# A Modified Greenberg Speed-flow Traffic Model

Mostafa Ghandehari and Siamak Ardekani
Civil Engineering Department, Box 19308
University of Texas at Arlington, TX, 76019

**Abstract**

A modified Greenberg speed-flow model is proposed. We assume speed is a logarithmic function of free-flow speed, concentration and a minimum constant density. Optimization techniques are used to maximize the flow. Taylor's series approximation is utilized to find concentration which maximizes the flow. Data calibration is discussed to validate the model.



**Introduction**

Greenberg's model of traffic flow was given in [1]. A description of Greenberg's model can be found in many textbooks, for example [2,4]. In a simplified gas dynamic model, the velocity u and the density k satisfy $u-\int k^{-1} c d_k =$ Constant, where c is the speed of sound. Greenberg used this relation, with c equals to a constant for his speed–density relation in traffic flow. Alone, the relation is meaningful for an unsteady wave motion. However in traffic flow, the relation was used assuming steady-state conditions. For traffic flow, a relation of the form

$$u=c\ln\frac{k_j}{k} \tag{1}$$

was found to fit certain observational data, where $k_j$ is the jam density or the density at which the traffic comes to a complete stop. We introduce a slight variation of equation (1), namely

$$u=c\ln\frac{k_j+k_0}{k+k_0} \tag{2}$$

The motivation for introducing equation (2) is that even under very light traffic conditions, there are always some vehicles on the road, represented by a non-zero minimum density, $k_o$. Using this modified model, we use the fundamental traffic flow relation q = u k to obtain equation (3) below:

$$q=ck\ln\frac{k_j+k_0}{k+k_0} \tag{3}$$

Equation (3) can be used to derive and estimate of the maximum flow or capacity by solving $\frac{dq}{dk}$=0. We use the product rule to obtain

$$-\ln\frac{k+k_0}{k_j+k_0}-\frac{k}{k+k_0}=0 \tag{4}$$

To obtain an analytical solution to equation (4), let

$$x = \frac{k_j - k}{k_j + k_0} \tag{5}$$

Note that $|x| < 1$. This condition is important for convergence.

We use (5) to obtain

$$k = k_j - (k_j + k_0)x. \tag{6}$$

After substituting (6) into (4) we derive

$$\ln(1 - x) + \frac{k}{k + k_0} = 0. \tag{7}$$

Then,

$$(k + k_0)\ln(1 - x) + k = 0 \tag{8}$$

Substituting (6) into (8) yields

$$[k_j - (k_j + k_0)x + k_0]\ln(1 - x) + k_j - (k_j + k_0)x = 0 \tag{9}$$

and

$$(k_j + k_0)(1 - x)\ln(1 - x) + k_j + k_0 - k_0 - (k_j + k_0)x = 0 \tag{10}$$

Dividing equation (10) by $(k_j + k_0)$ results in

$$(1 - x)\ln(1 - x) + (1 - x) - \frac{k_0}{k_j + k_0} = 0 \tag{11}$$

Now, we use Taylor series expansion for $\ln(1 - x)$ valid for $|x| < 1$ to obtain

$$(1 - x)(-x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots) - x + \frac{k_j}{(k_j + k_0)} = 0 \tag{12}$$

The above simplifies into

$$-x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots + x^2 + \frac{x^3}{2} + \frac{x^4}{3} + \ldots - x + \frac{k_j}{(k_j + k_0)} = 0. \tag{13}$$

Then we obtain

$$-2x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{12}+...+\frac{k_j}{(k_j+k_0)}=0 \tag{14}$$

The above power series can be used to approximate concentration $k_m$ to maximize flow. For example, for a linear approximation $x=\frac{k_j}{2(k_j+k_0)}$.

Then

$$k_m=\frac{k_j}{2}=0.5k_j \text{ ,if } k_0=0. \tag{15}$$

For quadratic approximation

$$x^2-4x+\frac{2k_j}{k_j+k_0}=0$$

Then, $x=2\pm\sqrt{4-\frac{2k_j}{(k_j+k_0)}}$

We need to use minus sign because of the fact that $x<1$.

Thus, $x=2-\sqrt{4-\frac{2k_j}{(k_j+k_0)}}$

Then, we use (6) to derive

$$k_m=k_j-(k_j+k_0)\left[2-\sqrt{4-\frac{2k_j}{(k_j+k_0)}}\right]. \tag{16}$$

We use (2) and (4) to find speed u at capacity given by $u=\frac{ck_m}{k_m+k_0}$. (17)

Note that if $k_0=0$, equation (16) reduces to $k_m\approx 0.4k_j$ compared to Greenberg's model $k_m=\frac{k_j}{e}\approx 0.368k_j$. If we use a fifth degree polynomial approximation, the result is very close to Greenberg's formula for $k_m$. Now we use an inequality to obtain a bound for $k_m$. The fact that $|x|<1$ implies

$$0=-2x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{12}+...+\frac{k_j}{(k_j+k_0)}<-2x+x^2\sum_{n=1}^{\infty}\frac{1}{n(n+1)}+\frac{k_j}{(k_j+k_0)}$$

We use telescoping sum to obtain $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1$.

