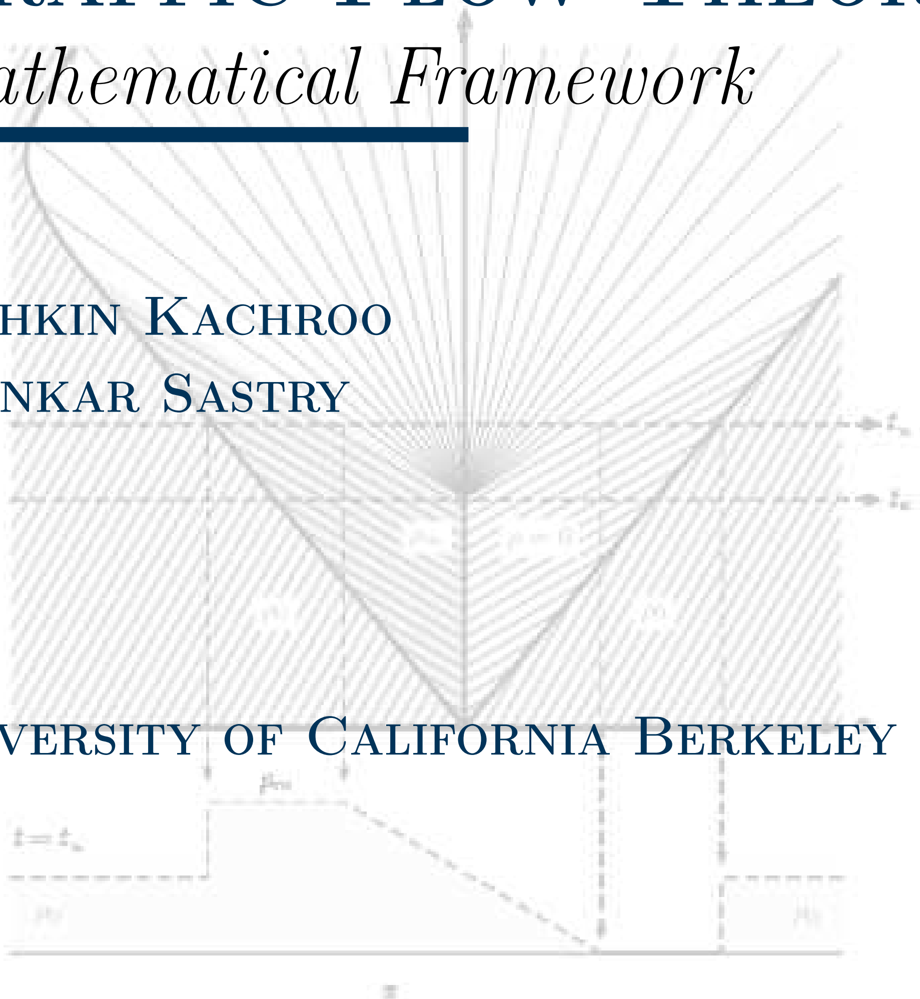
Traffic Flow Theory

Mathematical Framework



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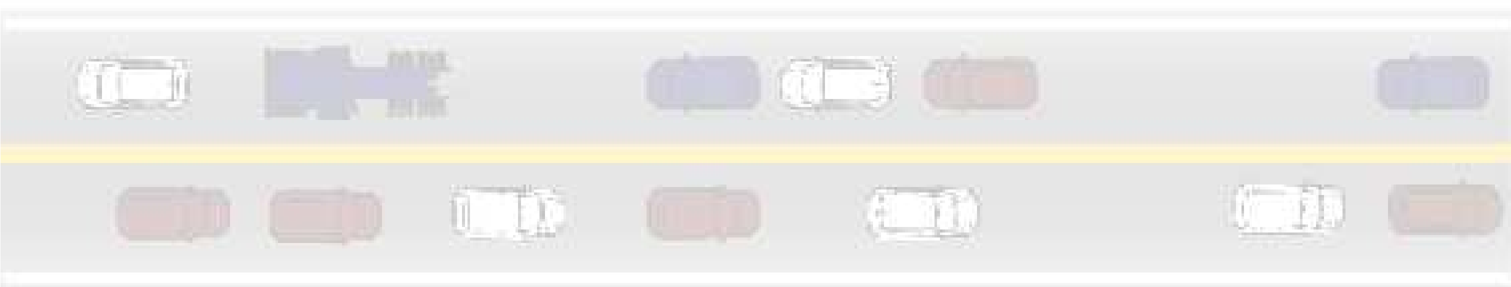

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DERIVATION OF CONSERVATION LAWS

In this chapter we review the derivation of the scalar and vector conservation laws. There are many references that give derivations for conservation laws, such as [Whi74], [CF99], [Lev92] and [Tor99].

2.1 Mass Conservation

Let us consider a section from distance x_1 to distance x_2 from some reference point on the x-axis (see Figure 2.1). Let this section contain a fluid with a scalar density field $\rho(t,x)$. Fluid enters this section from its left edge given by the flux (or flow) $q(x_1,t)$ and it leaves this section at its right edge at x_2 where the flux is given by $q(x_2,t)$. Flux is the product of density and speed of flow as shown in equation (2.1). For conservation of mass, the change in density in a section can happen only due to the fluxes at the boundary, which in this one dimensional case is at x_1 and x_2. Mathematically this statement can be written in integral or differential forms.

$$q(t,x) = \rho(t,x)\, v(t,x) \tag{2.1}$$

2.1.1 Mass Conservation in One Dimension

The mass in the section from $x = x_1$ to $x = x_2$ at time t is given by

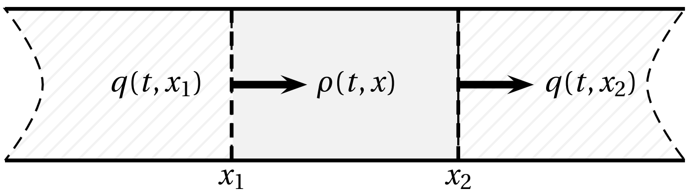

Figure 2.1: *Conservation of Mass*

$$\text{mass in } [x_1,x_2] \text{ at time t } = \int_{x_1}^{x_2} \rho(t,x)dx \tag{2.2}$$

The total mass that enters the section from the edge at $x = x_1$ is given by

$$\text{Inflow at } x_1 \text{ from time } t_1 \text{ to } t_2 = \int_{t_1}^{t_2} \rho(t,x_1)v(t,x_1)dt \tag{2.3}$$

Similarly, the total mass that leaves the section from the edge at $x = x_2$ is given by

$$\text{Outflow at } x_2 \text{ from time } t_1 \text{ to } t_2 = \int_{t_1}^{t_2} \rho(t,x_2)v(t,x_2)dt \tag{2.4}$$

The conservation law states that the change in mass in the section $[x_1,x_2]$ from time $[t_1,t_2]$ is equal to the mass that enters through the flux at x_1 from which the mass that exits through the flux at x_2 has been subtracted. This is stated below as the conservation law in the in the *first integral form.*

$$\int_{x_1}^{x_2} \rho(t_2,x)dx - \int_{x_1}^{x_2} \rho(t_1,x)dx = \int_{t_1}^{t_2} \rho(t,x_1)v(t,x_1)dt - \int_{t_1}^{t_2} \rho(t,x_2)v(t,x_2)dt \tag{2.5}$$

Alternately, this can also be written in the *second integral form* as:

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(t,x)dx = \rho(t,x_1)v(t,x_1) - \rho(t,x_2)v(t,x_2) \tag{2.6}$$

Equation (2.5) can be written as

$$\int_{x_1}^{x_2} \left[\rho(t_2,x) - \rho(t_1,x)\right] dx = \int_{t_1}^{t_2} \left[\rho(t,x_1)v(t,x_1) - \rho(t,x_2)v(t,x_2)\right] dx \tag{2.7}$$

If $\rho(t,x)$ and $v(t,x)$ are differentiable functions then we get

$$\rho(t_2,x)-\rho(t_1,x)=\int_{t_1}^{t_2}\frac{\partial}{\partial t}\rho(t,x)dt \tag{2.8}$$

and

$$\rho(t,x_2)v(t,x_2)-\rho(t,x_1)v(t,x_1)=\int_{x_1}^{x_2}\frac{\partial}{\partial x}(\rho(t,x)v(t,x))dx \tag{2.9}$$

Using equations (2.8) and (2.9) in (2.7) gives the following equation.

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2}\left\{\frac{\partial}{\partial t}\rho(t,x)+\frac{\partial}{\partial x}[\rho(t,x)v(t,x)]\right\}dtdx=0 \tag{2.10}$$

Since this must be satisfied for all intervals of time and x then it must be true that the following *differential form of the conservation law* is satisfied.

$$\frac{\partial}{\partial t}\rho(t,x)+\frac{\partial}{\partial x}[\rho(t,x)v(t,x)]=0 \tag{2.11}$$

In terms of the mass flux, this equation can be written as

$$\frac{\partial}{\partial t}\rho(t,x)+\frac{\partial}{\partial x}q(t,x)=0 \tag{2.12}$$

2.1.2 Mass Conservation in Two Dimensions

Consider the conservation law in two dimensions as shown in Figure 2.2. Here, the flow in the x-direction is q_1 and the flow in the y-direction is given by q_2. If $u(t,x,y)$ is the speed of the fluid in the x-direction at time (t,x,y), and $v(t,x,y)$ is the speed of the fluid in the y-direction at time (t,x,y), then we have the following two relationships for corresponding flows and speeds.

$$q_1(t,x)=\rho(t,x)u(t,x) \tag{2.13}$$

$$q_2(t,x)=\rho(t,x)v(t,x) \tag{2.14}$$

The development of conservation of mass in two dimensions follows along the same lines as the case of single dimension. The mass in the section from (x_1,y_1) to (x_2,y_2) at time t is given by

$$\text{mass in region}[(x_1,y_1)(x_2,y_2)]\text{ at time t }=\int_{x_1}^{x_2}\int_{y_1}^{y_2}\rho(t,x,y)dxdy \tag{2.15}$$

