Design of Self-supporting Surfaces

Abstract

Self-supporting masonry is one of the most ancient and at the same time most elegant ways of building curved shapes. Their analysis and modeling is a topic of geometry processing rather than classical continuum mechanics, because of the very geometric nature of failure of such structures. In this paper we use the thrust network method of analysis and present an iterative nonlinear optimization algorithm for efficiently approximating freeform shapes by self-supporting ones. This provides an interactive modeling tool for such shapes. The rich geometry of thrust networks which was first studied by Maxwell in the 1860s leads us to new viewpoints of discrete differential geometry: We find close connections between different objects such as a finite-element discretization of the Airy stress potential, perfect graph Laplacians, and computing admissible loads via curvatures of polyhedral surfaces.

CR Categories: I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling—Curve, surface, solid, and object representations;

Keywords: Self-supporting masonry structures, thrust networks, reciprocal force diagrams, discrete Laplace operators, isotropic geometry, mean curvature

Links: DL PDF

Figure 1: *A surface with many, irregularly placed holes almost never stands by itself; those that do are surprising and their stability is not obvious by inspection. The surface shown is produced by our algorithm which finds, for a given freeform shape, the nearest self-supporting surface.*

1 Introduction

Vaulted masonry structures are among the simplest and at the same time most elegant solutions for creating curved shapes in building construction. This is the reason why they have been an object of interest since antiquity, large non-convex examples being provided by gothic cathedrals. They continue to be an active topic of research in today's engineering community.

Our paper is concerned with a combined geometry+statics analysis of *self-supporting* masonry and with tools for the interactive modeling of freeform self-supporting structures. Here "self-supporting" means that the structure, considered as an arrangement of blocks (bricks, stones), holds together by itself, and additional support, additional chains and similar are present only during construction. Our analysis is based on the following assumptions, which follow the classic [Heyman 1966]:

Assumption 1: Masonry has no tensile strength, but the individual building blocks do not slip against each other (because of friction or mortar). On the other hand, their compressive strength is sufficiently high so that failure of the structure is by a sudden change in geometry, such as shown by Figure 2, and not by material failure.

Assumption 2 (The Safe Theorem): If a system of forces can be found which is in equilibrium with the load on the structure and which is contained within the masonry envelope then the structure will carry the loads, although the actually occurring forces may not be those postulated.

Our approach is twofold: We first give an overview of the continuous case of a smooth surface under stress which turns out to be governed by the so-called Airy stress function, at least locally. This mathematical model is called a membrane in the engineering literature and has been applied to the analysis of masonry before. The surface is self-supporting if and only if stresses are entirely compressive (i.e., the Airy function is convex). For computational purposes, stresses are discretized as a fictitious *thrust network* [Block and Ochsendorf 2007] contained in the masonry structure. This is a system of forces which together with the structure's deadload is in equilibrium. It can be interpreted as a finite element discretization of the continuous case, and it turns out to have very interesting geometry dating back to the work of J. C. Maxwell [1864], with the Airy stress function becoming a polyhdral surface directly related to a reciprocal force diagram. Our own contributions are the following:

Contributions.

- We connect the physics of self-supporting surfaces with vertical loads to the geometry of isotropic 3-space, with the direction of gravity as the distinguished direction (§2.3). Taking the convex Airy potential as unit sphere, one can express the equations governing self-supporting surfaces in terms of curvatures.
- We employ Maxwell's construction of polyhedral thrust networks and their reciprocal diagrams (§2.4), and give an interpretation of the equilibrium conditions in terms of discrete curvatures
- The graph Laplacian derived from a thrust network with compressive forces is a "perfect" one. We show how it appears in the analysis and establish a connection with mean curvatures which are otherwise defined for polyhedral surfaces.
- We present an optimization algorithm for efficiently finding a

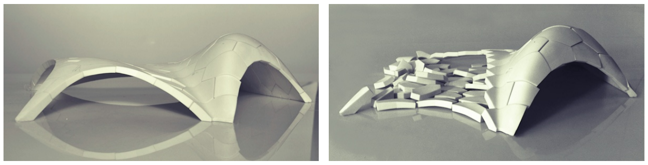

Figure 2: *Masonry fails via geometric catastrophe rather than material failure (models by Block Research Group, ETH Zürich).*

thrust network near a given arbitrary reference surface (§3), and build a tool for interactive design of self-supporting surfaces based on this algorithm (§4).

- We exploit the geometric relationships between a self-supporting surface and the stress potential in order to find particularly nice families of self-supporting surfaces, especially planar quadrilateral representations of thrust networks (§5).

- We demonstrate the versatility and applicability of our approach to the design and analysis of large-scale masonry and steel-glass structures.

Related Work. Unsupported masonry has been an active topic of research in the engineering community. The foundations for the modern approach were laid by Jacques Heyman [1966] and are available as the textbook [Heyman 1995]. A unifying view on polyhedral surfaces, compressive forces and corresponding "convex" force diagrams is presented by [Ash et al. 1988]. F. Fraternali [2002], [2010] established a connection between the continuous theory of stresses in membranes and the discrete theory of forces in thrust networks, by interpreting the latter as a certain nonconforming finite element discretization of the former.

Several authors have studied the problem of finding discrete compressive force networks contained within the boundary of masonry structures; early work in this area includes [Schek 1974], [Livesley 1992], and [O'Dwyer 1998]. Fraternali [2010] proposed solving for the structure's discrete stress surface, and examining its convex hull to study the structure's stability and susceptibility to cracking. Philippe Block's seminal thesis introduced the method of *Thrust Network Analysis*, which linearizes the form-finding problem by first seeking a reciprocal diagram of the top view, which guarantees equilibrium of horizontal forces, then solving for the heights that balance the vertical loads (see e.g. [Block and Ochsendorf 2007; Block 2009]). Recent work by Block and coauthors extends this method in the case where the reciprocal diagram is not unique; for different choices of reciprocal diagram, the optimal heights can be found using the method of least squares [Van Mele and Block 2011], and the search for the best such reciprocal diagram can be automated using a genetic algorithm [Block and Lachauer 2011].

Other approaches to the interactive design of self-supporting structures include modeling these structures as damped particle-spring systems [Kilian and Ochsendorf 2005; Barnes 2009], and mirroring the rich tradition in architecture of designing self-supporting surfaces using hanging chain models [Heyman 1998]. Alternatively, masonry structures can be represented by networks of rigid blocks [Whiting et al. 2009], whose conditions on the structural feasibility were incorporated into procedural modeling of buildings.

Algorithmic and mathematical methods relevant to this paper are work on the geometry of quad meshes with planar faces [Glymph et al. 2004; Liu et al. 2006], discrete curvatures for such meshes [Pottmann et al. 2007; Bobenko et al. 2010], in particular curvatures in isotropic geometry [Pottmann and Liu 2007]. Schiftner and Balzer [2010] discuss approximating a reference surface by quad mesh with planar faces, whose layout is guided by statics properties of that surface.

