# General Relativity Notes

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# Contents

# Part I

# Mathematics

# Chapter 1

# Tensor Calculus

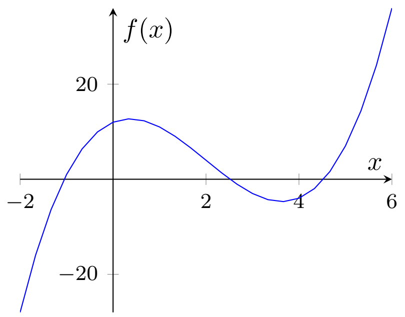
1. What is Tensor Calculus?

   **Definition.** Study of how tensors change over space

2. Why would you want to study it?
3. What do I need to start?

## 1.1 Multi-Variable Calculus

$f(x) = x^3 - 6x^2 + 4x + 12$




slope at $x = f'(x) = \frac{df}{dx}$

- Power Rule:

$$\frac{d}{dx} x^n = n x^{n-1}$$

- Exponential Rule:

$$\frac{d}{dx} e^x = e^x$$

- Trig Rule:

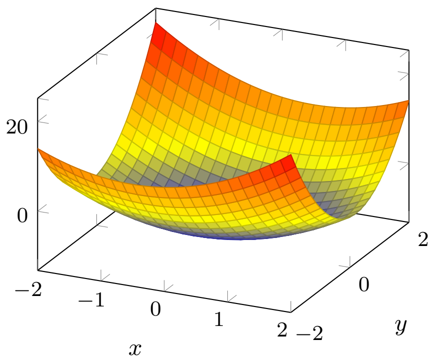
$$\frac{d}{dx} \sin x = \cos x$$
$$\frac{d}{dx} \cos x = -\sin x$$

- Sum Rule:
$$\frac{d}{dx}\left(f(x)+g(x)\right)=\frac{df}{dx}+\frac{dg}{dx}$$
- Product Rule:
$$\frac{d}{dx}(f(x)g(x))=\frac{df}{dx}g(x)+f(x)\frac{df}{dg}$$
- Chain Rule:
$$\frac{d}{dx}(f(g(x)))=\frac{df}{dg}\frac{dg}{dx}$$

*Remark* 1. $\frac{dx}{du}=\frac{du}{dx}$

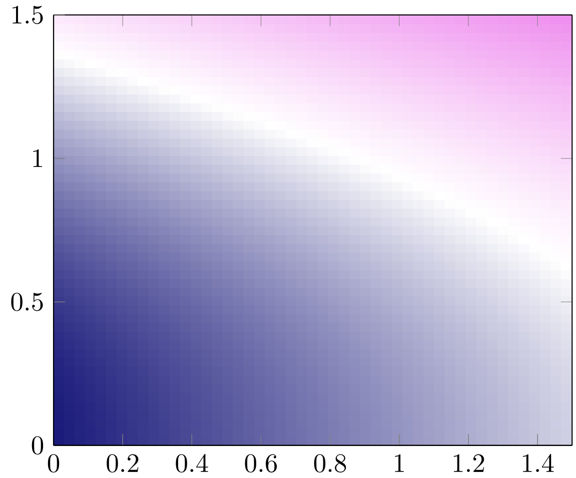
### 1.1.1 Derivative(multi-variable)

$f(x,y)=2x^2-xy+3y^2-10$




*Remark* 2. $\frac{dx}{du}\neq\frac{du}{dx}$

### 1.1.2 Chain Rule (multi-variable)



$$
\begin{aligned}
\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \\
\frac{df}{dt} &= \sum_i \frac{\partial f}{\partial q^i}\frac{\partial q^i}{\partial t} = \frac{\partial f}{\partial q^i}\frac{\partial q^i}{\partial t}
\end{aligned}
$$

### 1.1.3 Gradient of a function

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\boxed{df = \sum_i \frac{\partial f}{\partial q^i}dq^i = \frac{\partial f}{\partial q^i}dq^i}$$

### 1.1.4 Arc Length

$$
\begin{aligned}
arc\ length &= \int \left\| \frac{d\vec{R}}{dt} \right\| dt \\
&= \int \sqrt{\frac{d\vec{R}}{dt}\cdot\frac{d\vec{R}}{dt}}dt \\
&= \int \sqrt{\left(\frac{\partial \vec{R}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{R}}{\partial y}\frac{dy}{dt}\right)\cdot\left(\frac{\partial \vec{R}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{R}}{\partial y}\frac{dy}{dt}\right)}dt \\
&= \int \sqrt{\left(\frac{dx}{dt}\right)^2\left(\frac{\partial \vec{R}}{\partial x}\cdot\frac{\partial \vec{R}}{\partial x}\right) + 2\frac{dx}{dt}\frac{dy}{dt}\left(\frac{\partial \vec{R}}{\partial x}\cdot\frac{\partial \vec{R}}{\partial y}\right) + \left(\frac{dy}{dt}\right)^2\left(\frac{\partial \vec{R}}{\partial y}\cdot\frac{\partial \vec{R}}{\partial y}\right)}
\end{aligned}
$$

$$\left\| \frac{d\vec{R}}{dt} \right\| = \sqrt{\sum_i \sum_j \frac{dq^i}{dt}\frac{dq^j}{dt}\left(\frac{\partial \vec{R}}{\partial q^i}\cdot\frac{\partial \vec{R}}{\partial q^j}\right)} = \sqrt{\frac{dq^i}{dt}\frac{dq^j}{dt}\left(\frac{\partial \vec{R}}{\partial q^i}\cdot\frac{\partial \vec{R}}{\partial q^j}\right)}$$

### 1.1.5 Summary

- Multi-Variable Chain Rule

$$\boxed{\frac{df}{dt} = \sum_i \frac{\partial T}{\partial q^i}\frac{\partial q^i}{\partial t} = \frac{\partial T}{\partial q^i}\frac{\partial q^i}{\partial t}}$$

- Total Differential Formula

$$\boxed{df = \sum_i \frac{\partial f}{\partial q^i}dq^i = \frac{\partial f}{\partial q^i}dq^i}$$

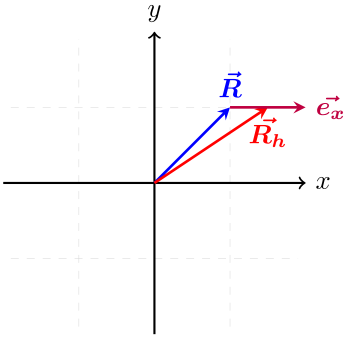
- Velocity Vector tangent to a Curve (magnitude)

$$\boxed{\left\|\frac{d\vec{R}}{dt}\right\| = \sqrt{\sum_i\sum_j \frac{dq^i}{dt}\frac{dq^j}{dt}\left(\frac{\partial\vec{R}}{\partial q^i}\cdot\frac{\partial\vec{R}}{\partial q^j}\right)} = \sqrt{\frac{dq^i}{dt}\frac{dq^j}{dt}\left(\frac{\partial\vec{R}}{\partial q^i}\cdot\frac{\partial\vec{R}}{\partial q^j}\right)}}$$

## 1.2 Cartesian and Polar Coordinate

$$\cos\theta = \frac{x}{r} \qquad\qquad x = r\cos\theta$$
$$\sin\theta = \frac{y}{r} \qquad\qquad y = r\sin\theta$$

$$\begin{aligned}\tan\theta &= \frac{y}{x}\\ \theta &= \arctan\left(\frac{y}{x}\right)\\ r &= \sqrt{x^2+y^2}\end{aligned}$$




$$\frac{\partial\vec{R}}{\partial x} = \lim_{h\to 0}\frac{\vec{R_h}(x+h,y)-\vec{R}(x,y)}{h} \tag{1.2.1}$$
$$\equiv \vec{e_x} \tag{1.2.2}$$

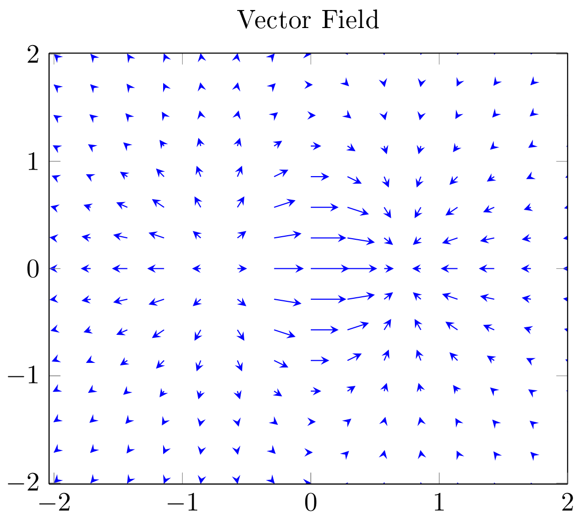
## 1.3 The Jacobian

$$\begin{aligned}\tilde{\vec{e_1}} &= 2\vec{e_1} + 1\vec{e_2}\\ \tilde{\vec{e_2}} &= -\frac{1}{2}\vec{e_1} + \frac{1}{4}\vec{e_2}\end{aligned}$$

$$F = \begin{bmatrix} 2 & -1/2 \\ 1 & 1/4 \end{bmatrix}$$

$$
\begin{aligned}
\tilde{\vec{e_r}} &= \quad \vec{e_x} + \quad \vec{e_y} \\
\tilde{\vec{e_\theta}} &= \quad \vec{e_x} + \quad \vec{e_y} \\
\frac{\partial \vec{R}}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial \vec{R}}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial \vec{R}}{\partial y} \\
\frac{\partial \vec{R}}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial \vec{R}}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial \vec{R}}{\partial y}
\end{aligned}
$$

## 1.4 Derivatives are Vectors




$\vec{R}$ is an position vector.
Tengent Vector to a Curve parametized by $\lambda$:

$$
\begin{aligned}
\vec{v} = v^1\vec{e_1} + v^2\vec{e_2} \qquad &= v^i\vec{e_i} \qquad = \tilde{v}^i\tilde{\vec{e_i}} \\
\frac{d\vec{R}}{d\lambda} = \frac{dx}{d\lambda}\frac{\partial \vec{R}}{\partial x} + \frac{dy}{d\lambda}\frac{\partial \vec{R}}{\partial y} &= \frac{dc^i}{d\lambda}\frac{\partial \vec{R}}{\partial c^i} = \frac{dp^i}{d\lambda}\frac{\partial \vec{R}}{\partial p^i}
\end{aligned}
$$

## 1.5 Derivative Transformation Rules (Contravariance)

$$
\begin{aligned}
\vec{v} = v^i\vec{e_i} \qquad &= \tilde{v}^i\tilde{\vec{e_i}} \\
\frac{d}{d\lambda} = \frac{dc^i}{d\lambda}\frac{\partial}{\partial c^i} &= \frac{dp^i}{d\lambda}\frac{\partial}{\partial p^i}
\end{aligned}
$$

$$\boxed{\begin{aligned}\tilde{\vec{e_j}} &= F^i_j \vec{e_i} \\ \vec{e_j} &= B^i_j \tilde{\vec{e_i}}\end{aligned}} \quad \boxed{\begin{aligned}\tilde{v^i} &= B^i_j v^i \\ v^i &= F^i_j \tilde{v^j}\end{aligned}}$$

$$\boxed{\begin{aligned}\frac{\partial}{\partial p^j} &= \frac{\partial c^i}{\partial p^j}\frac{\partial}{\partial c^i} \\ \frac{\partial}{\partial c^j} &= \frac{\partial p^i}{\partial c^j}\frac{\partial}{\partial p^i}\end{aligned}} \quad \boxed{\begin{aligned}\frac{dp^i}{d\lambda} &= \frac{\partial p^i}{\partial c^j}\frac{dc^j}{d\lambda} \\ \frac{dc^i}{d\lambda} &= \frac{\partial c^i}{\partial p^j}\frac{dp^j}{d\lambda}\end{aligned}}$$

Tangent Vector Space $T$ with point $p$ on surface $M$:

$$T_p M$$

## 1.6 Differentials are Covectors

$$\boxed{df\left(\vec{v}\right) \text{ is proportional to the steepness of } f \text{ in the direction of } \vec{v}}$$

$$\boxed{df\left(\vec{v}\right) \text{ is proportional to the length of } \vec{v}}$$

$$\boxed{df\left(\vec{v}\right) \text{tells us the rate of change of } f \text{ when moving at velocity } \vec{v}.}$$

$$\boxed{df\left(\vec{v}\right) \text{is the } \textit{directional derivative}.}$$

$$\boxed{df\left(\vec{v}\right)} = \nabla_{\vec{v}} f = D_{\vec{v}} f = \frac{\partial f}{\partial \vec{v}} = \boxed{\nabla f \cdot \vec{v}}$$

## 1.7 Covector Field Components

$$\text{Scalar Field } f \rightarrow_d \text{Covector Field } df$$

$$\begin{aligned}\frac{d}{d\lambda} &= \frac{dx}{d\lambda}\frac{\partial}{\partial x} + \frac{dy}{d\lambda}\frac{\partial}{\partial y} \\ df &= Adx + Bdy\end{aligned}$$

Dual basis(basis covector):

$$\boxed{\epsilon^i\left(\vec{e_j}\right) = \delta^i_j}$$

$$\boxed{dc^i\left(\frac{\partial}{\partial c^j}\right) = \left(\frac{\partial c^i}{\partial c^j}\right) = \delta^i_j}$$

$$\begin{aligned}
\frac{df}{d\lambda} &= \frac{\partial f}{\partial x}\frac{dx}{d\lambda} + \frac{\partial f}{\partial y}\frac{dy}{d\lambda} \\
df\left(\frac{d}{d\lambda}\right) &= \frac{\partial f}{\partial x}\frac{dx}{d\lambda} + \frac{\partial f}{\partial y}\frac{dy}{d\lambda} \\
df\left(\frac{d}{d\lambda}\right) &= \frac{\partial f}{\partial x}dx\left(\frac{d}{d\lambda}\right) + \frac{\partial f}{\partial y}dy\left(\frac{d}{d\lambda}\right) \\
df\left(\frac{d}{d\lambda}\right) &= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)\left(\frac{d}{d\lambda}\right) \\
df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy
\end{aligned}$$

$$\boxed{\begin{aligned}
&\alpha = \alpha_i \epsilon^i = \tilde{\alpha_i}\tilde{\epsilon^i} \\
&df = \frac{\partial f}{\partial c^i}dc^i = \frac{\partial f}{\partial p^i}dp^i
\end{aligned}}$$

