

Chromatic Number of the Plane

It doesn't matter how long my hair is or what colour my skin is or whether I'm a woman or a man. – John Lennon, English musician, singer and songwriter, 1940 - 1980

Problem. What is the smallest number of colours sufficient for colouring the plane in such a way that no two points of the same colour are unit distance apart? – Edward Nelson, 1950

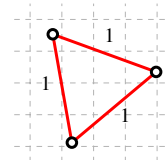
Edward Nelson was born on May 4, 1932, in Decatur, Georgia. He is a professor in the Mathematics Department at Princeton University.



(Source: Wikipedia)

Definition 0.1. Chromatic Number of the Plane. The smallest number of colours sufficient for colouring the plane in such a way that no two points of the same colour are unit distance apart is called the chromatic number of the plane and it is denoted by χ .

Question 0.2. Are two colours enough?



Proposition 0.3. $\chi \geq 3$.

Question 0.4. *Are three colours enough?*

Leo Moser was a professor of mathematics at the University of Alberta. William Moser was a professor of Mathematics at the University of Saskatchewan, the University of Manitoba and McGill University.

Be generous and patient as teachers, be active in projects which benefit the mathematical community and, above all, have as long and as happy a mathematical life as I have had, and am still having. –
W. Moser in 2003

(Source [MacTutor](#).)



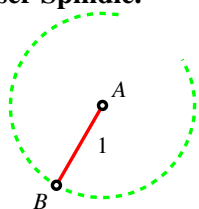
Leo Moser, 1921 – 1970



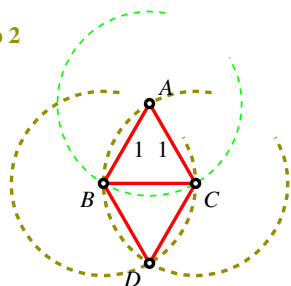
William Moser, 1927 – 2009

The Moser Spindle.

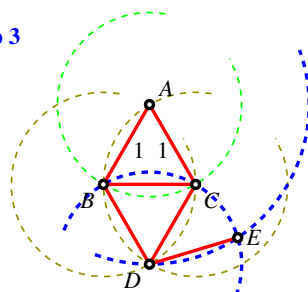
Step 1



Step 2



Step 3

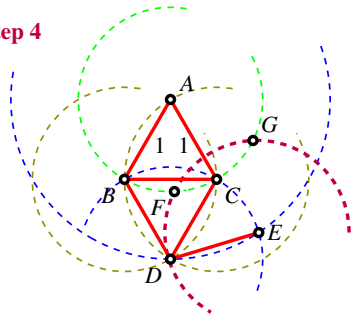


Start by choosing a point A in the plane and then draw a circle with the centre at A and radius 1. Denote this circle by C_1 . Next, choose a point B on the circle C_1 . Draw the line segment \overline{AB} .

Draw a circle with the centre at B and radius 1. Denote this circle by C_2 . Let C be the intersection point of C_1 and C_2 . Draw a circle, call it C_3 , with the centre at C and radius 1. Observe that the point A belongs to both C_2 and C_3 . Let D be the other intersection point of C_2 and C_3 . Draw the line segments \overline{AC} , \overline{BC} , \overline{BD} , and \overline{CD} . Observe that all those line segments are of length 1.

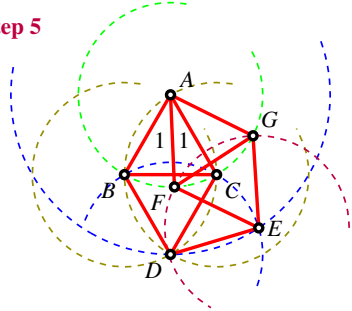
Draw a circle, call it C_4 , with the centre at D and radius 1. Draw a circle with the centre at A and passing through the point D . Denote this circle by C_5 . Next, choose a point E in the intersection of C_4 and C_5 . Draw the line segment \overline{DE} . Observe that $|\overline{DE}| = 1$.

