

# Design of Self-supporting Surfaces

## Abstract

Self-supporting masonry is one of the most ancient and elegant techniques for building curved shapes. Because of the very geometric nature of their failure, analyzing and modeling such structures is more a geometry processing problem than one of classical continuum mechanics. In this paper we use the thrust network method of analysis and present an iterative nonlinear optimization algorithm for efficiently approximating freeform shapes by self-supporting ones. The rich geometry of thrust networks leads us to close connections between different topics of discrete differential geometry, such as a finite-element discretization of the Airy stress potential, perfect graph Laplacians, and the problem of computing admissible loads via curvatures of polyhedral surfaces. This geometric viewpoint allows us, in particular, to remesh self-supporting shapes by self-supporting quad meshes with planar faces.

**CR Categories:** I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling—Curve, surface, solid, and object representations;

**Keywords:** Discrete differential geometry, architectural geometry, self-supporting masonry, thrust networks, reciprocal force diagrams, discrete Laplace operators, isotropic geometry, mean curvature

## 1 Introduction

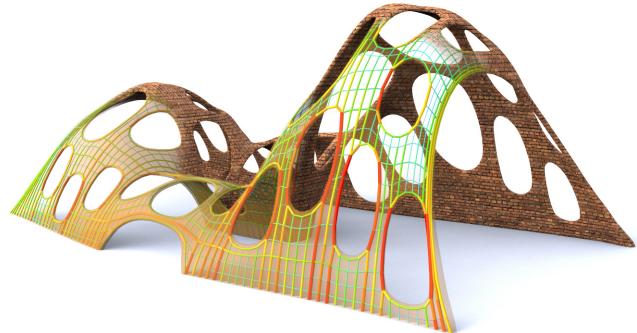
Vaulted masonry structures are among the simplest and at the same time most elegant solutions for creating curved shapes in building construction. For this reason they have been an object of interest since antiquity; large, non-convex examples of such structures include gothic cathedrals. They continue to be an active topic of research in today's engineering community.

Our paper is concerned with a combined geometry+statics analysis of *self-supporting* masonry and with tools for the interactive modeling of freeform self-supporting structures. Here “self-supporting” means that the structure, considered as an arrangement of blocks (bricks, stones), holds together by itself, with additional support present only during construction. Our analysis is based on the following assumptions, which follow the classic [Heyman 1966]:

*Assumption 1:* Masonry has no tensile strength, but the individual building blocks do not slip against each other (because of friction or mortar). On the other hand, their compressive strength is sufficiently high so that failure of the structure is by a sudden change in geometry and not by material failure.

*Assumption 2 (The Safe Theorem):* If a system of forces can be found which is in equilibrium with the load on the structure and which is contained within the masonry envelope then the structure will carry the loads, although the actual forces present may not be those postulated.

Our approach is twofold: We first give an overview of the continuous case of a smooth surface under stress, which turns out to be governed locally by the so-called Airy stress function. This mathematical model is called a membrane in the engineering literature and has been applied to the analysis of masonry before. The surface is self-supporting if and only if stresses are entirely compressive (i.e.,



**Figure 1:** Surfaces with irregularly placed holes almost never stand by themselves when built from bricks; for those that do, stability is not obvious by inspection. The surface shown is produced by finding the nearest self-supporting shape from a given freeform geometry. The image also illustrates the fictitious thrust network used in our algorithm, with edges' cross-section and coloring visualizing the magnitude of forces (warmer colors represent higher stresses.)

the Airy function is convex). For computational purposes, stresses are discretized as a fictitious *thrust network* [Block and Ochsendorf 2007] contained in the masonry structure; this network is a system of forces in equilibrium with the structure's deadload. It can be interpreted as a finite element discretization of the continuous case, and it turns out to have very interesting geometry, with the Airy stress function becoming a polyhedral surface directly related to a reciprocal force diagram.

While previous work in architectural geometry was mostly concerned with aspects of rationalization and purely geometric side-conditions which occur in freeform architecture, the focus of this paper is design with *statics* constraints. In particular, our contributions are the following:

## Contributions.

- We connect the physics of self-supporting surfaces with vertical loads to the geometry of isotropic 3-space, with the direction of gravity as the distinguished direction (§2.3). Taking the convex Airy potential as unit sphere, one can express the equations governing self-supporting surfaces in terms of curvatures.
- We consider the known constructions of polyhedral thrust networks and their reciprocal diagrams, and give an interpretation of the equilibrium conditions in terms of discrete curvatures (§2.4).
- The graph Laplacian derived from a thrust network with compressive forces is a “perfect” one (§2.2). We show how it appears in the analysis and establish a connection with mean curvatures which are otherwise defined for polyhedral surfaces.
- We present an optimization algorithm for efficiently finding a thrust network near a given arbitrary reference surface (§3), and build a tool for interactive design of self-supporting surfaces based on this algorithm (§4).
- We exploit the geometric relationships between a self-supporting surface and its stress potential in order to find particularly nice families of self-supporting surfaces, especially planar quadrilateral representations of thrust networks (§5).

- 87 • We demonstrate the versatility and applicability of our approach  
 88 to the design and analysis of large-scale masonry and steel-glass  
 89 structures.

147  $M(x, y)$  of  $2 \times 2$  symmetric positive semidefinite matrices satisfying  
 148

$$\operatorname{div}(M\nabla s) = F, \quad \operatorname{div} M = 0, \quad (1)$$

149 where the divergence operator  $\operatorname{div} \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = u_x + v_y$  is under-  
 150 stood to act on the columns of a matrix (see e.g. [Fraterno 2010],  
 151 [Giaquinta and Giusti 1985]).

152 The condition  $\operatorname{div} M = 0$  says that  $M$  is locally the Hessian of a  
 153 real-valued function  $\phi$  (the *Airy stress potential*): With the notation

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \iff \widehat{M} = \begin{pmatrix} m_{22} & -m_{12} \\ -m_{12} & m_{11} \end{pmatrix}$$

154 it is clear that  $\operatorname{div} M = 0$  is an integrability condition for  $\widehat{M}$ , so  
 155 locally there is a potential  $\phi$  with

$$\widehat{M} = \nabla^2 \phi, \quad \text{i.e.,} \quad M = \widehat{\nabla^2 \phi}.$$

156 If the domain  $\Omega$  is simply connected, this relation holds globally.  
 157 Positive semidefiniteness of  $M$  (or equivalently of  $\widehat{M}$ ) character-  
 158 izes *convexity* of the Airy potential  $\phi$ . The Airy function enters  
 159 computations only by way of its derivatives, so global existence is  
 160 not an issue.

161 *Remark:* Stresses at boundary points depend on the way the sur-  
 162 face is anchored: A fixed anchor means no condition, but a free  
 163 boundary with outer normal vector  $\mathbf{n}$  means  $\langle M\nabla s, \mathbf{n} \rangle = 0$ .

164 **Stress Laplacian.** Note that  $\operatorname{div} M = 0$  yields  $\operatorname{div}(M\nabla s) =$   
 165  $\operatorname{tr}(M\nabla^2 s)$ , which we like to call  $\Delta_\phi s$ . The operator  $\Delta_\phi$  is sym-  
 166 metric. It is elliptic (as a Laplace operator should be) if and only if  
 167  $M$  is positive definite, i.e.,  $\phi$  is strictly convex. The balance condi-  
 168 tion (1) may be written as  $\Delta_\phi s = F$ .