Hence,

$$x^2 - 2x + \frac{k_j}{(k_j + k_0)} > 0 \tag{18}$$

Now, we substitute (5) into (18) to obtain

$$\frac{(k_j - k)^2}{(k_j + k_0)^2} - 2\frac{k_j - k}{k_j + k_0} + \frac{k_j}{k_j + k_0} > 0. \tag{19}$$

Simplifying (19) yields

$$k_j^{\,2} - 2k_j k + k^2 - 2(k_j + k_0)(k_j - k) + k_j(k_j + k_0) > 0.$$

$$k^2 - k_j k_0 + 2kk_0 > 0.$$

$$k^2 + 2kk_0 - k_j k_0 > 0.$$

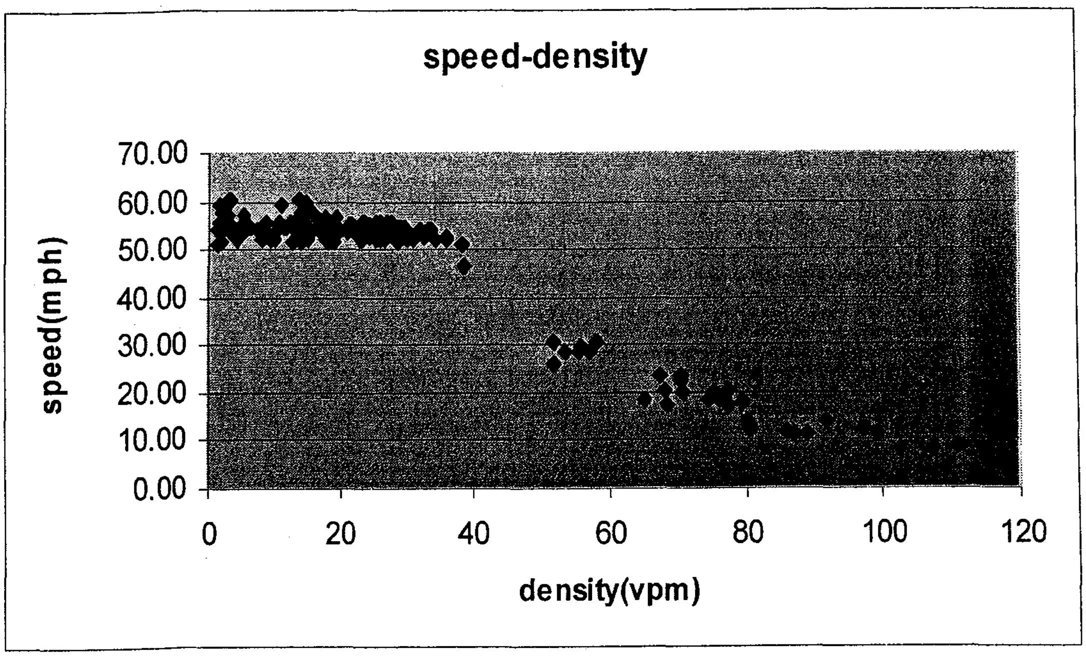
$$(k + k_0)^2 - k_0^{\,2} - k_j k_0 > 0.$$

$(k + k_0) \geq \sqrt{k_0 (k_0 + k_j)}$. Then

$$k_m \geq \sqrt{k_0^{\,2} + k_0 k_j} - k_0 \tag{20}$$

A regression model to calibrate the modified using data in [3] gives a correlation coefficient $R \approx 0.93$

The data was collected at the location of I-83 in Dallas area. A plot of spread sheet from density versus speed is given below.



[1] H. Greenberg, "An Analysis of Traffic Flow," Operation Research 7, 79 -85 (1959).

[2] D.C. Gazis, "Traffic Theory", Kluwer Academic, 2002.

[3] Ardekani et al, "Developing a Comprehensive Pricing Evaluation Model for Managed Lanes", Research Project No. 0-4818, University of Texas at Arlington, 2005.

[4] R. Haberman, "Mathematical Models; Mechanical Vibrations, Population Dynamics and Traffic Flow", Prentice-Hall, Englewood Cliffs, NJ, 1977.