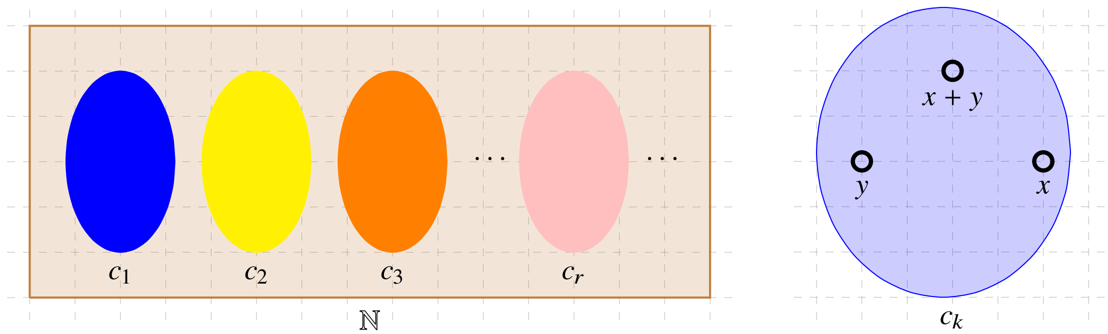
Schur's Theorem

1. **Quote.** The hardest thing to see is what is in front of your eyes.

   Johann Wolfgang von Goethe, German writer and politician, 1749 - 1832.

2. **Schur's Theorem.** If the set of positive integers $\mathbb{N}$ is finitely coloured then there exist x, y, z having the same colour such that

$$x + y = z.$$



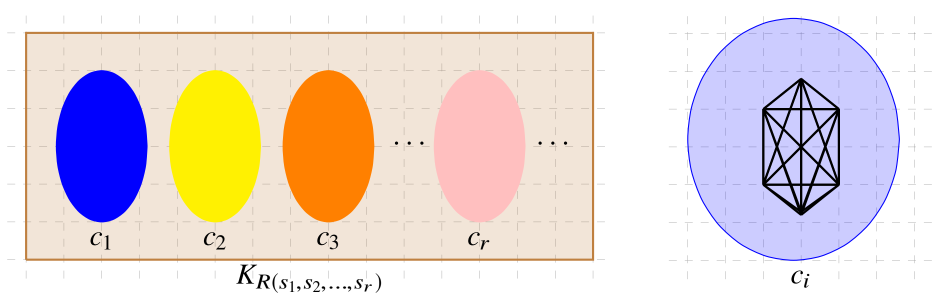
An r-colouring of $\mathbb{N}$. . . **. . . a monochromatic $\{x, y, x + y\}$**

Figure 1: Schur's Theorem: If the set of positive integers $\mathbb{N}$ is finitely coloured then there exist x, y, z having the same colour such that $x + y = z$.

   Note: A triple x, y, z that satisfies $x + y = z$ is called a Schur triple.

3. **Reminder:** The Ramsey number $R(s, t)$ is the minimum number n for which any edge 2-coloring of K_n, a complete graph on n vertices, in red and blue contains a red K_s or a blue K_t.

4. **Definition:** The Ramsey number $R(s_1, s_2, \ldots, s_r)$ is the minimum number n for which any edge r-colouring of K_n, a complete graph on n vertices, contains an i-monochromatic K_{s_i}, for some $i \in [1, r]$.

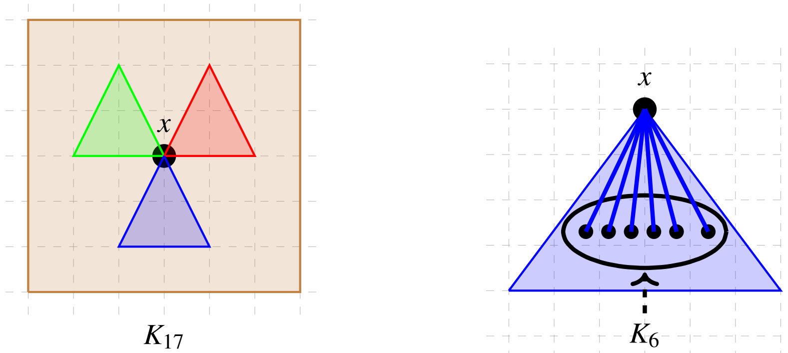


An r-colouring of $K_{R(s_1,s_2,\ldots,s_r)}$. . . **. . . an i-monochromatic K_{s_i}.**

