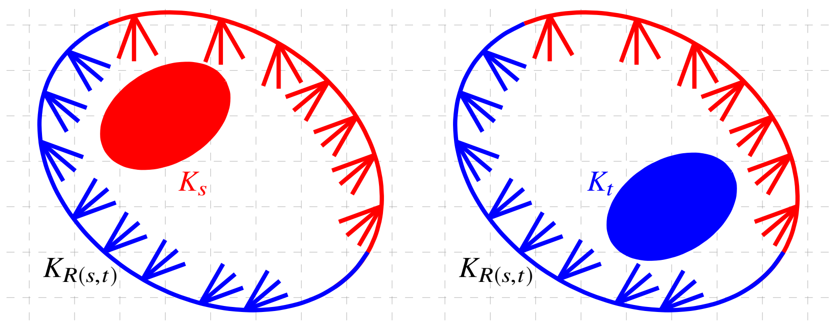
Ramsey's Theorem - Part 3

1. **Quote.** No finite point has meaning without an infinite reference point.

   Jean-Paul Sartre, French philosopher, playwright, novelist, screenwriter, political activist, biographer, and literary critic 1905 - 1980

2. **Reminder.** For any $s, t \in \mathbb{N}\backslash\{1\}$ the Ramsey number $R(s, t)$ exists and, for $s, t \geq 3$,

$$R(s, t) \leq R(s-1, t) + R(s, t-1).$$



There is a red K_s There is a blue K_t

Figure 1: $K_{R(s,t)}$: There is a red K_s or there is a blue K_t

3. **Infinite Case - Notation:**

   - The set of natural numbers:
$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$
   - For $r \in \mathbb{N}$ and any set X we define $X^{(r)}$ to be the set of all subsets on X with exactly r elements:
$$X^{(r)} = \{A \subset X : |A| = r\}.$$

   - For $k \in \mathbb{N}$ we define a k-colouring of $\mathbb{N}^{(r)}$ as a map from $\mathbb{N}^{(r)}$ to $\{1, 2, \ldots, k\}$:
$$c : \mathbb{N}^{(r)} \to \{1, 2, \ldots, k\}.$$

- If c is a k-colouring of $\mathbb{N}^{(r)}$ and $A \subset \mathbb{N}$ such that for all $x, y \in A^{(r)}$

$$c(x) = c(y)$$

we say that the set A is **monochromatic**.

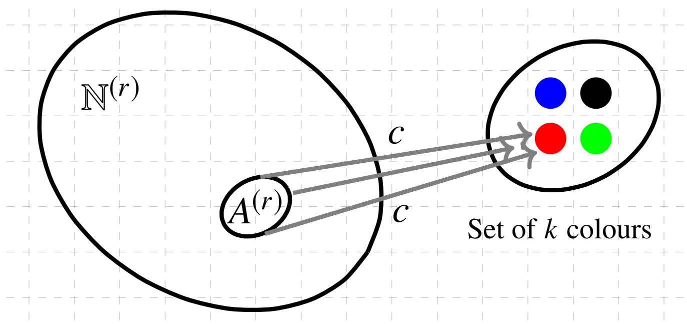


Figure 2: $A \subset \mathbb{N}$ is monochromatic!

4. **Ramsey's Theorem:** Whenever $\mathbb{N}^{(2)}$ is 2-coloured, there exist arbitrarily large monochromatic sets.

5. **Observation:** An infinite monochromatic set is much more than having arbitrarily large monochromatic sets.

   Example: Colour

$$\{1, 2\}, \{3, 4, 5\}, \{6, 7, 8, 9\}, \cdots$$

   i.e., colour red all edges within the sets above. Colour all other edges blue.

   Question: What about the existence of an *infinite* red set in this colouring?

6. **Ramsey Theorem - Two Colours - Infinite Case:** Whenever $\mathbb{N}^{(2)}$ is 2-coloured, there exists an infinite monochromatic set.