2 Self-supporting Surfaces

2.1 The Continuous Theory

We are here modeling masonry as a surface given by a height field $s(x, y)$ defined in some planar domain Ω. We assume that there are vertical loads $F(x, y)$ — usually F represents the structure's own weight. By definition this surface is self-supporting, if and only if there exists a field of compressive stresses which are in equilibrium with the acting forces. This is equivalent to existence of a field $M(x, y)$ of 2×2 symmetric positive semidefinite matrices satisfying

$$\operatorname{div}(M\nabla s) = F, \quad \operatorname{div} M = 0, \tag{1}$$

where the divergence operator $\operatorname{div}\binom{u(x,y)}{v(x,y)} = u_x + v_y$ is understood to act on the columns of a matrix (see e.g. [Fraternali 2010], [Giaquinta and Giusti 1985]).

The condition $\operatorname{div} M = 0$ says that M is essentially the Hessian of a real-valued function ϕ (the *Airy stress potential*): With the notation

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \iff \widehat{M} = \begin{pmatrix} m_{22} & -m_{12} \\ -m_{12} & m_{11} \end{pmatrix}$$

it is clear that $\operatorname{div} M = 0$ is an integrability condition for $\widehat{M}$, so locally there is a potential ϕ with

$$\widehat{M} = \nabla^2\phi, \quad \text{i.e.,} \quad M = \widehat{\nabla^2\phi}.$$

If the domain Ω is simply connected, this relation holds globally. Positive semidefiniteness of M (or equivalently of $\widehat{M}$) characterizes *convexity* of the Airy potential ϕ. The Airy function enters computations only by way of its derivatives, so global existence is not an issue.

Remark: Stresses at boundary points depend on the way the surface is anchored: A fixed anchor means no condition, but a free boundary with outer normal vector $\mathbf{n}$ means $\langle M\nabla s, \mathbf{n}\rangle = 0$.

Stress Laplacian. Note that $\operatorname{div} M = 0$ yields $\operatorname{div}(M\nabla s) = \operatorname{tr}(M\nabla^2 s)$, which we like to call $\Delta_\phi s$. The operator Δ_ϕ is symmetric. It is elliptic (as a Laplace operator should be) if and only if M is positive definite, i.e., ϕ is strictly convex. The balance condition (1) may be written as $\Delta_\phi s = F$.

2.2 Discrete Theory: Thrust Networks

We are discretizing a self-supporting surface by a polyhedral mesh $\mathcal{S} = (V, E, F)$ (see Figure 3). Loads are again vertical, and we discretize them as force densities F_i associated with vertices $\mathbf{v}_i$. The load acting on this vertex is then given by $F_i A_i$, where A_i is an area of influence (using a prime to indicate projection onto the xy plane, A_i is the area of the Voronoi cell of $\mathbf{v}_i'$ w.r.t. V'). We assume that stresses are carried by the edges of the mesh: the force exerted on the vertex $\mathbf{v}_i$ by the edge connecting $\mathbf{v}_i, \mathbf{v}_j$ is given by

$$w_{ij}(\mathbf{v}_j - \mathbf{v}_i), \quad \text{where} \quad w_{ij} = w_{ji} \geq 0.$$

The nonnegativity of the individual weights w_{ij} expresses the compressive nature of forces. The balance conditions at vertices then

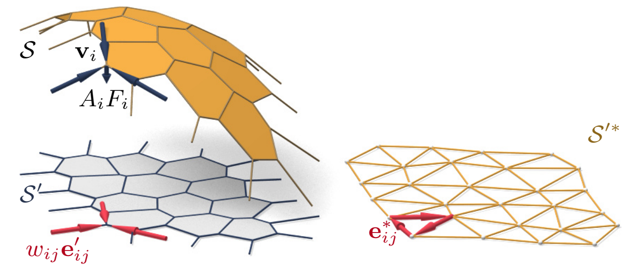

Figure 3: *A thrust network $\mathcal{S}$, with dangling edges indicating external forces (left). This network together with compressive forces which balance vertical loads A_iF_i projects onto a planar mesh $\mathcal{S}'$ with equilibrium compressive forces $w_{ij}\mathbf{e}'_{ij}$ in its edges. Rotating forces by 90° leads to the reciprocal force diagram $\mathcal{S}'^*$ (right).*

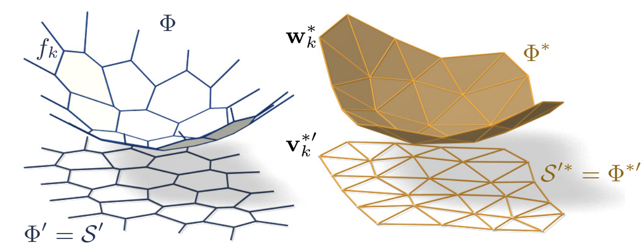

Figure 4: *Airy stress potential Φ and its polar dual Φ^*. Φ projects onto the same planar mesh as $\mathcal{S}$ does, while Φ^* projects onto the reciprocal force diagram. A primal face f_k lies in the plane $z = \alpha x + \beta y + \gamma \iff$ the corresponding dual vertex is $\mathbf{w}^*_k = (\alpha, \beta, -\gamma)$.*

read as follows: With $\mathbf{v}_i = (x_i, y_i, s_i)$ we have

$$\sum\nolimits_{j\sim i} w_{ij}(x_j - x_i) = \sum\nolimits_{j\sim i} w_{ij}(y_j - y_i) = 0, \quad (2)$$

$$\sum\nolimits_{j\sim i} w_{ij}(s_j - s_i) = A_iF_i. \quad (3)$$

A mesh equipped with edge weights in this way is a discrete *thrust network*. Invoking the safe theorem, we can state that a masonry structure is self-supporting, if we can find a thrust network with compressive forces which is entirely contained within the structure.

Reciprocal Diagram. Equations (2) have a geometric interpretation: With edge vectors

$$\mathbf{e}'_{ij} = \mathbf{v}'_j - \mathbf{v}'_i = (x_j, y_j) - (x_i, y_i),$$

Equation (2) asserts that vectors $w_{ij}\mathbf{e}'_{ij}$ form a closed cycle. Rotating them by 90 degrees, we see that likewise

$$\mathbf{e}'^*_{ij} = w_{ij} J \mathbf{e}'_{ij}, \quad \text{with} \quad J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right).$$

form a closed cycle (see Figure 3). If the mesh $\mathcal{S}$ is simply connected, there exists an entire *reciprocal diagram* $\mathcal{S}'^*$ which is a combinatorial dual of $\mathcal{S}$, and which has edge vectors $\mathbf{e}'^*_{ij}$. Its vertices are denoted by $\mathbf{v}'^*_i$.

