

# Mixed FEM stuff

## 1 Problem Setup

### 1.1 Continuous Problem

We begin with our continuous mixed energy. We write the energy as an integral over a reference domain  $\Omega \subset \mathbb{R}^d$ . For notation we denote vectors by bold lower case letters, matrices by upper-case letters, and scalars by lower case letters. Our mixed energy is a function of deformed coordinates,  $\mathbf{x} \in \mathbb{R}^d$ , symmetric deformation matrices  $S \in \mathbb{R}^{d \times d}$ , and Lagrange multipliers  $\Lambda \in \mathbb{R}^{d \times d}$ . Our continuous mixed variational is. **(Note: this is a funny semi-discretization - we've discretized in time so far – using Backwards Euler but not space - notice a lot of FE papers do the opposite to get an ODE first by discretizing in space first. No reason we need to start with this vs the other - right? Also we could propose the setup in the fully continuous setting first too - right?)**

$$\mathcal{L}(\mathbf{x}, S, \Lambda) = \int_{\Omega} \frac{\rho}{2h^2} \|\mathbf{x} - \tilde{\mathbf{x}}\|_2^2 + \Psi(S) - \Lambda : (R(\mathbf{x})S - F(\mathbf{x})) dV \quad (1)$$

where  $h$  is the timestep,  $\rho$  is the density, and  $\tilde{\mathbf{x}} = \mathbf{x}^t + h\dot{\mathbf{x}}^t + h^2\mathbf{f}_{\text{ext}}$ . In the mixed formulation, our strain energy density function  $\Psi$  is now a function of  $S$  rather than the deformed coordinates.  $R$  is the rotation matrix from the polar decomposition of the deformation gradient,  $F(\mathbf{x}) = R(\mathbf{x})S(\mathbf{x})$ . **Where specifically we have  $R(F(x))$  to indicate  $R$  is a projection of the deformation gradient onto  $\text{SO}(3)$ .**

### 1.2 Discretized Energy

We represent the domain by  $\mathcal{T} \equiv \{K_i\}_{i=1}^{n_e}$  consisting of a set of elements  $K_1, \dots, K_{n_e}$ . Per-element variables are represented by a subscript  $K$ . Our deformed coordinates are discretized on a nodal mesh, such as triangles or tetrahedra, and are stored in the vector  $\mathbf{x} \in \mathbb{R}^{dn}$  for  $n$  nodes. From this point on we'll assume we're working with linear tetrahedral elements in 3D, so we will have a single integration point per element and the integration weight just becomes the volume of the element,  $V_K$ . **( $V_K$  is just a scalar - right? If so can we make it lower case?)** Using an implicit euler time integrator, our discrete variational form for the timestep  $t + 1$  is

$$\mathbf{x}^{t+1}, \mathbf{s}^{t+1}, \lambda^{t+1} = \arg \min_{\mathbf{x}, \mathbf{s}, \lambda} \frac{1}{2h^2} \|\mathbf{x} - \tilde{\mathbf{x}}\|_M^2 + \sum_{K \in \mathcal{T}} (\Psi(\mathbf{s}_K) - \lambda_K \cdot (W_K \mathbf{s}_K - J_K \mathbf{x})) V_K. \quad (2)$$

There's several changes here compared to the continuous form so we explain them one-by-one. The first term, the kinetic energy term, is written now as a difference in the  $M$ -norm where  $M$  is the mass matrix for our mesh. Next we see that  $S$  is now a vector  $\mathbf{s} \in \mathbb{R}^{6n_e}$  where  $\mathbf{s}_K \in \mathbb{R}^6$  because there are only 6 degrees of freedom for a  $3 \times 3$  symmetric matrix. Similarly our Lagrange multipliers are

now  $\lambda \in \mathbb{R}^{9n_e}$  where each element's Lagrange multipliers are flattened into a vector  $\lambda_K$ . Another new piece is the matrix  $W_K$  which equals