## 1.8 Covector Field Transformation Rules(Covariance)

Basis Covectors(Contravariant):

$$\begin{aligned}
dp^i &= \frac{\partial p^i}{\partial c^j}dc^j \\
dc^i &= \frac{\partial c^i}{\partial p^j}dp^j
\end{aligned}$$

Covecotr Components(Covariant):

$$\begin{aligned}
\frac{\partial f}{\partial p^j} &= \frac{\partial c^i}{\partial p^j}\frac{\partial f}{\partial c^i} \\
\frac{\partial f}{\partial c^j} &= \frac{\partial p^i}{\partial c^j}\frac{\partial f}{\partial p^i}
\end{aligned}$$

## 1.9 The Metric Tensor and Arc Length

$$\begin{aligned}
\left\|\frac{d\vec{R}}{dt}\right\| &= \sqrt{\sum_i\sum_j \frac{dq^i}{dt}\frac{dq^j}{dt}\left(\frac{\partial \vec{R}}{\partial q^i}\cdot\frac{\partial \vec{R}}{\partial q^j}\right)} \\
&= \sqrt{\frac{dq^i}{dt}\frac{dq^j}{dt}\left(\frac{\partial \vec{R}}{\partial q^i}\cdot\frac{\partial \vec{R}}{\partial q^j}\right)} \\
&= \sqrt{\frac{dq^i}{dt}\frac{dq^j}{dt}g_{ij}}
\end{aligned}$$

## 1.10 The Metric Tensor in Curved Spaces

### 1.10.1 Extrinsic Geometry

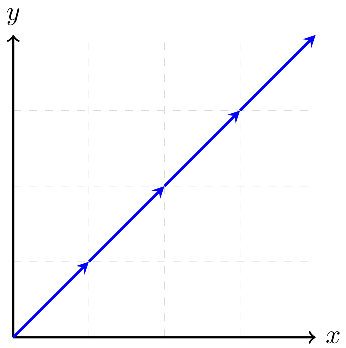
Map 2D plane into 3D coordinate

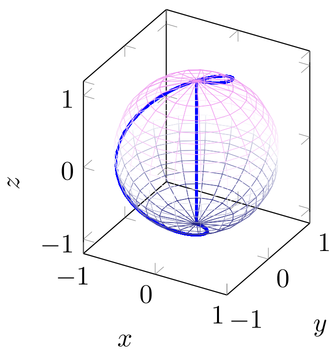
$$(u,v) \mapsto (X(u,v), Y(u,v), Z(u,v))$$

$$\begin{aligned} X &= \cos v \sin u \\ Y &= \sin v \sin u \\ Z &= \cos u \end{aligned}$$

**Example 3.** $\lambda \mapsto (u = \lambda, v = \lambda)$

$$\begin{aligned} X &= \cos \lambda \sin \lambda \\ Y &= \sin \lambda \sin \lambda \\ Z &= \cos \lambda \end{aligned}$$




$$\text{arc length} = \int \left\| \frac{d\vec{R}}{d\lambda} \right\| d\lambda$$

arc length$\mapsto$basis vector component

$$\begin{aligned}
\left\|\frac{d\vec{R}}{d\lambda}\right\|^2 &= \frac{d\vec{R}}{d\lambda}\cdot\frac{d\vec{R}}{d\lambda}\\
&= \left(\frac{dX}{d\lambda}\frac{\partial\vec{R}}{\partial X}+\frac{dY}{d\lambda}\frac{\partial\vec{R}}{\partial Y}+\frac{dZ}{d\lambda}\frac{\partial\vec{R}}{\partial Z}\right)\cdot\left(\frac{dX}{d\lambda}\frac{\partial\vec{R}}{\partial X}+\frac{dY}{d\lambda}\frac{\partial\vec{R}}{\partial Y}+\frac{dZ}{d\lambda}\frac{\partial\vec{R}}{\partial Z}\right)\\
&= \left(\frac{dX}{d\lambda}\right)^2\left(\frac{\partial\vec{R}}{\partial X}\cdot\frac{\partial\vec{R}}{\partial X}\right)+\left(\frac{dY}{d\lambda}\right)^2\left(\frac{\partial\vec{R}}{\partial Y}\cdot\frac{\partial\vec{R}}{\partial Y}\right)+\left(\frac{dZ}{d\lambda}\right)^2\left(\frac{\partial\vec{R}}{\partial Z}\cdot\frac{\partial\vec{R}}{\partial Z}\right)\\
&\quad+2\left(\frac{dX}{d\lambda}\cdot\frac{dY}{d\lambda}\right)\left(\frac{\partial\vec{R}}{\partial X}\cdot\frac{\partial\vec{R}}{\partial Y}\right)+2\left(\frac{dX}{d\lambda}\cdot\frac{dZ}{d\lambda}\right)\left(\frac{\partial\vec{R}}{\partial X}\cdot\frac{\partial\vec{R}}{\partial Z}\right)+2\left(\frac{dY}{d\lambda}\cdot\frac{dZ}{d\lambda}\right)\left(\frac{\partial\vec{R}}{\partial Y}\cdot\frac{\partial\vec{R}}{\partial Z}\right)\\
&= \left(\frac{dX}{d\lambda}\right)^2+\left(\frac{dY}{d\lambda}\right)^2+\left(\frac{dZ}{d\lambda}\right)^2\\
&= \begin{bmatrix}\frac{dX}{d\lambda} & \frac{dY}{d\lambda} & \frac{dZ}{d\lambda}\end{bmatrix}\begin{bmatrix}\frac{\partial\vec{R}}{\partial X}\cdot\frac{\partial\vec{R}}{\partial X} & \frac{\partial\vec{R}}{\partial X}\cdot\frac{\partial\vec{R}}{\partial Y} & \frac{\partial\vec{R}}{\partial X}\cdot\frac{\partial\vec{R}}{\partial Z}\\ \frac{\partial\vec{R}}{\partial Y}\cdot\frac{\partial\vec{R}}{\partial X} & \frac{\partial\vec{R}}{\partial Y}\cdot\frac{\partial\vec{R}}{\partial Y} & \frac{\partial\vec{R}}{\partial Y}\cdot\frac{\partial\vec{R}}{\partial Z}\\ \frac{\partial\vec{R}}{\partial Z}\cdot\frac{\partial\vec{R}}{\partial X} & \frac{\partial\vec{R}}{\partial Z}\cdot\frac{\partial\vec{R}}{\partial Y} & \frac{\partial\vec{R}}{\partial Z}\cdot\frac{\partial\vec{R}}{\partial Z}\end{bmatrix}\begin{bmatrix}\frac{dX}{d\lambda}\\ \frac{dY}{d\lambda}\\ \frac{dZ}{d\lambda}\end{bmatrix}\\
&= \begin{bmatrix}\frac{dX}{d\lambda} & \frac{dY}{d\lambda} & \frac{dZ}{d\lambda}\end{bmatrix}\begin{bmatrix}g_{ij}\end{bmatrix}\begin{bmatrix}\frac{dX}{d\lambda}\\ \frac{dY}{d\lambda}\\ \frac{dZ}{d\lambda}\end{bmatrix}\\
&= \begin{bmatrix}\frac{dX}{d\lambda} & \frac{dY}{d\lambda} & \frac{dZ}{d\lambda}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\frac{dX}{d\lambda}\\ \frac{dY}{d\lambda}\\ \frac{dZ}{d\lambda}\end{bmatrix}
\end{aligned} \tag{1.10.1}$$

$$\begin{aligned}
\frac{dX}{d\lambda} &= \cos(2\lambda)\\
\frac{dY}{d\lambda} &= \sin(2\lambda)\\
\frac{dZ}{d\lambda} &= -\sin(\lambda)
\end{aligned}$$

$$\begin{aligned}
\left\|\frac{d\vec{R}}{d\lambda}\right\|^2 &= \left(\frac{dX}{d\lambda}\right)^2+\left(\frac{dY}{d\lambda}\right)^2+\left(\frac{dZ}{d\lambda}\right)^2\\
&= 1+(\sin(\lambda))^2
\end{aligned}$$

$$\begin{aligned}
\text{arc length} &= \int_0^1\left\|\frac{d\vec{R}}{d\lambda}\right\|d\lambda\\
&= \int_0^1\sqrt{1+(\sin(\lambda))^2}d\lambda\\
&\approx 1.12389 \neq \sqrt{2}
\end{aligned}$$

### 1.10.2 Intrinsic Geometry

$$
\begin{aligned}
\left\|\frac{d\vec{R}}{d\lambda}\right\|^2 &= \frac{d\vec{R}}{d\lambda}\cdot\frac{d\vec{R}}{d\lambda} \\
&= \left(\frac{du}{d\lambda}\frac{\partial\vec{R}}{\partial u}+\frac{dv}{d\lambda}\frac{\partial\vec{R}}{\partial v}\right)\cdot\left(\frac{du}{d\lambda}\frac{\partial\vec{R}}{\partial u}+\frac{dv}{d\lambda}\frac{\partial\vec{R}}{\partial v}\right) \\
&= \left(\frac{du}{d\lambda}\right)^2\left(\frac{\partial\vec{R}}{\partial u}\cdot\frac{\partial\vec{R}}{\partial u}\right)+\left(\frac{dv}{d\lambda}\right)^2\left(\frac{\partial\vec{R}}{\partial v}\cdot\frac{\partial\vec{R}}{\partial v}\right)+2\left(\frac{du}{d\lambda}\cdot\frac{dv}{d\lambda}\right)\left(\frac{\partial\vec{R}}{\partial u}\cdot\frac{\partial\vec{R}}{\partial v}\right) \\
&= \begin{bmatrix}\frac{du}{d\lambda} & \frac{dv}{d\lambda}\end{bmatrix}\begin{bmatrix}\frac{\partial\vec{R}}{\partial u}\cdot\frac{\partial\vec{R}}{\partial u} & \frac{\partial\vec{R}}{\partial u}\cdot\frac{\partial\vec{R}}{\partial v} \\ \frac{\partial\vec{R}}{\partial v}\cdot\frac{\partial\vec{R}}{\partial u} & \frac{\partial\vec{R}}{\partial v}\cdot\frac{\partial\vec{R}}{\partial v}\end{bmatrix}\begin{bmatrix}\frac{du}{d\lambda} \\ \frac{dv}{d\lambda}\end{bmatrix} \\
&= \begin{bmatrix}\frac{du}{d\lambda} & \frac{dv}{d\lambda}\end{bmatrix}\begin{bmatrix}\vec{e_u}\cdot\vec{e_u} & \vec{e_u}\cdot\vec{e_v} \\ \vec{e_v}\cdot\vec{e_u} & \vec{e_v}\cdot\vec{e_v}\end{bmatrix}\begin{bmatrix}\frac{du}{d\lambda} \\ \frac{dv}{d\lambda}\end{bmatrix}
\end{aligned}
$$

$\frac{\partial\vec{R}}{\partial u}$ and $\frac{\partial\vec{R}}{\partial v}$ can be obtain by translating coordinate system to cartesian via chain rule

$$
\begin{aligned}
\frac{\partial\vec{R}}{\partial u} &= \vec{e_u} \\
&= \frac{dX}{du}\frac{\partial\vec{R}}{\partial X}+\frac{dY}{du}\frac{\partial\vec{R}}{\partial Y}+\frac{dZ}{du}\frac{\partial\vec{R}}{\partial Z} \\
&= \frac{dX}{du}\vec{e_X}+\frac{dY}{du}\vec{e_Y}+\frac{dZ}{du}\vec{e_Z} \\
&= \cos v\cos u\frac{\partial\vec{R}}{\partial X}+\sin v\cos u\frac{\partial\vec{R}}{\partial Y}-\sin u\frac{\partial\vec{R}}{\partial Z} \\
\frac{\partial\vec{R}}{\partial v} &= \vec{e_v} \\
&= \frac{dX}{dv}\frac{\partial\vec{R}}{\partial X}+\frac{dY}{dv}\frac{\partial\vec{R}}{\partial Y}+\frac{dZ}{dv}\frac{\partial\vec{R}}{\partial Z} \\
&= -\sin v\sin u\frac{\partial\vec{R}}{\partial X}+\cos v\sin u\frac{\partial\vec{R}}{\partial Y}
\end{aligned}
$$

$$
[g_{ij}] = \begin{bmatrix}\vec{e_u}\cdot\vec{e_u} & \vec{e_u}\cdot\vec{e_v} \\ \vec{e_v}\cdot\vec{e_u} & \vec{e_v}\cdot\vec{e_v}\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 0 & (\sin(u))^2\end{bmatrix}
$$

$$
\begin{aligned}
\left\|\frac{d\vec{R}}{d\lambda}\right\|^2 &= \begin{bmatrix}\frac{du}{d\lambda} & \frac{dv}{d\lambda}\end{bmatrix}\begin{bmatrix}\vec{e_u}\cdot\vec{e_u} & \vec{e_u}\cdot\vec{e_v} \\ \vec{e_v}\cdot\vec{e_u} & \vec{e_v}\cdot\vec{e_v}\end{bmatrix}\begin{bmatrix}\frac{du}{d\lambda} \\ \frac{dv}{d\lambda}\end{bmatrix} \\
&= \begin{bmatrix}\frac{du}{d\lambda} & \frac{dv}{d\lambda}\end{bmatrix}\begin{bmatrix}1 & 0 \\ 0 & (\sin(u))^2\end{bmatrix}\begin{bmatrix}\frac{du}{d\lambda} \\ \frac{dv}{d\lambda}\end{bmatrix} \\
&= \left(\frac{du}{d\lambda}\right)^2+(\sin(u))^2\left(\frac{dv}{d\lambda}\right)^2
\end{aligned}
$$

with example above:

$$\frac{du}{d\lambda} = \frac{dv}{d\lambda} = 1$$

$$\left\| \frac{d\vec{R}}{d\lambda} \right\|^2 = 1 + (\sin(u))^2$$

$$\begin{aligned}
\text{arc length} &= \int_0^1 \left\| \frac{d\vec{R}}{d\lambda} \right\| d\lambda \\
&= \int_0^1 \sqrt{1 + (\sin(\lambda))^2} d\lambda \\
&\approx 1.12389 \neq \sqrt{2}
\end{aligned}$$