Figure 2: Ramsey Theorem: If the the complete graph $K_{R(s_1,s_2,\ldots,s_r)}$ is r-coloured then, for some $i \in [1, r]$, there exists a complete graph K_{s_i} that is i-monochromatic.

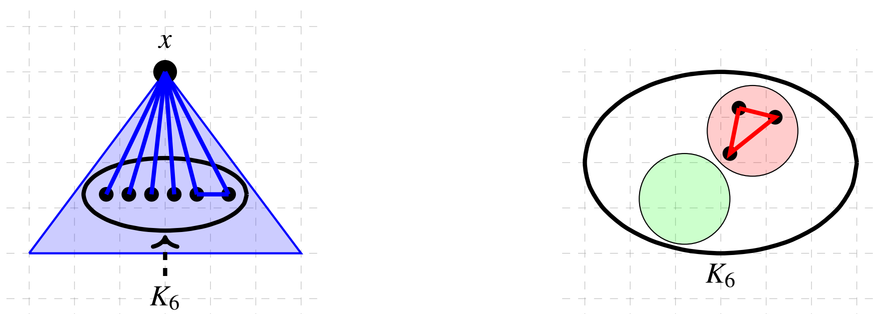
5. **Example.** $R(3,3,3) \leq 17$.

   Proof:



A 3-colouring of the edges of K_{17} with a vertex x. **There are at least 6 edges of the same colour, say blue, with the common vertex x.**

Figure 3: Use the the pigeonhole principle to conclude that if the edges of K_{17} are 3-coloured then each vertex is incident to at least six edges that are of the same colour.



Case 1: K_6 contains at least one blue edge. **Case 2: K_6 does not contain any blue edges.**

Figure 4: Two cases. . . Done!

6. **Question:** What is the meaning of $R(3,4,5,6)$? $R(3,3,3,3,3)$?

7. **Schur's Theorem.** If the set of positive integers $\mathbb{N}$ is finitely coloured then there exist x, y, z having the same colour such that

$$x + y = z,$$

e.a., there is a monochromatic Schur triple.

Proof: Let $c : \mathbb{N} \to [1, 2, \ldots, r]$ and let $M = R(\underbrace{3, 3, \ldots, 3}_{r})$.

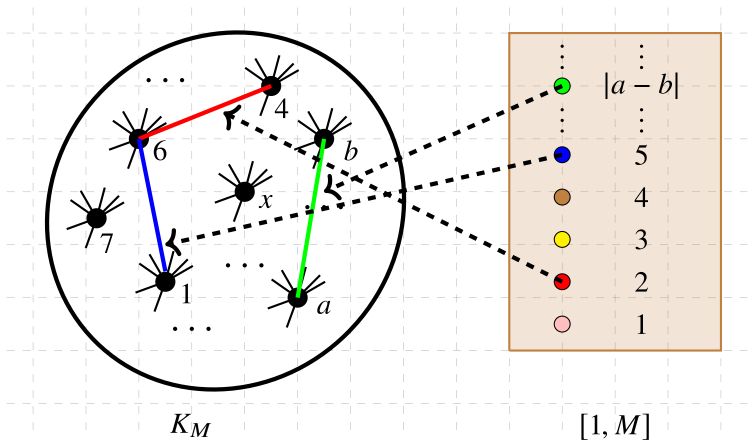


Figure 5: Denote vertices of K_M by $1, 2, \ldots, M$. For any $a, b \in [1, M]$, colour the edge $\{a, b\}$ by $c(|a - b|)$, where c is an r-colouring of $[1, M]$.