$$W_K s_K = \text{vec}(R_K S_K), \quad (3)$$

and so it is constructed from  $R_K$ .  $W_K$  is of size  $9 \times 6$  because it has the effect of performing a matrix multiplication between the rotation matrix and a symmetric matrix, then flattening the result. To simplify notation we just write  $W_K$  and  $R_K$  but to be more precise we would write  $W_K(R_K(\mathbf{x}))$  and  $R_K(F_K(\mathbf{x}))$  to indicate that the rotation is a function of the deformation gradient for the current positional configuration. **(More messy looking but I'd prefer to write those dependencies out explicitly in our energies to avoid bugs and misunderstanding downstream.)** Lastly we have rewritten the deformation gradient as

$$J_K \mathbf{x} = \text{vec}(F_K(\mathbf{x})) \quad (4)$$

where  $J_K \in \mathbb{R}^{9 \times dn}$  is a Jacobian matrix.

Next to make our energy more suitable for an iterative solver, we note that

$$\begin{aligned} \mathbf{x}^{k+1} &= \mathbf{x}^k + \delta \mathbf{x} \\ \mathbf{s}^{k+1} &= \mathbf{s}^k + \delta \mathbf{s} \\ \lambda^{k+1} &= \lambda^k + \delta \lambda, \end{aligned}$$

where  $k$  corresponds to the current iterate. With this we can instead solve for updates of our variables. Our discrete mixed energy becomes

$$\begin{aligned} E(\mathbf{x}^{k+1}, \mathbf{s}^{k+1}, \lambda^{k+1}) &= \frac{1}{2h^2} \|\mathbf{x}^k + \delta \mathbf{x} - \tilde{\mathbf{x}}\|_M^2 \\ &+ \sum_{K \in \mathcal{T}} \Psi(\mathbf{s}_K^k + \delta \mathbf{s}_K) V_K \\ &- \sum_{K \in \mathcal{T}} (\lambda_K^k + \delta \lambda_K) \cdot (W_K(\mathbf{s}_K^k + \delta \mathbf{s}_K) - J_K(\mathbf{x}^k + \delta \mathbf{x})) V_K. \end{aligned} \quad (5)$$

and  $W_K$  in the final term is a function of  $\mathbf{x}^k + \delta \mathbf{x}$ .

To simplify things for derivations we split up the three terms in this energy to get

$$E(\mathbf{x}, \mathbf{s}, \lambda) = E_M(\mathbf{x}) + E_\Psi(\mathbf{s}) - E_\Lambda(\mathbf{x}, \mathbf{s}, \lambda), \quad (6)$$

and to arrive at a Newton step we just need to linearize with respect to the  $\delta$  variables and construct our system of equations. The only hurdle to doing this is that  $E_\Psi$  is potentially high order, and  $E_\Lambda$  is nonlinear in  $\mathbf{x}$  from the rotations, so we need to form quadratic approximations of these energies. **Last two sentences are a bit contradictory I think? Newton process is no more/no less than just forming a quadratic approximation (not linear) via taylor expansion and then solving it. Probably good for us to confirm we're thinking of the same thing here before we get into the nitty gritty :). Also, for creating a Newton solver I'm not sure there's a need to create this bunch of extra  $\delta$  terms in the above and below? Probably makes it messier then we need it to be for sanity checking.**

## 2 Augmented Lagrangian

We rewrite our discrete Lagrangian as a saddle point problem and also introduce a quadratic penalty so that we have the augmented lagrangian:

$$\begin{aligned} \mathbf{x}^{t+1}, \mathbf{s}^{t+1}, \lambda^{t+1} = \arg \min_{\mathbf{x}, \mathbf{s}} \arg \max_{\lambda} & \frac{1}{2} \|\mathbf{x} - \tilde{\mathbf{x}}\|_M^2 + h^2 \sum_{K \in \mathcal{T}} \Psi(\mathbf{s}_K) V_K. \\ & + h^2 \left( \frac{\kappa}{2} \|W(\mathbf{x})\mathbf{s} - J\mathbf{x}\|_V^2 - \lambda^T V(W(\mathbf{x})\mathbf{s} - J\mathbf{x}) \right) \end{aligned} \quad (7)$$