### 1.10.3 Metric Tensor Field Execises

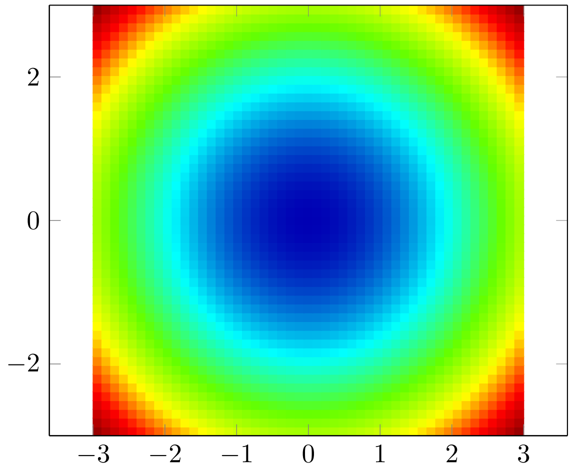
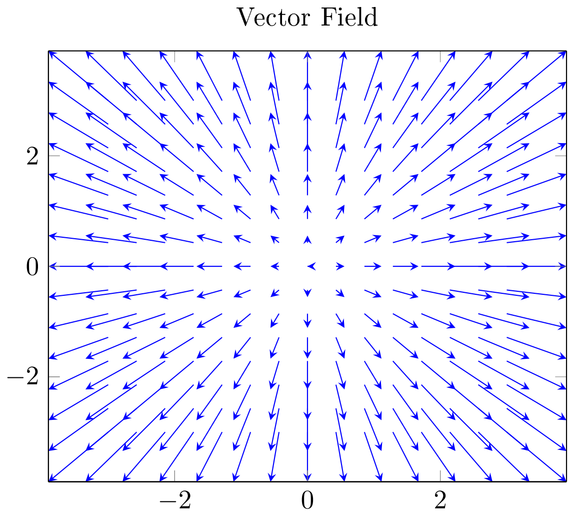
#### 1.10.3.1 Cylinder

#### 1.10.3.2 Saddle Surface

## 1.11 Gradient vs $d$ operator

$\nabla f$ “Del” $f$
aka “Gradient” of $f$
Vector Field

$df$ “dee” $f$
aka “Differential” of $f$
aka “Exterior Derivative” of $f$

<table>
<tr><td>$\nabla f$</td><td>$df$</td></tr>
<tr><td>“Del” $f$</td><td>“dee” $f$</td></tr>
<tr><td>“Gradient” of $f$</td><td>“Differential” of $f$</td></tr>
<tr><td></td><td>“Exterior Derivative” of $f$</td></tr>
<tr><td colspan="2">acts on scalar field<br></td></tr>
<tr><td>Vector Field</td><td>Covector Field(1-form)</td></tr>
<tr><td></td><td></td></tr>
</table>

Convert components $df$ and $\nabla f$:

$$(\nabla f)^i \, g_{ij} = \frac{\partial f}{\partial c^j}$$

$$(\nabla f)^k = \mathfrak{g}^{jk} \frac{\partial f}{\partial c^j}$$

$$\flat\,(\nabla f) = df$$

$$\sharp\,(df) = \nabla f$$

$$\boxed{df = \nabla f \cdot \_\_ = g_{ij} \, (\nabla f)^i \, dc^j}$$

## 1.12 Geodesics and Christoffel Symbols

**Definition 4.** In curved space, a straight path has zero tangential acceleration when travel along it at constant speed.

*Remark* 5. Geodesic curves are curves where the acceleration vector is normal to the surface.

$$\frac{d^2\vec{R}}{d\lambda^2} = \left(\frac{d^2\vec{R}}{d\lambda^2}\right)^{\text{normal}} + \underbrace{\left(\frac{d^2\vec{R}}{d\lambda^2}\right)^{\text{tangential}}}_{=0}$$

$$\text{Velocity Vector (always in tangent plane)} = \frac{d\vec{R}}{d\lambda} = \frac{du}{d\lambda}\frac{\partial \vec{R}}{\partial u} + \frac{dv}{d\lambda}\frac{\partial \vec{R}}{\partial v}$$

Acceleration Vector:

$$\begin{aligned}
\frac{d}{d\lambda}\left(\frac{\partial \vec{R}}{\partial \lambda}\right) &= \frac{d}{d\lambda}\left(\frac{du}{d\lambda}\frac{\partial \vec{R}}{\partial u} + \frac{dv}{d\lambda}\frac{\partial \vec{R}}{\partial v}\right) \\
&= \frac{d}{d\lambda}\left(\frac{du}{d\lambda}\frac{\partial \vec{R}}{\partial u}\right) + \frac{d}{d\lambda}\left(\frac{dv}{d\lambda}\frac{\partial \vec{R}}{\partial v}\right) \\
&= \frac{d^2u}{d\lambda^2}\frac{\partial \vec{R}}{\partial u} + \frac{du}{d\lambda} \underbrace{\left(\frac{d}{d\lambda}\frac{\partial \vec{R}}{\partial u}\right)}_{\boxed{\begin{aligned} &= \left(\frac{du}{d\lambda}\frac{\partial}{\partial u} + \frac{dv}{d\lambda}\frac{\partial}{\partial v}\right)\frac{\partial \vec{R}}{\partial u} \\ &= \frac{du}{d\lambda}\frac{\partial^2 \vec{R}}{\partial u^2} + \frac{dv}{d\lambda}\frac{\partial^2 \vec{R}}{\partial u \partial v}\end{aligned}}} + \frac{d^2v}{d\lambda^2}\frac{\partial \vec{R}}{\partial v} + \frac{dv}{d\lambda}\left(\frac{d}{d\lambda}\frac{\partial \vec{R}}{\partial v}\right) \\
&= \frac{d^2u}{d\lambda^2}\frac{\partial \vec{R}}{\partial u} + \frac{du}{d\lambda}\left(\frac{du}{d\lambda}\frac{\partial^2 \vec{R}}{\partial u^2} + \frac{dv}{d\lambda}\frac{\partial^2 \vec{R}}{\partial u \partial v}\right) \\
&\quad + \frac{d^2v}{d\lambda^2}\frac{\partial \vec{R}}{\partial v} + \frac{dv}{d\lambda}\left(\frac{du}{d\lambda}\frac{\partial^2 \vec{R}}{\partial v \partial u} + \frac{dv}{d\lambda}\frac{\partial^2 \vec{R}}{\partial v^2}\right) \\
&= \underbrace{\frac{d^2u}{d\lambda^2}\frac{\partial \vec{R}}{\partial u} + \frac{d^2v}{d\lambda^2}\frac{\partial \vec{R}}{\partial v}}_{\text{tangential}} \\
&\quad + \left(\frac{du}{d\lambda}\right)^2\frac{\partial^2 \vec{R}}{\partial u^2} + \frac{du}{d\lambda}\frac{dv}{d\lambda}\frac{\partial^2 \vec{R}}{\partial u \partial v} + \frac{dv}{d\lambda}\frac{du}{d\lambda}\frac{\partial^2 \vec{R}}{\partial v \partial u} + \left(\frac{dv}{d\lambda}\right)^2\frac{\partial^2 \vec{R}}{\partial v^2} \\
&= \underbrace{\frac{d^2u^i}{d\lambda^2}\frac{\partial \vec{R}}{\partial u^i}}_{\text{tangential}} + \frac{du^i}{d\lambda}\frac{du^j}{d\lambda}\frac{\partial^2 \vec{R}}{\partial u^i \partial u^j}
\end{aligned}$$

Denote $\frac{\partial^2 \vec{R}}{\partial u^i \partial u^j}$ in 3D using tangential basis vector $\frac{\partial \vec{R}}{\partial u^1}$, $\frac{\partial \vec{R}}{\partial u^2}$ and normal basis vector $\hat{n}$:

$$\boxed{\text{Second Fundamental Form } L_{ij} \text{ is the normal components of } \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j}}$$

$$\boxed{\text{Christoffel Symbol } \Gamma^k_{ij} \text{ is the tangential components of } \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j}}$$

$$\begin{aligned} \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j} &= \Gamma^1_{ij} \frac{\partial \vec{R}}{\partial u^1} + \Gamma^2_{ij} \frac{\partial \vec{R}}{\partial u^2} + L_{ij} \hat{n} \\ &= \Gamma^k_{ij} \frac{\partial \vec{R}}{\partial u^k} + L_{ij} \hat{n} \end{aligned} \tag{1.12.1}$$

$$\begin{aligned} \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j} \cdot \frac{\vec{R}}{\partial u^l} &= \left( \Gamma^k_{ij} \frac{\partial \vec{R}}{\partial u^k} + L_{ij} \hat{n} \right) \cdot \frac{\partial \vec{R}}{\partial u^l} \\ &= \Gamma^k_{ij} \frac{\partial \vec{R}}{\partial u^k} \cdot \frac{\partial \vec{R}}{\partial u^l} + \underbrace{L_{ij} \hat{n} \cdot \frac{\vec{R}}{\partial u^l}}_{\text{perpendicular}} \\ &= \Gamma^k_{ij} \frac{\partial \vec{R}}{\partial u^k} \cdot \frac{\partial \vec{R}}{\partial u^l} \\ &= \Gamma^k_{ij} g_{kl} \\ \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j} \cdot \frac{\vec{R}}{\partial u^l} \mathfrak{g}^{lm} &= \Gamma^k_{ij} g_{kl} \mathfrak{g}^{lm} \\ &= \Gamma^k_{ij} \delta^m_k \\ &= \Gamma^m_{ij} \end{aligned}$$

$$\boxed{\Gamma^m_{ij} = \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j} \cdot \frac{\vec{R}}{\partial u^l} \mathfrak{g}^{lm}}$$

$$\begin{aligned} \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j} \cdot \hat{n} &= \left( \Gamma^k_{ij} \frac{\partial \vec{R}}{\partial u^k} + L_{ij} \hat{n} \right) \cdot \hat{n} \\ &= L_{ij} \left( \hat{n} \cdot \hat{n} \right) \\ \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j} \cdot \frac{\vec{e_i} \times \vec{e_j}}{\|\vec{e_i} \times \vec{e_j}\|} &= L_{ij} \\ \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j} \cdot \frac{\frac{\partial \vec{R}}{\partial u^i} \times \frac{\partial \vec{R}}{\partial u^j}}{\left\| \frac{\partial \vec{R}}{\partial u^i} \times \frac{\partial \vec{R}}{\partial u^j} \right\|} &= L_{ij} \end{aligned}$$

$$\boxed{L_{ij} = \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j} \cdot \frac{\vec{e_i} \times \vec{e_j}}{\|\vec{e_i} \times \vec{e_j}\|} = \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j} \cdot \frac{\frac{\partial \vec{R}}{\partial u^i} \times \frac{\partial \vec{R}}{\partial u^j}}{\left\| \frac{\partial \vec{R}}{\partial u^i} \times \frac{\partial \vec{R}}{\partial u^j} \right\|}}$$

$$
\begin{aligned}
\frac{d}{d\lambda}\left(\frac{\partial\vec{R}}{\partial\lambda}\right) &= \underbrace{\frac{d^2u^i}{d\lambda^2}\frac{\partial\vec{R}}{\partial u^i}}_{\text{tangential}} + \frac{du^i}{d\lambda}\frac{du^j}{d\lambda}\frac{\partial^2\vec{R}}{\partial u^i\partial u^j} \\
&= \frac{d^2u^i}{d\lambda^2}\frac{\partial\vec{R}}{\partial u^i} + \frac{du^i}{d\lambda}\frac{du^j}{d\lambda}\left(\Gamma^k_{ij}\frac{\partial\vec{R}}{\partial u^k} + L_{ij}\hat{n}\right) 1.12.1 \\
&= \underbrace{\left(\frac{d^2u^k}{d\lambda^2} + \Gamma^k_{ij}\frac{du^i}{d\lambda}\frac{du^j}{d\lambda}\right)\frac{\partial\vec{R}}{\partial u^k}}_{\text{tantential component}} + \underbrace{L_{ij}\frac{du^i}{d\lambda}\frac{du^j}{d\lambda}}_{\text{normal component}}\hat{n}
\end{aligned}
$$

for geodesic curve, set tangential component to 0.

$$\boxed{\text{Geodesic Equation: } \frac{d^2u^k}{d\lambda^2} + \Gamma^k_{ij}\frac{du^i}{d\lambda}\frac{du^j}{d\lambda} = 0}$$

**Example 6.** Geodesic on Flat Plane

1. Normal position vector on flat plane

$$\vec{R}(u,v) = \vec{p} + u\vec{a} + v\vec{b}$$

2. Calculate partial derivative of variables

$$
\begin{aligned}
\frac{\partial\vec{R}}{\partial u} &= \vec{a} \\
\frac{\partial\vec{R}}{\partial v} &= \vec{b}
\end{aligned}
$$

3. Calculate second order derivative of variable $u$ and $v$.

$$\frac{\partial^2\vec{R}}{\partial u^2} = \frac{\partial^2\vec{R}}{\partial v^2} = \frac{\partial^2\vec{R}}{\partial u\partial v} = \frac{\partial^2\vec{R}}{\partial v\partial u} = 0$$

4. Calculate Christoffel Symbol $\Gamma^k_{ij}$.

$$\Gamma^k_{ij} = \underbrace{\frac{\partial^2\vec{R}}{\partial u^i\partial u^j}}_{\text{all are 0}} \cdot \frac{\vec{R}}{\partial u^l}\mathfrak{g}^{lk} = 0$$

Christoffel Symbol track basis vector changes from point to point, hence the zero in flat plane.