## 169 2.2 Discrete Theory: Thrust Networks

170 We discretize a self-supporting surface by a mesh  $\mathcal{S} = (V, E, F)$   
 171 (see Figure 2). Loads are again vertical, and we discretize them as  
 172 force densities  $F_i$  associated with vertices  $\mathbf{v}_i$ . The load acting on  
 173 this vertex is then given by  $F_i A_i$ , where  $A_i$  is an area of influence  
 174 (using a prime to indicate projection onto the  $xy$  plane,  $A_i$  is the  
 175 area of the Voronoi cell of  $\mathbf{v}'_i$  w.r.t.  $V'$ ). We assume that stresses  
 176 are carried by the edges of the mesh: the force exerted on the vertex  
 177  $\mathbf{v}_i$  by the edge connecting  $\mathbf{v}_i, \mathbf{v}_j$  is given by

$$w_{ij}(\mathbf{v}_j - \mathbf{v}_i), \quad \text{where } w_{ij} = w_{ji} \geq 0.$$

178 The nonnegativity of the individual weights  $w_{ij}$  expresses the com-  
 179 pressive nature of forces. The balance conditions at vertices then  
 180 read as follows: With  $\mathbf{v}_i = (x_i, y_i, s_i)$  we have

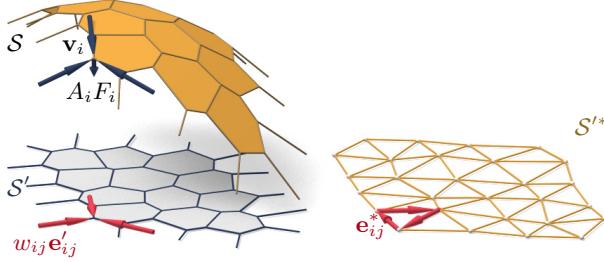
$$\sum_{j \sim i} w_{ij}(x_j - x_i) = \sum_{j \sim i} w_{ij}(y_j - y_i) = 0, \quad (2)$$

$$\sum_{j \sim i} w_{ij}(s_j - s_i) = A_i F_i. \quad (3)$$

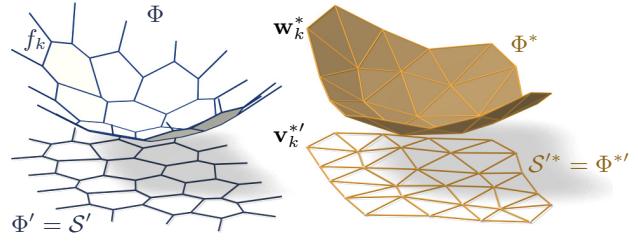
181 A mesh equipped with edge weights in this way is a discrete *thrust*  
 182 *network*. Invoking the safe theorem, we can state that a masonry  
 183 structure is self-supporting, if we can find a thrust network with  
 184 compressive forces which is entirely contained within the structure.

185 **Reciprocal Diagram.** Equations (2) have a geometric interpreta-  
 186 tion: with edge vectors

$$\mathbf{e}'_{ij} = \mathbf{v}'_j - \mathbf{v}'_i = (x_j, y_j) - (x_i, y_i),$$



**Figure 2:** A thrust network  $\mathcal{S}$  with dangling edges indicating external forces (left). This network together with compressive forces which balance vertical loads  $A_i F_i$  projects onto a planar mesh  $\mathcal{S}'$  with equilibrium compressive forces  $w_{ij} \mathbf{e}'_{ij}$  in its edges. Rotating forces by 90° leads to the reciprocal force diagram  $\mathcal{S}'^*$  (right).



**Figure 3:** Airy stress potential  $\Phi$  and its polar dual  $\Phi^*$ .  $\Phi$  projects onto the same planar mesh as  $\mathcal{S}$  does, while  $\Phi^*$  projects onto the reciprocal force diagram. A primal face  $f_k$  lies in the plane  $z = \alpha x + \beta y + \gamma \iff$  the corresponding dual vertex is  $\mathbf{w}_k^* = (\alpha, \beta, -\gamma)$ .

187 Equation (2) asserts that vectors  $w_{ij} \mathbf{e}'_{ij}$  form a closed cycle. Rotating them by 90 degrees, we see that likewise  
188

$$\mathbf{e}'_{ij}^* = w_{ij} J \mathbf{e}'_{ij}, \quad \text{with } J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

189 form a closed cycle (see Figure 2). If the mesh  $\mathcal{S}$  is simply connected,  
190 there exists an entire reciprocal diagram  $\mathcal{S}'^*$  which is a  
191 combinatorial dual of  $\mathcal{S}$ , and which has edge vectors  $\mathbf{e}'_{ij}^*$ . Its  
192 vertices are denoted by  $\mathbf{v}_i^*$ .

193 *Remark:* If  $\mathcal{S}'$  is a Delaunay triangulation, then the corresponding  
194 Voronoi diagram is an example of a reciprocal diagram.

195 **Polyhedral Stress Potential.** We can go further and construct a  
196 convex polyhedral “Airy stress potential” surface  $\Phi$  with vertices  
197  $\mathbf{w}_i = (x_i, y_i, \phi_i)$  combinatorially equivalent to  $\mathcal{S}$  by requiring that  
198 a primal face of  $\Phi$  lies in the plane  $z = \alpha x + \beta y + \gamma$  if and only if  
199  $(\alpha, \beta)$  is the corresponding dual vertex of  $\mathcal{S}'^*$  (see Figure 3). Ob-  
200 viously this condition determines  $\Phi$  up to vertical translation. For  
201 existence see [Ash et al. 1988]. The inverse procedure constructs  
202 a reciprocal diagram from  $\Phi$ . This procedure works also if forces  
203 are not compressive: we can construct an Airy mesh  $\Phi$  which has  
204 planar faces, but it will no longer be a convex polyhedron.

205 The vertices of  $\Phi$  can be interpolated by a piecewise-linear function  
206  $\phi(x, y)$ . It is easy to see that the derivative of  $\phi(x, y)$  jumps by the  
207 amount  $\|\mathbf{e}'_{ij}^*\| = w_{ij} \|\mathbf{e}'_{ij}\|$  when crossing over the edge  $\mathbf{e}'_{ij}$  at right  
208 angle, with unit speed. This identifies  $\Phi$  as the Airy polyhedron in-  
209 troduced by [Fraternali et al. 2002] as a finite element discretization  
210 of the continuous Airy function (see also [Fraternali 2010]).

211 If the mesh is not simply connected, the reciprocal diagram and  
212 the Airy polyhedron exist only locally. Our computations do not  
213 require global existence.

214 **Polarity.** Polarity with respect to the *Maxwell paraboloid*  $z =$   
215  $\frac{1}{2}(x^2 + y^2)$  maps the plane  $z = \alpha x + \beta y + \gamma$  to the point  $(\alpha, \beta, -\gamma)$ .  
216 Thus, applying polarity to  $\Phi$  and projecting the result  $\Phi^*$  into the  $xy$   
217 plane reconstructs the reciprocal diagram  $\Phi'^* = \mathcal{S}'^*$  (see Fig. 3).

218 **Discrete Stress Laplacian.** The weights  $w_{ij}$  may be used to de-  
219 fine a graph Laplacian  $\Delta_\phi$  which on vertex-based functions acts as

$$\Delta_\phi s(\mathbf{v}_i) = \sum_{j \sim i} w_{ij} (s_j - s_i).$$

220 This operator is a perfect discrete Laplacian in the sense of [War-  
221 detzky et al. 2007], since it is symmetric by construction, Equa-  
222 tion (2) implies linear precision for the planar “top view mesh”  $\mathcal{S}'$   
223 (i.e.,  $\Delta_\phi f = 0$  if  $f$  is a linear function), and  $w_{ij} \geq 0$  ensures

224 semidefiniteness and a maximum principle for  $\Delta_\phi$ -harmonic func-  
225 tions. Equation (3) can be written as  $\Delta_\phi s = AF$ .

226 Note that  $\Delta_\phi$  is well defined even when the underlying meshes are  
227 not simply connected.

### 2.3 Surfaces in Isotropic Geometry

229 It is worthwhile to reconsider the basics of self-supporting surfaces  
230 in the language of dual-isotropic geometry, which takes place in  $\mathbb{R}^3$   
231 with the  $z$  axis as a distinguished vertical direction. The basic ele-  
232 ments of this geometry are planes, having equation  $z = f(x, y) =$   
233  $\alpha x + \beta y + \gamma$ . The gradient vector  $\nabla f = (\alpha, \beta)$  determines the  
234 plane up to translation. A plane tangent to the graph of the function  
235  $s(x, y)$  has gradient vector  $\nabla s$ .

236 There is the notion of *parallel points*:  $(x, y, z) \parallel (x', y', z') \iff$   
237  $x = x', y = y'$ .

238 *Remark:* The Maxwell paraboloid is considered the unit sphere of  
239 isotropic geometry, and the geometric quantities considered above  
240 are assigned specific meanings: The forces  $\|\mathbf{e}'_{ij}\| = w_{ij} \|\mathbf{e}_{ij}\|$  are  
241 dihedral angles of the Airy polyhedron  $\Phi$ , and also “lengths” of  
242 edges of  $\Phi^*$ . We do not use this terminology in the sequel.