Polyhedral Stress Potential. We can go further and construct a convex polyhedral 'Airy stress potential' surface Φ with vertices $\mathbf{w}_i = (x_i, y_i, \phi_i)$ combinatorially equivalent to $\mathcal{S}$ by requiring that a primal face of Φ lies in the plane $z = \alpha x + \beta y + \gamma$ if and only if (α, β) is the corresponding dual vertex of $\mathcal{S}'^*$ (see Figure 4). Obviously this condition determines Φ up to vertical translation. For existence see [Ash et al. 1988]. The inverse procedure constructs a reciprocal diagram from Φ. This procedure obviously works also if forces are not compressive: we can construct an Airy mesh Φ which has planar faces, but it will no longer be a convex polyhedron.

The vertices of Φ can be interpolated by a piecewise-linear function $\phi(x, y)$. It is easy to see that the derivative of $\phi(x, y)$ jumps by the amount $\|\mathbf{e}'^*_{ij}\| = w_{ij}\|\mathbf{e}'_{ij}\|$, when crossing over the edge $\mathbf{e}'_{ij}$ at right angle, with unit speed. This identifies Φ as the Airy polyhedron introduced by [Fraternali et al. 2002] as a finite element discretization of the continuous Airy function (see also [Fraternali 2010]).

If the mesh is not simply connected, the reciprocal diagram and the Airy polyhedron exist only locally.

Polarity. Polarity with respect to the *Maxwell paraboloid* $z = \frac{1}{2}(x^2+y^2)$ maps the plane $z = \alpha x + \beta y + \gamma$ to the point $(\alpha, \beta, -\gamma)$. Thus, applying polarity to Φ and projecting the result Φ^* into the xy plane reconstructs the reciprocal diagram $\Phi^{*\prime} = \mathcal{S}'^*$ (see Fig. 4).

Discrete Stress Laplacian. The weights w_{ij} may be used to define a graph Laplacian Δ_ϕ which on vertex-based functions acts as

$$\Delta_\phi s(\mathbf{v}_i) = \sum\nolimits_{j\sim i} w_{ij}(s_j - s_i).$$

This operator is a perfect discrete Laplacian in the sense of [Wardetzky et al. 2007], since it is symmetric by construction, Equation (2) implies linear precision for the planar "top view mesh" $\mathcal{S}'$ (i.e., $\Delta_\phi f = 0$ if f is a linear function), and $w_{ij} \geq 0$ ensures semidefiniteness and a maximum principle for Δ_ϕ-harmonic functions. Equation (3) can be written as $\Delta_\phi s = AF$.

Note that Δ_ϕ is well defined also in case the underlying meshes are not simply connected.

2.3 Surfaces in Isotropic Geometry

It is worth wile to reconsider the basics of self-supporting surfaces in the language of dual-isotropic geometry, which takes place in $\mathbb{R}^3$ with the z axis as a distinguished vertical direction. The basic elements of this geometry are planes, having equation $z = f(x, y) = \alpha x + \beta y + \gamma$. The gradient vector $\nabla f = (\alpha, \beta)$ determines the plane up to translation. A plane tangent to the graph of the function $s(x, y)$ has gradient vector ∇s.

There is the notion of *parallel points*: $(x, y, z) \parallel (x', y', z') \iff x = x',\ y = y'$.

Generally speaking, in the differential geometry of surfaces one considers the *Gauss map* σ from a surface S to a convex unit sphere Φ by requiring that corresponding points have parallel tangent planes. Subsequently mean curvature H^{rel} and Gaussian curvature K^{rel} *relative to* Φ are computed from the derivative $d\sigma$. Classically Φ is the ordinary unit sphere $x^2 + y^2 + z^2 = 1$, so that σ maps each point its unit normal vector.

Computing Curvatures. In our setting, parallelity is a property of *points* rather than lines, and the Gauss map σ goes the other way, mapping the tangent planes of the unit sphere $z = \phi(x, y)$ to the corresponding tangent plane of the surface $z = s(x, y)$. If we know which point a plane is attached to, then it is determined by its gradient. So we simply write

$$\nabla\phi \overset{\sigma}{\longmapsto} \nabla s.$$

By moving along a curve $\mathbf{u}(t) = (x(t), y(t))$ in the parameter domain we get the first variation of tangent planes: $\frac{d}{dt}\nabla\phi|_{\mathbf{u}(t)} = (\nabla^2\phi)\dot{\mathbf{u}}$. This yields the derivative $(\nabla^2\phi)\dot{\mathbf{u}} \overset{d\sigma}{\longmapsto} (\nabla^2 s)\dot{\mathbf{u}}$, for all

$\dot{\mathbf{u}}$, and the matrix of $d\sigma$ is found as $(\nabla^2\phi)^{-1}(\nabla^2 s)$. By definition, curvatures of the surface s *relative* to ϕ are found as

$$K_s^{\text{rel}} = \det(d\sigma) = \tfrac{\det \nabla^2 s}{\det \nabla^2 \phi},$$
$$H_s^{\text{rel}} = \tfrac{1}{2}\operatorname{tr}(d\sigma) = \tfrac{1}{2}\operatorname{tr}\left(\tfrac{M}{\det \nabla^2\phi}\nabla^2 s\right) = \tfrac{\Delta_\phi s}{2\det \nabla^2 \phi}.$$

The Maxwell paraboloid $\phi_0(x,y) = \frac{1}{2}(x^2+y^2)$ is the canonical unit sphere of isotropic geometry, its Hessian equals E_2. Curvatures relative to ϕ_0 are not called "relative" and are denoted by the symbols H, K instead of $H^{\text{rel}}, K^{\text{rel}}$. The formulas above together with the observation $\operatorname{tr}(M\nabla^2\phi) = \operatorname{tr}(E_2) = 2$ immediately imply

$$K_s = \det \nabla^2 s,\ H_s = \frac{\Delta s}{2},\ K_s^{\text{rel}} = \frac{K_s}{K_\phi},\ H_s^{\text{rel}} = \frac{\Delta_\phi s}{2K_\phi} = \frac{\Delta_\phi s}{\Delta_\phi \phi}$$

Relation to Self-supporting Surfaces. Applying the definitions above to the convex Airy stress potential ϕ of a self-supporting surface, we rewrite the balance conditions (1) as

$$2K_\phi H_s^{\text{rel}} = F. \tag{4}$$

Let us draw some conclusions:

- Since $H_\phi^{\text{rel}} = 1$ we see that the load $F_\phi = 2K_\phi$ is admissible for the stress surface $\phi(x,y)$, which is hereby shown as self-supporting. The quotient of admissible loads yields $H_s^{\text{rel}} = F/F_\phi$.
- If the stress surface coincides with the Maxwell paraboloid, then *constant loads characterize constant mean curvature surfaces*, because we get $K_\phi = 1$ and $H_s = F/2$.
- If s_1, s_2 have the same stress potential ϕ, then $H_{s_1-s_2}^{\text{rel}} = H_{s_1}^{\text{rel}} - H_{s_2}^{\text{rel}} = 0$, so $s_1 - s_2$ is a (relative) minimal surface.