where we have  $W(\mathbf{x}) \in R^{9|\mathcal{T}| \times 6|\mathcal{T}|}$  as the block diagonal matrix with the  $W_K$  matrices as each block, and  $V$  is the element weights. To solve this we separate out the primal and dual variables and solve separately. Also, to simplify things for derivations we split up the three terms in this energy:

$$E(\mathbf{x}, \mathbf{s}, \lambda) = E_M(\mathbf{x}) + E_\Psi(\mathbf{s}) + E_\kappa(\mathbf{x}, \mathbf{s}) - E_\Lambda(\mathbf{x}, \mathbf{s}, \lambda). \quad (8)$$

## 3 Energy Derivatives

In the previous section we split our discrete mixed energy into three different terms, and before we write our final system of equations for each Newton step, we need to sort out the necessary derivatives and energy approximations.

### 3.1 Kinetic Energy

Or kinetic energy term is

$$E_M(\mathbf{x}) = \frac{1}{2} \|\mathbf{x} - \tilde{\mathbf{x}}\|_M^2 \quad (9)$$

$$(10)$$

and its derivative with respect to  $\mathbf{x}$  is

$$\left. \frac{\partial E_M}{\partial \mathbf{x}} \right|_{\mathbf{x}^k} = M(\mathbf{x}^k - \tilde{\mathbf{x}}), \quad (11)$$

and its hessian is

$$\left. \frac{\partial^2 E_M}{\partial \mathbf{x}^2} \right|_{\mathbf{x}^k} = M \quad (12)$$

### 3.2 Deformation Energy

The deformation energy is typically quadratic or some higher order function of  $\mathbf{s}$ , so we take a quadratic approximation of the energy about  $\mathbf{s}^k$ , so that for  $\mathbf{s} = \mathbf{s}^k + \delta \mathbf{s}$  we have

$$E_\Psi(\mathbf{s}) \approx \tilde{E}_\Psi(\mathbf{s}) = h^2 \sum_{K \in \mathcal{T}} \left( \Psi(\mathbf{s}_K^k) + \delta \mathbf{s}_K^T \frac{\partial \Psi}{\partial \mathbf{s}_K} + \frac{1}{2} \delta \mathbf{s}_K^T \frac{\partial^2 \Psi}{\partial \mathbf{s}_K^2} \delta \mathbf{s}_K \right) V_K \quad (13)$$

where the derivatives are evaluated at  $\mathbf{s}^k$ .

### 3.3 Constraint Energy

First we have the original constraint energy

$$E_\Lambda(\mathbf{x}, \mathbf{s}, \lambda) = h^2 \sum_{K \in \mathcal{T}} (\lambda_K \cdot (W_K \mathbf{s}_K - J_K \mathbf{x})) V_K, \quad (14)$$

and now we need to derive its derivatives. The main difficulty in these derivatives is dealing with the derivative of the rotation term. We do this by defining the rotation in terms of the SVD of the deformation gradient,

$$R(\mathbf{x}) = U_K(\mathbf{x}) V_K(\mathbf{x})^T \quad \text{where } F_K(\mathbf{x}) = U_K \Sigma_K V_K^T, \quad (15)$$

and

$$\frac{\partial R_K}{\partial F_{ij}} = \frac{\partial U_K}{\partial F_{ij}} V_K^T + U_K \frac{\partial V_K^T}{\partial F_{ij}}. \quad (16)$$

