B. Integral Calculas

* GAMMA FUNCTION (Γ):

$$\Gamma n \int_0^{\infty} e^{-x} x^{n-1} dx, \ n>0$$

It is denoted by Γn and is defined as

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx, n>0$$

* Properties of Gamma Functions:

1) The Alternate form of gamma function is;

$$\Gamma n = 2 \int_0^{\infty} e^{-x^2} x^{2n-1} dx$$

2) $\Gamma 1 = 1$; $= \int_0^{\infty} e^{-x} x^0 dx$

$$= \int_0^{\infty} e^{-x} dx$$

$$= [-e^{-x}]_0^{\infty}$$

$$= -[e^{-\infty} - e^0]$$

$$= -[0-1]$$

$$= 1$$

3) $\Gamma 0 = \infty$

4) $\Gamma \frac{1}{2} = \sqrt{\pi}$

5) $\Gamma n+1 = n \Gamma n$

$$\therefore \Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$\therefore$ replace n by $n+1$

$$\therefore \Gamma(n+1) = \int_0^\infty e^{x} x^{(n+1)-1} dx.$$

$$= \int_0^\infty e^{-x} x^{n}. dx$$

$$= \int_0^\infty x^n e^{-x} dx$$

$\therefore$ By using U.V. Rule;

$$= (-x^n e^{-x})_0^\infty + \int x^n - e^{-x} dx$$

$$= (0-0) + n \int_0^\infty x^{n-1} e^{-x} dx$$

$$= n\Gamma n, \forall n$$

6) $\Gamma(n+1) = n!$ if n is +ve integer

$$= n(n-1)\Gamma(n-1)$$
$$= n(n-1)(n-2)\Gamma(n-2)$$
$$= n(n-1)(n-2)(n-3)\Gamma(n-3)$$
$$= n(n-1)(n-2)(n-3)(n-4) \dots 3\Gamma 2$$
$$= n!$$

7) For negative fraction; $n\Gamma n = \frac{\Gamma(n+1)}{n}$

Eg. $\Gamma\frac{3}{2} = \Gamma\left(\frac{1}{2}+1\right)$

$$= \frac{1}{2}\Gamma\frac{1}{2}$$

$$= \frac{1}{2}\sqrt{\pi}$$

(using $\Gamma(n+1) = n\Gamma(n)$)

Eg 2 $\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right)$

$= \frac{3}{2}\Gamma\left(\frac{3}{2}\right)$

$= \frac{3}{2}\Gamma\left(\frac{1}{2}\right)\pi$

$= \frac{3}{4}\sqrt{\pi}$

Eg 3 $\Gamma\left(\frac{-5}{3}\right) = \Gamma n = \frac{\Gamma(n+1)}{n}$

$= \frac{\Gamma\left(\frac{-5}{3}+1\right)}{\frac{-5}{3}} = \frac{\Gamma\left(\frac{-2}{3}\right)}{\frac{-5}{3}}$

$= -\frac{3}{5}\Gamma\left(\frac{-2}{3}\right) = -\frac{3}{5}\frac{\Gamma\left(\frac{2}{3}+1\right)}{\frac{-2}{3}}$

$= \boxed{\frac{3}{10}\Gamma\left(\frac{1}{3}\right)}$

* Transformation of Gamma function:

1) $\int_0^\infty e^{-ky} y^{n-1} dy = \frac{\Gamma n}{k^n}$

As we have;

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

Put $x = ky$

$dx = k\,dy$

$x = 0 \quad ; \quad y = 0$

$x = \infty \quad ; \quad y = \infty$

$$\Gamma n = \int_0^\infty e^{-ky} (ky)^{n-1} k\, dy$$

$$\therefore \Gamma n = \int_0^\infty e^{-ky} k^{n-1} y^{n-1} dy$$

$$\therefore \Gamma n = k^n \int_0^\infty e^{-ky} y^{n-1} dy$$

$$\int_0^\infty e^{-ky} y^{n-1} dy = \frac{\Gamma n}{k^n}$$

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} \, dx$$

$$\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1} \, dx$$

$\Gamma 1 = 1$, $\Gamma 0 = \infty$, $\Gamma \frac{1}{2} = \sqrt{\pi}$

$\Gamma(n+1) = n\Gamma n$, $\forall n$

$= n!$, if n is +ve integer.

