Van der Waerden's Theorem - Part 2

1. **Quote.** Mathematics is really there for you to discover.

   Ron Graham, American mathematician, 1935 -

2. **Van der Waerden's Theorem - any number of colours, length 3:** Let $k \in \mathbb{N}$. Any k-colouring of positive integers contains a monochromatic 3-term arithmetic progression. Moreover, there is a natural number N such that any k colouring of the segment of positive integers $[1, N]$ contains a monochromatic 3-term arithmetic progression.

   (a) **Note:** The smallest N guaranteed by the theorem is annotated by $W(3, k)$.

   (b) **Proof - the main tool: Colour-focused arithmetic progressions and spikes:** Let c be a finite colouring of an interval of positive integers $[1, m]$ and $l, r \in \mathbb{N}$. We say that the set of l-term arithmetic progressions $A_1, A_2, \ldots, A_r$, i.e., for all $i \in [1, r]$ we have, for some $a_i, d_i \in \mathbb{N}$,

   $$A_i = \{a_i + jd_i : j \in [0, l-1]\}$$

   is *colour-focused* at $f \in \mathbb{N}$ if

   i. $A_i \subseteq [1, m]$ for each $i \in [1, r]$.

   ii. Each A_i is monochromatic.

   iii. If $i \neq j$ the A_i and A_j are not of the same colour.

   iv. $$a_1 + ld_1 = a_2 + ld_2 = \cdots = a_r + id_r = f.$$

   We call elements of a colour-focussed set **spikes**.

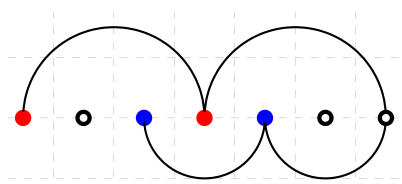

Figure 1: $\{1, 4\}$ and $\{3, 5\}$ are *colour-focussed* at 7.

(c) **Proof - a detail:** What happens when $r = k$?

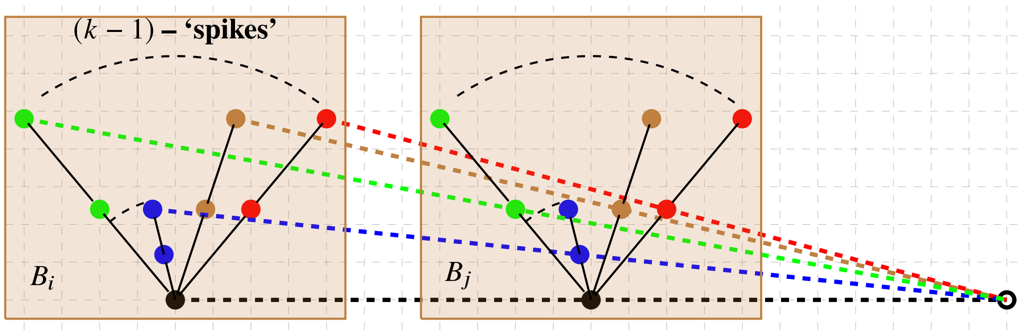

Figure 2: What happens when $r = k$? Do you see how a monochromatic 3-term arithmetic progression emerges?

3. **Baudet's Conjecture:** If the sequence of integers $1, 2, 3, \ldots$ is divided into two classes, at least one of the classes contains an arithmetic progression of l terms, no matter how large the length l is.

4. **Van der Waerden's Theorem - two colours, any length:** If the sequence of integers $1, 2, 3, \ldots$ is divided into two classes, at least one of the classes contains an arithmetic progression of l terms, no matter how large the length l is.

5. **Van der Waerden's Theorem - any number of colours, any length:** Let $l, k \in \mathbb{N}$. Any k-colouring of positive integers contains a monochromatic l-term arithmetic progression. Moreover, there is a natural number N such that any k-colouring of the segment of positive integers $[1, N]$ contains a monochromatic l-term arithmetic progression.

6. **Note:** The smallest N guaranteed by the theorem is annotated by $W(l, k)$. We have seen that $W(3, 2) = 9$ and that $W(3, k)$ exists for any $k \in \mathbb{N}$.