243 **Curvatures.** Generally speaking, in the differential geometry of  
244 surfaces one considers the *Gauss map*  $\sigma$  from a surface  $S$  to a con-  
245 vex unit sphere  $\Phi$  by requiring that corresponding points have par-  
246 allel tangent planes. Subsequently mean curvature  $H^{\text{rel}}$  and Gaus-  
247 sian curvature  $K^{\text{rel}}$  relative to  $\Phi$  are computed from the derivative  
248  $d\sigma$ . Classically  $\Phi$  is the ordinary unit sphere  $x^2 + y^2 + z^2 = 1$ , so  
249 that  $\sigma$  maps each point to its unit normal vector.

250 In our setting, parallelity is a property of *points* rather than planes,  
251 and the Gauss map  $\sigma$  goes the other way, mapping the tangent  
252 planes of the unit sphere  $z = \phi(x, y)$  to the corresponding tan-  
253 gent plane of the surface  $z = s(x, y)$ . If we know which point a  
254 plane is attached to, then it is determined by its gradient. So we  
255 simply write

$$\nabla \phi \xrightarrow{\sigma} \nabla s.$$

256 By moving along a curve  $\mathbf{u}(t) = (x(t), y(t))$  in the parameter  
257 domain we get the first variation of tangent planes:  $\frac{d}{dt} \nabla \phi|_{\mathbf{u}(t)} =$   
258  $(\nabla^2 \phi) \dot{\mathbf{u}}$ . This yields the derivative  $(\nabla^2 \phi) \dot{\mathbf{u}} \xrightarrow{d\sigma} (\nabla^2 s) \dot{\mathbf{u}}$ , for all  
259  $\dot{\mathbf{u}}$ , and the matrix of  $d\sigma$  is found as  $(\nabla^2 \phi)^{-1} (\nabla^2 s)$ . By definition,

260 curvatures of the surface  $s$  relative to  $\phi$  are found as

$$\begin{aligned} K_s^{\text{rel}} &= \det(d\sigma) = \frac{\det \nabla^2 s}{\det \nabla^2 \phi}, \\ H_s^{\text{rel}} &= \frac{1}{2} \text{tr}(d\sigma) = \frac{1}{2} \text{tr} \left( \frac{M}{\det \nabla^2 \phi} \nabla^2 s \right) = \frac{\Delta_\phi s}{2 \det \nabla^2 \phi}. \end{aligned}$$

261 The Maxwell paraboloid  $\phi_0(x, y) = \frac{1}{2}(x^2 + y^2)$  is the canonical  
262 unit sphere of isotropic geometry, with Hessian  $E_2$ . Curvatures relative  
263 to  $\phi_0$  are not called “relative” and are denoted by the symbols  
264  $H, K$  instead of  $H^{\text{rel}}, K^{\text{rel}}$ . The observation

$$\Delta_\phi \phi = \text{tr}(M \nabla^2 \phi) = \text{tr}(\widehat{\nabla^2 \phi} \nabla^2 \phi) = 2 \det \nabla^2 \phi$$

265 together with the formulas above implies

$$K_s = \det \nabla^2 s, \quad K_\phi = \det \nabla^2 \phi \implies H_s^{\text{rel}} = \frac{\Delta_\phi s}{2K_\phi} = \frac{\Delta_\phi s}{\Delta_\phi \phi}.$$

266 **Relation to Self-supporting Surfaces.** Summarizing the for-  
267 mulas above, we rewrite the balance condition (1) as

$$2K_\phi H_s^{\text{rel}} = \Delta_\phi s = F. \quad (4)$$

268 Let us draw some conclusions:

- Since  $H_\phi^{\text{rel}} = 1$  we see that the load  $F_\phi = 2K_\phi$  is admissible for the stress surface  $\phi(x, y)$ , which is hereby shown as self-supporting. The quotient of loads yields  $H_s^{\text{rel}} = F/F_\phi$ .
- If the stress surface coincides with the Maxwell paraboloid, then *constant loads characterize constant mean curvature surfaces*, because we get  $K_\phi = 1$  and  $H_s = F/2$ .
- If  $s_1, s_2$  have the same stress potential  $\phi$ , then  $H_{s_1-s_2}^{\text{rel}} = H_{s_1}^{\text{rel}} - H_{s_2}^{\text{rel}} = 0$ , so  $s_1 - s_2$  is a (relative) minimal surface.

## 2.4 Meshes in Isotropic Geometry

278 A general theory of curvatures of polyhedral surfaces with respect  
279 to a polyhedral unit sphere was proposed by [Pottmann et al. 2007;  
280 Bobenko et al. 2010], and its dual complement in isotropic geo-  
281 metry was elaborated on in [Pottmann and Liu 2007]. As illustrated by  
282 Figure 4, the mean curvature of a self-supporting surface  $S$  relative  
283 to its discrete Airy stress potential is associated with the vertices of  
284  $S$ . It is computed from areas and mixed areas of faces in the polar  
285 polyhedra  $S^*$  and  $\Phi^*$ :

$$H^{\text{rel}}(\mathbf{v}_i) = \frac{A_i(\mathcal{S}, \Phi)}{A_i(\Phi, \Phi)}, \quad \text{where}$$

$$A_i(\mathcal{S}, \Phi) = \frac{1}{4} \sum_{k: f_k \in 1\text{-ring}(\mathbf{v}_i)} \det(\mathbf{v}'_k, \mathbf{w}'_{k+1}) + \det(\mathbf{w}'_k, \mathbf{v}'_{k+1}).$$

286 The prime denotes the projection into the  $xy$  plane, and summation  
287 is over those dual vertices which are adjacent to  $\mathbf{v}_i$ . Replacing  $\mathbf{v}'_k$   
288 by  $\mathbf{w}'_k$  yields  $A_i(\Phi, \Phi) = \frac{1}{2} \sum \det(\mathbf{w}'_k, \mathbf{w}'_{k+1})$ .

289 **Proposition.** If  $\Phi$  is the Airy surface of a thrust network  $\mathcal{S}$ , then  
290 the mean curvature of  $\mathcal{S}$  relative to  $\Phi$  is computable as

$$H^{\text{rel}}(\mathbf{v}_i) = \frac{\sum_{j \sim i} w_{ij}(s_j - s_i)}{\sum_{j \sim i} w_{ij}(\phi_j - \phi_i)} = \frac{\Delta_\phi s}{\Delta_\phi \phi} \Big|_{\mathbf{v}_i}. \quad (5)$$

291 **Proof.** It is sufficient to show  $2A_i(\mathcal{S}, \Phi) = \sum_{j \sim i} w_{ij}(s_j - s_i)$ .

292 For that, consider edges  $\mathbf{e}'_1, \dots, \mathbf{e}'_n$  emanating from  $\mathbf{v}'_i$ . The dual  
293 cycles in  $\Phi^{**}$  and  $\mathcal{S}^{**}$  without loss of generality are given by ver-  
294 tices  $(\mathbf{v}'_1, \dots, \mathbf{v}'_n)$  and  $(\mathbf{w}'_1, \dots, \mathbf{w}'_n)$ , respectively. The latter  
295 has edges  $\mathbf{w}'_{j+1} - \mathbf{w}'_j = w_{ij} J \mathbf{e}'_j$  (indices modulo  $n$ ).

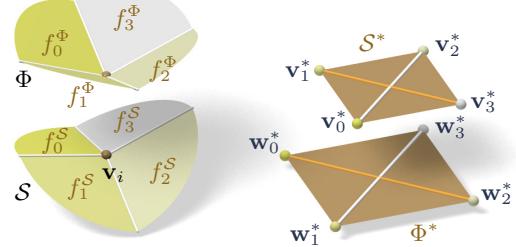


Figure 4: Mean curvature of a vertex  $\mathbf{v}_i$  of  $\mathcal{S}$ : Corresponding edges of the polar duals  $\mathcal{S}^*$ ,  $\Phi^*$  are parallel, and mean curvature according to [Pottmann et al. 2007] is computed from the vertices polar to faces adjacent to  $\mathbf{v}_i$ . For valence 4 vertices the case of zero mean curvature shown here is characterized by parallelity of non-corresponding diagonals of corresponding quads in  $\mathcal{S}^*$ ,  $\Phi^*$ .