5. Solve geodesic equation.

$$
\begin{aligned}
\frac{d^2u^k}{d\lambda^2} + \Gamma^k_{ij}\frac{du^i}{d\lambda}\frac{du^j}{d\lambda} &= 0 \\
\frac{d^2u^k}{d\lambda^2} &= 0
\end{aligned}
$$

Expand,

$$\text{Solve}\begin{cases}\frac{d^2u}{d\lambda^2} &= 0\\ \frac{d^2v}{d\lambda^2} &= 0\end{cases}$$

$$\int\int 0 d\lambda d\lambda = c_1\lambda + c_2$$

$$\begin{aligned}u &= k_u\lambda + u_0\\ v &= k_v\lambda + v_0\end{aligned}$$

6. Plug in function for $u$ and $v$, solve geodesic equation,

$$\begin{aligned}\vec{R}(u,v) &= \vec{p} + u\vec{a} + v\vec{b}\\ &= \vec{p} + (k_u\lambda + u_0)\vec{a} + (k_v\lambda + v_0)\vec{b}\\ &= \vec{p} + k_u\lambda\vec{a} + u_0\vec{a} + k_v\lambda\vec{b} + v_0\vec{b}\\ &= \underbrace{\left(\vec{p} + u_0\vec{a} + v_0\vec{b}\right)}_{\text{initial position}} + \underbrace{\lambda}_{\text{time}}\underbrace{\left(k_u\vec{a} + k_v\vec{b}\right)}_{\text{initial velocity}}\end{aligned}$$

**Example 7.** Geodesic on Sphere

1. Calculate partial derivative of variables

$$\begin{aligned}\frac{\partial\vec{R}}{\partial u} &= \frac{dX}{du}\frac{\partial\vec{R}}{\partial X} + \frac{dY}{du}\frac{\partial\vec{R}}{\partial Y} + \frac{dZ}{du}\frac{\partial\vec{R}}{\partial Z}\\ &= \frac{dX}{du}\vec{e_X} + \frac{dY}{du}\vec{e_Y} + \frac{dZ}{du}\vec{e_Z}\\ &= \cos v\cos u\frac{\partial\vec{R}}{\partial X} + \sin v\cos u\frac{\partial\vec{R}}{\partial Y} - \sin u\frac{\partial\vec{R}}{\partial Z}\\ \frac{\partial\vec{R}}{\partial v} &= \frac{dX}{dv}\frac{\partial\vec{R}}{\partial X} + \frac{dY}{dv}\frac{\partial\vec{R}}{\partial Y} + \frac{dZ}{dv}\frac{\partial\vec{R}}{\partial Z}\\ &= -\sin v\sin u\frac{\partial\vec{R}}{\partial X} + \cos v\sin u\frac{\partial\vec{R}}{\partial Y}\end{aligned}$$

2. Calculate metric tensor

$$[g_{ij}] = \begin{bmatrix}\frac{\partial\vec{R}}{\partial u}\cdot\frac{\partial\vec{R}}{\partial u} & \frac{\partial\vec{R}}{\partial u}\cdot\frac{\partial\vec{R}}{\partial v}\\ \frac{\partial\vec{R}}{\partial v}\cdot\frac{\partial\vec{R}}{\partial u} & \frac{\partial\vec{R}}{\partial v}\cdot\frac{\partial\vec{R}}{\partial v}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & (\sin(u))^2\end{bmatrix}$$

3. Calculate second order derivative along $u$ and $v$.

$$\begin{aligned}\frac{\partial^2\vec{R}}{\partial u^2} &= \frac{\partial}{\partial u}\frac{\partial\vec{R}}{\partial u} = -\cos v\sin u\frac{\partial\vec{R}}{\partial X} - \sin v\sin u\frac{\partial\vec{R}}{\partial Y} - \cos u\frac{\partial\vec{R}}{\partial Z}\\ \frac{\partial^2\vec{R}}{\partial v^2} &= \frac{\partial}{\partial v}\frac{\partial\vec{R}}{\partial v} = -\cos v\sin u\frac{\partial\vec{R}}{\partial X} - \sin v\sin u\frac{\partial\vec{R}}{\partial Y}\\ \frac{\partial}{\partial v}\left(\frac{\partial\vec{R}}{\partial u}\right) &= \frac{\partial}{\partial u}\left(\frac{\partial\vec{R}}{\partial v}\right) = -\sin v\cos u\frac{\partial\vec{R}}{\partial X} + \cos v\cos u\frac{\partial\vec{R}}{\partial Y}\end{aligned}$$

4. Calculate Christoffel Symbol $\Gamma^k_{ij}$.

$$\left[\mathfrak{g}^{\mathrm{ij}}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1/(\sin(u))^2 \end{bmatrix}$$

$$\frac{\partial \vec{R}}{\partial u}\frac{\partial^2 \vec{R}}{\partial u^2} = \frac{\partial \vec{R}}{\partial v}\frac{\partial^2 \vec{R}}{\partial u^2} = 0$$

$$\frac{\partial \vec{R}}{\partial u}\frac{\partial^2 \vec{R}}{\partial v^2} = -\cos u \sin u$$

$$\frac{\partial \vec{R}}{\partial u}\frac{\partial^2 \vec{R}}{\partial v \partial u} = \frac{\partial \vec{R}}{\partial u}\frac{\partial^2 \vec{R}}{\partial u \partial v} = 0$$

$$\frac{\partial \vec{R}}{\partial v}\frac{\partial^2 \vec{R}}{\partial v^2} = 0$$

$$\frac{\partial \vec{R}}{\partial v}\frac{\partial^2 \vec{R}}{\partial v \partial u} = \frac{\partial \vec{R}}{\partial v}\frac{\partial^2 \vec{R}}{\partial u \partial v} = \cos u \sin u$$

$$\Gamma^k_{ij} = \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j} \cdot \frac{\vec{R}}{\partial u^l}\mathfrak{g}^{lk}$$

$$\begin{aligned}
\Gamma^1_{ij} &= \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j} \cdot \frac{\vec{R}}{\partial u^l}\mathfrak{g}^{l1} \\
&= \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j} \cdot \frac{\vec{R}}{\partial u^1}\mathfrak{g}^{11} + \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j} \cdot \frac{\vec{R}}{\partial u^2}\mathfrak{g}^{21} \\
&= \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j} \cdot \frac{\vec{R}}{\partial u^1}\mathfrak{g}^{11} \\
&= \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j} \cdot \frac{\vec{R}}{\partial u^1} \\
\Gamma^2_{ij} &= \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j} \cdot \frac{\vec{R}}{\partial u^l}\mathfrak{g}^{l2} \\
&= \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j} \cdot \frac{\vec{R}}{\partial u^1}\mathfrak{g}^{12} + \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j} \cdot \frac{\vec{R}}{\partial u^2}\mathfrak{g}^{22} \\
&= \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j} \cdot \frac{\vec{R}}{\partial u^2}\mathfrak{g}^{22} \\
&= \frac{\partial^2 \vec{R}}{\partial u^i \partial u^j} \cdot \frac{\vec{R}}{\partial u^2}\left(\frac{1}{(\sin(u))^2}\right)
\end{aligned}$$

$$\frac{\partial \vec{R}}{\partial u}\frac{\partial^2 \vec{R}}{\partial v^2} = \frac{\partial \vec{R}}{\partial u^1}\frac{\partial^2 \vec{R}}{\partial u \partial u} = -\cos u \sin u$$

$$\frac{\partial \vec{R}}{\partial v}\frac{\partial^2 \vec{R}}{\partial v \partial u} = \frac{\partial \vec{R}}{\partial u^2}\frac{\partial^2 \vec{R}}{\partial u^1 \partial u^2} = \cos u \sin u$$

$$\Gamma^1_{22} = \frac{\partial^2 \vec{R}}{\partial u^2 \partial u^2} \cdot \frac{\vec{R}}{\partial u^1} = -\cos u \sin u$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{\partial^2 \vec{R}}{\partial u^1 \partial u^2} \cdot \frac{\vec{R}}{\partial u^2}\left(\frac{1}{(\sin(u))^2}\right) = \frac{\cos u \sin u}{(\sin(u))^2} = \frac{\cos u}{\sin u}$$

$$[\Gamma^1_{ij}] = \begin{bmatrix} 0 & 0 \\ 0 & -\cos u \sin u \end{bmatrix}$$
$$[\Gamma^2_{ij}] = \begin{bmatrix} 0 & {}^{\cos u}\!/\!_{\sin u} \\ {}^{\cos u}\!/\!_{\sin u} & 0 \end{bmatrix}$$

5. Solve geodesic equation.

$$\frac{d^2u^k}{d\lambda^2} + \Gamma^k_{ij}\frac{du^i}{d\lambda}\frac{du^j}{d\lambda} = 0$$

Expand,

$$\text{Solve}\begin{cases} \frac{d^2u^1}{d\lambda^2} + \Gamma^1_{22}\frac{du^i}{d\lambda}\frac{du^j}{d\lambda} & = 0 \\ \frac{d^2u^2}{d\lambda^2} + \Gamma^2_{12}\frac{du^1}{d\lambda}\frac{du^2}{d\lambda} + \Gamma^2_{21}\frac{du^2}{d\lambda}\frac{du^1}{d\lambda} & = 0 \end{cases}$$

Plug in Christoffel Symbols,

$$\frac{d^2u^1}{d\lambda^2} - \cos u \sin u \frac{du^i}{d\lambda}\frac{du^j}{d\lambda} = 0$$
$$\frac{d^2u^2}{d\lambda^2} + 2\frac{\cos u}{\sin u}\frac{du^1}{d\lambda}\frac{du^2}{d\lambda} = 0$$

## 1.13 Covariant Derivative

**Definition 8.** Covariant Derivative is a tool to understand the rate of change of vector (tensor) fields that takes changing basis vectors into account.

### 1.13.1 Flat Space Definition

$$\vec{v} = v^x\vec{e_x} + v^y\vec{e_y} = v^i\vec{e_i} \tag{1.13.1}$$
$$\vec{v} = v^x\frac{\partial\vec{R}}{\partial x} + v^y\frac{\partial\vec{R}}{\partial y} = v^i\frac{\partial\vec{R}}{\partial c^i} \tag{1.13.2}$$
$$\vec{v} = \tilde{v^r}\tilde{\vec{e_r}} + \tilde{v^\theta}\tilde{\vec{e_\theta}} = \tilde{v^i}\tilde{\vec{e_i}} \tag{1.13.3}$$
$$\vec{v} = \tilde{v^r}\frac{\partial\vec{R}}{\partial r} + \tilde{v^\theta}\frac{\partial\vec{R}}{\partial r} = \tilde{v^i}\frac{\partial\vec{R}}{\partial p^i} \tag{1.13.4}$$

**Example 9.** Cartesian Vector Field $\vec{v} = 2\vec{e_x} + 1\vec{e_y}$

$$\begin{aligned}\frac{\partial}{\partial x}(\vec{v}) &= \frac{\partial}{\partial x}(v^x\vec{e_x} + v^y\vec{e_y}) \\ &= \frac{\partial}{\partial x}(v^x\vec{e_x}) + \frac{\partial}{\partial x}(v^y\vec{e_y}) \\ &= \frac{\partial v^x}{\partial x}\vec{e_x} + v^x\underbrace{\frac{\partial}{\partial x}(\vec{e_x})}_{0} + \frac{\partial v^y}{\partial x}\vec{e_y} + v^y\underbrace{\frac{\partial}{\partial x}(\vec{e_y})}_{0} \\ &= \frac{\partial\overbrace{v^x}^{\text{const}}}{\partial x}\vec{e_x} + \frac{\partial v^y}{\partial x}\vec{e_y} \\ &= \vec{0}\end{aligned} \tag{1.13.5}$$

**Example 10.** Polar Vector Field $\vec{v} = 2\tilde{\vec{e_r}} + 1\tilde{\vec{e_\theta}}$

$$
\begin{aligned}
\frac{\partial}{\partial\theta}(\vec{v}) &= \frac{\partial}{\partial\theta}\left(\tilde{v^r}\tilde{\vec{e_r}} + \tilde{v^\theta}\tilde{\vec{e_\theta}}\right) \\
&= \frac{\partial}{\partial\theta}\left(\tilde{v^r}\tilde{\vec{e_r}}\right) + \frac{\partial}{\partial\theta}\left(\tilde{v^\theta}\tilde{\vec{e_\theta}}\right) \\
&= \frac{\partial\tilde{v^r}}{\partial\theta}\tilde{\vec{e_r}} + \tilde{v^r}\frac{\partial}{\partial\theta}\left(\tilde{\vec{e_r}}\right) + \frac{\partial\tilde{v^\theta}}{\partial\theta}\tilde{\vec{e_\theta}} + \tilde{v^\theta}\frac{\partial}{\partial\theta}\left(\tilde{\vec{e_\theta}}\right) \\
&= \underbrace{\frac{\partial\tilde{v^r}}{\partial\theta}\tilde{\vec{e_r}} + \frac{\partial\tilde{v^\theta}}{\partial\theta}\tilde{\vec{e_\theta}}}_{\text{change of components}} + \underbrace{\tilde{v^r}\frac{\partial}{\partial\theta}\left(\tilde{\vec{e_r}}\right) + \tilde{v^\theta}\frac{\partial}{\partial\theta}\left(\tilde{\vec{e_\theta}}\right)}_{\text{change of basis vector}}
\end{aligned}
\tag{1.13.6}
$$