The total mass that enters the section from the edge at $x=x_1$ is given by

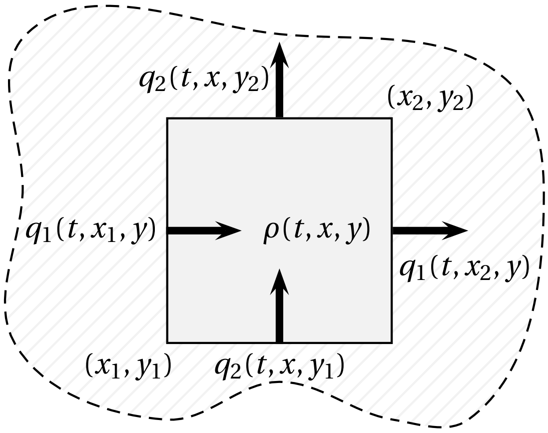

Figure 2.2: *Conservation of Mass in 2D*

$$\text{Inflow at } x_1 \text{ from time } t_1 \text{ to } t_2 = \int_{y_1}^{y_2}\int_{t_1}^{t_2} \rho(t,x_1,y)u(t,x_1,y)\,dt\,dy \tag{2.16}$$

Similarly, the total mass that leaves the section from the edge at $x = x_2$ is given by

$$\text{Outflow at } x_2 \text{ from time } t_1 \text{ to } t_2 = \int_{y_1}^{y_2}\int_{t_1}^{t_2} \rho(t,x_2,y)u(t,x_2,y)\,dt\,dy \tag{2.17}$$

The total mass that enters the section from the edge at $y = y_1$ is given by

$$\text{Inflow at } y_1 \text{ from time } t_1 \text{ to } t_2 = \int_{x_1}^{x_2}\int_{t_1}^{t_2} \rho(t,x,y_1)v(t,x,y_1)\,dt\,dx \tag{2.18}$$

Similarly, the total mass that leaves the section from the edge at $y = y_2$ is given by

$$\text{Outflow at } y_2 \text{ from time } t_1 \text{ to } t_2 = \int_{x_1}^{x_2}\int_{t_1}^{t_2} \rho(t,x,y_2)v(t,x,y_2)\,dt\,dx \tag{2.19}$$

The conservation law states that the change in mass in the section from time $[t_1,t_2]$ is equal to the exchange that takes place at the boundary of the section. This is stated below as the conservation law in the in the *first integral form* for two dimensions.

$$\begin{aligned}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\rho(t_2,x,y)dxdy-\int_{x_1}^{x_2}\int_{y_1}^{y_2}\rho(t_1,x,y)dxdy\\ =\int_{y_1}^{y_2}\int_{t_1}^{t_2}\rho(t,x_1,y)u(t,x_1,y)dydt+\int_{x_1}^{x_2}\int_{t_1}^{t_2}\rho(t,x,y_1)v(t,x,y_1)dxdt\\ -\int_{y_1}^{y_2}\int_{t_1}^{t_2}\rho(t,x_2,y)u(t,x_2,y)dydt-\int_{x_1}^{x_2}\int_{t_1}^{t_2}\rho(t,x,y_2)v(t,x,y_2)dxdt\end{aligned} \tag{2.20}$$

Alternately, this can also be written in the *second integral form* as:

$$\begin{aligned}\frac{d}{dt}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\rho(t_2,x,y)dxdy\\ =\int_{y_1}^{y_2}\rho(t,x_1,y)u(t,x_1,y)dy+\int_{x_1}^{x_2}\rho(t,x,y_1)v(t,x,y_1)dx\\ -\int_{y_1}^{y_2}\rho(t,x_2,y)u(t,x_2,y)dy-\int_{x_1}^{x_2}\rho(t,x,y_2)v(t,x,y_2)dx\end{aligned} \tag{2.21}$$

Equation (2.20) can be written as

$$\begin{aligned}\int_{x_1}^{x_2}\int_{y_1}^{y_2}[\rho(t_2,x,y)dxdy-\rho(t_1,x,y)]dxdy\\ =\int_{y_1}^{y_2}\int_{t_1}^{t_2}[\rho(t,x_1,y)u(t,x_1,y)-\rho(t,x_2,y)u(t,x_2,y)]dydt\\ +\int_{x_1}^{x_2}\int_{t_1}^{t_2}[\rho(t,x,y_2)v(t,x,y_2)-\rho(t,x,y_1)v(t,x,y_1)]dxdt\end{aligned} \tag{2.22}$$

If $\rho(t,x,y)$, $u(t,x,y)$ and $v(t,x,y)$ are differentiable functions then we get

$$\rho(t_2,x,y)-\rho(t_1,x,y)=\int_{t_1}^{t_2}\frac{\partial}{\partial t}\rho(t,x,y)dt \tag{2.23}$$

$$\rho(t,x_2,y)u(t,x_2,y)-\rho(t,x_1,y)u(t,x_1,y)=\int_{x_1}^{x_2}\frac{\partial}{\partial x}(\rho(t,x,y)u(t,x,y))dx \tag{2.24}$$

and

$$\rho(t,x,y_2)v(t,x,y_2)-\rho(t,x,y_1)v(t,x,y_1)=\int_{y_1}^{y_2}\frac{\partial}{\partial y}(\rho(t,x,y)v(t,x,y))dy \tag{2.25}$$

Using equations (2.23), (2.24) and (2.25) in (2.20) gives the following equation.

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2}\int_{t_1}^{t_2}\left\{\frac{\partial}{\partial t}\rho(t,x,y)+\frac{\partial}{\partial x}[\rho(t,x,y)u(t,x,y)]+\frac{\partial}{\partial y}[\rho(t,x,y)v(t,x,y)]\right\}dt\,dx\,dy=0 \tag{2.26}$$

Since this must be satisfied for all intervals of time, x and y then it must be true that the following *differential form of the conservation law* is satisfied.

$$\frac{\partial}{\partial t}\rho(t,x,y)+\frac{\partial}{\partial x}[\rho(t,x,y)u(t,x,y)]+\frac{\partial}{\partial y}[\rho(t,x,y)v(t,x,y)]=0 \tag{2.27}$$

or

$$\frac{\partial}{\partial t}\rho(t,x,y)+\nabla\cdot[\rho(t,x,y)v(t,x,y)]=0 \tag{2.28}$$

In terms of the mass flux, this equation can be written as

$$\frac{\partial}{\partial t}\rho(t,x,y)+\frac{\partial}{\partial x}q_1(t,x,y)+\frac{\partial}{\partial y}q_2(t,x,y)=0 \tag{2.29}$$

or

$$\frac{\partial}{\partial t}\rho(t,x,y)+\nabla\cdot q(t,x,y)=0 \tag{2.30}$$

2.1.3 Mass Conservation in n Dimensions

For the n-dimensional case, density is given by $\rho(t,x)$, velocity by $v(t,x)\in R^n$ and flux by $q(t,x)\in R^n$ where $x\in R^n$. The flux is given by

$$q(t,x)=\rho(t,x)v(t,x) \tag{2.31}$$

and the conservation law is given by

$$\frac{\partial}{\partial t}\rho(t,x)+\nabla\cdot q(t,x)=0 \tag{2.32}$$

2.2 Momentum Conservation

First we will study momentum conservation in one dimension, then followed by two dimensional and viscous cases.

2.2.1 Momentum Conservation in One Dimension

Let us consider a section in one dimension (see Figure 2.3). The momentum of the fluid in the section is given by the product of the density $\rho(t,x)$ and the velocity $v(t,x)$. Just as in the case of conservation of mass, the flux for momentum is given by the product of momentum and the velocity, i.e. $\rho(t,x)v^2(t,x)$. Now, according to Newton's second law (see [DHW04]), the change of momentum should be equal to the force applied. Force is equal to the product of pressure and area. Taking area to be of unit measurement in our problem, we get force to be $p(t,x_1)$ on the left edge, and $p(t,x_2)$ on the right.

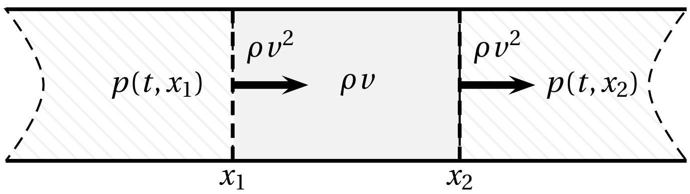

Figure 2.3: *Conservation of Momentum*

Applying Newton's law to the section, we obtain

$$\frac{\partial}{\partial t}[\rho(t,x)v(t,x)] + \frac{\partial}{\partial x}[\rho(t,x)v^2(t,x) + p(t,x)] = 0 \tag{2.33}$$

2.2.2 Momentum Conservation in Two Dimensions

There are two momentum fields in two dimensions. One is the momentum in the x direction (considered in 2.4) given by $\rho(t,x,y)u(t,x,y)$ and the other in the y direction given by $\rho(t,x,y)v(t,x,y)$, where $u(t,x,y)$ is the velocity in the x direction and $v(t,x,y)$ is the same in the y direction. We can derive the conservation of momentum in the x-direction as follows.

Momentum in the x-direction in the section is given by $\rho(t,x,y)u(t,x,y)$. The flux in the x direction is due to the velocity in x-direction given by $u(t,x,y)$ and is equal to the product of this velocity with the momentum. The flux is equal to $\rho(t,x,y)u^2(t,x,y)$. The flux in the y direction is due to the velocity in y-direction given by $v(t,x,y)$ and is equal to the product of this velocity with the momentum. The flux is equal to $\rho(t,x,y)u(t,x,y)v(t,x,y)$.