We now temporarily switch to indicial notation, and make  $K$  a superscript. Our derivative is a tensor  $\frac{\partial R_{ij}}{\partial F_{kl}} \in \mathbb{R}^{3 \times 3 \times 3 \times 3}$ . Since our degrees of freedom are all written as vectors, we redefine this tensor by flattening the first two indices, and modifying the last two dimensions so that it has the effect of multiplying by a symmetric matrix and flattening the result. This gives us the tensor  $\hat{W}_{ijk}^K \in \mathbb{R}^{9 \times 9 \times 6}$ . So in the same way that we have

$$W_{ij}^K s_j^K = \text{vec}(R_{kl}^K S_{lm}^K), \quad (17)$$

we also have

$$\hat{W}_{ijk}^K s_k^K = \frac{\partial \text{vec}(R_{lm}^K S_{mn}^K)}{\partial F_{(kl)}}. \quad (18)$$

The constraint energy in indicial notation for a single element is

$$E_\Lambda^K = h^2 \lambda_j^K (W_{jk}^K \mathbf{s}_k^K - J_{ij}^K \mathbf{x}_i), \quad (19)$$

and the relevant derivatives are

$$\frac{\partial E_\Lambda^K}{\partial \mathbf{x}_i} = J_{ij} (\hat{W}_{jkl}^K \mathbf{s}_l^K - I_{jk}) \lambda_k^K \quad (20)$$

$$\frac{\partial E_\Lambda^K}{\partial \mathbf{s}_i^K} = W_{ij}^K \lambda_j^K \quad (21)$$

$$\frac{\partial E_\Lambda^K}{\partial \lambda_i^K} = W_{ij}^K \mathbf{s}_j^K - J_{ij}^K \mathbf{x}_j \quad (22)$$

$$\frac{\partial^2 E_\Lambda^K}{\partial \lambda_i^K \partial \mathbf{s}_j^K} = W_{ij}^K \quad (23)$$

$$\frac{\partial^2 E_\Lambda^K}{\partial \mathbf{x}_i \partial \lambda_k^K} = J_{ij} (\hat{W}_{jkl}^K \mathbf{s}_l^K - I_{jk}) \quad (24)$$

$$\frac{\partial^2 E_\Lambda^K}{\partial \mathbf{x}_i \partial \mathbf{s}_l^K} = J_{ij} \hat{W}_{jkl}^K \lambda_k^K \quad (25)$$

$$\frac{\partial^2 E_\Lambda^K}{\partial \mathbf{x}_i^2} = \text{i'll get around to it} \quad (26)$$

$$(27)$$

where  $I_{jk}$  is a  $9 \times 9$  identity matrix. We omitted the  $h^2$  factor from the above derivatives to make things cleaner too.

### 3.4 Penalty Energy Derivatives

The last energy term is the penalty energy. We start with the equation

$$E_\kappa(\mathbf{x}, \mathbf{s}) = h^2 \frac{\mu}{2} (W(\mathbf{x})\mathbf{s} - J\mathbf{x})^T V (W(\mathbf{x})\mathbf{s} - J\mathbf{x}), \quad (28)$$

and switch to indicial notation in the same way we did for the constraint energy so that we have

$$E_\kappa(\mathbf{x}_i, \mathbf{s}_j) = h^2 \frac{\mu}{2} (W_{jk}\mathbf{s}_j - J_{ik}\mathbf{x}_i) V_{kl} (W_{lm}\mathbf{s}_m - J_{ln}\mathbf{x}_n). \quad (29)$$

Its derivatives are (again we omit the  $h^2$  factor)

$$\frac{\partial E_\kappa}{\partial \mathbf{s}_j} = \mu W_{jk} V_{kl} (W_{lm}\mathbf{s}_m - J_{ln}\mathbf{x}_n) \quad (30)$$

$$\frac{\partial E_\kappa}{\partial \mathbf{x}_i} = \mu \left( J_{im} \hat{W}_{mjk} \mathbf{s}_j - J_{ik} \right) V_{kl} (W_{lm}\mathbf{s}_m - J_{ln}\mathbf{x}_n) \quad (31)$$

$$\frac{\partial^2 E_\kappa}{\partial \mathbf{s}_j^2} = \mu W_{jk} V_{kk} W_{jk} \quad (32)$$