Step 4



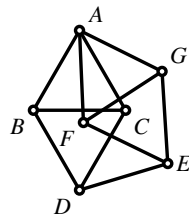
Draw a circle with the centre at E and radius 1. Denote this circle by C_6 . Let F and G be the intersection point of C_1 and C_6 .

Step 5



Draw the line segments \overline{AF} , \overline{AG} , \overline{EF} , and \overline{EG} . Observe that all those line segments are of length 1.

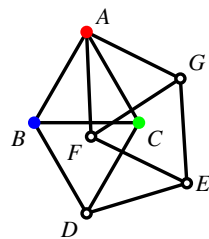
Step 6



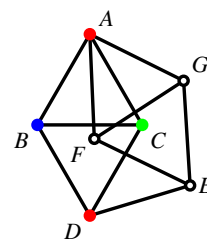
The Moser Spindle

Reminder: Question. Are three colours enough?

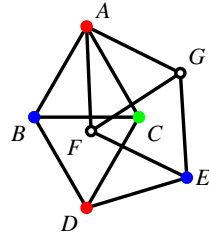
Toss the Moser Spindle on the red/blue/green coloured plane. Remember that every edge in the Moser Spindle is of length 1.



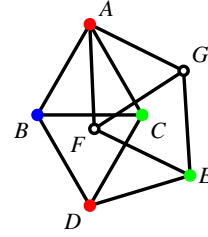
So far, so good ...



We are still fine ...



Trouble ...



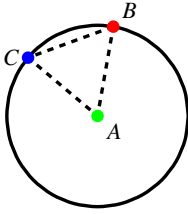
Trouble again ...

Proposition 0.5. $\chi \geq 4$.

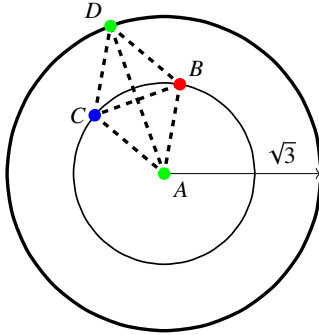
Question. Are three colours enough? (Again.)

Hugo Hadwiger in 1961: Consider a three colouring of the plane: $c : \Pi \rightarrow \{\bullet, \bullet, \bullet\}$. Hugo Hadwiger used the following construction to show that there must be two points, say, X and Y , such that

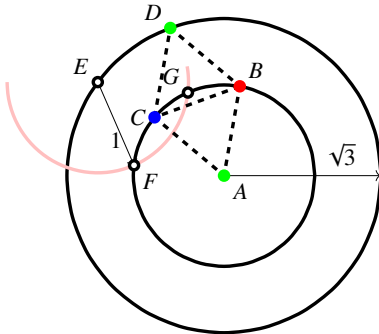
$$|\overline{XY}| = 1 \text{ and } c(X) = c(Y).$$



$$|\overline{AB}| = |\overline{AC}| = |\overline{BC}| = 1$$



$$|\overline{BD}| = |\overline{CD}| = 1$$



$$|\overline{DE}| = |\overline{EF}| = |\overline{EG}| = |\overline{FG}| = 1$$

Therefore... $\chi \geq 4$.

Start by choosing a point A in the plane and then draw a circle with the centre at A and radius 1. Denote this circle by C_1 . Suppose that $c(A) = \bullet$. If there is a green point on the circle C_1 , then we have two points coloured by the same colour that are one unit apart. Suppose that all points on the circle C_1 are coloured either blue or red. Next, choose a point B on the circle C_1 and suppose that $c(B) = \bullet$. There are two points on C_1 that are one unit apart from B . (Why?) If one of them is red, we are done. Suppose that both of them are blue and pick one of them. Call it C .