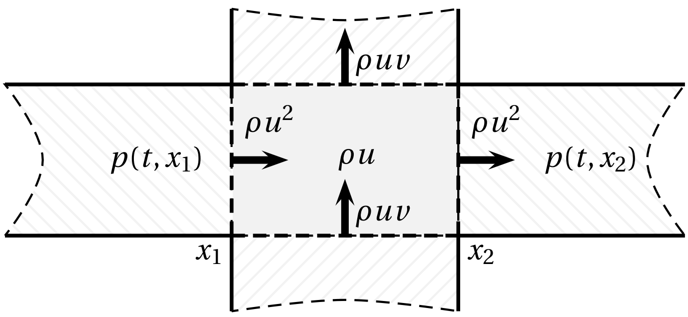

Figure 2.4: *Conservation of Momentum in the x-Direction*

According to Newton's law, total change in the linear momentum in the x-direction is equal to the force in the x-direction. The force comes from the pressure as in the one dimension case and we obtain

$$\frac{\partial}{\partial t}[\rho(t,x,y)u(t,x,y)]+\frac{\partial}{\partial x}[\rho(t,x,y)u^2(t,x,y)+p(t,x,y)]+\frac{\partial}{\partial y}\rho(t,x,y)u(t,x,y)v(t,x,y)=0 \tag{2.34}$$

Ignoring the dependencies on (t,x,y) we can write the momentum equation in the x- and y directions as follows.

$$\begin{aligned}\frac{\partial}{\partial t}[\rho u]+\frac{\partial}{\partial x}[\rho u^2+p]+\frac{\partial}{\partial y}\rho uv=0\\ \frac{\partial}{\partial t}[\rho v]+\frac{\partial}{\partial x}\rho uv+\frac{\partial}{\partial y}[\rho v^2+p]=0\end{aligned} \tag{2.35}$$

2.2.3 Momentum Equation with Viscosity

Let us study the two dimensional flow again where the fluid has shear and normal stresses including pressure (see Figure 2.5).

The total change in linear momentum in the x-directon is given by

$$\text{Change in Momentum in } x \text{ direction} = \frac{\partial}{\partial t}[\rho(t,x,y)u(t,x,y)]+\frac{\partial}{\partial x}\rho(t,x,y)u^2(t,x,y)$$

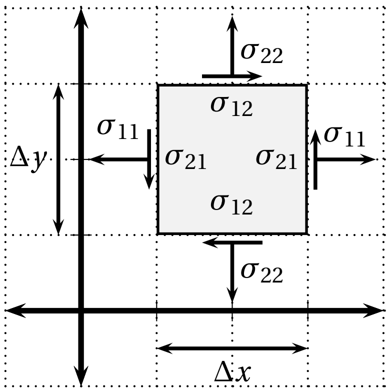

Figure 2.5: *Stresses on a Planar Fluid*

$$+\frac{\partial}{\partial y}\rho(t,x,y)u(t,x,y)v(t,x,y) \tag{2.36}$$

This should equal the force in x direction. The force is due to the normal and shear stresses in the same direction. The change in stress in x direction is

$$\text{Stress in } x \text{ direction} = \sigma_{11}(t,x+\Delta x,y)-\sigma_{11}(t,x,y)+$$

$$\sigma_{12}(t,x,y+\Delta y)-\sigma_{21}(t,x,y) \tag{2.37}$$

Taking appropriate limits as $\Delta x \to 0$, $\Delta y \to 0$ and matching with equation (2.36), we get

$$\begin{aligned}\frac{\partial}{\partial t}\rho u+\frac{\partial}{\partial x}\rho u^2+\frac{\partial}{\partial y}\rho uv &= \frac{\partial}{\partial x}\sigma_{11}+\frac{\partial}{\partial y}\sigma_{12}\\ \frac{\partial}{\partial t}\rho v+\frac{\partial}{\partial x}\rho uv+\frac{\partial}{\partial y}\rho v^2 &= \frac{\partial}{\partial x}\sigma_{21}+\frac{\partial}{\partial y}\sigma_{22}\end{aligned} \tag{2.38}$$

Now, pressure is the stress which is same in all directions. Hence, we can remove the pressure from the principle component of stresses as follows.

$$\begin{aligned}\sigma_{11} &= -p+\overline{\sigma_{11}}\\ \sigma_{22} &= -p+\overline{\sigma_{22}}\end{aligned} \tag{2.39}$$

Using (2.39) in (2.38), we get

$$
\begin{aligned}
\frac{\partial}{\partial t}\rho u + \frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\partial}{\partial y}\rho u v &= \frac{\partial}{\partial x}\overline{\sigma_{11}} + \frac{\partial}{\partial y}\sigma_{12} \\
\frac{\partial}{\partial t}\rho v + \frac{\partial}{\partial x}\rho u v + \frac{\partial}{\partial y}(\rho v^2 + p) &= \frac{\partial}{\partial x}\sigma_{21} + \frac{\partial}{\partial y}\overline{\sigma_{22}}
\end{aligned} \tag{2.40}
$$

Let us assume the following relationship between stress and strain

$$
\begin{aligned}
\overline{\sigma_{11}} &= \mu\frac{\partial}{\partial x}u \\
\sigma_{12} &= \mu\frac{\partial}{\partial y}u \\
\sigma_{21} &= \mu\frac{\partial}{\partial x}v \\
\overline{\sigma_{22}} &= \mu\frac{\partial}{\partial y}v
\end{aligned} \tag{2.41}
$$

In (2.41) we have taken μ to be the constant coefficient of viscosity. Now substituting (2.41) in (2.40), we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}\rho u + \frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\partial}{\partial y}\rho u v &= \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \\
\frac{\partial}{\partial t}\rho v + \frac{\partial}{\partial x}\rho u v + \frac{\partial}{\partial y}(\rho v^2 + p) &= \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2})
\end{aligned} \tag{2.42}
$$

Equation (2.42) can also be written as

$$
\frac{\partial}{\partial t}\begin{bmatrix} \rho u \\ \rho v \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho u^2 \\ \rho u v \end{bmatrix} + \frac{\partial}{\partial y}\begin{bmatrix} \rho u v \\ \rho v^2 \end{bmatrix} + \nabla p = \mu \Delta \begin{bmatrix} u \\ v \end{bmatrix} \tag{2.43}
$$

In equation (2.43), if we take μ to be zero, we obtain the non-viscous equation (2.35).

2.3 Energy Conservation

The derivation of energy conservation laws follows the same steps as the ones followed by conservation of mass and momentum. Energy flux in the x-direction is given by uE and in the y-direction by vE. Change of energy in unit time in a given direction is obtained from the power in that direction. Power is work done per unit time. Work is the inner-product (or dot product) of force and distance covered in that direction. Since force per unit area is pressure, power is given as a product of pressure and speed in the direction of interest. Hence, the conservation of energy in the two dimensional case is given as

$$\frac{\partial}{\partial t}E+\frac{\partial}{\partial x}[u(E+p)]+\frac{\partial}{\partial y}[v(E+p)]=0 \tag{2.44}$$

2.4 Combined Equations

Combining the equations (2.27), (2.35) and (2.44), we get the following equation.

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ E\end{bmatrix}+\frac{\partial}{\partial x}\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ u(E+p)\end{bmatrix}+\frac{\partial}{\partial y}\begin{bmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ v(E+p)\end{bmatrix}=0 \tag{2.45}$$

If we define the vector

$$U=\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ E\end{bmatrix} \tag{2.46}$$

and the corresponding vector flux in x-direction as

$$F_1=\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ u(E+p)\end{bmatrix} \tag{2.47}$$

in the y-direction as

$$F_2=\begin{bmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ v(E+p)\end{bmatrix} \tag{2.48}$$

then we can show the vector conservation law as

$$\frac{\partial U}{\partial t}+\nabla\cdot F=0 \tag{2.49}$$

F is the vector flux, whose component in the x-direction is F_1 and whose component in the y-direction is F_2.

2.4.1 Equation of State

Notice that in (2.45) there are four equations but five unknowns ($\rho, u, v, E,$ and p). Hence we need another equation for solvability of the system. For gases energy is the sum of kinetic energy and internal energy (e) as shown below.

$$E = \frac{1}{2}\rho(u_2 + v_2) + \rho e \tag{2.50}$$

The equation of state gives the formula for the internal energy in terms of pressure and density assuming chemical and thermodynamic equilibrium. The equation to be used depends on what type of gas it is.

Polytropic Gas

For an ideal gas with specific heat at constant volume given by c_v, the internal energy e is the following function of temperature.

$$e = c_v T \tag{2.51}$$

Temperature T is related to density ρ and pressure p by

$$p = R\rho T \tag{2.52}$$

where R is called the gas constant.

If a gas is kept at a constant volume as energy is added to it, the change in internal energy is given by

$$de = c_v dT \tag{2.53}$$

On the other hand if a gas is kept at a constant pressure as energy is added to it, some work is also done in increasing the volume. The change in internal energy is give by

$$d(e + \frac{p}{\rho}) = c_p dT \tag{2.54}$$

Enthalpy h is defined as

$$h = e + \frac{p}{\rho} \tag{2.55}$$

so that

$$h = c_p T \tag{2.56}$$

Using equations (2.56) and (2.51) in (2.52) gives

$$c_p - c_v = R \tag{2.57}$$

Using (2.51) and substituting T from (2.52), we get the following for the internal energy.

$$e = \frac{c_v P}{R\rho} \tag{2.58}$$

Using ratio of specific heats $\gamma = c_p / c_v$ and (2.57), we get

$$e = \frac{p}{(\gamma - 1)\rho} \tag{2.59}$$

Finally, substituting (2.59) into (2.50) gives the additional equation for the polytropic gas.

$$E = \frac{1}{2}\rho(u_2 + v_2) + \frac{p}{(\gamma - 1)} \tag{2.60}$$

Isothermal Flow

In the situation where the temperature of the gas is kept at a constant temperature T, energy is not conserved, and we can use the mass and momentum conservation equations only. Energy is not constant since external energy is required to keep the constant temperature. Since temperature is kept constant, because of equation (2.52) we obtain a linear relationship between pressure and density as

$$p = a^2 \rho \tag{2.61}$$

where $a = \sqrt{RT}$, T being the constant temperature. It can also be shown that a is the speed of sound (sound speed is given as the partial derivative of pressure with respect to density since sound travels as small disturbances in pressure). Using this, the system for isothermal flow for a two-dimensional flow becomes

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho u \\ \rho v \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho u \\ \rho u^2 + a^2 \rho \\ \rho u v \end{bmatrix} + \frac{\partial}{\partial y}\begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + a^2 \rho \end{bmatrix} = 0 \tag{2.62}$$

Isentropic Flow

Entropy (a measure of disorder in a system) is defined as

$$S = c_v \log(p/\rho^\gamma) + k \tag{2.63}$$

where k is a constant. Using equation (2.63) we can find an expression for pressure in terms of entropy and density as

$$p = \kappa \exp^{S/c_v} \rho^\gamma \tag{2.64}$$

where κ is a constant.