296 Without loss of generality  $\mathbf{v}_i = 0$ , so the vertex  $\mathbf{v}'_j$  by construction  
297 equals the gradient of the linear function  $\mathbf{x} \mapsto \langle \mathbf{v}'_j, \mathbf{x} \rangle$  defined by  
298 the properties  $\mathbf{e}'_{j-1} \mapsto s_{j-1} - s_i$ ,  $\mathbf{e}'_j \mapsto s_j - s_i$ . Corresponding  
299 edge vectors  $\mathbf{v}'_{j+1} - \mathbf{v}'_j$  and  $\mathbf{w}'_{j+1} - \mathbf{w}'_j$  are parallel, because  
300  $\langle \mathbf{v}'_{j+1} - \mathbf{v}'_j, \mathbf{e}'_j \rangle = (s_j - s_i) - (s_j - s_i) = 0$ . Expand  $2A_i(\mathcal{S}, \Phi)$ :

$$\begin{aligned} &\frac{1}{2} \sum \det(\mathbf{w}'_j, \mathbf{v}'_{j+1}) + \det(\mathbf{v}'_j, \mathbf{w}'_{j+1}) \\ &= \frac{1}{2} \sum \det(\mathbf{w}'_j - \mathbf{w}'_{j+1}, \mathbf{v}'_{j+1}) + \det(\mathbf{v}'_j, \mathbf{w}'_{j+1} - \mathbf{w}'_j) \\ &= \frac{1}{2} \sum \det(-w_{ij} J \mathbf{e}'_j, \mathbf{v}'_{j+1}) + \det(\mathbf{v}'_j, w_{ij} J \mathbf{e}'_j) \\ &= \sum w_{ij} \langle \mathbf{v}'_j, \mathbf{e}'_j \rangle = \sum w_{ij} (s_j - s_i). \end{aligned}$$

Here we have used  $\det(\mathbf{a}, J\mathbf{b}) = \langle \mathbf{a}, \mathbf{b} \rangle$ .  $\square$

In order to discretize (4), we also need a discrete Gaussian curvature, usually defined as a quotient of areas which correspond under the Gauss mapping. We define

$$K_\Phi(\mathbf{v}_i) = \frac{A_i(\Phi, \Phi)}{A_i},$$

where  $A_i$  is the Voronoi area of vertex  $\mathbf{v}'_i$  in the projected mesh  $\mathcal{S}'$  used in (3).

*Remark:* If the faces of the thrust network  $\mathcal{S}$  are not planar, the simple trick of introducing additional edges with zero forces in them makes them planar, and the theory is applicable. In the interest of space, we refrain from elaborating further.

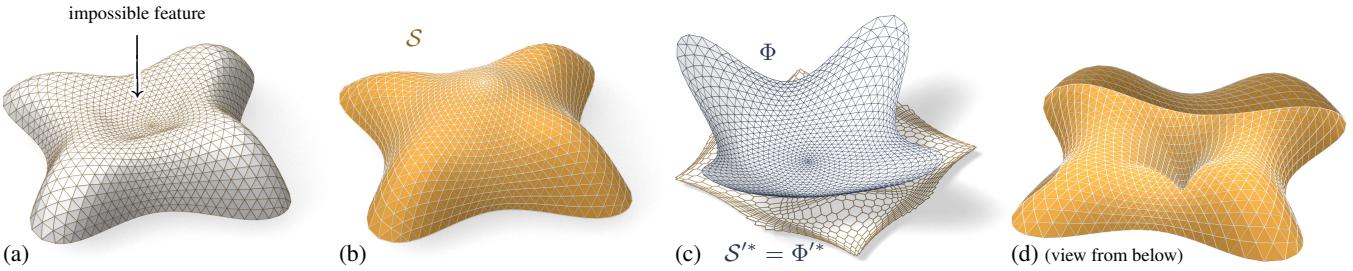
**Discrete Balance Equation.** The discrete version of the balance equation (4) reads as follows:

**Theorem.** A simply-connected mesh  $\mathcal{S}$  with vertices  $\mathbf{v}_i = (x_i, y_i, s_i)$  can be put into static equilibrium with vertical nodal forces  $A_i F_i$  if and only if there exists a combinatorially equivalent mesh  $\Phi$  with planar faces and vertices  $(x_i, y_i, \phi_i)$ , such that curvatures of  $\mathcal{S}$  relative to  $\Phi$  obey

$$2K_\Phi(\mathbf{v}_i) H^{\text{rel}}(\mathbf{v}_i) = F_i \quad (6)$$

at every interior vertex and every free boundary vertex  $\mathbf{v}_i$ .  $\mathcal{S}$  can be put into compressive static equilibrium if and only if there exists a convex such  $\Phi$ .

*Proof.* The relation between equilibrium forces  $w_{ij} \mathbf{e}_{ij}$  in  $\mathcal{S}$  and the polyhedral stress potential  $\Phi$  has been discussed above, and so has the equivalence “ $w_{ij} \geq 0 \iff \Phi$  convex” (see e.g. [Ash et al. 1988] for a survey of this and related results). It remains to show that Equations (2) and (6) are equivalent. This is



**Figure 5:** The top of the Lilium Tower (a) cannot stand as a masonry structure, because its central part is concave. Our algorithm finds a nearby self-supporting mesh (b) without this impossible feature. (c) shows the corresponding Airy mesh  $\Phi$  and reciprocal force diagram  $S'^*$ . (d) The user can edit the original surface, such as by specifying that the center of the surface is supported by a vertical pillar; and the self-supporting network adjusts accordingly.

326 the case because the proposition above implies  $2K(\mathbf{v}_i)H^{\text{rel}}(\mathbf{v}_i) =$   
 327  $2\frac{A_i(\Phi, \Phi)}{A_i}\frac{A_i(\Phi, \mathcal{S})}{A_i(\Phi, \Phi)} = \frac{1}{A_i}(\sum_{j \sim i} w_{ij}(s_j - s_i)) = \frac{1}{A_i}A_iF_i$ .  $\square$

328 **Existence of Discretizations.** When considering discrete thrust  
 329 networks as discretizations of continuous self-supporting surfaces,  
 330 the following question is important: For a given smooth surface  
 331  $s(x, y)$  with Airy stress function  $\phi$ , does there exist a polyhedral  
 332 surface  $\mathcal{S}$  in equilibrium approximating  $s(x, y)$ , whose top view  
 333 is a given planar mesh  $\mathcal{S}'$ ? We restrict our attention to triangle  
 334 meshes, where planarity of the faces of the discrete stress surface  $\Phi$   
 335 is not an issue. This question has several equivalent reformulations:

- Does  $\mathcal{S}'$  have a reciprocal diagram whose corresponding Airy polyhedron  $\Phi$  approximates the continuous Airy potential  $\phi$ ? (if the surfaces involved are not simply connected, these objects are defined locally).
- Does  $\mathcal{S}'$  possess a “perfect” discrete Laplace-Beltrami operator  $\Delta_\phi$  in the sense of Wardetzky et al. [2007] whose weights are the edge length scalars of such a reciprocal diagram?

343 From [Wardetzky et al. 2007] we know that perfect Laplacians ex-  
 344 ist only on regular triangulations which are projections of convex  
 345 polyhedra. On the other hand, previous sections show how to ap-  
 346 propriately re-triangulate: Let  $\Phi$  be a triangle mesh convex hull of  
 347 the vertices  $(x_i, y_i, \phi(x_i, y_i))$ , where  $(x_i, y_i)$  are vertices of  $\mathcal{S}'$ .  
 348 Then its polar dual  $\Phi^*$  projects onto a reciprocal diagram with pos-  
 349 itive edge weights, so  $\Delta_\phi$  has positive weights, and the vertices  
 350  $(x_i, y_i, s_i)$  of  $\mathcal{S}$  can be found by solving the discrete Poisson prob-  
 351 lem  $(\Delta_\phi s)_i = A_i F_i$ .