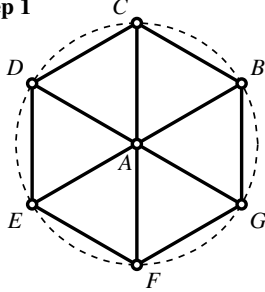
Draw a circle with the centre at A and radius $\sqrt{3}$. Denote this circle by C_2 . Let D be the intersection point of the circle C_2 and the line of symmetry of the line segment \overline{BC} that is not on the same side of the line BC as the point A . Observe that $|\overline{BD}| = |\overline{CD}| = 1$. (Why?) If the point D is coloured red or blue then we have two points that are one unit apart and of the same colour. Suppose that $c(D) = \bullet$.

Let E be a point on the circle C_2 such that $|\overline{DE}| = 1$. Draw a circle with the centre at E and radius 1. Denote this circle by C_3 . Observe that C_3 intersects the circle C_1 at two points, F and G and that $|\overline{FG}| = 1$. (Why?) Recall our assumption that all points on the circle C_1 are coloured or blue or red. Colour E by any of the three colours. What happens?

Question. Are three colours enough? (Again.)

Golomb Graph. By Solomon Golomb (1965).

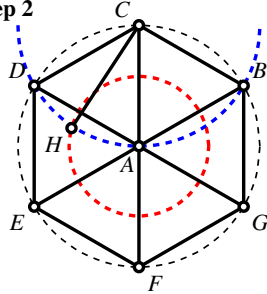
Step 1



$$|AB| = |BC| = |CD| = \dots = |GB| = 1$$

Draw a circle with the centre at A and radius 1. Denote this circle by C_1 . Let $BCDEFG$ be a regular hexagon inscribed in the circle C_1 .

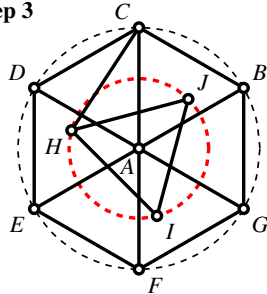
Step 2



$$|CH| = 1$$

Draw a circle with the centre at A and radius $\frac{\sqrt{3}}{3}$. Denote this circle by C_2 . Draw a circle with the centre at C and radius 1. Let H be an intersection point of C_2 and C_3 . Observe that $|CH| = 1$.

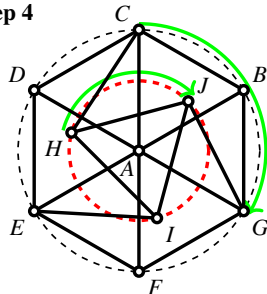
Step 3



$$|HI| = |HJ| = |IJ| = 1$$

Let $\triangle HIJ$ be an equilateral triangle inscribed in C_2 . Observe that, since the radius of C_2 equals to $\frac{\sqrt{3}}{3}$, $|HI| = |HJ| = |IJ| = 1$.

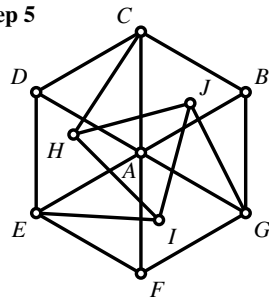
Step 4



$$|GJ| = |EI| = 1$$

Let r be the clockwise rotation by $\frac{2\pi}{3}$ with the centre at A . Then $r(C) = G$ and $r(H) = J$. Since r is an isometry it follows that $|GJ| = |CH| = 1$. Similarly, $|EI| = |CH| = 1$.

Step 5

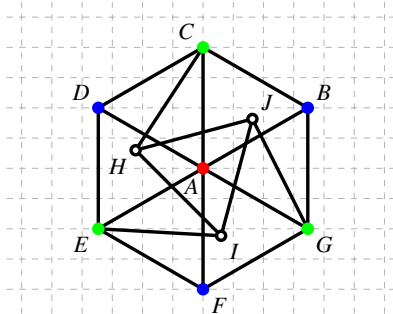


The Golomb Graph

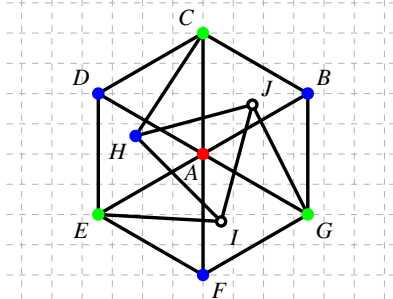
$$|\overline{GJ}| = |\overline{EI}| = 1$$

Question. Are three colours enough?