$$\frac{\partial^2 E_\kappa}{\partial \mathbf{x}_i \partial \mathbf{s}_j} = \mu \left( J_{im} \hat{W}_{mjk} V_{kl} (W_{lm}\mathbf{s}_m - J_{ln}\mathbf{x}_n) + \left( J_{im} \hat{W}_{mjk} \mathbf{s}_j - J_{ik} \right) V_{kl} W_{lj} \right) \quad (33)$$

$$\frac{\partial^2 E_\kappa}{\partial \mathbf{x}_i^2} = \mu \left( \underbrace{J_{im} \frac{\partial \hat{W}_{mjk} \mathbf{s}_j}{\partial \mathbf{x}_i} V_{kl} (W_{lm}\mathbf{s}_m - J_{ln}\mathbf{x}_n)}_{\text{Ignoring for now}} + \left( J_{im} \hat{W}_{mjk} \mathbf{s}_j - J_{ik} \right) V_{kl} \left( J_{im} \hat{W}_{mjl} \mathbf{s}_j - J_{il} \right) \right). \quad (34)$$

Right now we currently ignore the second derivative of the rotation for  $\frac{\partial^2 E_\kappa}{\partial \mathbf{x}_i^2}$ .

### 3.5 Primal Energy

For the primal variables, we take the quadratic approximation of the full energy by only taking variations in  $\mathbf{x}$  and  $\mathbf{s}$ . Also we define  $c_K(\mathbf{x}, \mathbf{s}) = W_K \mathbf{s}_K - J_K \mathbf{x}$  where  $W_K$  is a function of  $\mathbf{x}$ . This means that  $E_\Lambda^K(\mathbf{x}, \mathbf{s}, \lambda) = \lambda_K \cdot c_K(\mathbf{x}, \mathbf{s})$ . We then have a quadratic approximation of the entire energy that takes the form

$$\begin{aligned} E(\mathbf{x}, \mathbf{s}) &\approx \tilde{E}(\mathbf{x}, \mathbf{s}, \lambda^k) = E_M(\mathbf{x}^k) + E_\Psi(\mathbf{s}^k) + E_\kappa(\mathbf{x}, \mathbf{s}) - E_\Lambda(\mathbf{x}^k, \mathbf{s}^k, \lambda^k) \\ &\quad + \delta \mathbf{x}^T \left( \frac{\partial E_M}{\partial \mathbf{x}} + \frac{\partial E_\kappa}{\partial \mathbf{x}} - \frac{\partial \mathbf{c}}{\partial \mathbf{x}} \lambda^k \right) \\ &\quad + \delta \mathbf{s}^T \left( \frac{\partial E_\Psi}{\partial \mathbf{s}} + \frac{\partial E_\kappa}{\partial \mathbf{s}} - \frac{\partial \mathbf{c}}{\partial \mathbf{s}} \lambda^k \right) \\ &\quad + \delta \mathbf{x}^T \left( \frac{\partial^2 E_\kappa}{\partial \mathbf{x} \partial \mathbf{s}} - \frac{\partial}{\partial \mathbf{x}} \left( \frac{\partial \mathbf{c}}{\partial \mathbf{s}} \lambda^k \right) \right) \delta \mathbf{s} \\ &\quad + \frac{1}{2} \delta \mathbf{s}^T \left( \frac{\partial^2 E_\kappa}{\partial \mathbf{s}^2} + \frac{\partial^2 E_\Psi}{\partial \mathbf{s}^2} \right) \delta \mathbf{s} \\ &\quad + \frac{1}{2} \delta \mathbf{x}^T \left( \frac{\partial^2 E_M}{\partial \mathbf{x}^2} + \frac{\partial^2 E_\kappa}{\partial \mathbf{x}^2} - \frac{\partial}{\partial \mathbf{x}} \left( \frac{\partial \mathbf{c}}{\partial \mathbf{x}} \lambda^k \right) \right) \delta \mathbf{x} \end{aligned}$$

Now to setup our system of equations, we differentiate our energy with respect to the  $\delta$ -variables and set equal to zero. We switch back to vector notation and make  $K$  a subscript to simplify the notation.

