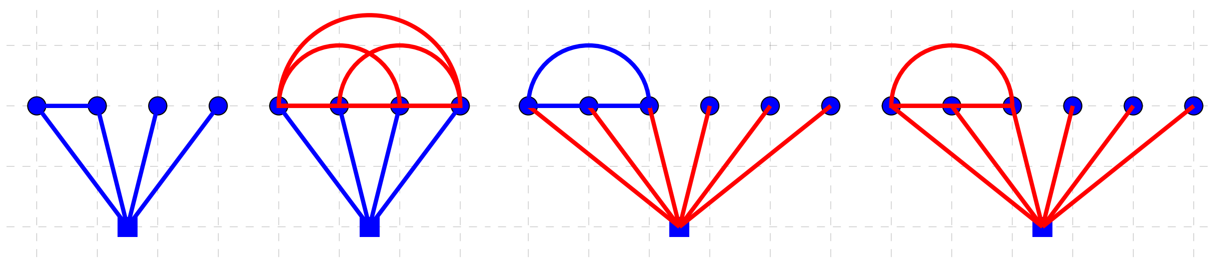


Figure 6: Step 1: Fix one vertex: ■. There are at least 6 red adjacent edges OR at least 4 blue adjacent edges



Case 1: At least four blue edges

Case 2: At least six red edges

Figure 7: Step 2 – Two cases

12. **Claim 2:** Any edge 2-colouring (blue and red) of K_9 yields or a red K_4 or a blue K_3.

Proof:

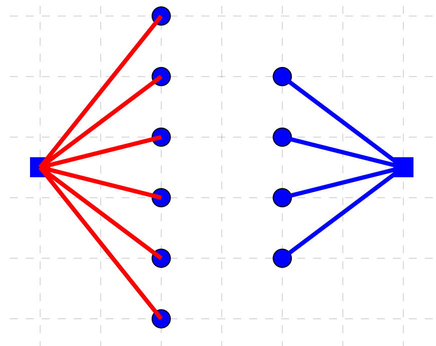


Figure 8: Step 1: If there is a vertex ■ with at least 6 red adjacent edges OR at least 4 blue adjacent edges - DONE

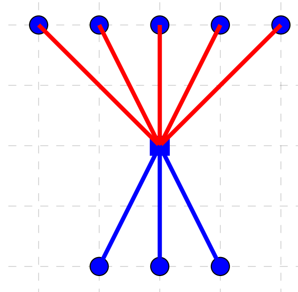


Figure 9: Step 2 – EVERY vertex ■ is adjacent with 5 red edges AND with 3 blue edges

How many blue edges altogether?

$$\frac{9 \cdot 3}{2} = 13.5$$

13. **Question.** Does every blue-red edge colouring of K_8 yield a red K_4 or a blue K_3?

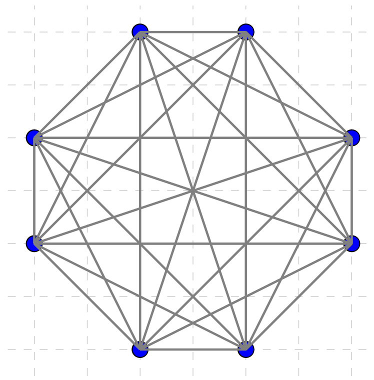


Figure 10: Find a blue-red edge colouring of K_8 with neither red K_4 nor blue K_3

14. **Ramsey's Theorem - Special Case.** Any blue-red edge colouring of K_9 yields a red K_4 or a blue K_3.

15. **Ramsey's Theorem - Special Case.** $R(4,3) = 9$.

16. **Ramsey's Theorem - Special Case.** $R(3,4) = 9$.

17. **Ramsey's Theorem - Special Case.** $R(4,4) \leq 18$.

    Proof: Consider a blue-red edge colouring of a K_{18}.

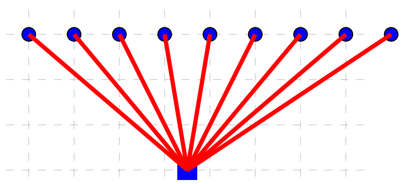
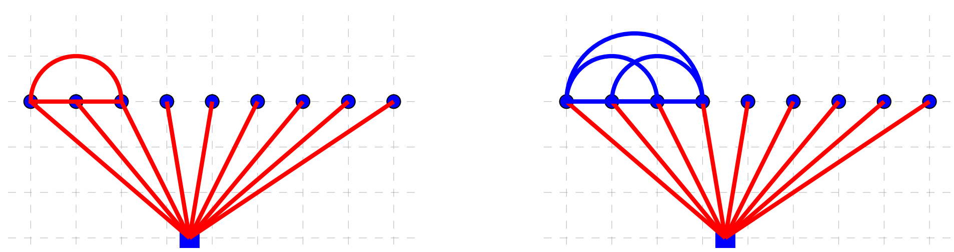


Figure 11: Step 1: Observe that each vertex ■ is adjacent to 9 edges of the same colour - say red



Case 1- There is a red K_3 n the induced K_9 Case 2: There is a blue K_4 in the induced K_9

Figure 12: Step 2 – Two cases

18. **Actually...** $R(4,4) = 18$

19. **Resources.**

(a) Theorem on Friends and Strangers - Wikipedia

(b) I. Leader, Friends and Strangers