7. **Proof:**

   (a) **Strategy:** We use induction on l.

   (b) **The base case:** We already know that $W(l,k)$ exists if $l \leq 3$ and $k \in \mathbb{N}$, i.e., that the claim of the theorem is true for $l = 1, 2, 3$.

   (c) **The inductive step:** Let $l \geq 4$ be such that $W(l-1,k)$ exists for all k.

      i. **Claim:** For all $r \leq k$, there exists a natural number M such that whenever $[1, M]$ is k-coloured, **either** there exists a monochromatic l-term arithmetic progression **or** there exist r coloured-focussed $(l-1)$-term arithmetic progressions.

         A. **The base case:** Let $r = 1$ and let $M = 2W(l-1,k)$. Any k-colouring of $[1, M]$ contains or a monochromatic l-term arithmetic progression or at least **one** coloured-focused $(l-1)$-term arithmetic progression focused at some $f \in [1, M]$.

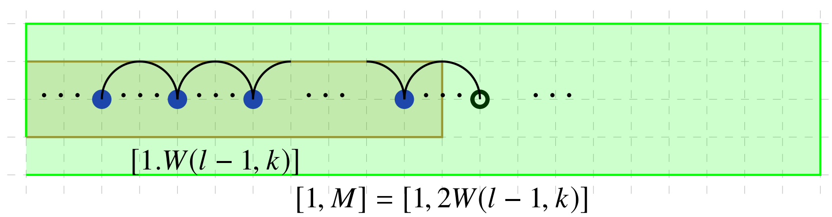

Figure 3: Any k-colouring of the set $[1, M]$ produces or a monochromatic l-term arithmetic progression or **one** coloured-focused $(l-1)$-term arithmetic progression.

B. **The inductive step:** Suppose that $r \in [2, k]$ is such that there is an M such that any k-colouring of $[1, M]$ contains a monochromatic l-term arithmetic progression or $r-1$ 'spikes', i.e., $r-1$ colour focussed $(l-1)$-term arithmetic progressions.

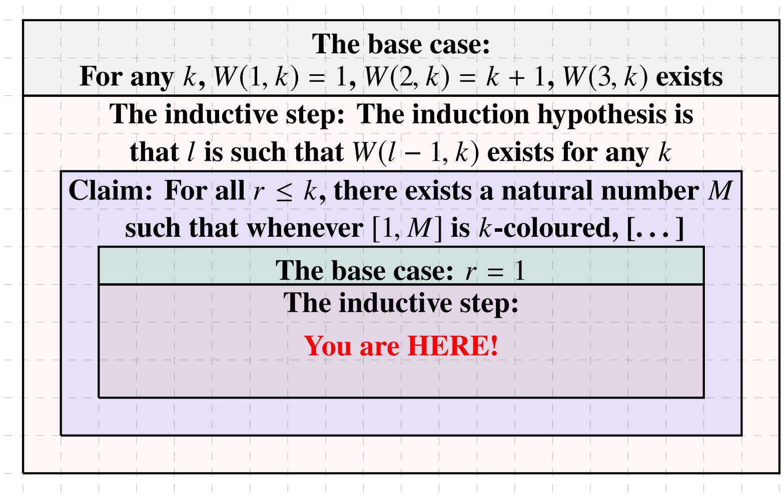

Figure 4: Where are you?

C. Observe that any k-colouring of $[1, 2M]$ contains or a monochromatic l-term arithmetic progression or at least $r-1$ coloured-focused $(l-1)$-term arithmetic progression focused at some $f \in [1, 2M]$.

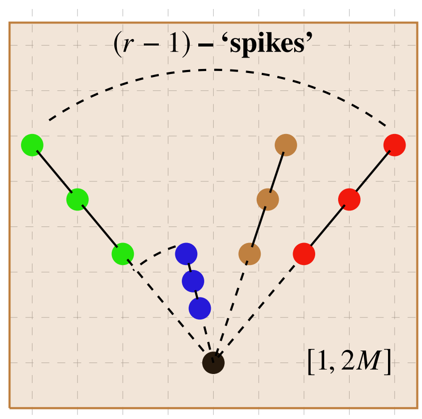

Figure 5: There are $r-1$ spikes.