Clearly, if entropy is constant, the equation of state is given by

$$p = \overline{\kappa} \rho^\gamma \tag{2.65}$$

where

$$\overline{\kappa} = \kappa \exp^{S/c_v} \tag{2.66}$$

Hence using equation (2.65) in the two dimensional system becomes

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho u \\ \rho v \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho u \\ \rho u^2 + \overline{\kappa}\rho^\gamma \\ \rho u v \end{bmatrix} + \frac{\partial}{\partial y}\begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + \overline{\kappa}\rho^\gamma \end{bmatrix} = 0 \tag{2.67}$$

It can also be shown by using the definition of entropy in equation (2.63) in conservation laws in the differential form that in the regions of smooth flow entropy is conserved, i.e.

$$S(t,x,y)_t + u(t,x,y)S(t,x,y)_x + v(t,x,y)S(t,x,y)_y = 0 \tag{2.68}$$

2.5 General Conservation

This section derives the conservation law in more general setting as presented in [JGZ01]. This general setting is illustrated in Figure 2.6.

The conservation law in general setting is given by

$$\frac{\partial u}{\partial t} + \nabla \cdot f = 0 \tag{2.69}$$

Conside a cell of volume V that has a boundary S. The volume contains material $\int u dV$ that changes over time because of flux f that flows only through the boundary. In time Δt the boundary moves to a new location changing the volume from V to $(V + \Delta V)$. The material inside the volume $(V + \Delta V)$ is $(u + \Delta u)$. To obtain the total change in u we get

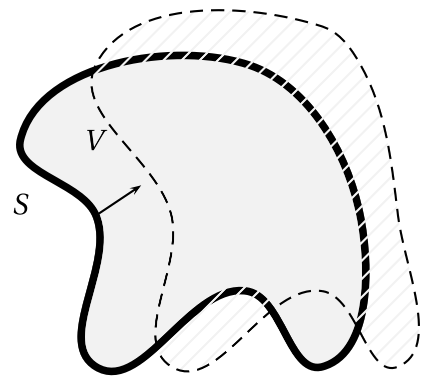

Figure 2.6: *Conservation in General Setting*

$$
\begin{aligned}
\int_{V+\Delta V}(u+\Delta u)dV - \int_V udV &= \int_V (u+\Delta u)dV + \int_{\Delta V}(u+\Delta u)dV - \int_V udV \\
&= \int_V udV + \int_V \Delta udV + \int_{\Delta V} udV + \int_{\Delta V} \Delta udV - \int_V udV \\
&= \int_V \Delta udV + \int_{\Delta V} udV + \int_{\Delta V} \Delta udV
\end{aligned} \tag{2.70}
$$

We neglect the last term on the right hand side since it involves second order differential terms. The second term shows the material that is in the differential volume. The new volume can be written in terms of the surface as $dV = S\Delta t$ and therefore, the second term can be written as

$$\int_{\Delta V} udV = \oint_S uv_n d(S\Delta t) \tag{2.71}$$

Here v_n is the outward normal component of the velocity at the surface. We can write equation (2.70) as

$$\int_{V+\Delta V}(u+\Delta u)dV - \int_V udV = \int_V \Delta udV + \oint_S uv_n d(S\Delta t) \tag{2.72}$$

Dividing both sides by Δt and taking $\Delta t \to 0$ we get

$$\frac{\partial}{\partial t}\int_V udV = \int_V \frac{\partial u}{\partial t}dV + \oint_S uv_n dS \tag{2.73}$$

Using the divergence theorem ([MT03]) and applying equation (2.69) in (2.73) we get the integral form of the conservation law for the moving boundary case as

$$\frac{\partial}{\partial t}\int_V u dV = \oint_S (f_n + u v_n) dS \tag{2.74}$$

Here f_n is the normal component of the flux. If the cell is stationary, then we get the following integral form of the conservation law.

$$\frac{\partial}{\partial t}\int_V u dV = \oint_S f_n dS \tag{2.75}$$

CHAPTER 3

TRAFFIC MODELS

In this chapter we review macroscopic traffic models and how they relate to conservation equations. We consider one-dimensional and two-dimensional vehicular and pedestrian traffic models. Traffic models can be microscopic (see [CHM58]), mesoscopic or macroscopic (see [Dag95], [May90]). Macroscopic models treat traffic as a continuum and these are the models of interest to this dissertation. Microscopic models treat each vehicle or pedestrian as an individual entity and treats acceleration as the control variable that depends on inter-vehicular or inter-pedestrian density (see [Ban95], [CHM58], [KO99b]). Mesoscopic models use kinetic models for traffic using Boltzmann equation from statistical mechanics (see [Pri71]).

3.1 Lighthill-Whitham-Richards Model

The LWR model, named after the authors in [LW55] and [Ric56], is a macroscopic one-dimensional traffic model. The conservation law for traffic in one dimension is given by

$$\frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x} f(t,x) = 0 \tag{3.1}$$

In this equation ρ is the traffic density (vehicles or pedestrians) and f is the flux which is the product of traffic density and the traffic speed v, i.e. $f = \rho v$. There are many models researchers have proposed for how the flux should be dependent on traffic conditions. This relationship is given by the *fundamental diagram.*

3.1.1 Greenshield's Model

Greenshield's model (see [Gre35]) uses a linear relationship between traffic density and traffic speed.

$$v(\rho) = v_f(1 - \frac{\rho}{\rho_m}) \tag{3.2}$$

where v_f is the free flow speed and ρ_m is the maximum density. Free flow speed is the speed of traffic when the density is zero. This is the maximum speed. The maximum density is the density at which there is a traffic jam and the speed is equal to zero. The flux function is concave as can be confirmed by noting the negative sign of the second derivative of flow with respect to density, i.e. $\partial^2 f/\partial\rho^2 < 0$. The fundamental diagram refers to the relationship that the traffic density ρ, traffic speed v and traffic flow f have with each other. These relationships are shown in Figure 5.9.

3.1.2 Greenberg Model

In this model (see [Gre59]) the speed-density function is given by

$$V(\rho) = v_f\, ln(\frac{\rho_m}{\rho}) \tag{3.3}$$

Greenberg fundamental diagram is shown in Figure 3.2.

3.1.3 Underwood Model

In the Underwood model (see [Und61]) the velocity-density function is represented by

$$V(\rho) = v_f \exp(\frac{-\rho}{\rho_m}) \tag{3.4}$$

Underwood fundamental diagram is shown in Figure 3.3.

3.1.4 Diffusion Model

Diffusion model is an extension of the Greenshield's model where the traffic speed depends not only on the traffic density but also on the density gradient. This models the driver behavior where changes in traffic density in the x-direction affect the traffic speed. The model is given by