Need to figure out change of basis vector,

$$
\frac{\partial}{\partial\theta}(\vec{v}) = \frac{\partial\tilde{v^r}}{\partial\theta}\tilde{\vec{e_r}} + \frac{\partial\tilde{v^\theta}}{\partial\theta}\tilde{\vec{e_\theta}} + \tilde{v^r}\frac{\partial}{\partial\theta}\left(\tilde{\vec{e_r}}\right) + \tilde{v^\theta}\frac{\partial}{\partial\theta}\left(\tilde{\vec{e_\theta}}\right) \tag{1.13.7}
$$

$$
\frac{\partial}{\partial r}(\vec{v}) = \frac{\partial\tilde{v^r}}{\partial r}\tilde{\vec{e_r}} + \frac{\partial\tilde{v^\theta}}{\partial r}\tilde{\vec{e_\theta}} + \tilde{v^r}\frac{\partial}{\partial r}\left(\tilde{\vec{e_r}}\right) + \tilde{v^\theta}\frac{\partial}{\partial r}\left(\tilde{\vec{e_\theta}}\right) \tag{1.13.8}
$$

convert to cartesian coordinates,

$$
\begin{aligned}
\tilde{\vec{e_r}} &= \frac{\partial x}{\partial r}\vec{e_x} + \frac{\partial x}{\partial r}\vec{e_y} \\
&= \cos\theta\vec{e_x} + \sin\theta\vec{e_y}
\end{aligned}
\tag{1.13.9}
$$

$$
\begin{aligned}
\tilde{\vec{e_\theta}} &= \frac{\partial x}{\partial\theta}\vec{e_x} + \frac{\partial x}{\partial\theta}\vec{e_y} \\
&= -r\sin\theta\vec{e_x} + r\cos\theta\vec{e_y}
\end{aligned}
\tag{1.13.10}
$$

$$
\begin{aligned}
\frac{\partial}{\partial\theta}\left(\tilde{\vec{e_r}}\right) &= \frac{\partial}{\partial\theta}(\cos\theta\vec{e_x} + \sin\theta\vec{e_y}) \\
&= \frac{\partial}{\partial\theta}(\cos\theta\vec{e_x}) + \frac{\partial}{\partial\theta}(\sin\theta\vec{e_y}) \\
&= \frac{\partial\cos\theta}{\partial\theta}\vec{e_x} + \cos\theta\frac{\partial\overbrace{\vec{e_x}}^{\text{const everywhere}}}{\partial\theta} + \frac{\partial\sin\theta}{\partial\theta}\vec{e_y} + \sin\theta\frac{\partial\vec{e_y}}{\partial\theta} \\
&= -\sin\theta\vec{e_x} + \cos\theta\vec{e_y} \\
&= \frac{\partial}{\partial\theta}\left(\tilde{\vec{e_r}}\right) = \frac{\partial}{\partial\theta}\frac{\partial\vec{R}}{\partial r} = \frac{\partial}{\partial r}\frac{\partial\vec{R}}{\partial\theta} = \frac{\partial}{\partial r}\left(\tilde{\vec{e_\theta}}\right)
\end{aligned}
\tag{1.13.11}
$$

$$
\begin{aligned}
\frac{\partial}{\partial r}\left(\tilde{\vec{e_r}}\right) &= \frac{\partial}{\partial r}(\cos\theta\vec{e_x} + \sin\theta\vec{e_y}) \\
&= \frac{\partial}{\partial r}(\cos\theta\vec{e_x}) + \frac{\partial}{\partial r}(\sin\theta\vec{e_y}) \\
&= \frac{\partial\cos\theta}{\partial r}\vec{e_x} + \cos\theta\frac{\partial\overbrace{\vec{e_x}}^{\text{const everywhere}}}{\partial r} + \frac{\partial\sin\theta}{\partial r}\vec{e_y} + \sin\theta\frac{\partial\vec{e_y}}{\partial r} \\
&= \vec{0}
\end{aligned}
\tag{1.13.12}
$$

$$\frac{\partial}{\partial \theta}\left(\tilde{\vec{e_\theta}}\right) = -r\cos\theta \vec{e_x} - \sin\theta \vec{e_y} \tag{1.13.13}$$

$$\begin{aligned} \vec{e_x} &= \frac{\partial r}{\partial x}\tilde{\vec{e_r}} + \frac{\partial \theta}{\partial x}\tilde{\vec{e_\theta}} \\ &= \left(\frac{x}{\sqrt{x^2+y^2}}\right)\tilde{\vec{e_r}} + \left(-\frac{y}{x^2+y^2}\right)\tilde{\vec{e_\theta}} \\ &= \left(\frac{x}{r}\right)\tilde{\vec{e_r}} + \left(-\frac{y}{r^2}\right)\tilde{\vec{e_\theta}} \\ &= \left(\frac{r\cos\theta}{r}\right)\tilde{\vec{e_r}} + \left(-\frac{r\sin\theta}{r^2}\right)\tilde{\vec{e_\theta}} \\ &= (\cos\theta)\,\tilde{\vec{e_r}} + \left(-\frac{\sin\theta}{r}\right)\tilde{\vec{e_\theta}} \end{aligned} \tag{1.13.14}$$

$$\begin{aligned} \vec{e_y} &= \frac{\partial r}{\partial y}\tilde{\vec{e_r}} + \frac{\partial \theta}{\partial y}\tilde{\vec{e_\theta}} \\ &= \left(\frac{y}{\sqrt{x^2+y^2}}\right)\tilde{\vec{e_r}} + \left(\frac{x}{x^2+y^2}\right)\tilde{\vec{e_\theta}} \\ &= \left(\frac{y}{r}\right)\tilde{\vec{e_r}} + \left(\frac{x}{r^2}\right)\tilde{\vec{e_\theta}} \\ &= (\sin\theta)\,\tilde{\vec{e_r}} + \left(\frac{\cos\theta}{r}\right)\tilde{\vec{e_\theta}} \end{aligned} \tag{1.13.15}$$

Plug in,

$$\frac{\partial}{\partial r}\left(\tilde{\vec{e_r}}\right) = 0 \tag{1.13.16}$$

$$\begin{aligned} \frac{\partial}{\partial \theta}\left(\tilde{\vec{e_r}}\right) = \frac{\partial}{\partial r}\left(\tilde{\vec{e_\theta}}\right) &= -\sin\theta\vec{e_x} + \cos\theta\vec{e_y} \\ &= -\sin\theta\left((\cos\theta)\,\tilde{\vec{e_r}} + \left(-\frac{\sin\theta}{r}\right)\tilde{\vec{e_\theta}}\right) \\ &\quad + \cos\theta\left((\sin\theta)\,\tilde{\vec{e_r}} + \left(\frac{\cos\theta}{r}\right)\tilde{\vec{e_\theta}}\right) \\ &= \frac{1}{r}\tilde{\vec{e_\theta}} \end{aligned} \tag{1.13.17}$$

$$\begin{aligned} \frac{\partial}{\partial \theta}\left(\tilde{\vec{e_\theta}}\right) &= -r\cos\theta\vec{e_x} - \sin\theta\vec{e_y} \\ &= -r\cos\theta\left((\cos\theta)\,\tilde{\vec{e_r}} + \left(-\frac{\sin\theta}{r}\right)\tilde{\vec{e_\theta}}\right) \\ &\quad - \sin\theta\left((\sin\theta)\,\tilde{\vec{e_r}} + \left(\frac{\cos\theta}{r}\right)\tilde{\vec{e_\theta}}\right) \\ &= -r\tilde{\vec{e_r}} \end{aligned} \tag{1.13.18}$$

Plug in,

$$\begin{aligned}
\frac{\partial}{\partial \theta}\left(\vec{v}\right) &= \frac{\partial \tilde{v}^r}{\partial \theta}\tilde{\vec{e}_r} + \frac{\partial \tilde{v}^\theta}{\partial \theta}\tilde{\vec{e}_\theta} + \tilde{v}^r \frac{\partial}{\partial \theta}\left(\tilde{\vec{e}_r}\right) + \tilde{v}^\theta \frac{\partial}{\partial \theta}\left(\tilde{\vec{e}_\theta}\right) \\
&= \frac{\partial \tilde{v}^r}{\partial \theta}\tilde{\vec{e}_r} + \frac{\partial \tilde{v}^\theta}{\partial \theta}\tilde{\vec{e}_\theta} + \tilde{v}^r \left(\frac{1}{r}\tilde{\vec{e}_\theta}\right) + \tilde{v}^\theta \left(-r\tilde{\vec{e}_r}\right) \\
&= \frac{1}{r}\tilde{\vec{e}_\theta}
\end{aligned} \tag{1.13.19}$$

$$\begin{aligned}
\frac{\partial}{\partial r}\left(\vec{v}\right) &= \frac{\partial \tilde{v}^r}{\partial r}\tilde{\vec{e}_r} + \frac{\partial \tilde{v}^\theta}{\partial r}\tilde{\vec{e}_\theta} + \tilde{v}^r \frac{\partial}{\partial r}\left(\tilde{\vec{e}_r}\right) + \tilde{v}^\theta \frac{\partial}{\partial r}\left(\tilde{\vec{e}_\theta}\right) \\
&= \frac{\partial \tilde{v}^r}{\partial r}\tilde{\vec{e}_r} + \frac{\partial \tilde{v}^\theta}{\partial r}\tilde{\vec{e}_\theta} + \tilde{v}^r \left(0\right) + \tilde{v}^\theta \left(\frac{1}{r}\tilde{\vec{e}_\theta}\right) \\
&= -r\tilde{\vec{e}_r} + \frac{2}{r}\tilde{\vec{e}_\theta}
\end{aligned} \tag{1.13.20}$$

$$\boxed{\text{Constant Components } \neq \text{ Constant Vector Field}}$$

$$\boxed{\frac{\partial \vec{e_j}}{\partial c^i} = \Gamma^1_{ij}\vec{e_1} + \Gamma^2_{ij}\vec{e_2} = \Gamma^k_{ij}\vec{e_k}} \tag{1.13.21}$$

$$\begin{aligned}
\frac{\partial}{\partial c^i}\left(\vec{v}\right) &= \frac{\partial}{\partial c^i}\left(v^j \vec{e_j}\right) \\
&= \underbrace{\frac{\partial v^j}{\partial c^i}\vec{e_j}}_{\text{components}} + \underbrace{v^j \frac{\partial \vec{e_j}}{\partial c^i}}_{\text{basis vectors}} \\
&= \frac{\partial v^j}{\partial c^i}\vec{e_j} + v^j \Gamma^k_{ij}\vec{e_k} \\
&= \frac{\partial v^k}{\partial c^i}\vec{e_k} + v^j \Gamma^k_{ij}\vec{e_k} \\
&= \left(\frac{\partial v^k}{\partial c^i} + v^j \Gamma^k_{ij}\right)\vec{e_k}
\end{aligned} \tag{1.13.22}$$

$$\text{Flat Space \& Cartesian Coord: } \Gamma^k_{ij} = 0 \tag{1.13.23}$$

$$\begin{aligned}
\frac{\partial}{\partial p^i}\left(\vec{v}\right) &= \frac{\partial}{\partial p^i}\left(\tilde{v^j}\tilde{\vec{e_j}}\right) \\
&= \underbrace{\frac{\partial \tilde{v^j}}{\partial p^i}\vec{e_j}}_{\text{components}} + \underbrace{\tilde{v^j} \frac{\partial \tilde{\vec{e_j}}}{\partial p^i}}_{\text{basis vectors}} \\
&= \frac{\partial \tilde{v^j}}{\partial p^i}\vec{e_j} + \tilde{v^j} \Gamma^k_{ij}\vec{e_k} \\
&= \frac{\partial \tilde{v^k}}{\partial p^i}\vec{e_k} + \tilde{v^j} \Gamma^k_{ij}\vec{e_k} \\
&= \left(\frac{\partial \tilde{v^k}}{\partial p^i} + \tilde{v^j} \Gamma^k_{ij}\right)\vec{e_k}
\end{aligned} \tag{1.13.24}$$

$$\frac{\partial}{\partial r}\left(\tilde{\vec{e_r}}\right) = \Gamma^1_{rr}\vec{e_1} + \Gamma^2_{rr}\vec{e_2} = \Gamma^k_{rr}\vec{e_k} = 0 \tag{1.13.25}$$
$$\Gamma^k_{rr} = 0 \tag{1.13.26}$$
$$\Gamma^r_{rr} = 0 \tag{1.13.27}$$
$$\Gamma^\theta_{rr} = 0 \tag{1.13.28}$$
$$\frac{\partial}{\partial \theta}\left(\tilde{\vec{e_\theta}}\right) = \Gamma^r_{\theta\theta}\vec{e_r} + \Gamma^\theta_{\theta\theta}\vec{e_\theta} = \Gamma^k_{\theta\theta}\vec{e_k} = -r\tilde{\vec{e_r}} \tag{1.13.29}$$
$$\Gamma^r_{\theta\theta} = -r \tag{1.13.30}$$
$$\Gamma^\theta_{\theta\theta} = 0 \tag{1.13.31}$$

### 1.13.2 Extrinsic

**Definition 11.** Coveriant Derivative $\nabla_{\vec{w}}\vec{v}$
the rate of change of vector field $\vec{v}$ in a direction $\vec{w}$ with the normal component subtracted.

$$\boxed{\Gamma^k_{ij} = \left(\frac{\partial \vec{e_j}}{\partial u^i} \cdot \vec{e_l}\right)\mathfrak{g}^{lk}} \tag{1.13.32}$$

### 1.13.3 Intrinsic

Levi-Civita Connection:

$$\boxed{\Gamma^m_{jk} = \frac{1}{2}\mathfrak{g}^{im}\left(\frac{\partial g_{ij}}{\partial u^k} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^i}\right)} \tag{1.13.33}$$

Parallel Transport:

$$\nabla_{\partial_i} T = 0$$

### 1.13.4 Abstract

$$\nabla_{\vec{e_i}}\vec{e_j} = \Gamma^k_{ij}\vec{e_k} \tag{1.13.34}$$
$$\nabla_{\partial_i}(a) = \frac{\partial a}{\partial u^i} \tag{1.13.35}$$
$$\nabla_{\partial_i}(\vec{v}) = \left(\frac{\partial v^k}{\partial u^i} + v^j\Gamma^k_{ij}\right)\vec{e_k} \tag{1.13.36}$$
$$\nabla_{\partial_i}(a) = \left(\frac{\partial a_k}{\partial u^i} - a_j\Gamma^j_{ik}\right)\epsilon^k \tag{1.13.37}$$
$$\nabla_{\partial_i}(g) = \left(\frac{\partial g_{rs}}{\partial u^i} - g_{ks}\Gamma^k_{ir} - g_{rk}\Gamma^k_{is}\right)(\epsilon^r \otimes \epsilon^s) \tag{1.13.38}$$

$$\boxed{\nabla_{\vec{w}}(T \otimes S) = (\nabla_{\vec{w}}T) \otimes S + T \otimes (\nabla_{\vec{w}}S)} \tag{1.13.39}$$