2.4 Meshes in Isotropic Geometry

A general theory of curvatures of polyhedral surfaces with respect to a polyhedral unit sphere was proposed by [Pottmann et al. 2007; Bobenko et al. 2010], and its dual complement in isotropic geometry was elaborated by [Pottmann and Liu 2007]. As illustrated by Figure 5, the mean curvature of a self-supporting surface $\mathcal{S}$ *relative* to its discrete Airy stress potential is associated with the vertices of $\mathcal{S}$. It is computed from areas and mixed areas of faces in the polar polyhedra $\mathcal{S}^*$ and Φ^*:

$$H^{\text{rel}}(\mathbf{v}_i) = \frac{A_i(\mathcal{S},\Phi)}{A_i(\Phi,\Phi)}, \quad \text{where}$$
$$A_i(\mathcal{S},\Phi) = \frac{1}{4}\sum_{k:\, f_k \in \text{1-ring}(\mathbf{v}_i)} \det(\mathbf{v}_k'^*, \mathbf{w}_{k+1}'^*) + \det(\mathbf{w}_k'^*, \mathbf{v}_{k+1}'^*).$$

The prime denotes the projection into the xy plane, and summation is over those dual vertices which are adjacent to $\mathbf{v}_i$. Replacing $\mathbf{v}_k^*$ by $\mathbf{w}_k^*$ yields $A_i(\Phi,\Phi) = \frac{1}{2}\sum \det(\mathbf{w}_k'^*, \mathbf{w}_{k+1}'^*)$.

Proposition. *If Φ is the Airy surface of a thrust network $\mathcal{S}$, then the mean curvature of $\mathcal{S}$ relative to Φ is computable as*

$$H^{\text{rel}}(\mathbf{v}_i) = \frac{\sum_{j\sim i} w_{ij}(s_j - s_i)}{\sum_{j\sim i} w_{ij}(\phi_j - \phi_i)} = \frac{\Delta_\phi s}{\Delta_\phi \phi}\Big|_{\mathbf{v}_i}. \tag{5}$$

Proof. It is sufficient to show the formula

$$2A_i(\mathcal{S},\Phi) = \sum_{j\sim i} w_{ij}(s_j - s_i).$$

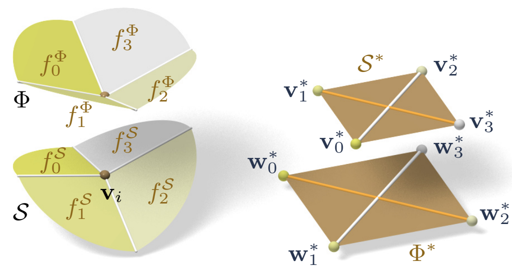

Figure 5: *Mean curvature of a vertex $\mathbf{v}_i$ of $\mathcal{S}$: Corresponding edges of the polar duals $\mathcal{S}^*$, Φ^* are parallel, and mean curvature according to [Pottmann et al. 2007] is computed from the vertices polar to faces adjacent to $\mathbf{v}_i$. For valence 4 vertices the case of zero mean curvature shown here is characterized by parallelity of non-corresponding diagonals of corresponding quads in $\mathcal{S}^*$, Φ^*.*

For that, consider edges $\mathbf{e}_1', \dots, \mathbf{e}_n'$ emanating from $\mathbf{v}_i'$. The dual cycles in $\Phi^{*\prime}$ and $\mathcal{S}^{*\prime}$ without loss of generality are given by vertices $(\mathbf{v}_1^{*\prime}, \dots, \mathbf{v}_n^{*\prime})$ and $(\mathbf{w}_1^{*\prime}, \dots, \mathbf{w}_n^{*\prime})$, respectively. The latter has edges $\mathbf{w}_{j+1}'^* - \mathbf{w}_j'^* = w_{ij}J\mathbf{e}_j'$ (indices modulo n).

Without loss of generality $\mathbf{v}_i = 0$, so the vertex $\mathbf{v}_j'^*$ by construction equals the gradient of the linear function $\mathbf{x} \mapsto \langle \mathbf{v}_j'^*, \mathbf{x}\rangle$ defined by the properties $\mathbf{e}_{j-1}' \mapsto s_{j-1} - s_i$, $\mathbf{e}_j' \mapsto s_j - s_i$. Corresponding edge vectors $\mathbf{v}_{j+1}'^* - \mathbf{v}_j'^*$ and $\mathbf{w}_{j+1}'^* - \mathbf{w}_j'^*$ are parallel, because $\langle \mathbf{v}_{j+1}' - \mathbf{v}_j', \mathbf{e}_j'\rangle = (s_j - s_i) - (s_j - s_i) = 0$. Expand $2A_i(\mathcal{S},\Phi)$:

$$\begin{aligned}
&\frac{1}{2}\sum \det(\mathbf{w}_j'^*, \mathbf{v}_{j+1}') + \det(\mathbf{v}_j', \mathbf{w}_{j+1}'^*)\\
&= \frac{1}{2}\sum \det(\mathbf{w}_j'^* - \mathbf{w}_{j+1}'^*, \mathbf{v}_{j+1}') + \det(\mathbf{v}_j', \mathbf{w}_{j+1}'^* - \mathbf{w}_j'^*)\\
&= \frac{1}{2}\sum \det(-w_{ij}J\mathbf{e}_j', \mathbf{v}_{j+1}') + \det(\mathbf{v}_j', w_{ij}J\mathbf{e}_j')\\
&= \sum \det(\mathbf{v}_j', w_{ij}J\mathbf{e}_j') = \sum w_{ij}\langle \mathbf{v}_j', \mathbf{e}_j'\rangle = \sum w_{ij}(s_j - s_i).
\end{aligned}$$

Here we have used $\det(\mathbf{a}, J\mathbf{b}) = \langle \mathbf{a}, \mathbf{b}\rangle$. $\square$

In order to discretize (4), we also need a discrete Gaussian curvature, which is usually defined as a quotient of areas which correspond under the Gauss mapping. We define

$$K_\Phi(\mathbf{v}_i) = \frac{A_i(\Phi,\Phi)}{A_i},$$

where A_i is the Voronoi area of vertex $\mathbf{v}_i'$ in the projected mesh $\mathcal{S}'$ used in (3).