$\Gamma n = \dfrac{\Gamma(n+1)}{n}$, if n is -ve

$$\int_0^\infty e^{-ky} y^{n-1} \, dy = \frac{\Gamma n}{k^n}$$

$$\int_0^\infty e^{-y^{1/n}} \, dy = n\Gamma n$$

$$\Gamma p \, \Gamma(1-p) = \frac{\pi}{\sin p\pi} \qquad 0 < p < 1$$

$$\Gamma\tfrac{1}{4}\,\Gamma\tfrac{3}{4} = \Gamma\tfrac{1}{4}\,\Gamma\left(1-\tfrac{1}{4}\right) = \frac{\pi}{\sin \pi/4} = \frac{\pi}{1/\sqrt{2}} = \sqrt{2}\,\pi$$

* Solve the following questions

1) $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$

Soln let

$$I = \int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$$

put $\sqrt{x} = t$ $\quad x = t^2$

$dx = 2t \cdot dt$

$x = 0$, $t = 0$

$x = \infty$, $t = \infty$

$$I = \int_0^\infty (t^2)^{1/4} e^{-t}\, 2t\, dt$$

$$= 2\int_0^\infty e^{-t}\, t^{1/2} \cdot t\, dt$$

$$= 2\int_0^\infty e^{-t}\, t^{3/2}\, dt$$

$$= 2\int_0^\infty e^{-t}\, t^{5/2-1}\, dt$$

$$= 2\,\Gamma\left(\frac{5}{2}\right)$$

$$= 2\,\Gamma\left(\frac{3}{2}+1\right)$$

2) $\int_0^\infty x^9 e^{-2x^2}\, dx$

Soln

$$I = \int_0^\infty x^9 e^{-2x^2}\, dx$$

put $2x^2 = t$ $\quad x^2 = \frac{t}{2}$

$x = \sqrt{\frac{t}{2}}$

$$dx = \frac{1}{\sqrt{2}} \frac{1}{2} t^{-1/2} dt$$

$x=0 \qquad t=0$

$x=\infty \qquad t=\infty$

$$I = \int_0^\infty \left(\sqrt{\frac{t}{2}}\right)^{9} e^{-t} \frac{1}{2\sqrt{2}} t^{-t/2} dt$$

$$= \frac{1}{2\sqrt{2}} \int_0^\infty \frac{(\sqrt{t})^9}{(\sqrt{2})^9} e^{-t} t^{-t/2} dt$$