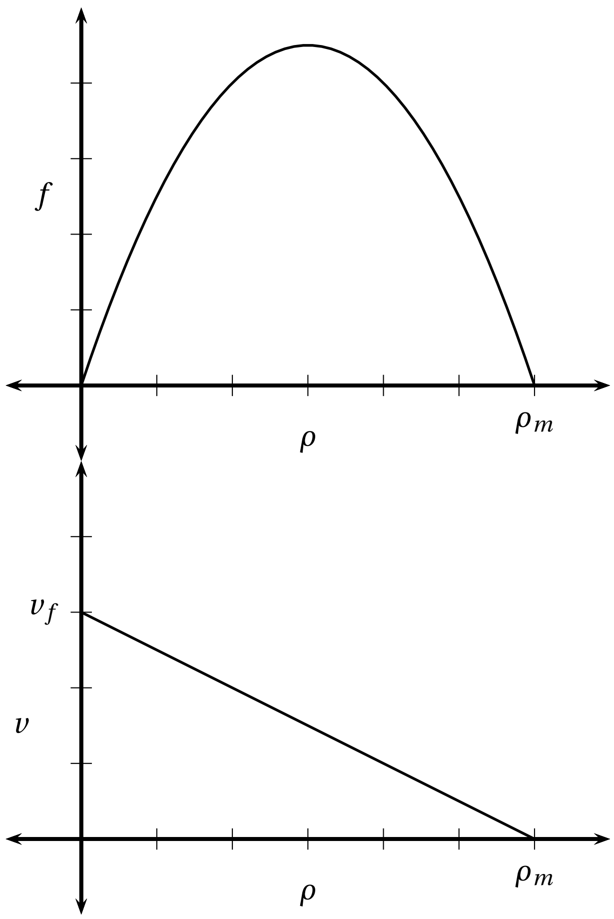

Figure 3.1: *Fundamental Diagram using Greenshield Model*

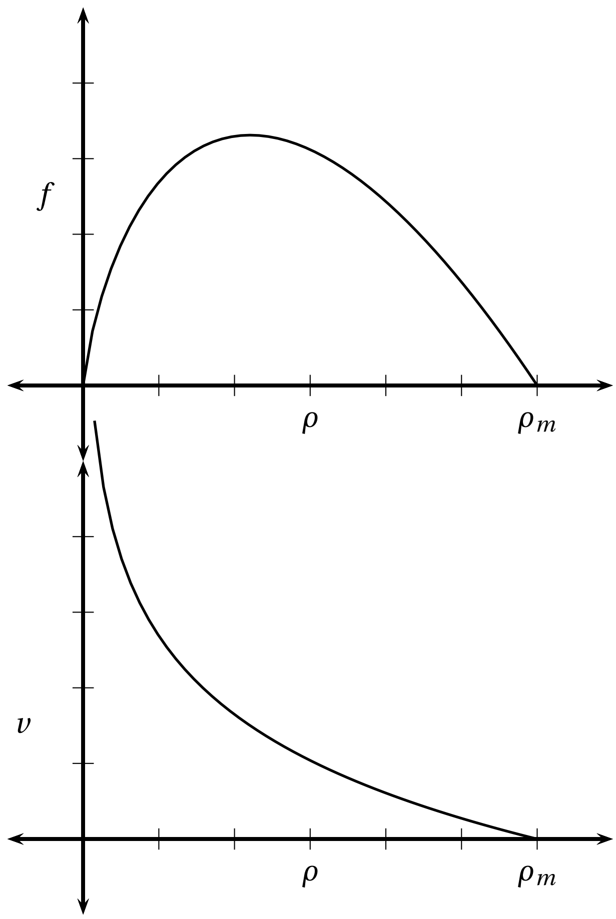

Figure 3.2: *Fundamental Diagram using Greenberg Model*

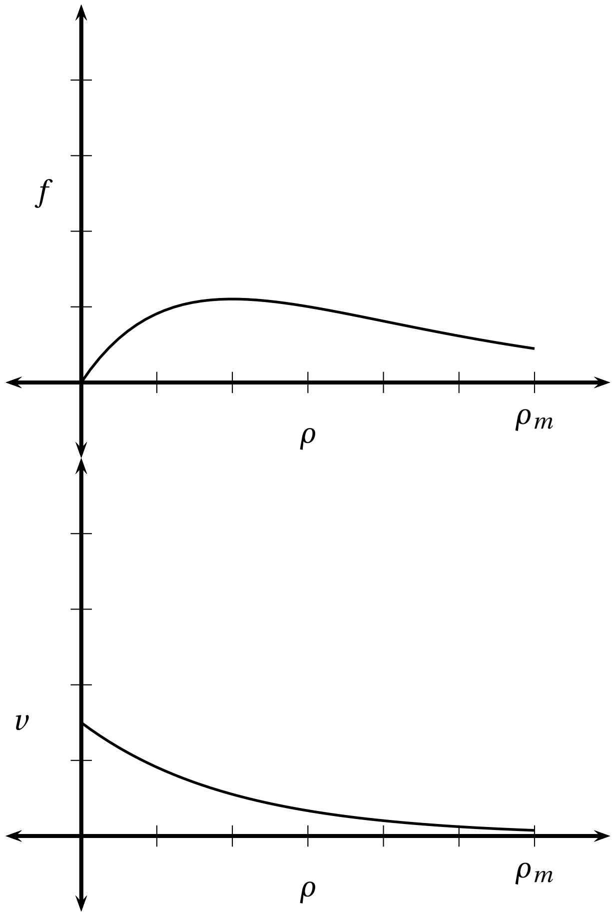

Figure 3.3: *Fundamental Diagram using Underwood Model*

$$V(\rho) = v_f(1 - \frac{\rho}{\rho_m}) - \frac{D}{\rho}(\frac{\partial \rho}{\partial x}) \tag{3.5}$$

where D is a diffusion coefficient given by $D = \tau\ v_r^2$, v_r is a random velocity, and τ is a relaxation parameter.

3.1.5 Other Models

There do exist other models such as Northwestern University model, Drew model, Pipes-Munjal model, and multi-regime models. The speed-density relationships for thse models are give below:

Northwestern University model

The speed-density relationship for this model [JSDM] is given by

$$V(\rho) = v_f \exp(-0.5\left(\frac{\rho}{\rho_0}\right)^2) \tag{3.6}$$

Drew model

The speed-density relationship for this model [Dre68] is given by

$$v(\rho) = v_f(1 - \left(\frac{\rho}{\rho_m}\right)^{(n+1)/2}) \tag{3.7}$$

Drew model is a generalization of other models such that taking different values for n in his model results in other models.

Pipes-Munjal Model

The speed-density relationship for this model [PE53] is given by

$$v(\rho) = v_f(1 - \left(\frac{\rho}{\rho_m}\right)^n) \tag{3.8}$$

This model is also a generalization of other models such that taking different values for n in this model results in other models.

Multi-regime Model

The speed-density relationship for this model can use different expressions in different regions [May90]. For instance it can use a constant speed in uncongested region and linear speed in the uncongested region.

$$v(\rho) = \begin{cases} v_f & \text{if } \rho < \frac{\rho_m}{2} \\ v_f(1 - \dfrac{\rho}{\rho_m}) & \text{otherwise} \end{cases} \tag{3.9}$$

The fundamental diagram for this model is shown in Figure 3.4.

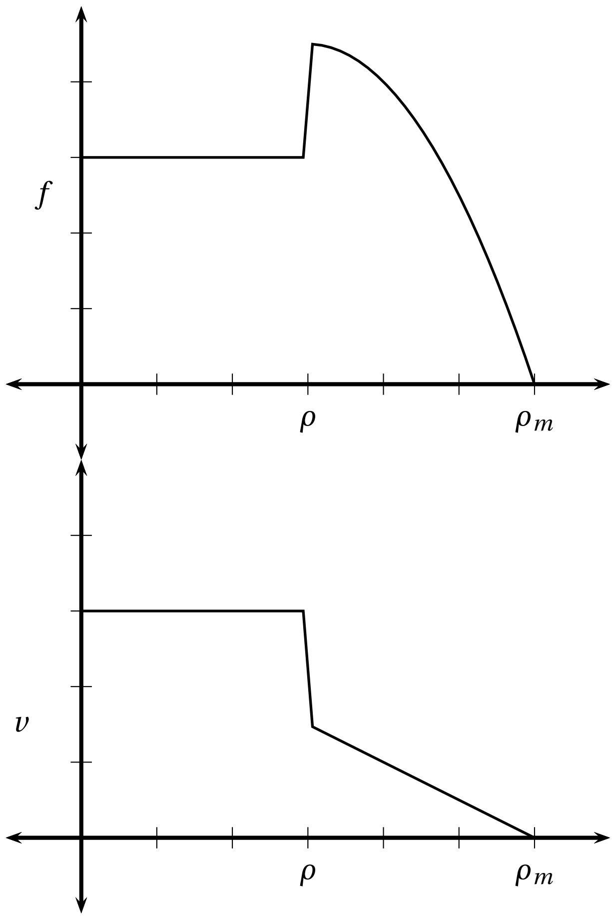

Figure 3.4: *Fundamental Diagram using Multi-regime Model*

3.1.6 LWR Models

Here we combine different fundamental relationships with the scalar conservation law.

LWR Model with Greenshields Flow

LWR model with Greenshields flow becomes

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x} v_f \rho (1 - \frac{\rho}{\rho_m}) = 0 \tag{3.10}$$

LWR Model with Greenberg Flow

LWR model with Greenwood flow becomes

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x} v_f \rho ln(\frac{\rho_m}{\rho}) = 0 \tag{3.11}$$

LWR Model with Underwoodwood Flow

LWR model with Underwood flow becomes

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x} v_f \rho \exp(\frac{-\rho}{\rho_m}) = 0 \tag{3.12}$$

LWR Model with Diffusion

LWR model with diffusive flow becomes the following viscous scalar conservation law.

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x} \left[v_f (1 - \frac{\rho}{\rho_m}) - \frac{D}{\rho} (\frac{\partial \rho}{\partial x}) \right] = 0 \tag{3.13}$$

3.2 Payne-Whitham Model

PW model or the Payne-Whitham model was proposed in the 1970s independently in [Pay71] and [Whi74]. It uses two PDEs to represent the traffic dynamics. In its most general form, the model takes the following form [PG06].

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ v_t + v\, v_x &= \frac{V(\rho) - v}{\tau} - \frac{\big(A(\rho)\big)_x}{\rho} + \mu \frac{v_{xx}}{\rho} \end{aligned} \tag{3.14}$$

Table 3.1 shows the different terms in this model. The first PDE is the conservation of traffic "mass" and the second tries to emulate the fluid momentum equation.

Term	**Meaning**
$V(\rho)$	Equilibrium Speed
τ	Relaxation Time
$(V(\rho)-v)/\tau$	Relaxation
$\left(A(\rho)\right)_x/\rho$	Anticipation
$\mu v_{xx}/\rho$	Viscosity

Table 3.1: Payne-Whitham Model Terms

The anticipation term is similar to the pressure term in fluids. In some specific models the term is taken as

$$A(\rho) = c_0^2 \rho \tag{3.15}$$

for some constant c_0. The relaxation term is there so that in equilibrium the speed follows the value $V(\rho)$. This could be chosen to be given by Greenshields formula or some other chosen formula. If we ignore the viscosity and use equation 3.15 then we get the PW model similar to isothermal flow as

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ v_t + v\, v_x &= \frac{V(\rho)-v}{\tau} - \frac{\left(c_0^2\rho\right)_x}{\rho} \end{aligned} \tag{3.16}$$

Equation (3.14) can be written in a conservation form by using the conservation of mass in the second equation to obtain

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ (\rho v)_t + \left(\rho v^2 + c_0^2\rho\right)_x &= \rho\frac{V(\rho)-v}{\tau} + \mu v_{xx} \end{aligned} \tag{3.17}$$

In the vector form this model becomes

$$u_t + f(u)_x = S \tag{3.18}$$

where

$$u = \begin{pmatrix} \rho \\ \rho v \end{pmatrix}, f(u) = \begin{pmatrix} \rho v \\ v^2 + c_0^2 \rho \end{pmatrix} \text{and } S = \begin{pmatrix} 0 \\ \rho\dfrac{V(\rho)-v}{\tau} + \mu v_{xx} \end{pmatrix} \tag{3.19}$$

We can write this in quasi-linear form as (see [Mor02])

$$u_t + A(u)u_x = S \tag{3.20}$$

where

$$A(u) = \frac{\partial f}{\partial u} = \begin{pmatrix} 0 & 1 \\ c_0^2 - v^2 & 2v \end{pmatrix} \tag{3.21}$$

The two eigenvalues of this matrix are

$$\lambda_1 = v + c_0 \text{ and } \lambda_2 = v - c_0 \tag{3.22}$$

The corresponding eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ v + c_0 \end{pmatrix} \text{and } v_2 = \begin{pmatrix} 1 \\ v - c_0 \end{pmatrix} \tag{3.23}$$

There has been some criticism of PW model, since it mimics the fluid behavior too closely especially the fact that it shows isotropic behavior, whereas the traffic behavior should be anisotropic. Isotropic models like the fluid models show that disturbances can travel in all directions the same way. However, for vehicular traffic that is moving forward the driver behavior should be affected by what happens in the front and not in the back. This deficiency has been overcome by other models, such as the AR and Zhang models presented next.