Discrete Balance Equation. The discrete version of the balance equation (4) reads as follows:

Theorem. *A simply-connected mesh $\mathcal{S}$ with vertices $\mathbf{v}_i = (x_i, y_i, s_i)$ can be put into static equilibrium with vertical forces "A_iF_i" if and only if there exists a combinatorially equivalent mesh Φ with planar faces and vertices (x_i, y_i, ϕ_i), such that curvatures of $\mathcal{S}$ relative to Φ obey*

$$2K_\Phi(\mathbf{v}_i)H^{\text{rel}}(\mathbf{v}_i) = F_i \tag{6}$$

at every interior vertex and every free boundary vertex $\mathbf{v}_i$. $\mathcal{S}$ can be put into compressive static equilibrium if and only if there exists a convex such Φ.

Proof. The relation between equilibrium forces $w_{ij}\mathbf{e}_{ij}$ in $\mathcal{S}$ and the polyhedral stress potential Φ has been discussed above, and

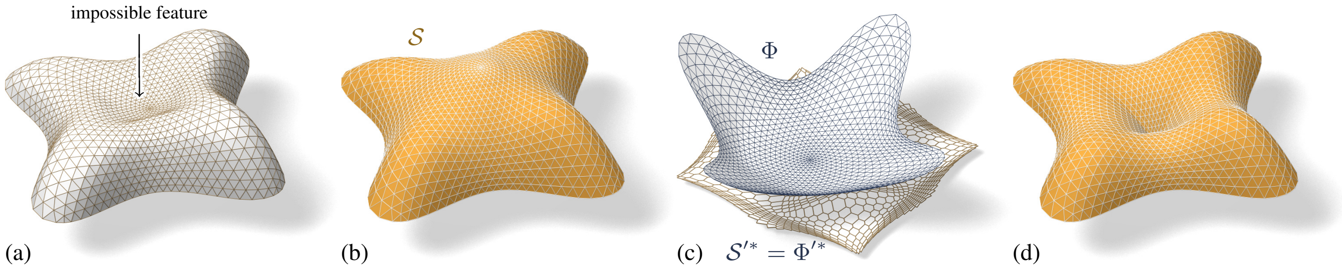

Figure 6: *The top of the Lilium Tower (a) cannot stand as a masonry structure, because of its central part is concave. Our algorithm finds a nearby self-supporting mesh (b) without this impossible feature. (c) shows the corresponding Airy mesh Φ and reciprocal force diagram $\mathcal{S}'^*$. (d) The user can edit the original surface, such as by specifying that the center of the surface is supported by a vertical pillar, and the self-supporting network adjusts accordingly*

so has the equivalence "$w_{ij} \geq 0 \iff \Phi$ convex" (see e.g. [Ash et al. 1988] for a survey of this and related results). It remains to show that Equations (2) and (6) are equivalent. This is the case because the proposition above implies $2K(\mathbf{v}_i)H^{\text{rel}}(\mathbf{v}_i) = 2\frac{A_i(\Phi,\Phi)}{A_i}\frac{A_i(\Phi,\mathcal{S})}{A_i(\Phi,\Phi)} = \frac{1}{A_i}(\sum_{j\sim i} w_{ij}(s_j - s_i)) = \frac{1}{A_i}A_iF_i$. $\square$

Existence of Discretizations. When considering discrete thrust networks as discretizations of continuous self-supporting surfaces, the following question is important: For a given smooth surface $s(x, y)$ with Airy stress function ϕ, does there exist a polyhedral surface $\mathcal{S}$ in equilibrium approximating $s(x, y)$, whose top view is a given planar mesh $\mathcal{S}'$? We restrict our attention to triangle meshes, where planarity of the faces of the discrete stress surface Φ is not an issue. This question has several equivalent reformulations:

- Does $\mathcal{S}'$ have a reciprocal diagram whose corresponding Airy polyhedron Φ approximates the continuous Airy potential ϕ? (if the surfaces involved are not simply connected, these objects are defined locally).
- Does $\mathcal{S}'$ possess a "perfect" discrete Laplace-Beltrami operator Δ_ϕ in the sense of Wardetzky et al. [2007] whose weights are the edge length scalars of such a reciprocal diagram?

From [Wardetzky et al. 2007] we know that perfect Laplacians exist only on regular triangulations which are projections of convex polyhedra. On the other hand, previous sections show how to appropriately re-triangulate:Let Φ be a triangle mesh convex hull of the vertices $(x_i, y_i, \phi(x_i, y_i))$, where (x_i, y_i) are vertices of $\mathcal{S}'$. Then its polar dual Φ^* projects onto a reciprocal diagram with positive edge weights, so Δ_ϕ has positive weights, and the vertices (x_i, y_i, s_i) of $\mathcal{S}$ can be found by solving the discrete Poisson problem $(\Delta_\phi s)_i = A_iF_i$, which yields a mesh approximating $s(x, y)$.

Assuming the discrete operator Δ_ϕ approximates its continuous counterpart, we conclude: *A smooth self-supporting surface can be approximated by a discrete self-supporting triangular mesh for any sampling of the surface.*

3 Thrust Networks from Reference Meshes

Consider now the problem of taking a given reference mesh, say $\mathcal{R}$, and finding a combinatorially equivalent mesh $\mathcal{S}$ in static equilibrium approximating $\mathcal{R}$. The loads on $\mathcal{S}$ include user-prescribed loads as well as the dead load caused by the mesh's own weight. Conceptually, finding $\mathcal{S}$ amounts to minimizing some formulation of distance between $\mathcal{R}$ and $\mathcal{S}$, subject to constraints (2), (3), and $w_{ij} \geq 0$. For any choice of distance this minimization will be a nonlinear, non-convex, inequality-constrained variational problem that cannot be efficiently solved in practice. Instead we propose a staggered optimization algorithm:

0. Start with an initial guess $\mathcal{S} = \mathcal{R}$.
1. Estimate the self-load on the vertices of $\mathcal{S}$, using their current positions.
2. Fixing $\mathcal{S}$, fit an associated stress surface Φ.
3. Alter positions $\mathbf{v}_i$ to improve the fit.
4. Repeat from Step 1 until convergence.

Step 1: Estimating Self-Load. The dead load due to the surface's own weight depends not only on the top view of $\mathcal{S}$, but also on the surface area of its faces. To avoid adding nonlinearity to the algorithm, we estimate the load coefficients F_i at the beginning of each iteration, and assume they remain constant until the next iteration. We estimate the load "A_iF_i" associated with each vertex by calculating its Voronoi area on each of its incident faces, and then multiplying by a user-specified surface density ρ.