D. Consider the interval of positive integers $[1, 2M \cdot W(l-1, k^{2M})]$. (How do we know that $W(l-1, k^{2M})$ exists?)

Divide this interval into $W(l-1, k^{2M})$ consecutive blocks B_i, $1 \leq i \leq W(l-1, k^{2M})$, of length $2M$.

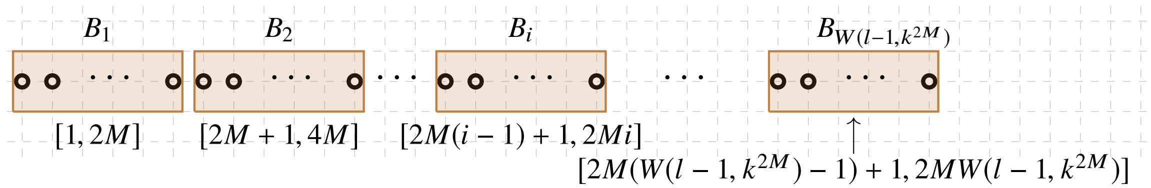

Figure 6: The interval $[1, 2M \cdot W(l-1, k^{2M})]$ is divided into $W(l-1, k^{2M})$ consecutive blocks B_i, $1 \leq i \leq W(l-1, k^{2M})$, of length $2M$.

E. Why $W(l-1, k^{2M})$?

F. Suppose that c is a k-colouring of $[1, 2M \cdot W(l-1, k^{2M})]$ that does not contain a monochromatic l-term arithmetic progression.
Each block B_i is k-coloured in one of the possible k^{2M} ways.

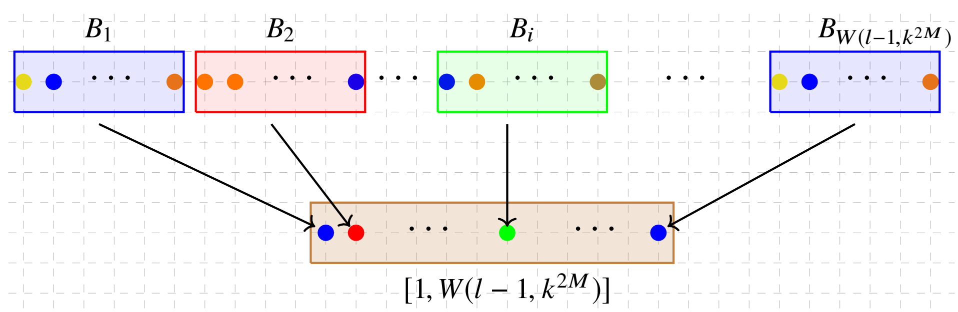

Figure 7: The k-colouring c of $[1, 2M \cdot W(l-1, k^{2M})]$ induces a k^{2M}-colouring of $[1, W(l-1, k^{2M})]$.