352 Assuming the discrete  $\Delta_\phi$  approximates its continuous counter-  
 353 part, this yields a mesh approximating  $s(x, y)$ , and we conclude: A  
 354 *smooth self-supporting surface can be approximated by a discrete*  
 355 *self-supporting triangular mesh for any sampling of the surface.*

### 3 Thrust Networks from Reference Meshes

357 Consider now the problem of taking a given reference mesh, say  
 358  $\mathcal{R}$ , and finding a combinatorially equivalent mesh  $\mathcal{S}$  in static equi-  
 359 librium approximating  $\mathcal{R}$ . The loads on  $\mathcal{S}$  include user-prescribed  
 360 loads as well as the dead load caused by the mesh’s own weight.  
 361 Conceptually, finding  $\mathcal{S}$  amounts to minimizing some formulation  
 362 of distance between  $\mathcal{R}$  and  $\mathcal{S}$ , subject to constraints (2), (3), and  
 363  $w_{ij} \geq 0$ . For any choice of distance this minimization will be a  
 364 nonlinear, non-convex, inequality-constrained variational problem.  
 365 Our experience with black-box solvers is that they perform very  
 366 well for surfaces without complex geometry or for polishing refer-  
 367 ence meshes close to self-supporting, but fail to converge in rea-  
 368 sonable time for more complicated shapes such as the swiss cheese  
 369 example (Fig. 1). We therefore propose the following specialized,

370 staggered linearization for solving the optimization problem:

0. Start with an initial guess  $\mathcal{S} = \mathcal{R}$ .
1. Estimate the self-load on the vertices of  $\mathcal{S}$ , using their current positions.
2. Fixing  $\mathcal{S}$ , locally fit an associated stress surface  $\Phi$ .
3. Alter positions  $\mathbf{v}_i$  to improve the fit.
4. Repeat from Step 1 until convergence.

377 This staggered approach shares the several advantages of solving  
 378 the full nonlinear problem: a nearby self-supporting surface is  
 379 found given only a suggested reference shape, without needing to  
 380 single one of the many possible top view reciprocal diagrams or  
 381 needing to specify boundary tractions – these are found automati-  
 382 cally during optimization. Although providing an initial top view  
 383 graph with good combinatorics remains important, by not fixing the  
 384 top view our approach allows the thrust network to slide both ver-  
 385 tically and tangentially to the ground, essential to finding faithful  
 386 thrust networks for surfaces with free boundary conditions.

387 **Step 1: Estimating Self-Load.** The dead load due to the sur-  
 388 face’s own weight depends not only on the top view of  $\mathcal{S}$ , but also  
 389 on the surface area of its faces. To avoid adding nonlinearity to  
 390 the algorithm, we estimate the load coefficients  $F_i$  at the beginning  
 391 of each iteration, and assume they remain constant until the next  
 392 iteration. We estimate the load  $A_i F_i$  associated with each vertex  
 393 by calculating its Voronoi surface area on each of its incident faces  
 394 (note that this surface area is distinct from  $A_i$ , the vertex’s Voronoi  
 395 area on the top view), and then multiplying by a user-specified sur-  
 396 face density  $\rho$ .

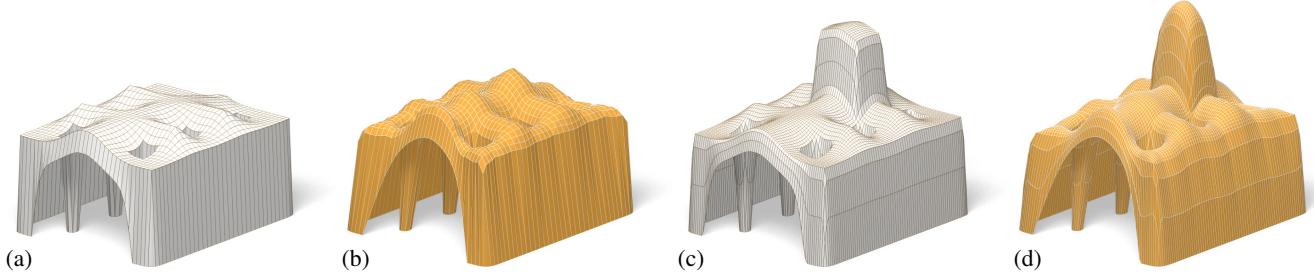
397 **Step 2: Fit a Stress Surface.** In this step, we fix  $\mathcal{S}$  and try to  
 398 fit a stress surface  $\Phi$  subordinate to the top view  $\mathcal{S}'$  of the primal  
 399 mesh. We do so by searching for dihedral angles between the faces  
 400 of  $\Phi$  which minimize, in the least-squares sense, the error in force  
 401 equilibrium (6) and local integrability of  $\Phi$ . Doing so is equivalent  
 402 to minimizing the squared residuals of Equations (3) and (2), re-  
 403 spectively, with the positions held fixed. We define the *equilibrium*  
 404 *energy*

$$E = \sum_i \left\| \begin{pmatrix} 0 \\ 0 \\ A_i F_i \end{pmatrix} - \sum_{j \sim i} w_{ij} (\mathbf{v}_j - \mathbf{v}_i) \right\|^2, \quad (7)$$

405 where the outer sum is over the interior and free boundary vertices,  
 406 and we solve

$$\min_{w_{ij}} E, \quad \text{s.t. } 0 \leq w_{ij} \leq w_{\max}. \quad (8)$$

407 Here  $w_{\max}$  is an optional maximum weight we are willing to assign  
 408 (to limit the amount of stress in the surface). This convex, sparse,



**Figure 6:** The user-designed reference mesh (a) is not self-supporting, but our algorithm finds a nearby perturbation of the reference surface (b) that is in equilibrium. As the user makes edits to the reference surface (c), the thrust network automatically adjusts (d).

Example	Figure	Vertices	Edges	Time (s)	Iterations	Max Rel Error
Top of Lilium Tower	Fig. 5b	1201	3504	21.6	9	$4.2 \times 10^{-5}$
Top of Lilium Tower (with pillar)	Fig. 5d	1200	3500	26.5	10	$8.5 \times 10^{-5}$
Freeform Structure with Two Pillars	Fig. 7	1535	2976	17.0	21	$2.7 \times 10^{-5}$
Swiss Cheese	Fig. 9	2358	4302	19.5	9	$3.0 \times 10^{-4}$
Brick Domes	Fig. 8	752	2165	8.0	9	$5.8 \times 10^{-5}$
Structural Glass	Fig. 16	527	998	5.7	25	$2.4 \times 10^{-5}$

**Table 1:** Numerical details about the examples throughout this paper. We show the -clock time needed by an Intel Xeon 2.3GHz desktop PC with 4 GB of RAM to find a self-supporting thrust network and associated stress surface from the example’s reference mesh; we also give the number of outer iterations of the four steps in (§3). The maximum relative error is the dimensionless relative error in force equilibrium defined by  $\max_i \|A_i F_i - \sum_{j \sim i} w_{ij}(\mathbf{v}_j - \mathbf{v}_i)\| / \|A_i F_i\|$ , where the maximum is taken over interior vertices  $\mathbf{v}_i$ .

409 box-constrained least-squares problem [Friedlander 2007] always  
 410 has a solution. If the objective is 0 at this solution, the faces of  $\Phi$   
 411 locally integrate to a stress surface satisfying (6), and this  $\Phi$  certifies  
 412 that  $\mathcal{S}$  is self-supporting – we are done. Otherwise,  $\mathcal{S}$  is not self-  
 413 supporting and its vertices must be moved.

439 **Implementation Details.** Solving the weighted least-squares  
 440 problem of Step 3 amounts to solving a sparse, symmetric linear  
 441 system. While the MINRES algorithm [Paige and Saunders 1975]  
 442 is likely the most robust algorithm for solving this system, in prac-  
 443 tice we have observed that the method of conjugate gradients works  
 444 well despite the potential ill-conditioning of the objective matrix.