Toss the Golomb graph on the red/blue/green coloured plane. Recall that every edge in the Golomb graph is of length 1.



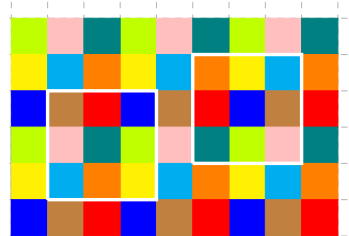
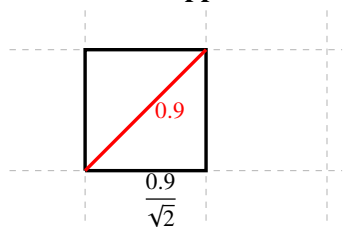
Suppose that the point A is coloured red and that each triangle that has A as a vertex is rainbow.



If H is coloured green, then there are two points, C and H , at the unit distance coloured by the same colour. Suppose that the point H is coloured blue. What are our options for colouring I and J ? What if we assume that the point H is coloured red?

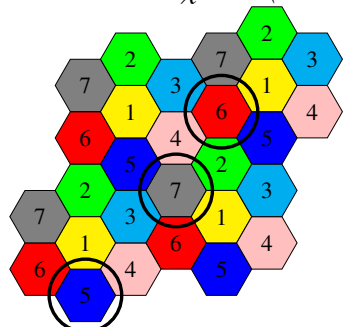
Therefore... $\chi \geq 4$.

What About Upper Bounds?



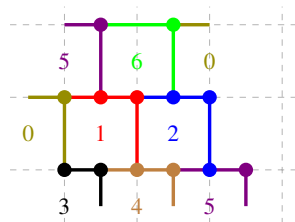
Therefore... $\chi \leq 9$.

Theorem 0.6. $\chi \leq 7$. (Hadwiger, 1961)



Proposition 0.7. $\chi \leq 7$. (Szekely, 1983)

3	4	5	6	0	1	2	3	4	5
5	6	0	1	2	3	4	5	6	0
1	2	3	4	5	6	0	1	2	3
3	4	5	6	0	1	2	3	4	5
6	0	1	2	3	4	5	6	0	1



Consider the grid in which the length of the side of each square in the grid equals $\frac{0.9}{\sqrt{2}}$. Observe that the length of the diagonal of each grid cell is $d = 0.9$.

A 9-colouring in which all neighbours of each square are of different colours. Is it possible to find two points of the same colour that are one unit apart?

Hugo Hadwiger in 1961: A 7-colouring of a tessellation of the plane by regular hexagons, with diameter slightly less than one. Observe that each hexagon is surrounded by hexagons of a different colour.

7-colouring by László Székely: Start with a row of squares of diagonal 1, with cyclically alternating colours from 0 to 6 of the squares. Obtain the consecutive rows of coloured squares by shifting the previous row by 2.5 squares. Upper and right boundaries are included in each square, except for the upper left and lower right corner.

Closer look: Upper and right boundaries are included in each square, except for the upper left and lower right corner.