G. Any k^{2M}-colouring of $[1, W(l-1, k^{2M})]$ contains a monochromatic $(l-1)$-term arithmetic progression.

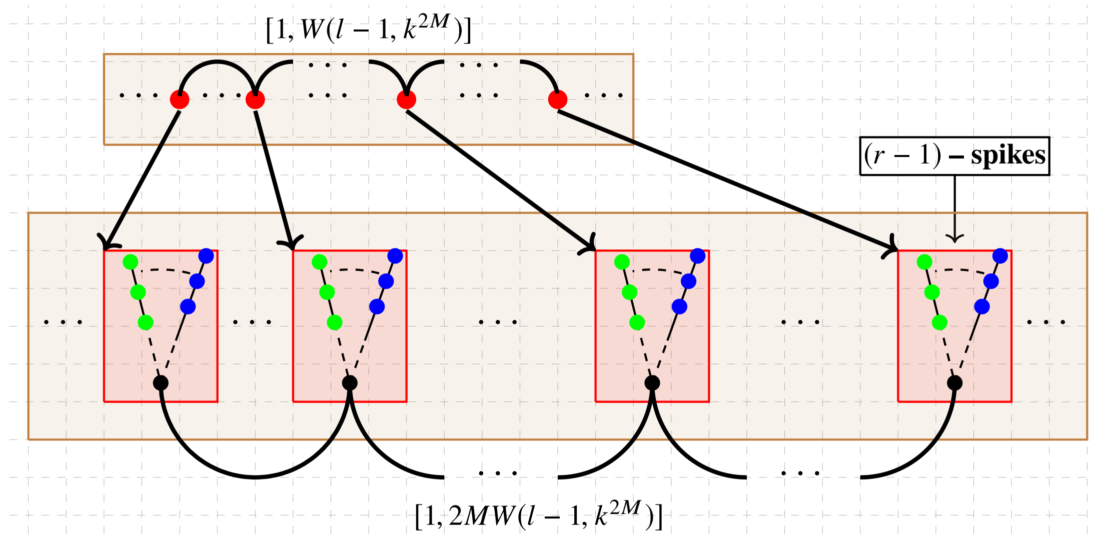

Figure 8: The k^{2M}-colouring of $[1, W(l-1, k^{2M})]$ induced by the colouring c contains a monochromatic $(l-1)$-term arithmetic progression. This means that there are $l-1$ blocks B_{i_j}, $1 \le j \le l-1$, that are coloured by c in the same way and they are equally spaced between each other.

H. Every B_{i_j}, $1 \le j \le l-1$:
   - is k-coloured the same way
   - contains $r-1$ spikes (monochromatic $(l-1)$-term arithmetic progressions) together with their focus. Note that there are no two spikes of the same colour (by definition!) and that the focus is of a different colour. (Why?)

I. The key step! The rth spike appears!

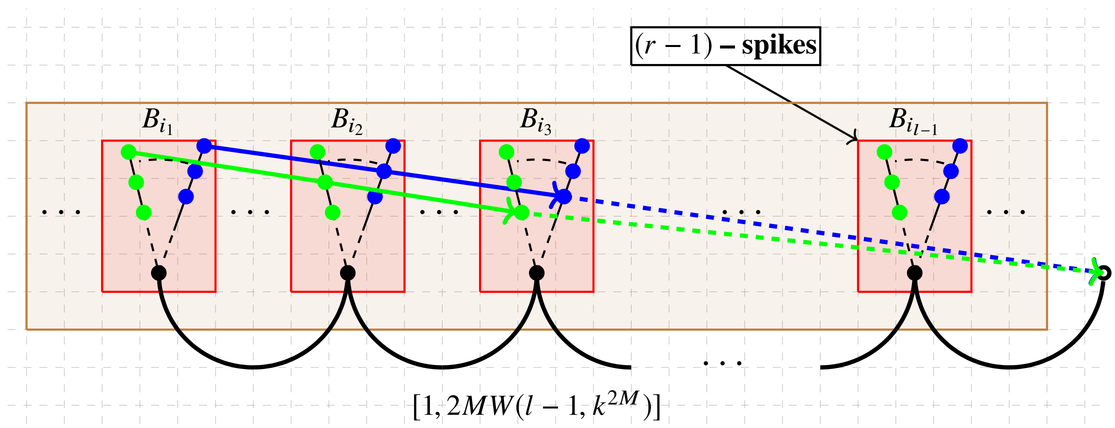

Figure 9: The k^{2M}-colouring of $[1, W(l-1, k^{2M})]$ induced by the colouring c contains a monochromatic $(l-1)$-term arithmetic progression. This means that there are $l-1$ blocks B_{i_j}, $1 \le j \le l-1$, that are coloured by c in the same way and they are equally spaced between each other.

Take a closer look:

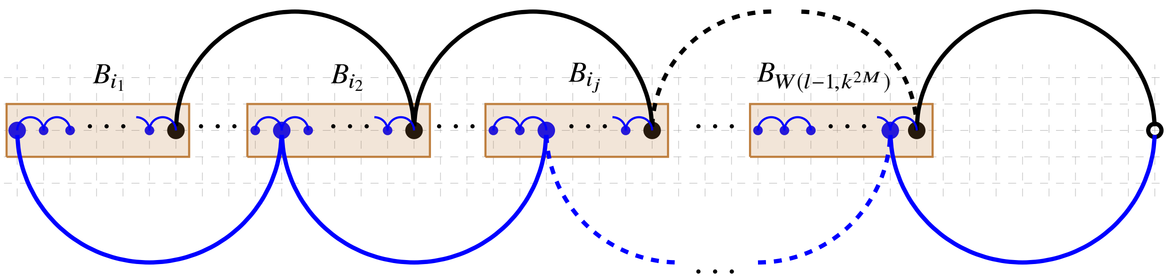

Figure 10: Do you see how $r-1$ initial spikes generate r new spikes?

ii. Where are we?

