The Pigeonhole Principle

1. **Fact.** There are three kinds of mathematicians: Those who know how to count and those who don't.

2. **Pigeonhole Principle:** Suppose you have k pigeonholes and n pigeons to be placed in them. If $n > k$ then at least one pigeonhole contains at least two pigeons.

   (Attributed to German mathematician Dirichlet, 1805-1859.)

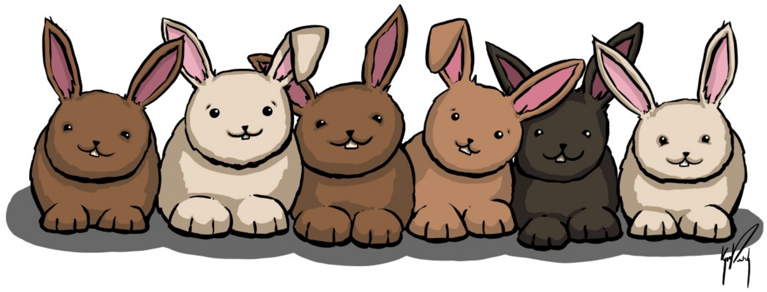
(a) 7 pigeons and 9 pigeonholes (b) 10 pigeons and 9 pigeonholes

Figure 1: Pigeons and Pigeonholes

3. **Pigeonhole Principle.** Under the surface, the pigeonhole principle reflects one of the questions that has been a source of debate among logicians and philosophers since the last quarter of the 19th century: "How are natural numbers individuated? That is, what is our most basic way of singling out a natural number for reference in language or in thought?"

   One of those *basic ways of singling out a natural number* is the notion of cardinal numbers: two sets are assigned the same cardinal number if there is a one–to–one correspondence between them. In this setting, natural numbers are cardinal numbers of non–empty finite sets, i.e. sets that are not in a one–to–one correspondence with any of their proper subsets.

   Let n be the cardinal number of a finite set A and let k is the cardinal number of a finite set B. If A contains a proper subset A' with the cardinal number equal to k, then we say that $n > k$. In other words, $n > k$ means that for any function $f : A \longrightarrow B$, there are $x, y \in A$, $x \neq y$, and $z \in B$ such that $f(x) = f(y) = z$. Or, in the terms of the pigeonhole principle, at least two pigeons (x and y) belong to the same pigeonhole (z).



4. **Example.** Show that among any 5 numbers one can find 2 numbers so that their difference is divisible by 4.

5. **Example.** Consider a chess board with two of the diagonally opposite corners removed. Is it possible to cover the board with pieces of domino whose size is exactly two board squares?

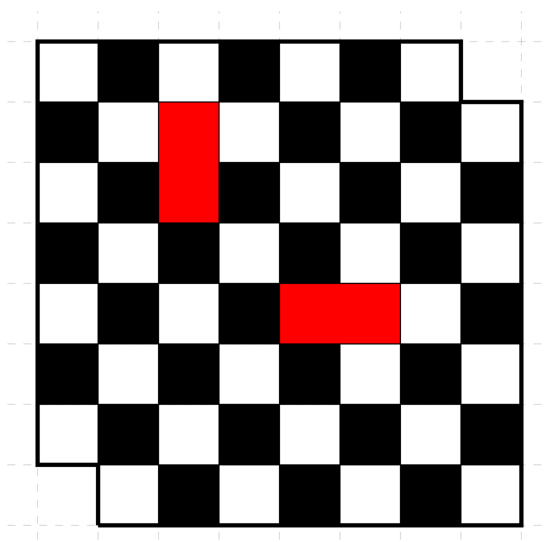


Figure 2: Two dominos on a chess board with two of the diagonally opposite corners removed

6. **Example:** There are 5 points in a square of side length 2. Prove that at least two of them are with the distance at most $\sqrt{2}$.

7. **Example.** A grid of 27 points in the plane is given.

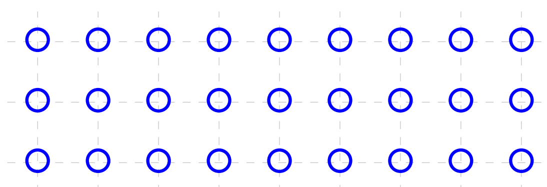


Figure 3: 27 points in the plane; 3 rows and 9 columns

Each point is coloured red or black. Prove that there exists a monochromatic rectangle, i.e., a rectangle with all four vertices of the same colour.

8. **Generalized Pigeonhole Principle:** If n pigeons are sitting in k pigeonholes, where $n > k$, then there is at least one pigeonhole with at least $\lceil \frac{n}{k} \rceil$ pigeons and at least one pigeonhole containing not more than $\lfloor \frac{n}{k} \rfloor$ pigeons.

9. **Example:** There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, how many different rooms will be needed?

10. Consider the set

$$A = \{1, 11, 111, 1111, \ldots\},$$

the set that contains all natural numbers whose decimal expression uses only the digit 1.

Prove that the set A contains an element that is divisible by 2024.