Step 2: Fit a Stress Surface. In this step, we fix $\mathcal{S}$ and try to fit a stress surface Φ subordinate to the top view $\mathcal{S}'$ of the primal mesh. We do so by searching for dihedral angles between the faces of Φ which minimize, in the least-squares sense, the error in force equilibrium (6) and local integrability of Φ. Doing so is equivalent to minimizing the squared residuals of Equations (3) and (2), respectively, with the positions held fixed. Defining the *equilibrium energy*

$$E = \sum_i \left\| \begin{pmatrix} 0 \\ 0 \\ A_iF_i \end{pmatrix} - \sum_{j\sim i} w_{ij}(\mathbf{v}_j - \mathbf{v}_i) \right\|^2 \qquad (7)$$

where the outer sum is over the interior and free boundary vertices, we solve

$$\min_{w_{ij}} E, \quad \text{s.t. } 0 \leq w_{ij} \leq w_{\max}. \qquad (8)$$

Here $w_{\max}$ is an optional maximum weight we are willing to assign (to limit the amount of stress in the surface). This convex, sparse, box-constrained least-squares problem [Friedlander 2007] always has a solution. If the objective is 0 at this solution, the faces of Φ locally integrate to a stress surface satisfying (6), and so Φ certifies that $\mathcal{S}$ is self-supporting – we are done. Otherwise, $\mathcal{S}$ is not self-supporting and its vertices must be moved.

Step 3: Alter Positions. In the previous step we fit as best as possible a stress surface Φ to $\mathcal{S}$. There are two possible kinds of error with this fit: the faces around a vertex (equivalently, the reciprocal diagram) might not close up; and the resulting stress forces might not be exactly in equilibrium with the loads. These errors can be decreased by modifying the top view and heights of $\mathcal{S}$, respectively. It is possible to simply solve for new vertex positions

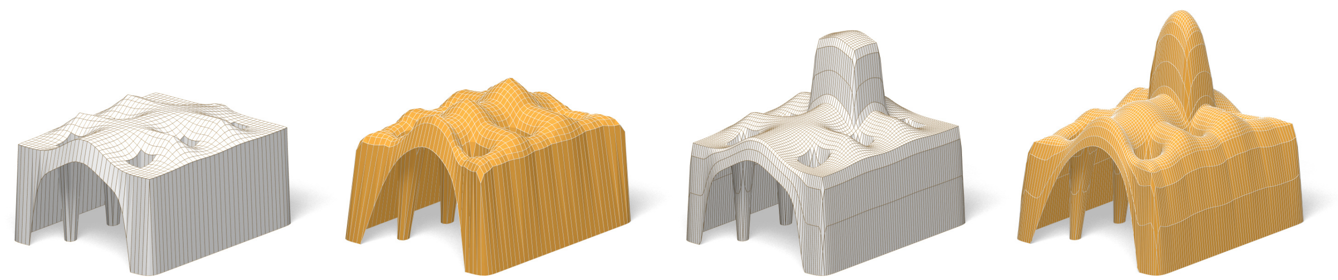

Figure 7: *The user-designed reference mesh (left) is not self-supporting, but our algorithm finds a nearby perturbation of the reference surface (middle-left) that is in equilibrium. As the user makes edits to the reference surface (middle-right), the thrust network automatically adjusts (right).*

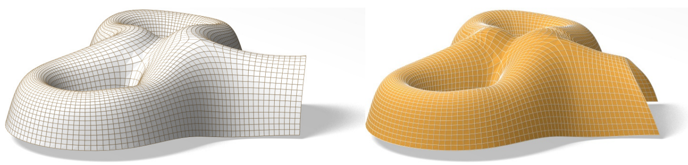

Figure 8: *A freeform surface (left) needs adjustments around the entrance arch and between the two pillars in order to be self-supporting; our algorithm finds the nearby surface in equilibrium (right) that incorporates these changes.*

that put $\mathcal{S}$ in static equilibrium, since Equations (2) and (3) with w_{ij} fixed form a square linear system that is typically nonsingular.

While this approach would yield a self-supporting $\mathcal{S}$, this mesh is often far from the reference mesh $\mathcal{R}$, since any local errors in the stress surface from Step 2 amplify into global errors in $\mathcal{S}$. We propose instead to look for new positions that decrease the imbalance in the stresses and loads, while also penalizing drift away from the reference mesh:

$$\min_{\mathbf{v}} E + \alpha \sum_i \langle \mathbf{n}_i, \mathbf{v}_i - \mathbf{v}_i^0 \rangle^2 + \beta \|\mathbf{v} - \mathbf{v}_P^0\|^2,$$

where $\mathbf{v}_i^0$ is the position of the i-th vertex at the start of this step of the optimization, $\mathbf{n}_i$ is the starting vertex normal (computed as the average of the incident face normals), $\mathbf{v}_P^0$ is the projection of $\mathbf{v}^0$ onto the reference mesh, and $\alpha > \beta$ are penalty coefficients that are decreased every iteration of Steps 1–3 of the algorithm. The second term allows $\mathcal{S}$ to slide over itself (if doing so improves equilibrium) but penalizes drift in the normal direction. The third term, weaker than the second, regularizes the optimization by preventing large drift away from the reference surface or excessive tangetial sliding.

Implementation Details. Solving the weighted least-squares problem of Step 3 amounts to solving a sparse, symmetric linear system. While the MINRES algorithm [Paige and Saunders 1975] is likely the most robust algorithm for solving this system, in practice we have observed that the method of conjugate gradients works well despite the potential ill-conditioning of the objective matrix.

Limitations. This algorithm is not guaranteed to always converge; this fact is not surprising from the physics of the problem (if the boundary of the reference mesh encloses too large of a region, $w_{\max}$ is set too low, and the density of the surface too high, a thrust network in equilibrium simply does not exist – the vault is too ambitious and cannot be built to stand; pillars are needed.)

We can, however, make a few remarks. Step 2 always decreases the equilibrium energy E of Equation (7) and Step 3 does as well as $\beta \to 0$. Moreover, as $\alpha \to 0$ and $\beta \to 0$, Step 3 approaches a linear system with as many equations as unknowns; if this system has full rank, its solution sets $E = 0$. These facts suggest that the algorithm should generally converge to a thrust network in equilibrium, provided that Step 1 does not increase the loads by too much at every iteration, and this is indeed what we observe in practice. One case where this assumption is guaranteed to hold is if the thickness of the surface is allowed to freely vary, so that it can be chosen so that the surface has uniform density over the top view.

If the linear system in Step 3 is singular and infeasible, the algorithm can stall at $E > 0$. This failure occurs, for instance, when an interior vertex has height z_i lower than all of its neighbors, and Step 2 assigns all incident edges to that vertex a weight of zero: clearly no amount of moving the vertex or its neighbors can bring the vertex into equilibrium. We avoid such degenerate configurations by bounding weights slightly away from zero in (8), trading increased robustness for slight smoothing of the resulting surface.