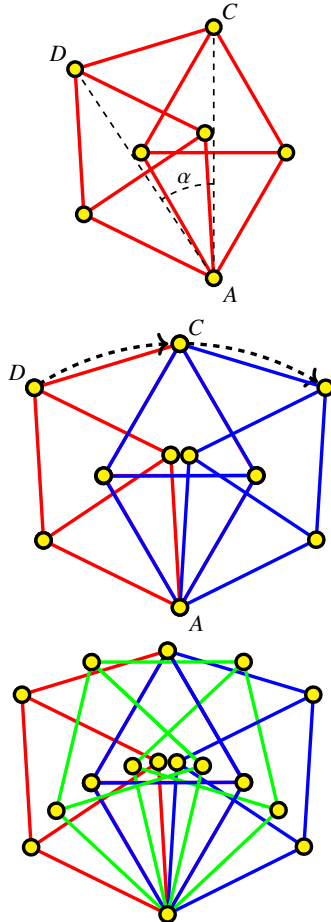
Theorem 0.8. $\chi \geq 5$. (de Grey, 2018)

“In seeking graphs that can serve as M in our construction, we focus on graphs that contain a high density of Moser spindles. The motivation for exploring such graphs is that a spindle contains two pairs of vertices distance $\sqrt{3}$ apart, and these pairs cannot both be monochromatic. Intuitively, therefore, a graph containing a high density of interlocking spindles might be constrained to have its monochromatic $\sqrt{3}$ -apart vertex pairs distributed rather uniformly (in some sense) in any 4-colouring. Since such graphs typically also contain regular hexagons of side-length 1, one might be optimistic that they could contain some such hexagon that does not contain a monochromatic triple in any 4-colouring of the overall graph, since such a triple is always an equilateral triangle of edge $\sqrt{3}$ and thus constitutes a locally high density, i.e. a departure from the aforementioned uniformity, of monochromatic $\sqrt{3}$ -apart vertex pairs.” – de Grey, Aubrey D.N.J. (2018), “The Chromatic Number of the Plane Is at least 5”, *Geombinatorics*, 28: 5–18, [arXiv:1804.02385](https://arxiv.org/abs/1804.02385).



Aubrey de Grey, 1963 –

One of de Grey’s tools: Multiple tightly linked Moser spindles:



Consider a Moser spindle and observe that (A, C) and (A, D) are two pairs of vertices distance $\sqrt{3}$ apart. Also, observe that the measure of the angle $\angle DAC$ is $\alpha = \arccos\left(\frac{5}{6}\right) \approx 33.56^\circ$.

Rotate the Moser spindle through α about the point A to obtain another Moser spindle. Observe that the two spindles share four vertices and five edges.

An additional clockwise rotation by $\frac{\alpha}{2}$ produces “three tightly linked Moser spindles.”

de Grey's Proof. Initially, de Grey constructed a graph with 20425 vertices. He “developed a custom program” to test this graph for the existence of monochromatic points with the unit distance under a 4-colouring. In de Grey's words, “This algorithm was implemented in Mathematica 11 on a standard MacBook Air and terminated in only a few minutes.”

Follow up. In his original paper, de Grey describes a construction of graph G with 1581 vertices that yields, under any 4-colouring, a pair of monochromatic points one unit apart. Again in de Grey's words, “Happily, G has turned out to be within the reach of standard SAT solvers.”

On August 3, 2019, as part of the [Polymath16](#) project, Jean Parts posted an image of a unit distance graph with 510 vertices and 2508 edges that confirms that $\chi \geq 5$.

Question 0.9. 1. *Is it possible to further reduce the size of the “good” graph?*

2. *How to find a human-verifiable proof that $\chi \geq 5$?*

Three facts:

1. Computing has become an instrumental part of mathematical research.
2. Mathematical research has become increasingly collaborative. See, for example, the [Polymath Project](#).
3. Ron Graham offered \$250 for a proof that $\chi \leq 6$.

Resources.

1. Soifer, A. The Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of its Creators, Springer, New York, 2008, pp 13-20.
2. [Hadwiger-Nelson problem - Wikipedia](#)
3. [Chromatic Number of the Plane by Cut The Knot](#)
4. [Open problems in Euclidean Ramsey Theory by Ron Graham and Eric Tressler](#)
5. [The chromatic number of the plane is at least 5 by Aubrey D.N.J. de Grey](#)