$$= \frac{1}{2(\sqrt{2})^{10}} \int_0^\infty (\sqrt{t})^9 e^{-t} t^{-1/2} dt$$

$$= \frac{1}{2(\sqrt{2})^{10}} \int_0^\infty e^{-t} (t^{1/2})^9 t^{-1/2} dt$$

$$= \frac{1}{2^6} \int_0^\infty e^{-t} t^4 dt$$

$$= \frac{1}{2^6} \int_0^\infty e^{-t} t^{5-1} dt$$

$$= \frac{1}{2^6} \Gamma 5$$

$$= \frac{1}{26} 4!$$

$$= \frac{1}{64} (24)$$

$$= \frac{3}{8}$$

3) $\int_0^\infty e^{-x^4} dx$

Soln $I = \int_0^\infty e^{-x^4} dx$

put $x^4 = t$

$x = \sqrt[4]{t} = t^{1/4}$

$dx = \frac{1}{4} t^{-3/4} dt$

$x = 0 \quad t = 0$
$x = \infty \quad t = \infty$

$I = \int_0^\infty (e^{-t}) \frac{1}{4} t^{-3/4} dt$

$I = \frac{1}{4} \int_0^\infty e^{-t} t^{-3/4} dt$

$= \frac{1}{4} \int_0^\infty e^{-t} t^{\frac{1}{4}-1} dt$

$= \frac{1}{4} \Gamma \frac{1}{4}$

4) $\int_0^1 (x \log x) dx$

Soln Let

$I = \int_0^1 (x \log x)^4 dx$

$= \int_0^1 x^4 (\log x)^4 dx$

put $\log x = -t$

$x = e^{-t}$

$dx = -e^{-t} dt$

$x = 0 \qquad t = -\infty$

$x = 1 \qquad t = 0$

$$I = \int_{\infty}^{0} e^{(-t)^4} (-t)^4 (-e^{-t}) dt$$

$$= - \int_{\infty}^{0} e^{-4t} t^4 e^{-t} dt$$

$$= - \int_{\infty}^{0} e^{-5t} t^4 dt$$

$$= \int_{0}^{\infty} e^{-5t} t^4 dt$$

$$= \int_{0}^{\infty} e^{-5t} t^{5-1} dt$$

We have,

$$\int_{0}^{\infty} e^{-ky} y^{n-1} dy = \frac{\Gamma n}{k^n}$$

Here, $k = 5$, $n = 5$

$$\therefore \quad = \frac{\Gamma 5}{5^5}$$

$$= \frac{4!}{5^5}$$

$$= \frac{24}{5^5}$$

5) $\displaystyle\int_{0}^{1} \frac{x}{\sqrt{\log(\frac{1}{x})}} dx$

Soln

$$I = \int_{0}^{1} \frac{x}{\sqrt{\log(\frac{1}{x})}} dx$$

put $\log x = -t$ $\qquad \log\left(\frac{1}{x}\right) = t$

$x = +e^{-t}$

$dx = -e^{-t}\,dt$

$x = 0 \;,\; t = \infty$

$x = 1 \;,\; t = 0$

$$I = \int_{\infty}^{0} \frac{e^{t}}{\sqrt{t}} (e^{-t})\,dt$$

$$= -\int_{\infty}^{0} e^{-t}\, t^{-1/2}\, e^{-t}\,dt$$

$$= \int_{0}^{\infty} e^{-2t}\, t^{-1/2}\,dt$$

$k = 2, \qquad n = \frac{1}{2}$

$$\frac{\Gamma(1/2)}{2^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{2}}$$

$$= \sqrt{\frac{\pi}{2}}$$ //

Q) evaluate $\int_0^\infty \frac{dx}{3^{4x^2}}$

Sol^n: let

$$I = \int_0^\infty \frac{dx}{3^{4x^2}}$$

put $3^{4x^2} = e^t$

$$\log 3^{4x^2} = \log e^t$$

$$4x^2 \log 3 = t$$

$$x^2 = \frac{t}{4\log 3}$$

$$x = \frac{\sqrt{t}}{\sqrt{4\log 3}}$$

$$x = \frac{\sqrt{t}}{2\sqrt{\log 3}}$$

$$dx = \frac{1}{2\sqrt{\log 3}} \frac{1}{2} t^{1/2-1}$$

$$dx = \frac{1}{4\sqrt{\log 3}} t^{-1/2} dt$$

$x = 0$; $t = 0$

$x = \infty$; $t = \infty$

$$I = \int_0^\infty \frac{1}{e^t} \frac{1}{4\sqrt{\log 3}} t^{-1/2} dt$$

$$I = \int_0^\infty e^{-t} \frac{1}{4\sqrt{\log 3}} t^{-1/2} dt$$

$$I = \frac{1}{4\sqrt{\log 3}} \int_0^\infty e^{-t}\, t^{-1/2}\, dt$$

$$= \frac{1}{4\sqrt{\log 3}} \Gamma\frac{1}{2}$$

$$I = \frac{1}{4\sqrt{\log 3}} \sqrt{\pi}$$

2) $\int_0^\infty \frac{x^2}{3x^2}\, dx$

Sol^n Let $I = \int_0^\infty \frac{x^2}{3x^2}\, dx$

put $3x^2 = e^t$

$\cancel{x^2 = \frac{e^t}{3}}$ $\quad \log 3^{x^2} = \log e^t$

$x^2 \log 3 = t$

$$x^2 = \frac{t}{\log 3}$$

$$x = \frac{\sqrt{t}}{\sqrt{\log 3}}$$

$$dx = \frac{1}{\sqrt{\log 3}} \frac{1}{2} t^{-1/2} dt$$

$$dx = \frac{1}{2\sqrt{\log 3}} t^{-1/2} dt$$

$x = 0$; $t = 0$

$x = \infty$; $t = \infty$

$$I = \int_0^\infty \frac{1}{e^t} \frac{t}{\log 3} \frac{1}{2\sqrt{\log 3}} t^{-1/2} dt$$