# Chapter 2

# Spacetime

## 2.1 Spherical Coordinates

$$
\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \\
z &= r\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial r} &= \vec{e_r} \\
&= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} + \frac{\partial z}{\partial r}\frac{\partial}{\partial z} \\
&= \sin\theta\cos\phi\vec{e_x} + \sin\theta\sin\phi\vec{e_y} + \cos\theta\vec{e_z}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \theta} &= \vec{e_\theta} \\
&= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial}{\partial z} \\
&= r\cos\theta\cos\phi\vec{e_x} - r\cos\theta\sin\phi\vec{e_y} - r\sin\theta\vec{e_z}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial \phi} &= \vec{e_\phi} \\
&= \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \phi}\frac{\partial}{\partial z} \\
&= -r\sin\theta\sin\phi\vec{e_x} + r\sin\theta\cos\phi\vec{e_y}
\end{aligned}
$$

$$
\begin{aligned}
\vec{e_r}\cdot\vec{e_r} &= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta \\
&= \sin^2\left(\cos^2\phi + \sin^2\phi\right) + \cos^2\theta \\
&= \sin^2\phi + \cos^2\phi \\
&= 1
\end{aligned}
$$

$$\begin{aligned}\vec{e_\theta} \cdot \vec{e_\theta} &= r^2 \cos^2\theta \cos^2\phi + r^2 \cos^2\theta \sin^2\phi + r^2 \sin^2\theta \\ &= r^2 \cos^2 \left(\cos^2\phi + \sin^2\phi\right) + r^2 \sin^2\theta \\ &= r^2 \cos^2 + r^2 \sin^2\theta \\ &= r^2\end{aligned}$$

$$\begin{aligned}\vec{e_\phi} \cdot \vec{e_\phi} &= r^2 \sin^2\theta \sin^2\phi + r^2 \sin^2\theta \cos^2\phi \\ &= r^2 \sin\theta^2 \left(\cos^2\phi + \sin^2\phi\right) \\ &= r^2 \sin\theta^2\end{aligned}$$

Metric tensor for spherical coordinates$(r, \theta, \phi)$:

$$\left[g_{ij}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin\theta^2 \end{bmatrix}$$

$$\begin{aligned}\left\|\frac{d\vec{R}}{d\lambda}\right\|^2 = ds^2 &= \begin{bmatrix} \frac{du}{d\lambda} & \frac{dv}{d\lambda} \end{bmatrix} \begin{bmatrix} \vec{e_u} \cdot \vec{e_u} & \vec{e_u} \cdot \vec{e_v} \\ \vec{e_v} \cdot \vec{e_u} & \vec{e_v} \cdot \vec{e_v} \end{bmatrix} \begin{bmatrix} \frac{du}{d\lambda} \\ \frac{dv}{d\lambda} \end{bmatrix} \\ &= du^i g_{ij} du^j\end{aligned}$$

Line element for cartesian coordinates:

$$\left\|\frac{d\vec{R}}{d\lambda}\right\|^2 = ds^2 = dx^2 + dy^2 + dz^2$$

which $\left[g_{ij}\right]$ for cartesian coords is $\left[\delta^i_j\right]$.
Line element for sperical coordinates:

$$\left\|\frac{d\vec{R}}{d\lambda}\right\|^2 = ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin\theta^2 d\phi^2$$

## 2.2 Schwarzschild Geodesic

A static space-time is one for which,

1. All components of $g_{\mu\nu}$ are independent of $t$
2. The line element $ds^2$is invariant under the transformation $t \to -t$

A space-time satisfies the first but not the second is called a stationary space-time.

- Metric has the form,

$$\left[g_{\mu\nu}\right] = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix} = \left[g_{\nu\mu}\right]$$

- Line element or interval is $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$.

- Given the symmetry of the situation we choose the spherical polar coordinates,
$$x^\mu = \left(x^0, x^1, x^2, x^3\right) = (ct, r, \theta, \phi)$$

- Stationary of the metric requires,
$$\frac{g_{\mu\nu}}{\partial x^0} = \frac{g_{\mu\nu}}{\partial t} = 0 \tag{2.2.1}$$

- Static nature of this space-time requires invariance of the line element under a reverse of time, $t \to -t$.

- For $g_{00} \to dt^2 = (-dt)^2$,but for $g_{01} \to dtdr \neq -dtdr$,$g_{02} \to dtd\theta \neq -dtd\theta$,$g_{03} \to dtd\phi \neq -dtd\phi$,we get,
$$\left[g_{\mu\nu}\right] = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & g_{12} & g_{13} \\ 0 & g_{21} & g_{22} & g_{23} \\ 0 & g_{31} & g_{32} & g_{33} \end{bmatrix} \tag{2.2.2}$$

- For $g_{11} \to dr^2 = (-dr)^2$,but for $g_{12} \to drd\theta \neq -drd\theta$,$g_{13} \to drd\phi \neq -drd\phi$, we get,
$$\left[g_{\mu\nu}\right] = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & g_{23} \\ 0 & 0 & g_{32} & g_{33} \end{bmatrix} \tag{2.2.3}$$

- For $g_{22} \to d\theta^2 = (-d\theta)^2$,but for $g_{23} \to d\theta d\phi \neq -d\theta d\phi$, we get,
$$\left[g_{\mu\nu}\right] = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{bmatrix} \tag{2.2.4}$$

- So line element become,
$$ds^2 = g_{00}dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\phi^2 \tag{2.2.5}$$

$$\left[g_{\mu\nu}\right] = \begin{bmatrix} -\left(1 - \frac{2GM}{c^2 r}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2GM}{c^2 r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix}$$

$$ds^2 = c^2 d\tau^2 = c^2\left(1 - \frac{2MG}{c^2 r}\right)dt^2 - \left(1 - \frac{2MG}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin\theta^2 d\phi^2$$

- The tangent vector to a particle's world-line is its four velocity, $u^\mu$,
$$t^\mu = \frac{dx^\mu}{d\lambda} = u^\mu$$

- For parallel transport,

$$\frac{d\vec{t}}{d\lambda} = 0$$

- For particle with mass travel along a time-like world-line, square of the magnitude of its four velocity is given by,

$$g_{\mu\nu}t^\mu t^\nu = g_{\mu\nu}u^\mu u^\nu = u^\mu u_\mu = -c^2$$

- For massless particle travel along null geodesics,

$$u^\mu u_\mu = 0$$

- Derive,

$$\begin{aligned}
-c^2 &= g_{\mu\nu}u^\mu u^\nu = g_{00}u^0u^0 + g_{11}u^1u^1 + g_{22}u^2u^2 + g_{33}u^3u^3 \\
-c^2 &= g_{\mu\nu}u^\mu u^\nu = \\
&= -\left(1 - \frac{2GM}{c^2 r}\right)u^0u^0 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1}u^1u^1 + r^2u^2u^2 + r^2\sin\theta^2u^3u^3 \\
-c^2 &= \left(1 - \frac{2MG}{c^2 r}\right)c\frac{dt}{d\tau}c\frac{dt}{d\tau} - \left(1 - \frac{2MG}{c^2 r}\right)^{-1}\frac{dr}{d\tau}\frac{dr}{d\tau} \\
&\quad - r^2\frac{d\theta}{d\tau}\frac{d\theta}{d\tau} - r^2\sin\theta^2\frac{d\phi}{d\tau}\frac{d\phi}{d\tau}
\end{aligned}$$

- From killing vector we have,

$$\begin{aligned}
\left(1 - \frac{2GM}{c^2 r}\right)c^2\frac{dt}{d\tau} &= \frac{E}{m_0} \\
\frac{dt}{d\tau} &= \frac{E}{m_0c^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1} \\
\left(\frac{dt}{d\tau}\right)^2 &= \frac{E^2}{m_0^2c^4}\left(1 - \frac{2GM}{c^2 r}\right)^{-2} \\
c^2\left(\frac{dt}{d\tau}\right)^2 &= \frac{E^2}{m_0^2c^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-2} \\
r^2\sin^2\theta\frac{d\phi}{d\tau} &= \frac{L}{m_0} \\
\frac{d\phi}{d\tau} &= \frac{L}{m_0r^2\sin^2\theta} \\
\left(\frac{d\phi}{d\tau}\right)^2 &= \frac{L^2}{m_0^2r^4\sin^4\theta}
\end{aligned}$$

- Substitude,

$$
\begin{aligned}
-c^2 &= \left(1-\frac{2MG}{c^2r}\right)c\frac{dt}{d\tau}c\frac{dt}{d\tau}-\left(1-\frac{2MG}{c^2r}\right)^{-1}\frac{dr}{d\tau}\frac{dr}{d\tau} \\
&\quad -r^2\frac{d\theta}{d\tau}\frac{d\theta}{d\tau}-r^2\sin\theta^2\frac{d\phi}{d\tau}\frac{d\phi}{d\tau} \\
-c^2 &= \left(1-\frac{2MG}{c^2r}\right)\frac{E^2}{m_0^2c^2}\left(1-\frac{2GM}{c^2r}\right)^{-2} \\
&\quad -\left(1-\frac{2MG}{c^2r}\right)^{-1}\frac{dr}{d\tau}\frac{dr}{d\tau} \\
&\quad -r^2\frac{d\theta}{d\tau}\frac{d\theta}{d\tau}-r^2\sin\theta^2\frac{L^2}{m_0^2r^4\sin^4\theta} \\
-c^2 &= \frac{E^2}{m_0^2c^2}\left(1-\frac{2GM}{c^2r}\right)^{-1}-\left(1-\frac{2MG}{c^2r}\right)^{-1}\frac{dr}{d\tau}\frac{dr}{d\tau} \\
&\quad -r^2\frac{d\theta}{d\tau}\frac{d\theta}{d\tau}-\frac{L^2}{m_0^2r^2\sin^2\theta}
\end{aligned}
$$

- Setting $\theta=\frac{\pi}{2}$ and $d\theta=0$,

$$
\begin{aligned}
-c^2 &= \frac{E^2}{m_0^2c^2}\left(1-\frac{2GM}{c^2r}\right)^{-1}-\left(1-\frac{2MG}{c^2r}\right)^{-1}\frac{dr}{d\tau}\frac{dr}{d\tau} \\
&\quad -r^2\frac{d\theta}{d\tau}\frac{d\theta}{d\tau}-\frac{L^2}{m_0^2r^2\sin^2\theta} \\
-c^2 &= \frac{E^2}{m_0^2c^2}\left(1-\frac{2GM}{c^2r}\right)^{-1}-\left(1-\frac{2MG}{c^2r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2-\frac{L^2}{m_0^2r^2} \\
\left(\frac{dr}{d\tau}\right)^2 &= \frac{E^2}{m_0^2c^2}-\left(1-\frac{2MG}{c^2r}\right)\left(\frac{L^2}{m_0^2r^2}+c^2\right)
\end{aligned}
$$

$$
\boxed{\left(\frac{dr}{d\tau}\right)^2 = \frac{E^2}{m_0^2c^2}-\left(1-\frac{2MG}{c^2r}\right)\left(\frac{L^2}{m_0^2r^2}+c^2\right)}
$$

- For pure radial motion $\frac{d\phi}{d\tau}=0\Rightarrow L=0$, gives,

$$
\left(\frac{dr}{d\tau}\right)^2=\frac{E^2}{m_0^2c^2}-c^2+\frac{2MG}{r}
$$

- to get acceration $\frac{d^2r}{d\tau^2}$,

$$
\begin{aligned}
\left(\frac{dr}{d\tau}\right)^2 &= \frac{E^2}{m_0^2c^2} - c^2 + \frac{2MG}{r} \\
\frac{d}{d\tau}\left(\frac{dr}{d\tau}\right)^2 &= \frac{d}{d\tau}\left(\frac{E^2}{m_0^2c^2} - c^2 + \frac{2MG}{r}\right) \\
\frac{d^2r}{d\tau^2}\frac{dr}{d\tau} + \frac{dr}{d\tau}\frac{d^2r}{d\tau^2} &= \frac{d}{d\tau}\left(\frac{2MG}{r}\right) \\
2\frac{d^2r}{d\tau^2}\frac{dr}{d\tau} &= \frac{d}{d\tau}\left(\frac{2MG}{r}\right) \\
&= \frac{\frac{d(2MG)}{d\tau}\frac{1}{r} - 2MG\frac{dr}{d\tau}}{r^2} \\
2\frac{d^2r}{d\tau^2}\frac{dr}{d\tau} &= \frac{-2MG}{r^2}\frac{dr}{d\tau} \\
\frac{d^2r}{d\tau^2} &= \frac{-MG}{r^2}
\end{aligned}
$$

- Energy of particle per unit mass at the point of release from $\infty$ where $r = r_\infty$and $\frac{dr}{d\tau} = 0$,

$$
\begin{aligned}
\left(\frac{dr}{d\tau}\right)^2 &= \frac{E^2}{m_0^2c^2} - c^2 + \frac{2MG}{r_\infty} \\
0 &= \frac{E^2}{m_0^2c^2} - c^2 + \frac{2MG}{r_\infty} \\
\frac{E^2}{m_0^2c^2} &= c^2 - \frac{2MG}{r_\infty} \\
\left(\frac{E}{m_0}\right)^2 &= c^2\left(c^2 - \frac{2MG}{r_\infty}\right)
\end{aligned}
$$

- plugging in $\frac{E}{m_0}$,

$$
\begin{aligned}
\left(\frac{dr}{d\tau}\right)^2 &= \frac{E^2}{m_0^2c^2} - c^2 + \frac{2MG}{r} \\
\left(\frac{dr}{d\tau}\right)^2 &= \left(c^2 - \frac{2MG}{r_\infty}\right) - c^2 + \frac{2MG}{r} \\
\left(\frac{dr}{d\tau}\right)^2 &= -\frac{2MG}{r_\infty} + \frac{2MG}{r} \\
\left(\frac{dr}{d\tau}\right)^2 &= 2MG\left(\frac{1}{r} - \frac{1}{r_\infty}\right)
\end{aligned}
$$

- From killing vector we have $\frac{E}{m_0} = \left(1 - \frac{2GM}{c^2r}\right)c^2\frac{dt}{d\tau}$

$$
\left(1 - \frac{2GM}{c^2r}\right)^{-1}\frac{E}{m_0c} = c\frac{dt}{d\tau}
$$

- For four velocity of radial motion partical,

$$u^{\mu} = \frac{dx^{\mu}}{d\tau} = \left(c\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, 0\right)$$
$$= \left(\left(1 - \frac{2GM}{c^2 r}\right)^{-1} \frac{E}{m_0 c}, -\sqrt{\frac{E^2}{m_0^2 c^2} - c^2 + \frac{2MG}{r}}, 0, 0\right)$$

## 2.3 Circular orbit for massive particle

- Time-like four velocity

$$g_{\mu\nu} u^{\mu} u^{\nu} = u^{\mu} u_{\mu} = -c^2$$

$$-c^2 = \left(1 - \frac{2MG}{c^2 r}\right) c\frac{dt}{d\tau} c\frac{dt}{d\tau} - \left(1 - \frac{2MG}{c^2 r}\right)^{-1} \frac{dr}{d\tau}\frac{dr}{d\tau}$$
$$- r^2 \frac{d\theta}{d\tau}\frac{d\theta}{d\tau} - r^2 \sin^2\theta \frac{d\phi}{d\tau}\frac{d\phi}{d\tau}$$