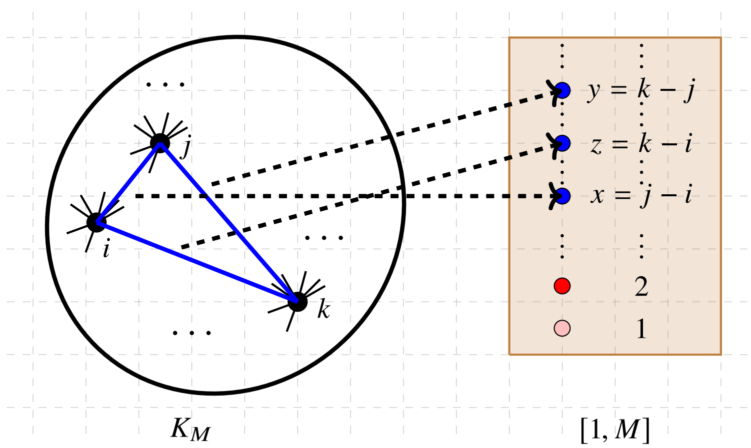


Figure 6: There is a monochromatic triangle with vertices $i < j < k$. (Why?) Take $x = k - j$, $y = j - i$, and $z = k - i$. Done! (Do you see why?)

8. **Actually . . . Schur's Theorem.** For any $r \in \mathbb{N}$ there is a natural number M such that any r- colouring of $[1, M]$ contains x, y, z having the same colour such that

$$x + y = z.$$

The least M with such property is called a **Schur number** and it is detonated by $s(r)$.

9. **Example.** What is $s(2)$?

   (a) Can you 2-colour, say in red and blue, the interval of positive integers $[1, 4]$ and avoid monochromatic Schur triples? Note that $1, 1, 2$ and $2, 2, 4$ are Schur triples.

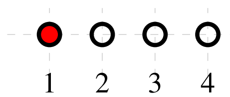
   

   Figure 7: $s(2) > 4$

   (b) Can you 2-colour, say in red and blue, the interval of positive integers $[1, 5]$ and avoid monochromatic Schur triples? Note that $1, 1, 2$ and $2, 2, 4$ are Schur triples.

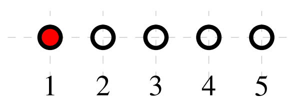
   

   Figure 8: $s(2) = 5$

10. **Known Schur Numbers.**

$$s(1) = 2, s(2) = 5, s(3) = 14, s(4) = 45.$$

11. **Time Machine.**

(a) In 1637 Fermat scribbled into the margins of his copy of *Arithmetica* by Diophantus, that

> It is impossible for a cube to be the sum of two cubes, a fourth power to be the sum of two fourth powers, or in general for any number that is a power greater than the second to be the sum of two like powers. I have discovered a truly marvellous demonstration of this proposition that this margin is too narrow to contain.

The margin note became known as Fermat's Last Theorem. It was proved by Andrew Wiles in 1995.

Figure 9: Pierre de Fermat, 1601 -1665

Figure 10: Andrew Wiles, 1953-

(b) In 1916 Schur proved the following:

> Let $n > 1$. Then, for all primes $p > s(n)$, the congruence
>
> $$x^n + y^n \equiv z^n (\text{mod } p)$$
>
> has a solution in the integers, such that p does not divide xyz.

(c) Fact:

> For any odd prime p, the multiplicative group
>
> $$\mathbb{Z}_p^* = \mathbb{Z}/p\mathbb{Z} = \{1, 2, \ldots, p-1\}$$
>
> is *cyclic*.

i. Example: Take $p = 5$. Then $\mathbb{Z}_5^* = \{1, 2, 3, 4\}$ and the multiplication is given by