7. **Proof.** We colour elements of $\mathbb{N}^{(2)}$ red and blue:

$$c : \mathbb{N}^{(2)} \to \{\bullet, \bullet\},$$

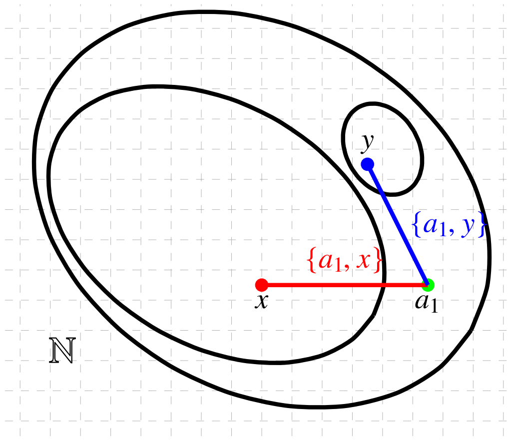


Figure 3: Step 1: Pick $a_1 \in \mathbb{N}$. Look at $\{a_1, x\}$ and $\{a_1, y\}$, $x, y \in \mathbb{N}$.

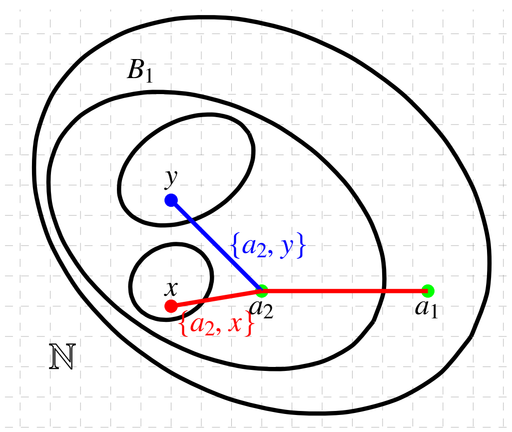


Figure 4: Step 2: Say that $B_1 = \{x \in \mathbb{N} : \{a_1, x\}\}$ is infinite. Pick $a_2 \in B_1$. Look at $\{a_2, x\}$ and $\{a_2, y\}$, $x, y \in B_1$.

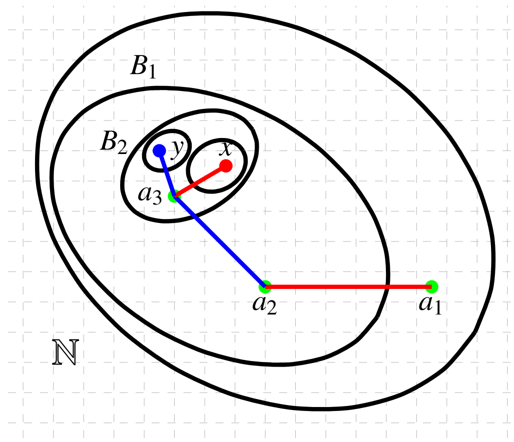


Figure 5: Step 3: Say that $B_2 = \{y \in B_1 : \{a_2, x\}\}$ is infinite. Pick $a_3 \in B_2$. Look at $\{a_3, x\}$ and $\{a_3, y\}$, $x, y \in B_2$.

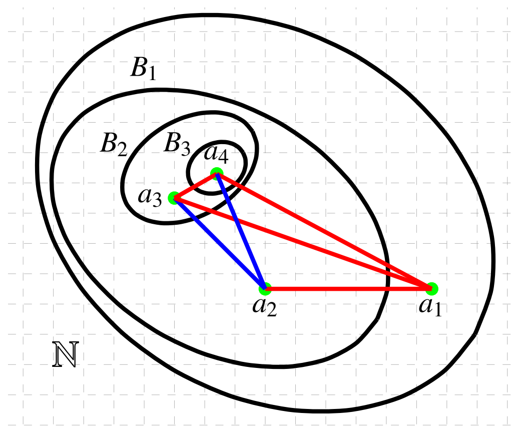


Figure 6: Step 4: Say that $B_3 = \{x \in B_2 : \{a_3, x\}\}$ is infinite. Pick $a_4 \in B_3$. Look at $\{a_4, x\}$ and $\{a_4, y\}$, $x, y \in B_3$. Note that $\{a_1, a_2\}$, $\{a_1, a_3\}$, $\{a_1, a_4\}$, $\{a_2, a_3\}$, $\{a_2, a_4\}$, and $\{a_3, a_4\}$.