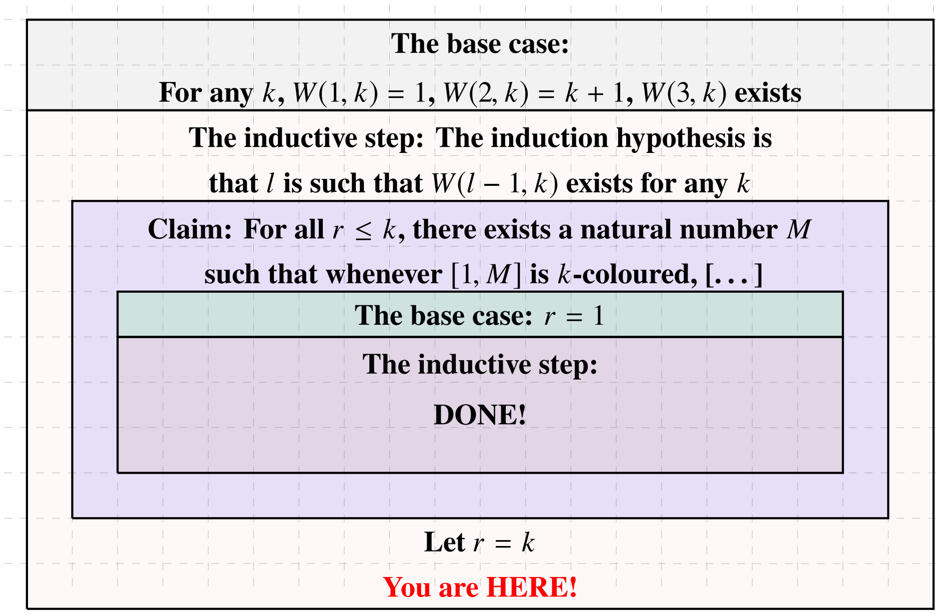

Figure 11: Almost there!

iii. Let $r = k$:

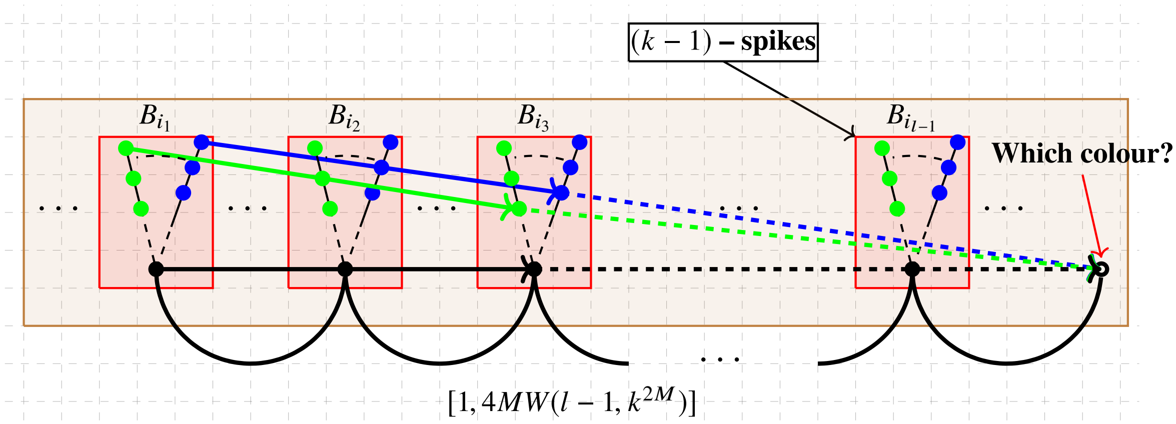

Figure 12: Done!