4 Results

Interactive Design of Self-Supporting Surfaces. The optimization algorithm described in the previous section forms the basis of an interactive design tool for self-supporting surfaces. Users manipulate a mesh representing a reference surface, and the computer searches for a nearby thrust network in equilibrium (see e.g. Figure 7). Fitting this thrust network does not require that the user specify boundary tractions, and although the top view of the reference mesh is used as an initial guess for the top view of the thrust network, the search is not restricted to this top view. The features of the design tool include:

- Handle-based 3D editing of the reference mesh using Laplacian coordinates [Lipman et al. 2004; Sorkine et al. 2003] to extrude vaults, insert pillars, and apply other deformations to the reference mesh. Handle-based adjustments of the heights, keeping the top view fixed, and deformation of the top view, keeping the heights fixed, are also supported. The thrust network adjusts interactively to fit the deformed positions, giving the usual visual feedback about the effects of her edits on whether or not the surface can stand.
- Specification of boundary conditions. Points of contact between the reference surface and the ground or environment are specified by "pinning" vertices of the surface, specifying that the thrust network must coincide with the reference mesh at this point, and relaxing the condition that forces must be in equilibrium there.
- Interactive adjustment of surface density ρ, external loads, and maximum permissible stress per edge w_{max}, with visual feedback of how these parameters affect the fitted thrust net-

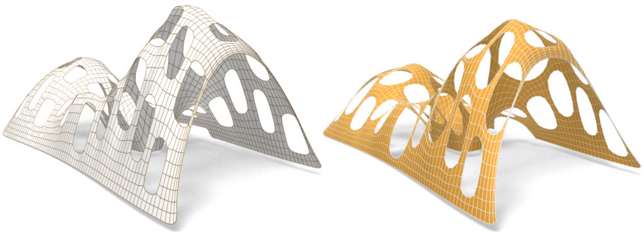

Figure 9: *A mesh with holes (left) requires large deformations to both the top view and heights to render it self-supporting (right).*

work.

- Upsampling of the thrust network through Catmull-Clark subdivision [Catmull and Clark 1978] and polishing of the resulting refined thrust network using optimization (§3).
- Visualization of the stress surface $\mathcal{R}$ dual to the thrust network and corresponding reciprocal diagram.

Example: Vault with Pillars. As an example of the design and optimization workflow, consider a rectangular vault with six pillars, free boundary conditions along one edge, fixed boundary conditions along the others, and a tower extruded from the top of the surface (see Figure 7). This surface is neither convex nor simply connected, and exhibits a mix of boundary conditions, none of which cause our algorithm any difficulty; it finds a self-supporting thrust network near the designed reference mesh. The user is now free to make edits to the reference mesh, and the thrust network adapts to these edits, providing the user feedback on whether these designs are physically realizable.

Example: Top of the Lilium Tower. Consider the top portion of the steel-glass exterior surface of the Lilium Tower, which is currently being built in Warszaw (see Figure 6). This surface contains a concave part with local minimum in its interior and so cannot possibly be self-supporting. Given this surface as a reference mesh, our algorithm constructs a nearby thrust network in equilibrium without the impossible feature. The user can then explore how editing the reference mesh – adding a pillar, for example – affects the thrust network and its deviation from the reference surface.

Example: Freeform Structure with Two Pillars. Suppose an architect's experience and intuition has permitted the design a freeform surface (see Figure 8) that is nearly self-supporting. Our algorithm reveals those edits needed to make the structure sound – principally around the entrance arch, and the area between the two pillars.

Example: Swiss Cheese. Cutting holes in a self-supporting surface interrupts force flow lines and causes dramatic global changes to the surface stresses, often to the point that the surface is no longer in equilibrium. Whether a given surface with many such holes can stand is far from obvious. Figures 9 show such an implausible and unstable surface; our optimization finds a nearby, equally implausible but stable surface without difficulty (see Figures 1 and 9, right).

5 Special Self-Supporting Surfaces

PQ Meshes. Meshes with *planar* faces are of particular interest in architecture, so in this section we discuss how to remesh a given thrust network in equilibrium such that it becomes a quad mesh with planar faces (again in equilibrium). For this purpose we first demonstrate how to find a quad mesh $\mathcal{S}$ with vertices $\mathbf{v}_{ij} = (x_{ij}, y_{ij}, s_{ij})$ which approximates a given continuous surface $s(x, y)$ equipped with an equilibrium stress potential $\phi(x, y)$.

It is known that $\mathcal{S}$ must approximately follow a network of conjugate curves in the surface (see e.g. [Liu et al. 2006]). We can derive this condition in an elementary way as follows: Using a Taylor expansion, we compute the volume of the convex hull of the quadrilateral $\mathbf{v}_{ij}$, $\mathbf{v}_{i+1,j}$, $\mathbf{v}_{i+1,j+1}$, $\mathbf{v}_{i,j+1}$, assuming the vertices lie exactly on the surface $s(x, y)$. This results in

$$\text{vol} = \tfrac{1}{6}\det(\mathbf{a}_1, \mathbf{a}_2, (\mathbf{a}_1)^T\, \nabla^2 s\, \mathbf{a}_2) + \cdots,$$
$$\text{where } \mathbf{a}_1 = \binom{x_{i+1,j}-x_{ij}}{y_{i+1,j}-y_{ij}}, \quad \mathbf{a}_2 = \binom{x_{i,j+1}-x_{ij}}{y_{i,j+1}-y_{ij}},$$

and the dots indicate higher order terms. We see that planarity requires $(\mathbf{a}_1)^T\, \nabla^2 s\, \mathbf{a}_2 = 0$. In addition to the mesh $\mathcal{S}$ approximating the surface $s(x, y)$, the corresponding polyhedral Airy surface Φ must approximate $\phi(x, y)$; thus we get the conditions

$$(\mathbf{a}_1)^T\, \nabla^2 s\ \mathbf{a}_2 = (\mathbf{a}_1)^T\, \nabla^2 \phi\ \mathbf{a}_2 = 0.$$

$\mathbf{a}_1, \mathbf{a}_2$ are therefore eigenvectors of $(\nabla^2\phi)^{-1}\nabla^2 s$. In view of §2.3, $\mathbf{a}_1, \mathbf{a}_2$ indicate the principal directions of the surface $s(x, y)$ relative to $\phi(x, y)$ (see Figure 10).

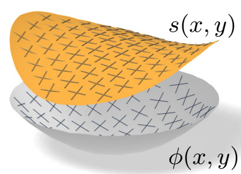

Figure 10: *Planar quad remeshing of a self-supporting surface $s(x, y)$ with stress potential ϕ is guided by the principal curvature directions of s relative to ϕ (found from eigenvectors of $(\nabla^2\phi)^{-1}\nabla^2 s$).*

In the discrete case, where s, ϕ are not given as continuous surfaces, but are represented by a mesh in equilibrium and its Airy mesh, we use the techniques of Schiftner [2007] and Cohen-Steiner and Morvan [2003] to approximate the Hessians $\nabla^2 s$, $\nabla^2\phi$, compute principal directions as eigenvectors of $(\nabla^2\phi)^{-1}\nabla^2 s$, and subsequently find meshes $\mathcal{S}, \Phi$ approximating s, ϕ which follow those directions. Global optimization now makes $\mathcal{S}, \Phi$ a valid thrust network with discrete stress potential. Convexity of Φ ensures that $\mathcal{S}$ is self-supporting.