Continue. . .

Summary: We obtain an infinite sequence of natural numbers

$$a_1, a_2, a_3, \ldots$$

and an infinite sequence of sets

$$\mathbb{N} \supseteq B_1 \supseteq B_2 \supseteq B_3 \supseteq \ldots$$

with the property that, for any $i \in \mathbb{N}$

(a) B_i is an infinite set

(b) $a_{i+1} \in B_i$

(c) $c(\{a_i, a_{i+1}\}) = c(\{a_i, a_{i+2}\}) = c(\{a_i, a_{i+3}\}) = \ldots.$

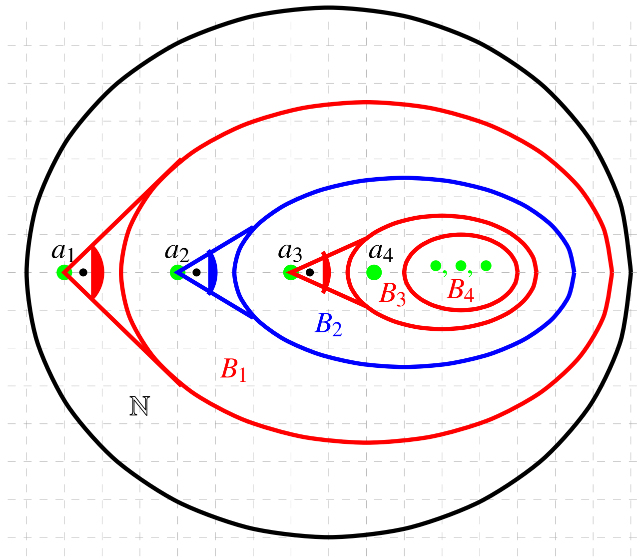


Figure 7: Conclusion: There must be an infinite number of a_i's that see only red or an infinite number of a_i's that see only blue.

8. **Example.** Let

$$a_1, a_2, a_3, \ldots$$

be a sequence of mutually distinct real numbers. Prove that that it contains a monotone subsequence.

Figure 8: Any sequence of mutually distinct real numbers contains a monotone subsequence.

9. **Ramsey Theorem Infinite Case.** Let $k, r \in \mathbb{N}$. Whenever $\mathbb{N}^{(r)}$ is k–coloured, there exists an infinite monochromatic set, where, by an *infinite monochromatic set* we mean an infinite subset T of $\mathbb{N}$ such that $T^{(r)}$ is monochromatic.

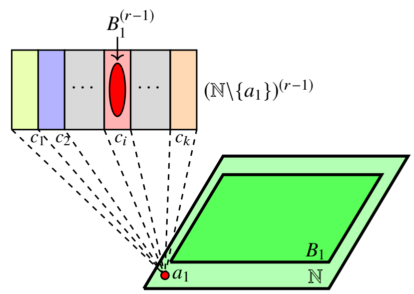
   Proof: Let a k–colouring $C : \mathbb{N}^{(r)} \longrightarrow \{c_1, c_2, \ldots, c_k\}$ be fixed.

   The proof is by induction on r. If $r = 1$, then $\mathbb{N}^{(1)} = \{\{n\} : n \in \mathbb{N}\}$, Since C is a finite colouring, by the pigeonhole principle at least one of the colour classes is infinite. Hence, there is $i \in [1, k]$ such that the monochromatic set $\{n \in \mathbb{N} : C(n) = c_i\}$ is infinite.

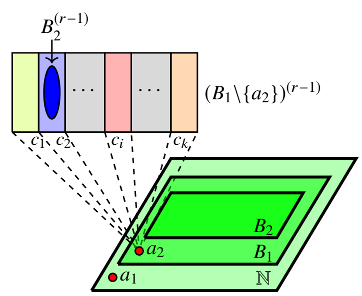
   Let $r \geq 2$ be such that for any k–colouring of $\mathbb{N}^{(r-1)}$ there exists an infinite monochromatic set. Observe that this assumption implies that, if B is any infinite subset of $\mathbb{N}$ and if $B^{(r-1)}$ is k–coloured, then B contains an infinite monochromatic subset.