3.2.1 Characteristic Variables

Using the eigenvectors from the quasilinear form the system of PDEs can be diagonalized, so that the system of PDEs transforms into two scalar PDEs. To perform these steps, we start with the quasilinear form where any source terms have been ignored. The analysis on characteristic variables for various macroscopic traffic models is adapted from [Mor02].

$$u_t + A(u)u_x = 0 \tag{3.24}$$

Let λ_1 and λ_2 be the two distinct real eigenvalues and v_1 and v_2 be their corresponding *independent right* eigenvectors. Construct a square matrix whose columns are these eigenvectors (see any linear algebra textbook such as [SHFS02] and [Lax96]).

$$X_R = [v_1 | v_2] \tag{3.25}$$

From linear algebra $X_R X_R^{-1} = I$ and also $X_R^{-1} A X_R = \Gamma$, where Γ is the diagonal matrix consisting of the eigenvalues as the diagonal terms. Pre-multiplying equation (3.24) with X_R^{-1} and using $X_R X_R^{-1} = I$, we obtain

$$X_R^{-1} u_t + \Gamma X_R^{-1} u_x = 0 \tag{3.26}$$

Now, if we define

$$R_t = X^{-1} u_t \text{ and } R_x = X^{-1} u_x, \tag{3.27}$$

we can write the quasilinear system 3.24 as

$$R_t + \Gamma R_x = 0 \tag{3.28}$$

The characteristic variables r_1 and r_2 are constant along the characteristics $dx/dt = \lambda_1, \lambda_2$ respectively, where

$$R = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \tag{3.29}$$

3.2.2 Characteristic Variables for Payne-Whitham Model

For the Payne-Whitham model from equations (3.23) and (3.22) we get

$$X_R = \begin{pmatrix} 1 & 1 \\ v + c_0 & v - c_0 \end{pmatrix} \tag{3.30}$$

and

$$\Gamma = \begin{pmatrix} v + c_0 & 0 \\ 0 & v - c_0 \end{pmatrix} \tag{3.31}$$

For equation (3.27) here we obtain

$$R_t = \begin{pmatrix} \dfrac{\rho}{2c_0}(v_t + c_0(\ln\rho)_t) \\ -\dfrac{\rho}{2c_0}(v_t - c_0(\ln\rho)_t) \end{pmatrix} \tag{3.32}$$

Similarly,

$$R_x = \begin{pmatrix} \dfrac{\rho}{2c_0}(v_x + c_0(\ln\rho)_x) \\ -\dfrac{\rho}{2c_0}(v_x - c_0(\ln\rho)_x) \end{pmatrix} \tag{3.33}$$

We can define a matrix M such that $R_t = M\bar{R}_t$. Consequently, $R_x = M\bar{R}_x$. Then

$$M = \frac{\rho}{2c_0}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.34}$$

and

$$\bar{R} = \begin{pmatrix} v_t + c_0 \ln\rho \\ v_t - c_0 \ln\rho \end{pmatrix} \tag{3.35}$$

Now, equation (3.28) becomes

$$M(\bar{R}_t + \Gamma\bar{R}_x) = 0 \tag{3.36}$$

When $\rho \neq 0$ M is invertible and so we can solve

$$\bar{R}_t + \Gamma\bar{R}_x = 0 \tag{3.37}$$

Using notation

$$\bar{R} = \begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \end{pmatrix} \tag{3.38}$$

we can obtain the following by inverting equation 3.35.

$$\begin{pmatrix} v \\ \rho \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\bar{r}_1 + \bar{r}_2) \\ \exp\left(\frac{\bar{r}_1 - \bar{r}_2}{2c_0}\right) \end{pmatrix} \tag{3.39}$$

3.3 Aw-Rascle Model

A new model in [AR00] and improved in [Ras02] is designed to model the anisotropic traffic behavior. The following is the Aw-Rascle or AR model where we have added the relaxation term.

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ \left[v + p(\rho)\right]_t + v\left[(v + p(\rho))\right]_x &= \frac{V(\rho) - v}{\tau} \end{aligned} \tag{3.40}$$

where $V(\rho)$ is the equilbrium speed generally taken as Greenshields relationship. The pressure term is usually taken as

$$p(\rho) = c_0^2 \rho^\gamma \tag{3.41}$$

where $\gamma > 0$ and $c_0 = 1$.

For further analysis, we will ignore the relaxation term. For smooth solutions system (3.40) is equivalent to the following system that is obtained by multiplying the first equation by $p'(\rho)$ in (3.40) and then adding that to the second equation. That operation leads to the model in the following form.

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ v_t + \left[v - \rho p'(\rho)\right] v_x &= 0 \end{aligned} \tag{3.42}$$

The AR model in conservation form is given below.

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ \left[\rho(v + p(\rho))\right]_t + \left[\rho v(v + p(\rho))\right]_x &= 0 \end{aligned} \tag{3.43}$$

Now, we define a new variable $m = \rho(v + p(\rho))$, so that the model can be written as

$$\begin{aligned} \rho_t + (m - \rho p)_x &= 0 \\ m_t + \left[\frac{m^2}{\rho} - mp\right]_x &= 0 \end{aligned} \tag{3.44}$$

In the vector form this model becomes

$$u_t + f(u)_x = 0 \tag{3.45}$$

where

$$u = \begin{pmatrix} \rho \\ m \end{pmatrix}, \text{and } f(u) = \begin{pmatrix} m - \rho p \\ \dfrac{m^2}{\rho} - mp \end{pmatrix} \tag{3.46}$$

We can write this vector form in the quasi-linear form and obtain the eigenvalues and eigenvectors for the system. The quasilinear form is

$$u_t + A(u)u_x = 0 \tag{3.47}$$

where

$$A(u) = \frac{\partial f}{\partial u} = \begin{pmatrix} -(\gamma+1)p & 1 \\ -\dfrac{m^2}{\rho^2} - \dfrac{\gamma p m}{\rho} & \frac{2m}{\rho} - p \end{pmatrix} \tag{3.48}$$

The two eigenvalues of this matrix are

$$\lambda_1 = v \text{ and } \lambda_2 = v - \gamma p \tag{3.49}$$

The corresponding eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ v + (\gamma+1)p \end{pmatrix} \text{and } v_2 = \begin{pmatrix} 1 \\ v+p \end{pmatrix} \tag{3.50}$$

3.3.1 Characteristic Variables for Aw-Rascle Model

For the Aw-Rascle model we get

$$X_R = \begin{pmatrix} 1 & 1 \\ v + (\gamma+1)p & v+p \end{pmatrix} \tag{3.51}$$

and

$$\Gamma = \begin{pmatrix} v & 0 \\ 0 & v - \gamma p \end{pmatrix} \tag{3.52}$$

For equation 3.27 here we obtain

$$R_t = \begin{pmatrix} \frac{\rho}{\gamma p}(v_t + p_t) \\ -\frac{\rho}{\gamma p} v_t \end{pmatrix} \tag{3.53}$$

Similarly,

$$R_x = \begin{pmatrix} \frac{\rho}{\gamma p}(v_x + p_x) \\ -\frac{\rho}{\gamma p} v_x \end{pmatrix} \tag{3.54}$$

We can define matrix M such that $R_t = M\bar{R}_t$. Consequently, $R_x = M\bar{R}_x$. Then

$$M = \frac{\rho}{\gamma p}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.55}$$

and

$$\bar{R} = \begin{pmatrix} v+p \\ v \end{pmatrix} \tag{3.56}$$

Now, equation 3.28 becomes

$$M(\bar{R}_t + \Gamma\bar{R}_x) = 0 \tag{3.57}$$

When $\frac{\rho}{\gamma p} \neq 0$ M is invertible and so we can solve

$$\bar{R}_t + \Gamma \bar{R}_x = 0 \tag{3.58}$$

Using notation

$$\bar{R} = \begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \end{pmatrix} \tag{3.59}$$

we can obtain the following by inverting equation 3.56.

$$\begin{pmatrix} v \\ \rho \end{pmatrix} = \begin{pmatrix} \bar{r}_2 \\ (\bar{r}_1 - \bar{r}_2)^{1/\gamma} \end{pmatrix} \tag{3.60}$$

3.4 Zhang Model

We present here another model [Zha98], [Zha02] that retains the anisotropic traffic property, because its momentum equation is derived from a microscopic car-following model. The Zhang model is given by the following set of PDEs.