445 **Step 3: Alter Positions.** In the previous step we fit as best as  
 446 possible a stress surface  $\Phi$  to  $\mathcal{S}$ . There are two possible kinds of  
 447 error with this fit: the faces around a vertex (equivalently, the recip-  
 448 ical diagram) might not close up; and the resulting stress forces  
 449 might not be exactly in equilibrium with the loads. These errors  
 450 can be decreased by modifying the top view and heights of  $\mathcal{S}$ , re-  
 451 spectively. It is possible to simply solve for new vertex positions  
 452 that put  $\mathcal{S}$  in static equilibrium, since Equations (2) and (3) with  
 453  $w_{ij}$  fixed form a square linear system that is typically nonsingular.  
 454

455 While this approach would yield a self-supporting  $\mathcal{S}$ , this mesh is  
 456 often far from the reference mesh  $\mathcal{R}$ , since any local errors in the  
 457 stress surface from Step 2 amplify into global errors in  $\mathcal{S}$ . We pro-  
 458 pose instead to look for new positions that decrease the imbalance  
 459 in the stresses and loads, while also penalizing drift away from the  
 460 reference mesh:  
 461

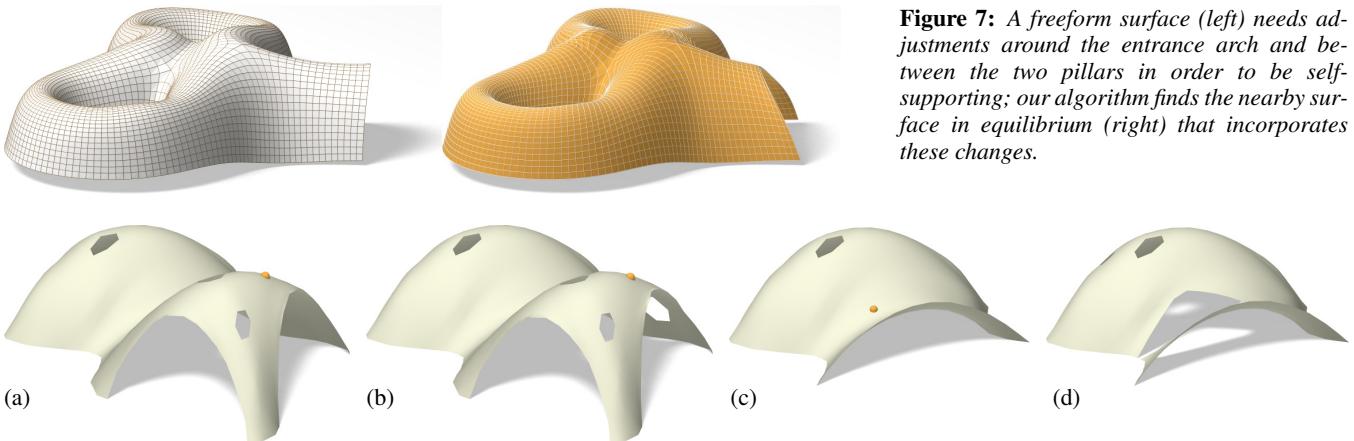
$$\min_{\mathbf{v}} E + \alpha \sum_i \langle \mathbf{n}_i, \mathbf{v}_i - \mathbf{v}_i^0 \rangle^2 + \beta \|\mathbf{v} - \mathbf{v}_P^0\|^2,$$

462 where  $\mathbf{v}_i^0$  is the position of the  $i$ -th vertex at the start of this step  
 463 of the optimization,  $\mathbf{n}_i$  is the starting vertex normal (computed as  
 464 the average of the incident face normals),  $\mathbf{v}_P^0$  is the projection of  $\mathbf{v}^0$   
 465 onto the reference mesh, and  $\alpha > \beta$  are penalty coefficients that are  
 466 decreased every iteration of Steps 1–3 of the algorithm. The second  
 467 term allows  $\mathcal{S}$  to slide over itself (if doing so improves equilibrium)  
 468 but penalizes drift in the normal direction. The third term, weaker  
 469 than the second, regularizes the optimization by preventing large  
 470 drift away from the reference surface or excessive tangential slid-  
 471 ing.  
 472

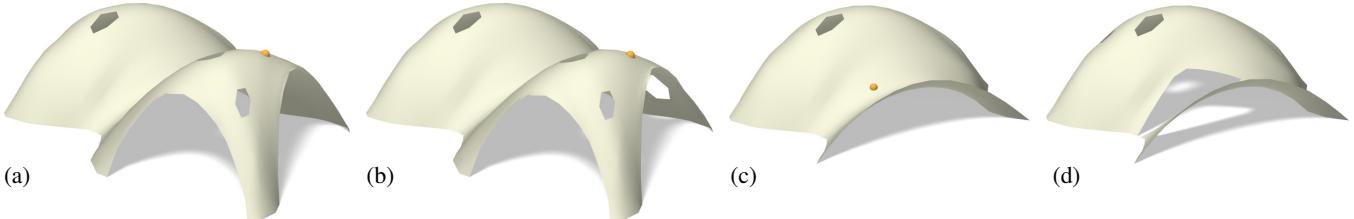
473 **Limitations.** This algorithm is not guaranteed to always con-  
 474 verge; this fact is not surprising from the physics of the problem  
 475 (if the boundary of the reference mesh encloses too large of a re-  
 476 gion,  $w_{\max}$  is set too low, and the density of the surface too high,  
 477 a thrust network in equilibrium simply does not exist – the vault is  
 478 too ambitious and cannot be built to stand; pillars are needed.)

479 We can, however, make a few remarks. Step 2 always decreases the  
 480 equilibrium energy  $E$  of Equation (7) and Step 3 does as well as  
 481  $\beta \rightarrow 0$ . Moreover, as  $\alpha \rightarrow 0$  and  $\beta \rightarrow 0$ , Step 3 approaches a lin-  
 482 ear system with as many equations as unknowns; if this system has  
 483 full rank, its solution sets  $E = 0$ . These facts suggest that the algo-  
 484 rithm should generally converge to a thrust network in equilibrium,  
 485 provided that Step 1 does not increase the loads by too much at ev-  
 486 ery iteration, and this is indeed what we observe in practice. One  
 487 case where this assumption is guaranteed to hold is if the thickness  
 488 of the surface is allowed to freely vary, so that it can be chosen so  
 489 that the surface has uniform density over the top view.  
 490

491 If the linear system in Step 3 is singular and infeasible, the algo-  
 492 rithm can stall at  $E > 0$ . This failure occurs, for instance, when  
 493 an interior vertex has height  $z_i$  lower than all of its neighbors, and  
 494 Step 2 assigns all incident edges to that vertex a weight of zero:  
 495 clearly no amount of moving the vertex or its neighbors can bring  
 496 the vertex into equilibrium. We avoid such degenerate configura-  
 497 tions by bounding weights slightly away from zero in (8), trading  
 498 increased robustness for slight smoothing of the resulting surface.  
 499 Attempting to optimize meshes that have self-intersecting top views  
 500 (i.e., aren’t height fields), have too many impossible features, or are  
 501 insufficiently supported by fixed boundary points can also result in  
 502



**Figure 7:** A freeform surface (left) needs adjustments around the entrance arch and between the two pillars in order to be self-supporting; our algorithm finds the nearby surface in equilibrium (right) that incorporates these changes.



**Figure 8:** Destruction sequence. We simulate removing small parts of masonry (their location is shown by a yellow ball) and the falling off of further pieces which are no longer supported after removal. For this example, removing a certain small number of single bricks does not affect stability (a,b). Removal of material at a certain point (yellow ball in (b)) will cause a greater part of the structure to collapse, as seen in (c). (d) shows the result after one more removal (all images show the respective thrust networks, not the reference surface).

473 errors and instability.

## 4 Results

475 **Interactive Design of Self-Supporting Surfaces.** The optimization algorithm described in the previous section forms the basis of an interactive design tool for self-supporting surfaces. Users 476 manipulate a mesh representing a reference surface, and the computer 477 searches for a nearby thrust network in equilibrium (see e.g. 478 Figure 6). Features of the design tool include:

- 481 • Handle-based 3D editing of the reference mesh using Laplacian 482 coordinates [Lipman et al. 2004; Sorkine et al. 2003] to 483 extrude vaults, insert pillars, and apply other deformations to 484 the reference mesh. Handle-based adjustments of the heights, 485 keeping the top view fixed, and deformation of the top view, 486 keeping the heights fixed, are also supported. The thrust 487 network adjusts interactively to fit the deformed positions, giving 488 the usual visual feedback about the effects of edits on whether 489 or not the surface can stand.
- 490 • Specification of boundary conditions. Points of contact 491 between the reference surface and the ground or environment 492 are specified by “pinning” vertices of the surface, specifying 493 that the thrust network must coincide with the reference mesh 494 at this point, and relaxing the condition that forces must be in 495 equilibrium there.
- 496 • Interactive adjustment of surface density  $\rho$ , external loads, 497 and maximum permissible stress per edge  $w_{\max}$ , with visual 498 feedback of how these parameters affect the fitted thrust 499 network.
- 500 • Upsampling of the thrust network through Catmull-Clark 501 subdivision and polishing of the resulting refined thrust network 502 using optimization (§3).
- 503 • Visualization of the stress surface dual to the thrust network 504 and corresponding reciprocal diagram.