   We build an infinite C–monochromatic set in the following way.

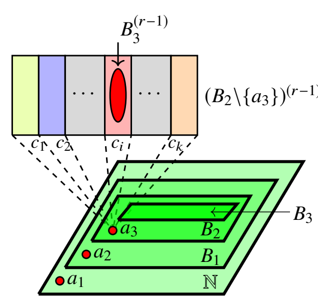
   Step 1: Pick $a_1 \in \mathbb{N}$. Let a k–colouring C_1 of $(\mathbb{N}\setminus\{a_1\})^{(r-1)}$ be defined by $C_1(F) = C(F \cup \{a_1\})$. By the induction hypothesis, $\mathbb{N}\setminus\{a_1\}$ contains an infinite C_1–monochromatic set. Call this set B_1.



   Step 2: Pick $a_2 \in B_1$. Let a k–colouring C_2 of $(B_1\setminus\{a_2\})^{(r-1)}$ be defined by $C_2(F) = C(F \cup \{a_2\})$. By the induction hypothesis, $B_1\setminus\{a_2\}$ contains an infinite C_2–monochromatic set. Call this set B_2.



   Step 3: Pick $a_3 \in B_2$. Let a k–colouring C_3 of $(B_2\setminus\{a_3\})^{(r-1)}$ be defined by $C_3(F) = C(F \cup \{a_3\})$. By the induction hypothesis, $B_2\setminus\{a_3\}$ contains an infinite C_3–monochromatic set. Call this set B_3.

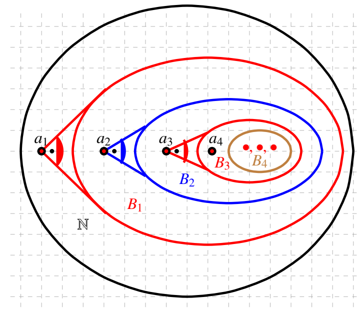


   Observe that, for any $F \in B_1^{(r-1)}$, any $F' \in B_2^{(r-2)}$, and any $F'' \in B_3^{(r-3)}$, $C(\{a_1\} \cup F) = C(\{a_1, a_2\} \cup F') = C(\{a_1, a_2, a_3\} \cup F'')$. Also, for any $G \in B_2^{(r-1)}$ and any $G' \in B_3^{(r-2)}$, $C(\{a_2\} \cup G) = C(\{a_2, a_3\} \cup G')$.

   Continue...

   Through this process we obtain an infinite sequence of natural numbers $a_1, a_2, a_3, \ldots$ and an infinite sequence of infinite sets $\mathbb{N} \supseteq B_1 \supseteq B_2 \supseteq B_3 \supseteq \ldots$ with the property that, for any $i \in \mathbb{N}$, $\{a_{i+1}, a_{i+2}, a_{i+3}, \ldots\} \subset B_i$, and, for any $F = \{a_{i_1}, a_{i_2}, \ldots, a_{i_{r-1}}\}, H = \{a_{j_1}, a_{j_2}, \ldots, a_{j_{r-1}}\} \in B_{i+1}^{(r-1)}$, $C(\{a_i\} \cup F\}) = C(\{a_i\} \cup H)$.

In other words, each a_i sees all $(r-1)$–sets of elements of the sequence that come after it, i.e. all sets $\{a_{j_1}, a_{j_2}, \ldots, a_{j_{r-1}}\} \in B_{i+1}^{(r-1)}$, in the *same* colour.



By the pigeonhole principle, at least one of sets $C_j = \{i \in \mathbb{N} : a_i \text{ sees only the colour } c_j\}$, $j \in [1, k]$, must be infinite. Therefore, there is an infinite C–monochromatic set.

Since C was an arbitrarily k–colouring of $\mathbb{N}^{(r)}$, this completes the induction step and the proof of the theorem.

10. **Example.** Let

$$a_1, a_2, a_3, \ldots$$

be a sequence of real numbers. Prove that that it contains either constant or strictly monotonic subsequence.