- For circular motion $\theta = \frac{\pi}{2}$ and $d\theta = 0$,

$$-c^2 = -\left(1 - \frac{2MG}{c^2 r}\right) c\frac{dt}{d\tau} c\frac{dt}{d\tau} + \left(1 - \frac{2MG}{c^2 r}\right)^{-1} \frac{dr}{d\tau}\frac{dr}{d\tau} + r^2 \frac{d\phi}{d\tau}\frac{d\phi}{d\tau}$$

- From killing vector we have,

$$\frac{dt}{d\tau} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \frac{E}{m_0 c^2} \tag{2.3.1}$$
$$\frac{d\phi}{d\tau} = \frac{L}{m_0 r^2 \sin^2\theta} = \frac{L}{m_0 r^2} \tag{2.3.2}$$

- Plug in,

$$-c^2 = \frac{E^2}{m_0^2 c^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1} - \left(1 - \frac{2MG}{c^2 r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 - \frac{L^2}{m_0^2 r^2}$$

- Subtitude

$$\frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \frac{dr}{d\phi}\frac{L}{m_0 r^2}$$

- Plug in,

$$
\begin{aligned}
-c^2 &= -\left(1-\frac{2GM}{c^2r}\right)^{-1}\frac{E^2}{m_0^2c^2}+\left(1-\frac{2MG}{c^2r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2+\frac{L^2}{m_0^2r^2}\\
-c^2 &= \left(1-\frac{2GM}{c^2r}\right)^{-1}\left(-\frac{E^2}{m_0^2c^2}-\left(\frac{dr}{d\phi}\right)^2\frac{L^2}{m_0^2r^4}\right)-\frac{L^2}{m_0^2r^2}\\
-c^2\left(1-\frac{2GM}{c^2r}\right) &= -\frac{E^2}{m_0^2c^2}+\left(\frac{dr}{d\phi}\right)^2\frac{L^2}{m_0^2r^4}+\frac{L^2}{m_0^2r^2}\left(1-\frac{2GM}{c^2r}\right)\\
\frac{E^2}{m_0^2c^2} &= c^2\left(1-\frac{2GM}{c^2r}\right)+\left(\frac{dr}{d\phi}\right)^2\frac{L^2}{m_0^2r^4}+\frac{L^2}{m_0^2r^2}\left(1-\frac{2GM}{c^2r}\right)\\
\frac{E^2}{m_0^2c^2}\frac{m_0^2r^4}{L^2} &= \left(\frac{dr}{d\phi}\right)^2+\left(1-\frac{2GM}{c^2r}\right)\frac{m_0^2r^4c^2}{L^2}+r^2\left(1-\frac{2GM}{c^2r}\right)\\
\frac{E^2}{m_0^2c^2}\frac{m_0^2r^4}{L^2} &= \left(\frac{dr}{d\phi}\right)^2+\frac{m_0^2r^4c^2}{L^2}-\frac{2GMm_0^2r^3}{L^2}+r^2-\frac{2GMr}{c^2}\\
\frac{E^2}{m_0^2c^2}\frac{m_0^2r^4}{L^2}-\frac{m_0^2r^4c^2}{L^2} &= \left(\frac{dr}{d\phi}\right)^2-\frac{2GMm_0^2r^3}{L^2}+r^2-\frac{2GMr}{c^2}\\
\frac{m_0^2r^4}{L^2}\left(\frac{E^2}{m_0^2c^2}-c^2\right) &= \left(\frac{dr}{d\phi}\right)^2-\frac{2GMm_0^2r^3}{L^2}+r^2-\frac{2GMr}{c^2}
\end{aligned}
$$

- Substitude $r=\frac{1}{u}$, we get $\frac{dr}{d\phi}=\frac{dr}{du}\frac{du}{d\phi}=-\frac{1}{u^2}\frac{du}{d\phi}$,
- Plug in,

$$
\begin{aligned}
\frac{m_0^2r^4}{L^2}\left(\frac{E^2}{m_0^2c^2}-c^2\right) &= \left(\frac{dr}{d\phi}\right)^2-\frac{2GMm_0^2r^3}{L^2}+r^2-\frac{2GMr}{c^2}\\
\frac{m_0^2}{L^2u^4}\left(\frac{E^2}{m_0^2c^2}-c^2\right) &= \left(-\frac{1}{u^2}\frac{du}{d\phi}\right)^2-\frac{2GMm_0^2}{L^2u^3}+\frac{1}{u^2}-\frac{2GM}{c^2u}\\
\frac{m_0^2}{L^2u^4}\left(\frac{E^2}{m_0^2c^2}-c^2\right) &= \frac{1}{u^4}\left(\frac{du}{d\phi}\right)^2+\frac{1}{u^2}-\frac{2GMm_0^2}{L^2u^3}-\frac{2GM}{c^2u}\\
\frac{m_0^2}{L^2}\left(\frac{E^2}{m_0^2c^2}-c^2\right) &= \left(\frac{du}{d\phi}\right)^2+u^2-\frac{2GMm_0^2}{L^2}u-\frac{2GM}{c^2}u^3
\end{aligned}
$$

- Differentiate both sides with respect to $\phi$, we get accerleration in $\phi$,

$$
\begin{aligned}
\frac{d}{d\phi}\left[\frac{m_0^2}{L^2}\left(\frac{E^2}{m_0^2c^2}-c^2\right)\right] &= \frac{d}{d\phi}\left[\left(\frac{du}{d\phi}\right)^2+u^2-\frac{2GMm_0^2}{L^2}u-\frac{2GM}{c^2}u^3\right]\\
0 &= 2\frac{du}{d\phi}\frac{d^2u}{d\phi^2}+2u\frac{du}{d\phi}-\frac{2GM}{c^2}3u^2\frac{du}{d\phi}-\frac{2GMm_0^2}{L^2}\frac{du}{d\phi}\\
0 &= \frac{du}{d\phi}\frac{d^2u}{d\phi^2}+u\frac{du}{d\phi}-\frac{GM}{c^2}3u^2\frac{du}{d\phi}-\frac{GMm_0^2}{L^2}\frac{du}{d\phi}\\
0 &= \frac{d^2u}{d\phi^2}+u-\frac{GM}{c^2}3u^2-\frac{GMm_0^2}{L^2}
\end{aligned}
$$

- Orbital motion equation,

$$\boxed{\frac{d^2u}{d\phi^2} + u = \frac{GMm_0^2}{L^2} + \frac{3GM}{c^2}u^2}$$

- For circular motion $u$ is constant and $\frac{dr}{d\tau} = 0$,

$$\begin{aligned}
u &= \frac{GMm_0^2}{L^2} + \frac{3GM}{c^2}u^2 \\
u - \frac{3GM}{c^2}u^2 &= \frac{GMm_0^2}{L^2} \\
L^2 &= \frac{GMm_0^2}{u - \frac{3GMu^2}{c^2}} \\
L^2 &= \frac{GMm_0^2c^2}{uc^2 - 3GMu^2}
\end{aligned}$$

- Substitude back $r$ for $u$,

$$L^2 = \frac{GMm_0^2c^2r^2}{rc^2 - 3GM}$$

- For $\frac{dr}{d\tau}$, we have,

$$\begin{aligned}
\left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{2MG}{c^2r}\right)\left(\frac{L^2}{m_0^2r^2} + c^2\right) &= \frac{E^2}{m_0^2c^2} \\
\left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{m_0^2r^2} + c^2 - \frac{2MG}{c^2r}\frac{L^2}{m_0^2r^2} - \frac{2MG}{c^2r}c^2 &= \frac{E^2}{m_0^2c^2} \\
\left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{m_0^2r^2}\left(1 - \frac{2MG}{c^2r}\right) - \frac{2MG}{r} &= \frac{E^2}{m_0^2c^2} - c^2 \\
\left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{m_0^2r^2}\left(1 - \frac{2MG}{c^2r}\right) - \frac{2MG}{r} &= c^2\left(\frac{E^2}{m_0^2c^4} - 1\right)
\end{aligned}$$

- Setting $\frac{dr}{d\tau} = 0$,

$$\frac{L^2}{m_0^2r^2}\left(1 - \frac{2MG}{c^2r}\right) - \frac{2MG}{r} = c^2\left(\frac{E^2}{m_0^2c^4} - 1\right)$$

- Plug in $L^2$,

$$\begin{aligned}
\frac{1}{m_0^2r^2}\frac{GMm_0^2c^2r^2}{rc^2 - 3GM}\left(1 - \frac{2MG}{c^2r}\right) - \frac{2MG}{r} &= c^2\left(\frac{E^2}{m_0^2c^4} - 1\right) \\
\frac{E^2}{m_0^2c^4} &= \frac{\left(c^2r - 2GM\right)^2}{c^2r\left(c^2r - 3GM\right)}
\end{aligned}$$

- The energy of the particle sets a limit for bound orbits and so must satisfy,

$$E = m_0c^2$$

- So,

$$\begin{aligned}
\frac{E^2}{m_0^2 c^4} &= \frac{\left(c^2 r - 2GM\right)^2}{c^2 r\left(c^2 r - 3GM\right)} \\
1 &= \frac{\left(c^2 r - 2GM\right)^2}{c^2 r\left(c^2 r - 3GM\right)} \\
\left(c^2 r - 2GM\right)^2 &= c^2 r\left(c^2 r - 3GM\right) \\
c^4 r^2\left(1 - \frac{2GM}{c^2 r}\right)^2 &= c^4 r^2\left(1 - \frac{3GM}{c^2 r}\right) \\
\left(1 - \frac{2GM}{c^2 r}\right)^2 &= \left(1 - \frac{3GM}{c^2 r}\right)
\end{aligned}$$

## 2.4 Circular orbit for massless particle

- null geodesic

$$g_{\mu\nu} u^\mu u^\nu = u^\mu u_\mu = 0$$

- In the schwarzschild geometry we have,

$$\begin{aligned}
0 = &-\left(1 - \frac{2MG}{c^2 r}\right)\left(c\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2MG}{c^2 r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 \\
&+ r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2 \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2
\end{aligned}$$

- For circular motion in equatorial plane, $\theta = \frac{\pi}{2}$ and $d\theta = 0$,

$$0 = -\left(1 - \frac{2MG}{c^2 r}\right)\left(c\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2MG}{c^2 r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2$$

- From killing vector we have,

$$\frac{dt}{d\lambda} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\frac{E}{m_0 c^2} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\frac{E}{c^2} \tag{2.4.1}$$

$$\frac{d\phi}{d\lambda} = \frac{L}{m_0 r^2 \sin^2\theta} = \frac{L}{m_0 r^2} = \frac{L}{r^2} \tag{2.4.2}$$

- Plug in,

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) = \frac{E^2}{c^2}$$

- Subtitude

$$\frac{dr}{d\lambda} = \frac{dr}{d\phi}\frac{d\phi}{d\lambda} = \frac{dr}{d\phi}\frac{L}{r^2}$$

- we get,

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) = \frac{E^2}{c^2}$$
$$\left(\frac{dr}{d\phi}\frac{L}{r^2}\right)^2 + \frac{L^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) = \frac{E^2}{c^2}$$
$$\left(\frac{dr}{d\phi}\right)^2 + r^2 - \frac{2GM}{c^2} r = \frac{r^4}{L^2}\frac{E^2}{c^2}$$

- Substitude $r = \frac{1}{u}$, we get $\frac{dr}{d\phi} = \frac{dr}{du}\frac{du}{d\phi} = -\frac{1}{u^2}\frac{du}{d\phi}$,
- Plug in,

$$\left(-\frac{1}{u^2}\frac{du}{d\phi}\right)^2 + \frac{1}{u^2} - \frac{2GM}{c^2 u} = \frac{E^2}{L^2 c^2 u^4}$$
$$\left(\frac{du}{d\phi}\right)^2 + u^2 - \frac{2GM}{c^2} u^3 = \frac{E^2}{L^2 c^2}$$

- Differentiate both sides with respect to $\phi$, we get accerleration in $\phi$,

$$\frac{d}{d\phi}\left[\left(\frac{du}{d\phi}\right)^2 + u^2 - \frac{2GM}{c^2} u^3\right] = \frac{d}{d\phi}\left(\frac{E^2}{L^2 c^2}\right)$$

- Orbital motion equation,

$$\frac{d^2 u}{d\phi^2} + u = \frac{3GM}{c^2} u^2$$

- For circular motion $u$ is constant and $\frac{dr}{d\lambda} = 0$,

$$u = \frac{3GM}{c^2} u^2$$
$$1 = \frac{3GM}{c^2} u$$
$$1 = \frac{3GM}{c^2}\frac{1}{r}$$
$$r = \frac{3GM}{c^2}$$

- For circular motion $r$ is constant, get infomation about angular momentum $L$,

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) = \frac{E^2}{c^2}$$
$$\frac{L^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) = \frac{E^2}{c^2}$$

- Plug in $r$ above,

$$\frac{L^2}{r^2}\left(1-\frac{2GM}{c^2 r}\right)=\frac{E^2}{c^2}$$
$$L=\frac{\sqrt{27}EGM}{c^3}$$

- Plug in $L$,

$$\left(\frac{dr}{d\lambda}\right)^2+\frac{L^2}{r^2}\left(1-\frac{2GM}{c^2 r}\right)=\frac{E^2}{c^2}$$
$$\left(\frac{dr}{d\lambda}\right)^2+\frac{27E^2G^2M^2}{c^6r^2}\left(1-\frac{2GM}{c^2 r}\right)=\frac{E^2}{c^2}$$
$$\left(\frac{dr}{d\lambda}\right)^2+\frac{27E^2G^2M^2}{c^6}\left(\frac{1}{r^2}-\frac{2GM}{c^2 r^3}\right)=\frac{E^2}{c^2}$$
$$\underbrace{\left(\frac{dr}{d\lambda}\right)^2}_{\text{kinetic}}+\underbrace{V_{\text{eff}}}_{\text{potential}}=\underbrace{\frac{E^2}{c^2}}_{\text{total energy}}$$