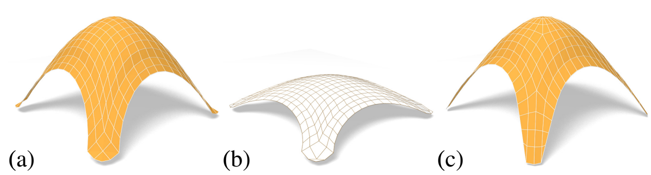

Figure 11: *Directly enforcing planarity of the faces of even a very simple self-supporting quad-mesh vault (a) results in a surface far removed from the original design (b). Starting instead from a remeshing of the surface with edges following relative principal curvature directions yields a self-supporting, PQ mesh far more faithful to the original (c).*

Note that the relative principal curvature directions give the *unique* curve network along which a planar quad discretization of a self-supporting surface is possible. Taking an arbitrary non-planar quad mesh and attempting naive, simultaneous enforcement of planarity and static equilibrium does not yield good results, as shown in Figure 11. Figures 12 and 13 further illustrate the result of applying this procedure to self-supporting surfaces.

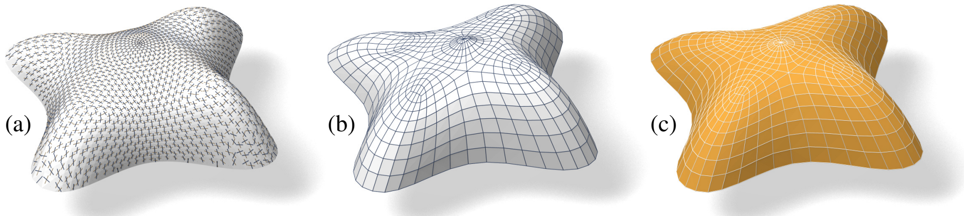

Figure 12: *Planar quad remeshing of the "Lilium tower" surface of Figure 6. (b) Principal directions. (c) Quad mesh guided by principal directions is close to planar and close to self-supporting. (c) Small changes make it self-supporting.*

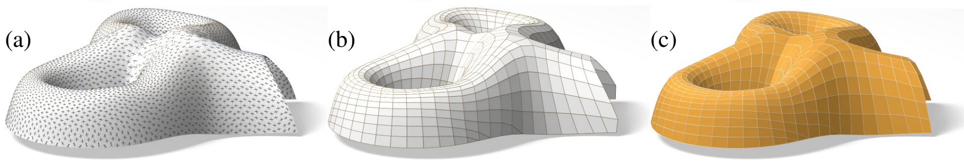

Figure 13: *Planar quad remeshing of the surface of Figure 8, analogous to Figure 12.*

Koenigs Meshes. UNFINISHED Consider a self-supporting thrust network $\mathcal{S}$ and corresponding Airy mesh Φ. Both $\mathcal{S}$ and Φ are elements of the linear space of meshes which project onto $\mathcal{S}'$. Any such mesh has vertices $\mathbf{v}_i = (x_i, y_i, z_i)$, where the choice $z_i = s_i$ leads to $\mathcal{S}$ and $z_i = \phi_i$ leads to Φ. We already know that the vertical loads $A_i F_i$ which put $\mathcal{S}$ into equilibrium are computed as $AF = \Delta_\phi s$.

We ask: Which perturbations $\mathcal{S} + \mathcal{R}$, having z coordinates $z_i = s_i + r_i$, support the *same* vertical loads as $\mathcal{S}$ does? Such a mesh must satisfy $\Delta_\phi(s+r) = \Delta_\phi s$, so $\Delta_\phi r = 0$. This linear system is easily solved in principle.

There is a nice explicit geometric construction of all such harmonic functions in the case of quad meshes: Equation (6) immediately leads to $H^{\mathrm{rel}}_{\mathcal{S}} = H^{\mathrm{rel}}_{\mathcal{S}+\mathcal{R}}$, which is equivalent to

$$H^{\mathrm{rel}}_{\mathcal{R}} = 0.$$

So $\mathcal{R}$ is a *minimal surface*. Recall that $H^{\mathrm{rel}}_{\mathcal{R}}$ is the mean curvature of $\mathcal{R}^*$ with respect to the Gauss image Φ^* in the sense of [Pottmann et al. 2007], where the star indicates the polar polyhedron. We conclude that $\mathcal{R}^*$ is constructed from Φ^* by the condition of *parallel non-corresponding diagonals*, which is also called Christoffel duality, and which can be seen in Figure 5. This condition determines $\mathcal{R}^*$ uniquely up to translation and scaling; thus $\mathcal{R}$ is unique up to scaling of z coordinates and adding linear functions.

In general, however, $\mathcal{R}$ does not exist, since only the so-called *Koenigs meshes* possess a Christoffel dual.

An interesting special case occurs if Φ is an *isotropic Koebe mesh*, i.e., a PQ mesh whose edges touch the Maxwell paraboloid. Since Φ approximates the Maxwell paraboloid, we get $2K(\mathbf{v}_i)H^{\mathrm{rel}}(\mathbf{v}_i) \approx 1$ which means Φ is self-supporting for unit load. Applying the Christoffel dual construction described above yields a family a "minimal" mesh $\mathcal{R}$ and a family of meshes $\Phi + \alpha\mathcal{R}$ which are self-supporting for unit load. Figure 14 shows an example.

6 Conclusion and Future Work

TODO Some ideas:

- Non-manifold surfaces
- Non-vertical loads
- Adaptive remeshing

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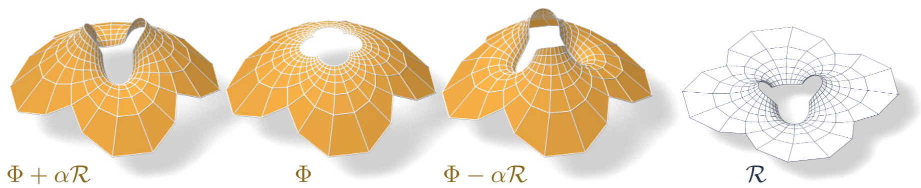

Figure 14: *A 'Koebe' mesh Φ is self-supporting for unit dead load. An entire family of self-supporting meshes with the same top view is defined by $\mathcal{S}_\alpha = \Phi + \alpha\mathcal{R}$, where $\mathcal{R}$ is chosen as Φ's Christoffel-dual.*

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