### 3.6 Equation for $\delta\mathbf{x}$

Last we differentiate  $\tilde{E}$  with respect to  $\delta\mathbf{x}$  (neglecting the  $\frac{\partial^2 \mathbf{c}_K}{\partial \mathbf{x}^2}$  term for now):

$$\begin{aligned}
\frac{\partial \tilde{E}}{\partial \delta \mathbf{x}} &= \frac{\partial E_M}{\partial \mathbf{x}} + \frac{\partial E_\kappa}{\partial \mathbf{x}} - \frac{\partial E_\Lambda}{\partial \mathbf{x}} + \left( \frac{\partial^2 E_\kappa}{\partial \mathbf{x} \partial \mathbf{s}} - \frac{\partial}{\partial \mathbf{x}} \left( \frac{\partial \mathbf{c}}{\partial \mathbf{s}} \lambda^k \right) \right) \delta \mathbf{s} + \left( \frac{\partial^2 E_M}{\partial \mathbf{x}^2} + \frac{\partial^2 E_\kappa}{\partial \mathbf{x}^2} \right) \delta \mathbf{x} \\
&= M(\mathbf{x}^k - \tilde{\mathbf{x}}) + h^2 \mu J^T (\hat{W}^k \cdot \mathbf{s}^k - I) V (W^k \mathbf{s}^k - J \mathbf{x}^k) - h^2 J^T (\hat{W}^k \cdot \mathbf{s}^k - I) V \lambda^k \\
&\quad + h^2 \mu \left( J^T \hat{W} \cdot (V (W^k \mathbf{s}^k - J \mathbf{x}^k)) + J^T (\hat{W}^k \cdot \mathbf{s}^k - I) V W^k \right) \delta \mathbf{s} \\
&\quad - h^2 \left( J^T V (\hat{W}^k \cdot \lambda^k) \right) \delta \mathbf{s} \\
&\quad + \left( M + h^2 \mu J^T (\hat{W}^k \cdot \mathbf{s}^k - I) V (\hat{W}^k \cdot \mathbf{s}^k - I)^T J \right) \delta \mathbf{x}.
\end{aligned} \tag{35}$$

To simplify things we define the variables

$$G = J^T (\hat{W}^k \cdot \mathbf{s}^k - I) \tag{36}$$

$$\mathbf{r} = (W^k \mathbf{s}^k - J \mathbf{x}^k), \tag{37}$$

and the equation the equation for  $\delta\mathbf{x}$  is now

$$\begin{aligned}
\frac{\partial \tilde{E}}{\partial \delta \mathbf{x}} &= \underbrace{M(\mathbf{x}^k - \tilde{\mathbf{x}}) + h^2 \mu G V (\mathbf{r} - \lambda^k)}_{\mathbf{g}_\mathbf{x}} \\
&\quad + \underbrace{h^2 \left( \mu \left( J^T V \hat{W} \cdot (V \mathbf{r}) + G V W^k \right) - J^T (\hat{W}^k \cdot \lambda^k) \right)}_{\tilde{J}^T} \delta \mathbf{s} \\
&\quad + \underbrace{\left( M + h^2 \mu G V G^T \right)}_{H_\mathbf{x}} \delta \mathbf{x}. \\
&= \mathbf{g}_\mathbf{x} + \tilde{J}^T \delta \mathbf{s} + H_\mathbf{x} \delta \mathbf{x}
\end{aligned} \tag{38}$$