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ v_t + \left[v + \rho V'(\rho)\right] v_x &= \frac{V(\rho) - v}{\tau} \end{aligned} \tag{3.61}$$

Ignoring the relaxation term, the conservation form of this model becomes

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ \left[\rho(v - V(\rho))\right]_t + \left[\rho v(v - V(\rho))\right]_x &= 0 \end{aligned} \tag{3.62}$$

Now, we define a new variable $m = \rho(v - V(\rho))$, so that the model can be written as

$$\begin{aligned} \rho_t + (m - \rho P)_x &= 0 \\ m_t + \left[\frac{m^2}{\rho} - mP\right]_x &= 0 \end{aligned} \tag{3.63}$$

In the vector form this model becomes

$$u_t + f(u)_x = 0 \tag{3.64}$$

where

$$u = \begin{pmatrix} \rho \\ m \end{pmatrix}, \text{and } f(u) = \begin{pmatrix} m + \rho V(\rho) \\ \dfrac{m^2}{\rho} + mV(\rho) \end{pmatrix} \tag{3.65}$$

We can write this vector form in the quasi-linear form and obtain the eigenvalues and eigenvectors for the system. The quasilinear form is

$$u_t + A(u)u_x = 0 \tag{3.66}$$

where

$$A(u) = \frac{\partial f}{\partial u} = \begin{pmatrix} \rho V'(\rho) + V(\rho) & 1 \\ -\dfrac{m^2}{\rho^2} + mV'(\rho) & \dfrac{2m}{\rho} + V(\rho) \end{pmatrix} \tag{3.67}$$

The two eigenvalues of this matrix are

$$\lambda_1 = v \text{ and } \lambda_2 = v + \rho V'(\rho) \tag{3.68}$$

The corresponding eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ v - V(\rho) - \rho V'(\rho) \end{pmatrix} \text{and } v_2 = \begin{pmatrix} 1 \\ v - V(\rho) \end{pmatrix} \tag{3.69}$$

3.4.1 Characteristic Variables for Zhang Model

For the Zhang model we get

$$X_R = \begin{pmatrix} 1 & 1 \\ v - V(\rho) - \rho V'(\rho) & v - V(\rho) \end{pmatrix} \tag{3.70}$$

and

$$\Gamma = \begin{pmatrix} v & 0 \\ 0 & v + \rho V'(\rho) \end{pmatrix} \tag{3.71}$$

For equation (3.27) here we obtain

$$R_t = \begin{pmatrix} \dfrac{-1}{V'(\rho)} \left(\dfrac{m}{\rho}\right)_t \\ \dfrac{1}{V'(\rho)} \left(\dfrac{m}{\rho} + V(\rho)\right)_t \end{pmatrix} \tag{3.72}$$

Similarly,

$$R_x = \begin{pmatrix} \dfrac{-1}{V'(\rho)}\left(\dfrac{m}{\rho}\right)_x \\ \dfrac{1}{V'(\rho)}\left(\dfrac{m}{\rho}+V(\rho)\right)_x \end{pmatrix} \tag{3.73}$$

We can define matrix M such that $R_t = M\bar{R}_t$. Consequently, $R_x = M\bar{R}_x$. Then

$$M = \frac{1}{V'(\rho)}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.74}$$

and

$$\bar{R} = \begin{pmatrix} \dfrac{m}{\rho} \\ \dfrac{m}{\rho}+V(\rho) \end{pmatrix} \tag{3.75}$$

Now, equation (3.28) becomes

$$M(\bar{R}_t + \Gamma \bar{R}_x) = 0 \tag{3.76}$$

When $\rho \neq 0$ M is invertible and so we can solve

$$\bar{R}_t + \Gamma \bar{R}_x = 0 \tag{3.77}$$

Using notation

$$\bar{R} = \begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \end{pmatrix} \tag{3.78}$$

we can obtain the following by inverting equation 3.75.

$$\begin{pmatrix} V(\rho) \\ m \end{pmatrix} = \begin{pmatrix} \bar{r}_2 - \bar{r}_1 \\ \rho\,\bar{r}_1 \end{pmatrix} \tag{3.79}$$

3.5 Pedestrian and Control Models in One Dimension

There is one major difference between vehicular traffic and pedestrian traffic. In vehicular traffic if we use the LWR model, traffic density fixes the value of traffic speed. However, in pedestrian flow, just knowing the traffic density does not fix the pedestrian speed. The actual speed depends on the function that the pedestrians are performing. For example, if pedestrians are inside a museum or in a school their movement is dependent on the activity that is taking place. If however, the pedestrians are all trying to exit from a corridor, then their speed becomes a function of density just like the vehicular traffic. Notice that even in a single corridor, people could be moving in both directions at different places, but vehicular traffic on a highway or street lane is unidirectional. The models (such as Greenshields) only have to provide the speed based on density, since the direction of travel is fixed. If we introduce a time-varying scalar field that abstracts the activity that is taking place for pedestrians we can modify the vehicular traffic model to get pedestrian models. For distributed traffic control problems, this field will be used as the control variable.

3.5.1 LWR Pedestrian Model with Greenshields Flow

In order to convert the LWR model with Greenshields flow into a pedestrian model, we can make the free-flow speed to be the scalar control field. This is a very natural choice, since if we consider the case when there is only a single pedestrian, then according to Greenshields model, the speed would be the constant free-flow speed. A pedestrian could be going in the positive or negative direction and the magnitude would be in the closed interval [0, v_m] where v_m is a constant maximum possible speed. The model then becomes

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x} v_f(t,x)\rho(1-\frac{\rho}{\rho_m}) = 0 \tag{3.80}$$

where $v_f(t,x) \in [-v_m, v_m]$.

3.5.2 Payne-Whitham Pedestrian Model with Greenshields Flow

We can convert the Payne Whitham model by making the $V(\rho)$ term change with time and space. We can use Greenshields relationship combined with this to produce the time-dependent scalar field. The model then becomes

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ v_t + v\, v_x &= \frac{V(t,x,\rho) - v}{\tau} - \frac{\big(A(\rho)\big)_x}{\rho} + \mu \frac{v_{xx}}{\rho} \end{aligned} \tag{3.81}$$

where

$$V(t,x,\rho) = v_f(t,x)\rho(1-\frac{\rho}{\rho_m}) \tag{3.82}$$

The control scalar field for the movement becomes $v_f(t,x)$.

3.5.3 Aw-Rascle Pedestrian Model with Greenshields Flow

The Aw-Rascle model with the relaxation term can be used for controlled traffic. The model with the control term is presented below.

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ \left[v + p(\rho)\right]_t + v\left[(v + p(\rho))\right]_x &= \frac{V(t,x,\rho) - v}{\tau} \end{aligned} \tag{3.83}$$

The control variable is the equilbrium velocity term, which combined with Greenshields model can be taken as

$$V(t,x,\rho) = v_f(t,x)\rho(1 - \frac{\rho}{\rho_m}) \tag{3.84}$$

3.5.4 Zhang Pedestrian Model with Greenshields Flow

Zhang model with the relaxation term can be also used similarly for controlled traffic. The model with the control term is presented below.

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ v_t + \left[v + \rho V'(\rho)\right] v_x &= \frac{V(\rho) - v}{\tau} \end{aligned} \tag{3.85}$$

The control variable is the equilbrium velocity term, which combined with Greenshields model can be taken as

$$V(t,x,\rho) = v_f(t,x)\rho\left(1 - \frac{\rho}{\rho_m}\right) \tag{3.86}$$

CHAPTER 4

CONSERVATION LAW SOLUTIONS

This chapter presents different notions of solutions of conservation laws.

4.1 Method of Characteristics

We can use method of characteristics to solve quasilinear partial differential equations which allows us to convert the PDE into ordinary differential equations. As an example, consider

$$u_t + u u_x = 0 \tag{4.1}$$

If $u = u(t,x)$ solves (4.1), let $x = x(t)$ solve ODE

$$\dot{x}(t) = u(t, x(t)) \tag{4.2}$$

Set $z(t) = u(t,x(t))$. Then $\dot{z}(t) = 0$. Notice that

$$\frac{d}{dt} u(t, x(t)) = u_t + \frac{dx}{dt} u_x \tag{4.3}$$

Using equation (4.3) with equation (4.1) gives

$$\frac{du}{dt} = 0, \ \frac{dx}{dt} = u \tag{4.4}$$

This shows that we can use the initial data $u_0(x)$ to propagate the solution in the (t,x) plane.

In general, if we have

$$u_t + h(t,x,u)u_x = g(t,x,u) \tag{4.5}$$

The characteristics would give us

$$\frac{du}{dt} = g(t,x,u)\,,\ \frac{dx}{dt} = h(t,x,u) \tag{4.6}$$

As a solution of equation 4.1 consider the initial data given in Figure 4.1.

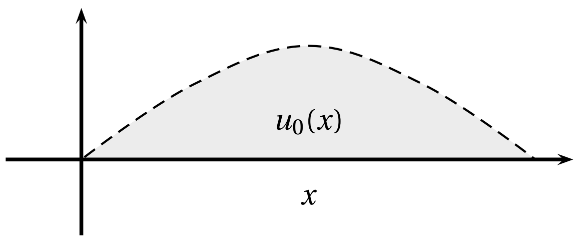

Figure 4.1: *Initial Data*

As we see in equation(4.4) the slope of the characteristics in the (x,t)-plane is equal to the value of u. This value is tranferred on the characteristic line on which this value is constant. The slopes are illustrated in Figure 4.2.

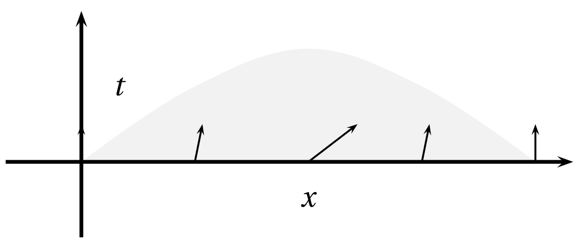

Figure 4.2: *Characteristic Slopes*

We can find the solution $u(t,x)$ after some time by following the characteristics, as shown in Figure 4.3

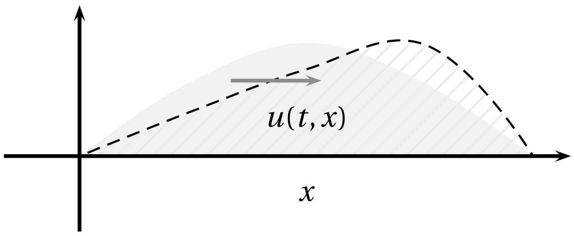

Figure 4.3: *Solution after some time*

4.1.1 Characteristics in Two Dimensions

We can use the method of characteristics to solve quasilinear partial differential equations in two (or more) dimensions as well which allows us to convert the PDE into a set of ordinary differential equations. As an example, consider

$$u_t + f_1(t,x,y)u_x + f_2(t,x,y)u_y = h(t,x,y) \tag{4.7}$$

The left hand side of this equation can be considered a directional derivative as before where in the two dimensional case now

$$\frac{Du(t,x(t),y(t))}{Dt} = u_t + \frac{dx}{dt}u_x + \frac{dy}{dt}u_y \tag{4.8}$$

Using equation (4.8) with equation (4.7) gives

$$\frac{du}{dt} = h(t,x,y)\,,\ \frac{dx}{dt} = f_1(t,x,y) \text{ and } \frac{dy}{dt} = f_2(t,x,y) \tag{4.9}$$

4.1.2 Characteristics for a System

Consider the first-order quasilinear system of equations

$$u_t + f(u)_x = 0 \tag{4.10}$$

where $u : R^+ \times R \to R^n$ and $f : R^n \to R^n$ is smooth.