505 **Examples.** *Vault with Pillars:* As an example of the design and 506 optimization workflow, consider a rectangular vault with six pillars, 507 free boundary conditions along one edge, fixed boundary conditions 508 along the others, and a tower extruded from the top of the surface 509 (see Figure 6). This surface is neither convex nor simply connected, 510 and exhibits a mix of boundary conditions, none of which cause 511 our algorithm any difficulty; it finds a self-supporting thrust 512 network near the designed reference mesh. The user is now free to

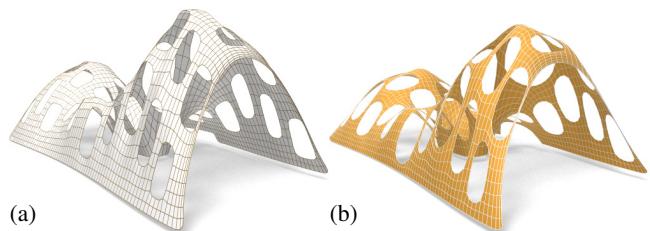
513 make edits to the reference mesh, and the thrust network adapts to 514 these edits, providing the user feedback on whether these designs 515 are physically realizable.

516 *Example: Top of the Lilium Tower.* Consider the top portion of the 517 steel-glass exterior surface of the Lilium Tower, which is currently 518 being built in Warszaw (see Figure 5). This surface contains a 519 concave part with local minimum in its interior and so cannot possibly 520 be self-supporting. Given this surface as a reference mesh, our 521 algorithm constructs a nearby thrust network in equilibrium without 522 the impossible feature. The user can then explore how editing the 523 reference mesh – adding a pillar, for example – affects the thrust 524 network and its deviation from the reference surface.

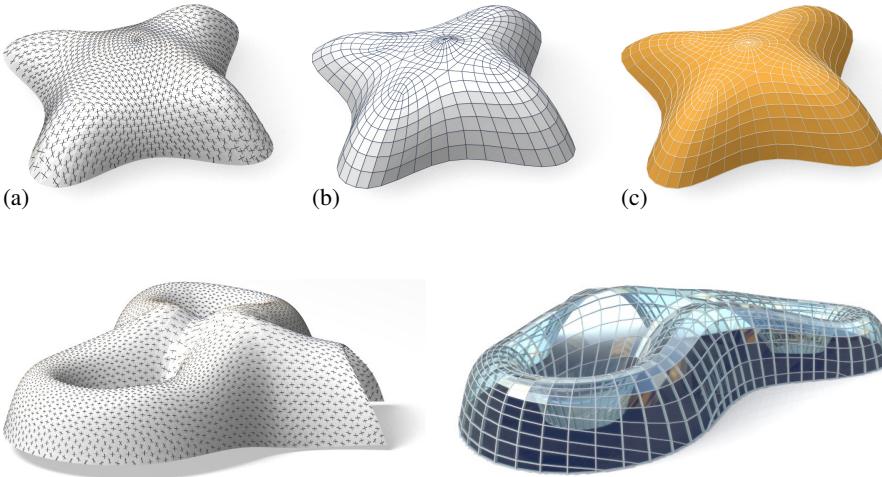
525 *Example: Freeform Structure with Two Pillars.* Suppose an architect’s 526 experience and intuition has permitted the design of a nearly 527 self-supporting freeform surface (Figure 7). Our algorithm reveals 528 those edits needed to make the structure sound – principally around 529 the entrance arch, and the area between the two pillars.

530 *Example: Destruction Sequence.* In Figure 8 we simulate removing 531 parts of masonry and the falling off of further pieces which are no 532 longer supported after removal. This is done by deleting the 1- 533 neighborhood of a vertex and solving for a new thrust network in 534 compressive equilibrium close to the original reference surface. We 535 delete those parts of the network which deviate too much and are 536 no longer contained in the masonry hull, and iterate.

537 *Example: Swiss Cheese.* Cutting holes in a self-supporting surface 538 interrupts force flow lines and causes dramatic global changes to 539 the surface stresses, often to the point that the surface is no longer 540 in equilibrium. Whether a given surface with many such holes can



**Figure 9:** A mesh with holes (a) requires large deformations to both the top view and heights to render it self-supporting (b)

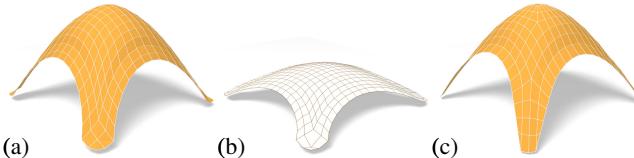


**Figure 11:** Planar quad remeshing of the surface of Figure 7. Left: Relative principal directions. Center: The result of optimization is a self-supporting PQ mesh, which guides a moment-free steel/glass construction. Right: Interior view.

stand is far from obvious. Figure 9 shows such an implausible and unstable surface; our optimization finds a nearby, equally implausible but stable surface without difficulty (see Figures 1 and 9).

## 5 Special Self-Supporting Surfaces

**PQ Meshes.** Meshes with *planar* faces are of particular interest in architecture, so in this section we discuss how to remesh a given thrust network in equilibrium such that it becomes a quad mesh with planar faces (again in equilibrium). If this mesh is realized as a steel-glass construction, it is self-supporting in its beams alone, with no forces exerted on the glass (this is the usual manner of using glass). The beams constitute a self-supporting structure which is in perfect force equilibrium (without moments in the nodes) if only the deadload is applied.



**Figure 12:** Directly enforcing planarity of the faces of even a very simple self-supporting quad-mesh vault (a) results in a surface far removed from the original design (b). Starting instead from a remeshing of the surface with edges following relative principal curvature directions yields a self-supporting, PQ mesh far more faithful to the original (c).

Taking an arbitrary non-planar quad mesh and attempting naive, simultaneous enforcement of planarity and static equilibrium – either by staggering a planarity optimization step every outer iteration, or adding a planarity penalty term to the position update – does not yield good results, as shown in Figure 12. Indeed, as we will see later in this section, such a planar perturbation of a thrust network is not expected to generally exist.

Consider a planar quad mesh  $\mathcal{S}$  with vertices  $\mathbf{v}_{ij} = (x_{ij}, y_{ij}, s_{ij})$  which approximates a given continuous surface  $s(x, y)$ . It is known that  $\mathcal{S}$  must approximately follow a network of conjugate curves in the surface (see e.g. [Liu et al. 2006]). We can derive this condition in an elementary way as follows: Using a Taylor expansion, we

**Figure 10:** Planar quad remeshing of the “Lilium tower” surface of Figure 5. (a) Principal directions which are found as eigenvectors of  $(\nabla^2\phi)^{-1}\nabla^2s$ . (b) Quad mesh guided by principal directions is almost planar and almost self-supporting. (c) Small changes achieve both properties.



compute the volume of the convex hull of the quadrilateral  $\mathbf{v}_{ij}$ ,  $\mathbf{v}_{i+1,j}$ ,  $\mathbf{v}_{i+1,j+1}$ ,  $\mathbf{v}_{i,j+1}$ , assuming the vertices lie exactly on the surface  $s(x, y)$ . This results in

$$\text{vol} = \frac{1}{6} \det(\mathbf{a}_1, \mathbf{a}_2) \cdot ((\mathbf{a}_1)^T \nabla^2 s \mathbf{a}_2) + \dots,$$

where  $\mathbf{a}_1 = \begin{pmatrix} x_{i+1,j} - x_{ij} \\ y_{i+1,j} - y_{ij} \end{pmatrix}$ ,  $\mathbf{a}_2 = \begin{pmatrix} x_{i,j+1} - x_{ij} \\ y_{i,j+1} - y_{ij} \end{pmatrix}$ ,

and the dots indicate higher order terms. We see that planarity requires  $(\mathbf{a}_1)^T \nabla^2 s \mathbf{a}_2 = 0$ . In addition to the mesh  $\mathcal{S}$  approximating the surface  $s(x, y)$ , the corresponding polyhedral Airy surface  $\Phi$  must approximate  $\phi(x, y)$ ; thus we get the conditions

$$(\mathbf{a}_1)^T \nabla^2 s \mathbf{a}_2 = (\mathbf{a}_1)^T \nabla^2 \phi \mathbf{a}_2 = 0.$$

$\mathbf{a}_1, \mathbf{a}_2$  are therefore eigenvectors of  $(\nabla^2\phi)^{-1}\nabla^2s$ . In view of §2.3,  $\mathbf{a}_1, \mathbf{a}_2$  indicate the principal directions of the surface  $s(x, y)$  relative to  $\phi(x, y)$ .