- We have effective potential

$$V_{\text{eff}}=\frac{27E^2G^2M^2}{c^6}\left(\frac{1}{r^2}-\frac{2GM}{c^2 r^3}\right)$$

- A minimum occurs at,

$$\frac{V_{\text{eff}}}{dr}=0$$
$$\frac{27E^2G^2M^2}{c^6}\left(\frac{-2}{r^3}-\frac{6GM}{c^2 r^4}\right)=0$$
$$r=\frac{3GM}{c^2}$$

## 2.5 Light deflection

- Motion equation in equatorial plane,

$$\left(\frac{dr}{d\lambda}\right)^2+\frac{L^2}{r^2}\left(1-\frac{2GM}{c^2 r}\right)=\frac{E^2}{c^2}$$
$$\frac{1}{L^2}\left(\frac{dr}{d\lambda}\right)^2+\frac{1}{r^2}\left(1-\frac{2GM}{c^2 r}\right)=\frac{1}{L^2}\frac{E^2}{c^2}$$
$$\frac{1}{L^2}\left(\frac{dr}{d\lambda}\right)^2+V_{\text{eff}}(r)=\frac{1}{L^2}\frac{E^2}{c^2}$$
$$\frac{1}{L}\left(\frac{dr}{d\lambda}\right)=\pm\sqrt{\left(\frac{E}{Lc}\right)^2-V_{\text{eff}}(r)}$$

- From killing vector we have,

$$\frac{dt}{d\lambda} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \frac{E}{m_0 c^2} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \frac{E}{c^2} \tag{2.5.1}$$

$$\frac{d\phi}{d\lambda} = \frac{L}{m_0 r^2 \sin^2\theta} = \frac{L}{m_0 r^2} = \frac{L}{r^2} \tag{2.5.2}$$

$$L = r^2 \frac{d\phi}{d\lambda} \tag{2.5.3}$$

- Plug in,

$$\frac{1}{L}\left(\frac{dr}{d\lambda}\right) = \sqrt{\left(\frac{E}{Lc}\right)^2 - V_{\text{eff}}(r)}$$

$$\frac{1}{r^2}\frac{d\lambda}{d\phi}\frac{dr}{d\lambda} = \sqrt{\left(\frac{E}{Lc}\right)^2 - V_{\text{eff}}(r)}$$

$$\frac{d\phi}{dr} = \frac{1}{r^2\sqrt{\left(\frac{E}{Lc}\right)^2 - V_{\text{eff}}(r)}}$$

- Substitude $r = \frac{1}{u}$, we get $\frac{dr}{d\phi} = \frac{dr}{du}\frac{du}{d\phi} = -\frac{1}{u^2}\frac{du}{d\phi}$,

$$\frac{d\phi}{dr} = \frac{1}{r^2\sqrt{\left(\frac{E}{Lc}\right)^2 - V_{\text{eff}}(r)}}$$

$$-u^2\frac{d\phi}{du} = \frac{u^2}{\sqrt{\left(\frac{E}{Lc}\right)^2 - V_{\text{eff}}}}$$

$$\frac{d\phi}{du} = \frac{-1}{\sqrt{\left(\frac{E}{Lc}\right)^2 - V_{\text{eff}}}}$$

$$-u^2\frac{d\phi}{du} = \frac{u^2}{\sqrt{\left(\frac{E}{Lc}\right)^2 - u^2\left(1 - \frac{2GMu}{c^2}\right)}}$$

$$\frac{d\phi}{du} = \frac{-1}{\sqrt{\left(\frac{E}{Lc}\right)^2 - u^2\left(1 - \frac{2GMu}{c^2}\right)}}$$

$$\frac{d\phi}{du} = \frac{-1}{\sqrt{\left(\frac{E}{Lc}\right)^2 - u^2 f(u)}}$$

- For large $r$, $f(u) = f\left(\frac{1}{r}\right) \approx 1$,

$$\frac{d\phi}{du} = \frac{1}{\sqrt{\left(\frac{E}{Lc}\right)^2 - u^2}}$$

- Substitude $b = \frac{Lc}{E}$,

$$\frac{d\phi}{du} = \frac{1}{\sqrt{\left(\frac{1}{b}\right)^2 - u^2}}$$

- Integrate,

$$
\begin{aligned}
\frac{d\phi}{du} &= \frac{1}{\sqrt{\left(\frac{1}{b}\right)^2 - u^2}} \\
\int d\phi &= \int \frac{du}{\sqrt{\left(\frac{1}{b}\right)^2 - u^2}} \\
\phi - \phi_0 &= \sin^{-1} \frac{u}{1/b} \\
\sin(\phi - \phi_0) &= bu = \frac{b}{r} \\
b &= r \sin(\phi - \phi_0) \\
r &= \pm \frac{b}{\sin(\phi - \phi_0)}
\end{aligned}
$$

- The impact parameter and effective potential determine the path of photon,

$$
\frac{d\phi}{du} = \frac{1}{\sqrt{\left(\frac{E}{Lc}\right)^2 - u^2 f(u)}} = \frac{1}{\sqrt{\left(\frac{1}{b}\right)^2 - V_{\text{eff}}(r)}}
$$

- When $r$ is $\frac{3GM}{c^2}$ the only circular orbit,

$$
V_{\text{eff}}(r) = V_{\text{eff}}\left(\frac{3GM}{c^2}\right) = \frac{c^4}{27G^2M^2}
$$

- Motion equation govens the shape of the orbit in general is,

$$
\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2}u^2
$$

- Relationship between $u, b, \phi$ is,

$$
u = \frac{\sin\phi}{b}
$$

- Add a small perturbation to this form,

$$
u = \frac{\sin\phi}{b} + \Delta u
$$

- Substitude in,

$$
\begin{aligned}
\frac{d^2u}{d\phi^2} + u &= \frac{3GM}{c^2}u^2 \\
\frac{d^2}{d\phi^2}\left(\frac{\sin\phi}{b} + \Delta u\right) + \frac{\sin\phi}{b} + \Delta u &= \frac{3GM}{c^2}\left(\frac{\sin\phi}{b} + \Delta u\right)^2 \\
\frac{d^2}{d\phi^2}\left(\frac{\sin\phi}{b} + \Delta u\right) + \frac{\sin\phi}{b} + \Delta u &= \frac{3GM}{c^2}\left(\frac{\sin\phi}{b} + \Delta u\right)^2 \\
-\frac{\sin\phi}{b} + \frac{d^2\Delta u}{d\phi^2} + \frac{\sin\phi}{b} + \Delta u &= \frac{3GM}{c^2}\left(\frac{\sin^2\phi}{b^2} + 2\frac{\sin\phi}{b}\Delta u + \Delta u^2\right) \\
\frac{d^2\Delta u}{d\phi^2} + \Delta u &= \frac{3GM}{c^2}\left(\frac{\sin^2\phi}{b^2} + \underbrace{2\frac{\sin\phi}{b}\Delta u + \Delta u^2}_{\text{leave out these small terms}}\right) \\
\frac{d^2\Delta u}{d\phi^2} + \Delta u &= \frac{3GM}{c^2b^2}\sin^2\phi
\end{aligned}
$$

## 2.6 Do it again

- Equation of motion,

$$
\begin{aligned}
\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}\left(1 - \frac{2GM}{c^2r}\right) &= \frac{E^2}{c^2} \\
\frac{dr}{d\lambda} &= \sqrt{\frac{E^2}{c^2} - \frac{L^2}{r^2}\left(1 - \frac{2GM}{c^2r}\right)} \\
L = r^2\frac{d\phi}{d\lambda} \Rightarrow \frac{1}{L} &= \frac{1}{r^2}\frac{d\lambda}{d\phi} \\
\frac{d\lambda}{d\phi} &= \frac{r^2}{L}
\end{aligned}
$$

- $\frac{dr}{d\phi} = \frac{dr}{d\lambda}\frac{d\lambda}{d\phi}$

$$
\begin{aligned}
\frac{dr}{d\lambda}\frac{d\lambda}{d\phi} &= \frac{r^2}{L}\sqrt{\frac{E^2}{c^2} - \frac{L^2}{r^2}\left(1 - \frac{2GM}{c^2r}\right)} \\
\frac{dr}{d\phi} &= \frac{r^2}{L}\sqrt{\frac{E^2}{c^2} - \frac{L^2}{r^2}\left(1 - \frac{2GM}{c^2r}\right)} \\
\frac{d\phi}{dr} &= \frac{L}{r^2\sqrt{\frac{E^2}{c^2} - \frac{L^2}{r^2}\left(1 - \frac{2GM}{c^2r}\right)}} \\
d\phi &= \frac{Ldr}{r^2\sqrt{\frac{E^2}{c^2} - \frac{L^2}{r^2}\left(1 - \frac{2GM}{c^2r}\right)}}
\end{aligned}
$$

- Substitude $r = \frac{1}{u}$,

$$d\phi = \frac{-Ldu}{\sqrt{\frac{E^2}{c^2} - L^2u^2\left(1 - \frac{2GMu}{c^2}\right)}}$$
$$d\phi = \frac{-Ldu}{\sqrt{\frac{E^2}{c^2} - L^2u^2\left(1 - \frac{2GMu}{c^2}\right)}}$$

## 2.7 Ray tracing Schwarzschild

- null geodesic

$$g_{\mu\nu}u^\mu u^\nu = u^\mu u_\mu = 0$$

- In the schwarzschild geometry we have,

$$\begin{aligned}0 = &-\left(1 - \frac{2MG}{c^2r}\right)\left(c\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2MG}{c^2r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 \\ &+ r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\end{aligned}$$

- For circular motion in equatorial plane, $\theta = \frac{\pi}{2}$ and $d\theta = 0$,

$$0 = -\left(1 - \frac{2MG}{c^2r}\right)\left(c\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2MG}{c^2r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2$$
$$0 = -\left(1 - \frac{r_s}{r}\right)\left(c\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{r_s}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2$$

- From killing vector we have,

$$\frac{dt}{d\lambda} = \left(1 - \frac{2GM}{c^2r}\right)^{-1}\frac{E}{m_0c^2} = \left(1 - \frac{2GM}{c^2r}\right)^{-1}\frac{E}{c^2} \tag{2.7.1}$$
$$\frac{dt}{d\lambda} = \left(1 - \frac{2GM}{c^2r}\right)^{-1}\frac{E}{m_0c^2} = \left(1 - \frac{r_s}{r}\right)^{-1}\frac{E}{c^2} \tag{2.7.2}$$
$$\frac{d\phi}{d\lambda} = \frac{L}{m_0r^2\sin^2\theta} = \frac{L}{m_0r^2} = \frac{L}{r^2} \tag{2.7.3}$$

- Plug in,

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{E^2}{c^2} - \frac{L^2}{r^2}\left(1 - \frac{r_s}{r}\right)$$

- Subtitude

$$\frac{dr}{d\lambda} = \frac{dr}{dt}\frac{dt}{d\lambda} = \frac{dr}{dt}\frac{E}{1 - \frac{r_s}{r}c^2}$$

- Set,

$$\zeta = \frac{r_s}{r}$$
$$I = \frac{L}{Er_s}$$

## 2.8 Killing vector

$$E = mc^2$$
$$p = mv$$

in time direction

$$p = mc$$
$$p_0 = \frac{E}{c}$$
$$p_0 = \frac{mc^2}{c}$$
$$p_0 = mc$$

- Killing vectors obey the condition,

$$\frac{d}{d\tau}\left(\vec{K}^t \cdot \vec{u}\right) = 0$$

- Momentum per unit of mass for particle is $\vec{p} = m_0\vec{u} = \vec{u}$ where $m_0 = 1$, four-velocity,

$$\frac{\vec{p}}{m_0} = \frac{m_0\vec{u}}{m_0} = \vec{u}$$

- So,

$$\frac{d}{d\tau}\left(\vec{K}^t \cdot \vec{u}\right) = 0$$
$$\frac{d}{d\tau}\left(\vec{K}^t \cdot \frac{\vec{p}}{m_0}\right) = 0$$

- For time direction,

$$
\begin{aligned}
\frac{d}{d\tau}\left(\vec{K}^t \cdot \vec{u}\right) &= 0 \\
\frac{d}{d\tau}\left(\vec{K}^t \cdot \frac{\vec{p}}{m_0}\right) &= 0 \\
\frac{d}{d\tau}\left(g_{00}K^0u^0\right) &= 0 \\
\frac{d}{d\tau}\left(g_{00}u^0\right) &= 0 \\
\frac{d}{d\tau}\left(g_{00}\frac{p^0}{m_0}\right) &= 0 \\
\frac{d}{d\tau}\left(\frac{p_0}{m_0}\right) &= 0 \\
\frac{d}{d\tau}\left(g_{00}u^0\right) &= 0 \\
\frac{d}{d\tau}\left(\left(1-\frac{2GM}{c^2r}\right)c\frac{dt}{d\tau}\right) &= 0 = \frac{d}{d\tau}\left(\frac{p_0}{m_0}\right) = \frac{d}{d\tau}\left(\frac{E}{m_0c}\right) \\
\frac{E}{m_0c} &= \left(1-\frac{2GM}{c^2r}\right)c\frac{dt}{d\tau} \\
\frac{E}{m_0} &= \left(1-\frac{2GM}{c^2r}\right)c^2\frac{dt}{d\tau} \\
\frac{dt}{d\tau} &= \left(1-\frac{2GM}{c^2r}\right)^{-1}\frac{E}{m_0c^2}
\end{aligned}
$$

- For $\phi$ direction,

$$
\begin{aligned}
\frac{d}{d\tau}\left(\vec{K}^t \cdot \vec{u}\right) &= 0 \\
\frac{d}{d\tau}\left(g_{\phi\phi}K^\phi u^\phi\right) &= 0 \\
\frac{d}{d\tau}\left(r^2\sin^2\theta m_0u^3\right) &= 0 \\
\frac{d}{d\tau}\left(r^2\sin^2\theta m_0\frac{d\phi}{d\tau}\right) &= 0
\end{aligned}
$$

- Which $r^2\sin^2\theta\frac{d\phi}{d\tau} = \frac{L}{m_0}$ angular momentum is constant.

$$
\begin{aligned}
r^2\sin^2\theta\frac{d\phi}{d\tau} &= \frac{L}{m_0} \\
\frac{d\phi}{d\tau} &= \frac{L}{m_0r^2\sin^2\theta}
\end{aligned}
$$