$$I = \int_0^\infty e^{-t} \frac{1}{2(\log 3)^{3/2}} t^{1/2} dt$$

$$I = \frac{1}{2(\log 3)^{3/2}} \int_0^\infty e^{-t} t^{1/2} dt$$

$$I = \frac{1}{2(\log 3)^{3/2}} \int_0^\infty e^{-t} t^{3/2-1} dt$$

$$I = \frac{1}{2(\log 3)^{3/2}} \Gamma\frac{3}{2}$$

$$I = \frac{1}{2(\log 3)^{3/2}} \Gamma\left(\frac{1}{2}+1\right)$$

$$= \frac{1}{2(\log 3)^{3/2}} \frac{1}{2} \Gamma\frac{1}{2}$$

$$= \frac{1}{4(\log 3)^{3/2}} \sqrt{\pi} \; //$$

3) $\int_0^\infty \frac{x^a}{a^x} dx$

Sol^n let $I = \int_0^\infty \frac{x^a}{a^x} dx$

put $a^x = e^t$

$x \log a = t$

$x = \frac{t}{\log a}$

$$dx = \frac{1}{\log a} dt$$

$$I = \int_0^\infty \left(\frac{t}{\log a}\right)^a \frac{1}{e^t} \left(\frac{1}{\log a}\right) dt$$

$$I = \int_0^\infty \frac{(t)^a}{(\log a)^a} \frac{1}{\log a} \frac{1}{e^t} dt$$

$$I = \int_0^\infty e^{-t} (t)^a \frac{1}{(\log a)^{a+1}} dt$$

$$I = \frac{1}{(\log a)^{a+1}} \int_0^\infty e^{-t} t^a dt$$

$$I = \frac{1}{(\log a)^{a+1}} \int_0^\infty e^{-t} t^{(a+1)-1} dt$$

$$I = \frac{1}{(\log a)^{a+1}} \Gamma(a+1)$$

Q4) $\int_0^\infty \frac{x^3}{3^x} dx$

5) Show that $\frac{2^n \Gamma(n+1/2)}{\sqrt{\pi}} = 1 \cdot 3 \cdot 5 \ldots (2n-1)$

Soln Consider

$$\Gamma\left(n+\frac{1}{2}\right) = \left(n+\frac{1}{2}-1\right) \Gamma\left(n+\frac{1}{2}-1\right)$$

$$= \left(n-\frac{1}{2}\right) \Gamma\left(n-\frac{1}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\Gamma\left(n-\frac{5}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right) \cdots \frac{3}{2} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right) \cdots \frac{3}{2}\,\frac{1}{2}\sqrt{\pi}$$

$$= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\left(\frac{2n-5}{2}\right) \cdots \frac{3}{2}\,\frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)(2n-3)(2n-5)\cdots 3 \cdot 1 \cdot \sqrt{\pi}}{2^n}$$

$$\frac{2^n\,\Gamma\left(n+1/2\right)}{\sqrt{\pi}} = 1 \cdot 3 \cdot 5 \cdots (2n-5)(2n-3)(2n-1)$$

Q6) Evaluate $\int_0^\infty x^n e^{-x^m}\,dx$

Sol^n ~~put~~ let $I = \int_0^\infty x^n e^{-x^m}\,dx$

put

$$x^m = t$$

$$x = t^{1/m}$$

$$dx = \frac{1}{m} t^{1/m - 1}\,dt$$

$$dx = \frac{1}{m} t^{\frac{1-m}{m}}\,dt$$

$x=0 \quad ; \quad t=0$

$x=\infty \quad ; \quad t=\infty$

$$I=\int_0^\infty \left(t^{\frac{1}{m}}\right)^n \cdot e^{-t} \frac{1}{m} t^{\frac{1-m}{m}} \, dt .$$

$$I=\frac{1}{m}\int_0^\infty e^{-t} \, t^{\frac{n}{m}} \, t^{\frac{1-m}{m}} \, dt$$

$$I=\frac{1}{m}\int_0^\infty e^{-t} \, t^{\frac{n+1-m}{m}} \, dt$$

$$I=\frac{1}{m}\int_0^\infty e^{-t} \, t^{\frac{n+1}{m}-1} \, dt$$

$$I=\frac{1}{m}\Gamma\left(\frac{n+1}{m}\right) //$$