$\cdot$	1	2	3	4
1				
2				
3				
4				

ii. Also,

$$\begin{aligned} \mathbb{Z}_5^* &= \{2, 2^2, 2^3, 2^4\} = \{2, 4, 3, 1\} \\ \mathbb{Z}_7^* &= \{3, 3^2, 3^3, 3^4, 3^5, 3^6\} = \{3, 2, 6, 4, 5, 1\} \end{aligned}$$

iii. In general, for any odd prime p there is $q \in \{1, 2, \ldots, p-1\}$ such that

$$\mathbb{Z}_p^* = \{q, q^2, \ldots, q^{p-1}\}.$$

(d) **Theorem (Schur, 1916):** Let $n > 1$. Then, for all primes $p > s(n)$, the congruence

$$x^n + y^n \equiv z^n (\text{mod } p)$$

has a solution in the integers, such that p does not divide xyz.
Proof:

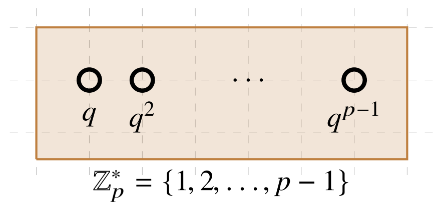


Figure 11: $\mathbb{Z}_p^* = \{1, 2, \ldots, p-1\} = \{q, q^2, \ldots, q^{p-1}\}$

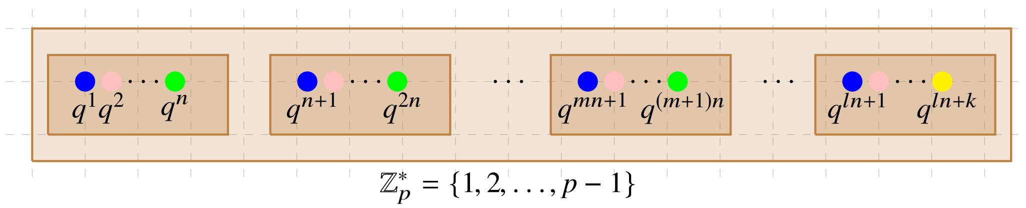


Figure 12: An n-colouring of $\{1, 2, \ldots, p-1\}$, $p > s(n)$

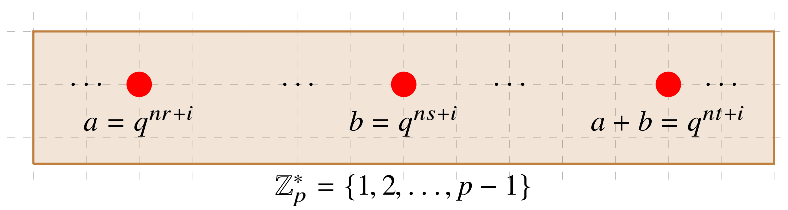


Figure 13: There is a monochromatic Schur triple!

From

$$\begin{aligned} 0 &= a + b - (a + b) \\ &\equiv q^{nr+i} + q^{ns+i} - q^{nt+i} \pmod{p} \\ &\equiv q^i (q^{nr} + q^{ns} - q^{nt}) \pmod{p} \end{aligned}$$

we conclude that

$$p | q^i (q^{nr} + q^{ns} - q^{nt}).$$

Since $0 \le i < n < s(n) \le p-1$ it follows that $p \nmid q^i$. Therefore

$$p | (q^{nr} + q^{ns} - q^{nt})$$

or, what is the same

$$q^{nr} + q^{ns} - q^{nt} \equiv 0 \pmod{p}.$$

By taking $x = q^r$, $y = q^s$, and $z = q^t$ we obtain

$$x^n + y^n \equiv z^n \pmod{p}.$$

12. **Exercise.** Let P be the set of points in the plane $x + y - z = 0$ whose coordinates are positive integers. Let an r-colouring of the set of positive integers be given.

    For each $(a, b, c) \in P$, do the following. If a, b, c are of the same colour, then colour (a, b, c) with that colour. Otherwise, mark (a, b, c) with an X.

    Questions:

    (a) Can all of the points be marked with an X?

    (b) Can we tell if, under any given finite colouring, the plane must contain an infinite number of coloured points?

    (c) Same for the plane $x + y - 2z = 0$.