In the discrete case, where  $s, \phi$  are not given as continuous surfaces, but are represented by a mesh in equilibrium and its Airy mesh, we use the techniques of Schiftner [2007] and Cohen-Steiner and Morvan [2003] to approximate the Hessians  $\nabla^2s, \nabla^2\phi$ , compute principal directions as eigenvectors of  $(\nabla^2\phi)^{-1}\nabla^2s$ , and subsequently find meshes  $\mathcal{S}, \Phi$  approximating  $s, \phi$  which follow those directions. Global optimization can now polish  $\mathcal{S}, \Phi$  to a valid thrust network with discrete stress potential, where before it failed: we do so by taking the planarity energy  $\sum_f (2\pi - \theta_f)^2$ , where the sum runs over faces and  $\theta_f$  is the sum of the interior angles of face  $f$ , linearizing it at every iteration, and adding it to the objective function of the position update (Step 3). Convexity of  $\Phi$  ensures that  $\mathcal{S}$  is self-supporting.

Note that for each  $\Phi$ , the relative principal curvature directions give the *unique* curve network along which a planar quad discretization of a self-supporting surface is possible. Other networks lead to results like the one shown by Figure 12. Figures 10 and 11 further illustrate the result of applying this procedure to self-supporting surfaces.

*Remark:* When remeshing a given shape by planar quad meshes, we know that the circular and conical properties require that the mesh follows the ordinary, Euclidean principal curvature directions [Liu et al. 2006]. It is remarkable that the self-supporting property in a

similar manner requires us to follow certain *relative* principal directions. Practitioners' observations regarding the beneficial statics properties of principal directions can be explained by this analogy, because the relative principal directions are close to the Euclidean ones, if the stress distribution is uniform and  $\|\nabla s\|$  is small.

**Koenigs Meshes.** Given a self-supporting thrust network  $\mathcal{S}$  with stress surface  $\Phi$ , we ask the question: Which vertical perturbation  $\mathcal{S} + \mathcal{R}$  is self-supporting, with the same loads as  $\mathcal{S}$ ? As to notation, all involved meshes  $\mathcal{S}, \mathcal{R}, \Phi$  have the same top view, and arithmetic operations refer to the respective  $z$  coordinates  $s_i, r_i, \phi_i$  of vertices.

The condition of equal loads then is expressed as  $\Delta_\phi(s + r) = \Delta_\phi s$  in terms of Laplacians or as  $H_{\mathcal{S}}^{\text{rel}} = H_{\mathcal{S} + \mathcal{R}}^{\text{rel}}$  in terms of mean curvature, and is equivalent to

$$\Delta_\phi r = 0, \quad \text{i.e.,} \quad H_{\mathcal{R}}^{\text{rel}} = 0.$$

So  $\mathcal{R}$  is a *minimal surface* relative to  $\Phi$ . While in the triangle mesh case there are enough degrees of freedom for nontrivial solutions, the case of planar quad meshes is more intricate: Polar polyhedra  $\mathcal{R}^*$ ,  $\Phi^*$  have to be Christoffel duals of each other [Pottmann and Liu 2007], as illustrated by Figure 4. Unfortunately not all quad meshes have such a dual; the condition is that the mesh is *Koenigs*, i.e., the derived mesh formed by the intersection points of diagonals of faces again has planar faces [Bobenko and Suris 2008].



**Figure 13:** A “Koebe” mesh  $\Phi$  is self-supporting for unit dead load. An entire family of self-supporting meshes with the same top view is defined by  $\mathcal{S}_\alpha = \Phi + \alpha\mathcal{R}$ , where  $\mathcal{R}$  is chosen as  $\Phi$ 's Christoffel-dual.

**Koebe meshes.** An interesting special case occurs if  $\Phi$  is a Koebe mesh of isotropic geometry, i.e., a PQ mesh whose edges touch the Maxwell paraboloid. Since  $\Phi$  approximates the Maxwell paraboloid, we get  $2K(\mathbf{v}_i)H^{\text{rel}}(\mathbf{v}_i) \approx 1$  and  $\Phi$  consequently is self-supporting for unit load. Applying the Christoffel dual construction described above yields a minimal mesh  $\mathcal{R}$  and a family of meshes  $\Phi + \alpha\mathcal{R}$  which are self-supporting for unit load (see Figure 13).

## 6 Conclusion and Future Work

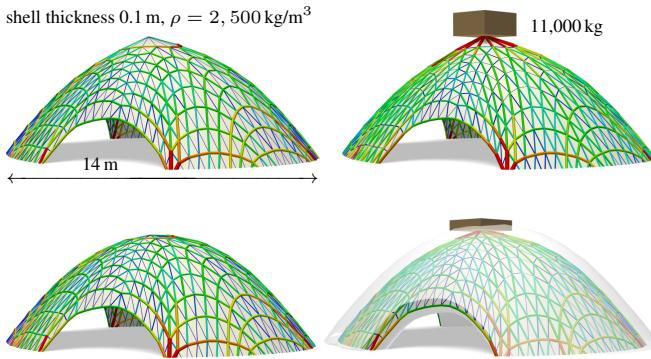
**Conclusion.** This paper builds on relations between statics and geometry, some of which have been known for a long time, and connects them with newer methods of discrete differential geometry, such as discrete Laplace operators and curvatures of polyhedral surfaces. We were able to find efficient ways of modeling self-supporting freeform shapes, and provide architects and engineers with an interactive tool for evaluating the statics of freeform geometries. The self-supporting property of a shape is directly relevant for freeform masonry. The actual thrust networks we use for computation are relevant e.g. for steel constructions, where equilibrium of deadload forces implies absence of moments. This theory and accompanying algorithms thus constitute a new contribution to architectural geometry, connecting statics and geometric design.

**Future Work.** There are several directions of future research. One is to incorporate non-manifold meshes, which occur naturally when e.g. supporting walls are introduced. It is also obvious that non-vertical loads, e.g. wind load, play a role. There are also some directions to pursue in improving the algorithms, for instance adaptive remeshing in problem areas. Probably the interesting connections between statics properties and geometry are not yet exhausted, and we would like to propose the *geometrization* of problems as a strategy for their solution.

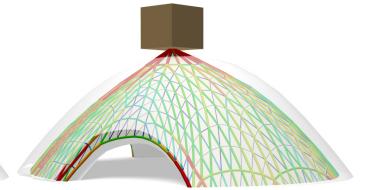
**Acknowledgements.** This work was very much inspired by Philippe Block's plenary lecture at the 2011 Symposium on Geometry Processing in Lausanne. Several illustrations (the maximum load example of Figure 14 and the destruction sequence of Figure 8) have real-world analogues on his web page [Block 2011].

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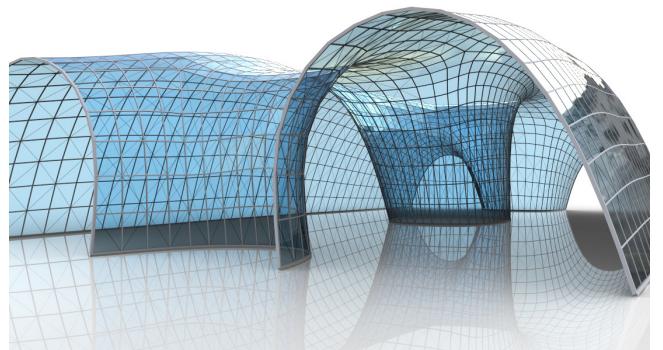


**Figure 14:** Stability Test. Left: Coloring and cross-section of edges visualize the magnitude of forces in a thrust network which is in equilibrium with this dome’s dead load. Right: When an additional load is applied, there exists a corresponding compressive thrust network which is still contained in the masonry hull of the original dome. This implies stability of the dome under that load.



**Figure 15:** Stability test similar to Figure 14, but with a shell thickness of 1 m, in order to better visualize the way the thrust network starts to leave the masonry hull as the load increases. Additional loads are 0 kg, 5,000 kg, 10,000 kg, and 20,000 kg, resp., from left to right.

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**Figure 16:** Glass as a structural element can support stresses up to, say, 30 MPa. We propose steel/glass constructions which utilize the structural properties of glass by first solving for a self-supporting thrust network such that forces do not exceed the maximum values, and subsequent remeshing of this surface by a planar quad mesh (not necessarily self-supporting itself). Since this surface is very close to a self-supporting shape, joints will experience low bending and torsion moments.