### 3.7 Equation for $\delta\mathbf{s}$

We start by differentiating  $\tilde{E}$  with respect to  $\delta\mathbf{s}$ :

$$\begin{aligned}
\frac{\partial \tilde{E}}{\partial \delta \mathbf{s}} &= \frac{\partial E_\Psi}{\partial \mathbf{s}} + \frac{\partial E_\kappa}{\partial \mathbf{s}} - \frac{\partial E_\Lambda}{\partial \mathbf{s}} + \left( \frac{\partial^2 E_\kappa}{\partial \mathbf{x} \partial \mathbf{s}} - \frac{\partial}{\partial \mathbf{x}} \left( \frac{\partial \mathbf{c}}{\partial \mathbf{s}} \lambda^k \right) \right) \delta \mathbf{x} + \left( \frac{\partial^2 E_\Psi}{\partial \mathbf{s}^2} + \frac{\partial^2 E_\kappa}{\partial \mathbf{s}^2} \right) \delta \mathbf{s} \\
&= h^2 \left( \frac{\partial \Psi}{\partial \mathbf{s}} + \mu W^{kT} V (W^k \mathbf{s}^k - J \mathbf{x}^k) - W^{kT} V \lambda^k \right. \\
&\quad + \mu \left( J^T \hat{W} \cdot (V (W^k \mathbf{s}^k - J \mathbf{x}^k)) + J^T (\hat{W}^k \cdot \mathbf{s}^k - I) V W^k \right) \delta \mathbf{x} \\
&\quad - \left( J^T (\hat{W}^k \cdot \lambda^k) \right) \delta \mathbf{x} \\
&\quad \left. + \left( \frac{\partial^2 \Psi}{\partial \mathbf{s}^2} + \mu W^{kT} V W^k \right) \delta \mathbf{s} \right),
\end{aligned} \tag{39}$$

and now substituting in some variables:

$$\begin{aligned}
\frac{\partial \tilde{E}}{\partial \delta \mathbf{s}} &= h^2 \underbrace{\left( \frac{\partial \Psi}{\partial \mathbf{s}} + \mu W^{kT} V(\mathbf{r} - \lambda^k) \right)}_{\mathbf{g}_s} \\
&\quad + h^2 \underbrace{\left( \mu \left( J^T V \hat{W} \cdot (V \mathbf{r}) + G V W^k \right) - J^T (\hat{W}^k \cdot \lambda^k) \right)^T}_{\tilde{J}} \delta \mathbf{x} \\
&\quad + h^2 \underbrace{\left( \frac{\partial^2 \Psi}{\partial \mathbf{s}^2} + \mu W^{kT} V W^k \right)}_{H_s} \delta \mathbf{s} \\
&= \mathbf{g}_s + \tilde{J} \delta \mathbf{x} + H_s \delta \mathbf{s}.
\end{aligned} \tag{40}$$

### 3.8 Dual Energy

To get an update rule for  $\lambda$  we further modify our energy by adding a quadratic penalty  $\frac{1}{2} \|\lambda - \lambda^k\|^2$ . Holding  $\mathbf{x}$  and  $\mathbf{s}$  fixed, our energy in terms of only  $\lambda$  is

$$E(\lambda) = \lambda^T V \mathbf{c}(\mathbf{x}^k, \mathbf{s}^k) + \frac{1}{2\kappa} \|\lambda - \lambda^k\|_V^2 \tag{41}$$

For  $\lambda = \lambda^k + \delta \lambda$ , and differentiating with respect to  $\delta \lambda$  we have

$$\delta \lambda = -\kappa \mathbf{c}(\mathbf{x}^k, \mathbf{s}^k) \tag{42}$$

### 3.9 System of Equations

Using the equations for  $\delta \mathbf{s}$  and  $\delta \mathbf{x}$ , we can now form a system of equations:

$$\begin{pmatrix} H_{\mathbf{x}} & \tilde{J}^T \\ \tilde{J} & H_s \end{pmatrix} \begin{pmatrix} \delta \mathbf{x} \\ \delta \mathbf{s} \end{pmatrix} = \begin{pmatrix} -\mathbf{g}_{\mathbf{x}} \\ -\mathbf{g}_s \end{pmatrix}. \tag{43}$$