Definition 4.1. A curve $t \mapsto x(t)$ is a characteristic curve for (4.10) whose solution is $u(t,x)$ if the following matrix is singular.

$$\frac{dx}{dt}I - \nabla f(u(t,x)) \tag{4.11}$$

□

4.2 Classical or Strong Solutions

For a scalar conservation law

$$u_t + f'(u)u_x = 0 \tag{4.12}$$

with initial condition

$$u(x,0) = u_0(x), \tag{4.13}$$

strong or classical solution is defined below for $f : R \mapsto R$ smooth and continuous $u_0(x)$.

Definition 4.2. We say that $u(t,x) : (R^+ \times R) \mapsto R$ is a classical solution of the Cauchy problem if $u(t,x) \in C^1\,(R^+ \times R)$ and (4.12) with (4.13) is satisfied. □

We have the following theorem for strong solutions for the scalar conservation law (4.12) (see [RR04]).

Theorem 4.3. *Any C^1 solution of the single conservation law* (4.12) *for sufficiently smooth flux $f(u)$ is constant along its characteristics that must satisfy*

$$\frac{dx}{dt} = f'(u(t,x(t))) \tag{4.14}$$

⊡

4.3 Weak Solutions

In this section we use method of characteristics to see that even for smooth initial conditions the strong solutions cannot be extended in time indefinitely. In fact, even smooth initial conditions can lead to discontinuous solutions in finite time. Therefore, we will need a notion of solutions that is more general than the notion of strong solutions. Let us illustrate this blowup of solutions next.

4.3.1 Blowup of Solutions

To see how smooth initial solutions blow up, let us consider the scalar traffic model.

$$\frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x} f(\rho) = 0 \tag{4.15}$$

where

$$f(\rho) = v(\rho)\rho \tag{4.16}$$

and

$$v(\rho) = v_f(1 - \frac{\rho}{\rho_m}) \tag{4.17}$$

In quasilinear form we write equation 4.15 as

$$\frac{\partial}{\partial t}\rho(t,x) + f'(\rho)\frac{\partial}{\partial x}\rho(t,x) = 0 \tag{4.18}$$

Combining equations (4.16) and (4.17), we get

$$f(\rho) = v_f \rho(1 - \frac{\rho}{\rho_m}) \tag{4.19}$$

From equation (4.19) we obtain the *characteristic speed* by differentiating.

$$f'(\rho) = v_f(1 - 2\frac{\rho}{\rho_m}) \tag{4.20}$$

The characteristic speed is the value obtained from the slope of the fundamental diagram at the given density as shown in Figure 4.4

Now, let us consider the initial traffic conditions that are shown in Figure 4.5.

The characteristics in the (t,x)-plane starting at initial time are shown in Figure 4.6.

We can see that after some finite time the characteristics intersect. That would mean that at a single point (t,x) there are multiple possible values for ρ. If we propagate the initial curve, we can see that the traffic density gets a discontinuity as shown in Figure 4.7.

We need to allow solutions that can have discontinuities as shown in Figure 4.7. However the integral form the conservation law should still be satisfied. Hence, we define weak solutions for conservation laws.

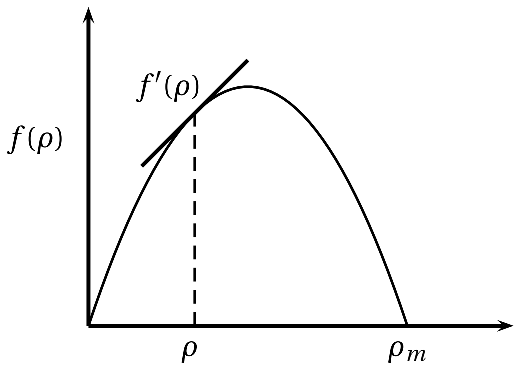

Figure 4.4: *Characteristic Speed*

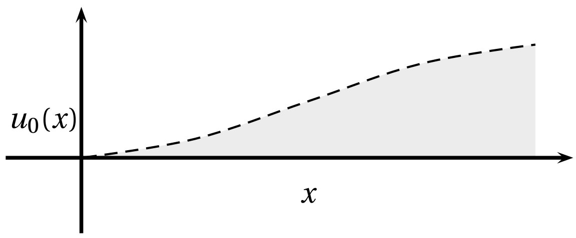

Figure 4.5: *Initial Conditions*

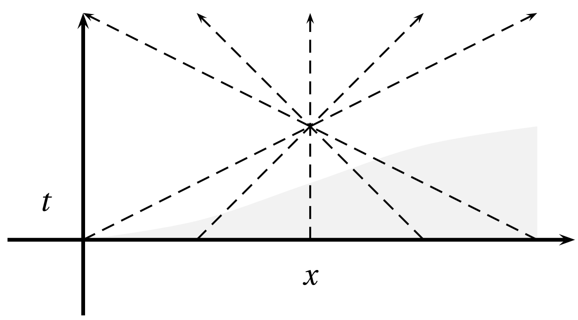

Figure 4.6: *Characteristics*

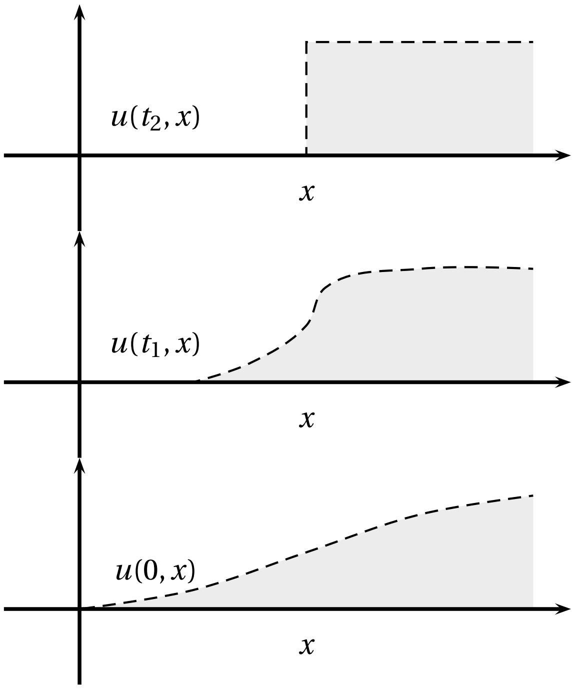

Figure 4.7: *Initial Conditions Propagating*

Implicit Solution

Another way to observe singularities in solutions of the conservation law (4.12) is to view the following implicit solution of the conservation law (see [Smo94]). This solution comes from following characteristic back from (t, x) to a point when $t = 0$.

$$u(t, x) = u_0(x - tf'(u(t, x))) \tag{4.21}$$

where $u(0, x) = u_0(x)$ is the initial smooth data.

Using the implicit function theorem (see [Rud76]) and performing

$$\frac{D}{Dt}[u(t, x) - u_0(x - tf'(u(t, x)))] = 0 \tag{4.22}$$

we get

$$u_t = -\frac{f'(u)u_0'}{1 + f''(u)u_0' t} \tag{4.23}$$

Performing

$$\frac{D}{Dx}[u(t,x) - u_0(x - tf'(u(t,x)))] = 0 \tag{4.24}$$

we get

$$u_x = \frac{u_0'}{1 + f''(u)u_0' t} \tag{4.25}$$

This shows that if $u_0' < 0$ at some point both u_t and u_x become unbounded when $(1 + f''(u)u_0' t) \to 0$

4.3.2 Generalized Solutions

For a conservation law

$$u_t + f(u)_x = 0 \tag{4.26}$$

with initial condition

$$u(x,0) = u_0(x), \tag{4.27}$$

where $u_0(x) \in L^1_{loc}(R;R^n)$, solution in the distributional sense is defined below for smooth vector field $f : R^n \to R^n$ (see [Bre05]).

Definition 4.4. A measurable locally integrable function $u(t,x)$ is a solution in the distributional sense of the Cauchy problem ((5.40)) if for every $\phi \in C_0^\infty(R^+ \times R) \mapsto R^n$

$$\iint_{R^+ \times R} [u(t,x)\,\phi_t(t,x) + f(u(t,x))\,\phi_x(t,x)]\,dx\,dt + \int_R u_0(x)\,\phi(x,0)\,dx = 0 \tag{4.28}$$

□

Weak Solutions

A measurable locally integrable function $u(t,x)$ is a weak solution in the distributional sense of the Cauchy problem ((5.40)) if it is a distributional solution in the open strip $(0,T) \times \mathbb{R}$, satisfies the initial condition (5.41) and if u is continuous as a function from $[0,T]$ into L^1_{loc}. We require $u(t,x) = u(t,x^+)$ and

$$\lim_{t\to 0} \int_R |u(t,x) - u_0(x)|\,dx = 0 \tag{4.29}$$

Every weak solution is also a generalized solution but a generalized solution is not necessarily a weak solution. To see this we can take a generalized solution and make the value of the solution zero at initial time (i.e. on a set of measure zero). This would still be a generalized solution to the problem, but would not be a weak solution.

4.3.3 Generalized Solution Property

Generalized solutions have a nice convergence property that is stated and proved here (see [Bre05]).

Lemma 4.5. *If u_n is a sequence of distributional solutions to the conservation law* (5.40)*, then*

1. $(u_n \to u,\ f(u_n) \to f(u)$ *in* $L^1_{loc}) \Rightarrow u$ *is a solution of the conservation law* (5.40).
2. $(u_n \to u$ *in* L^1_{loc} *and if all* u_n *take values in a compact set*$) \Rightarrow u$ *is a solution of the conservation law* (5.40).

$\boxdot$

Proof. 1. Assume that $u_n \to u$, and $f(u_n) \to f(u)$ in L^1_{loc} and that $\phi \in C^1_0$ then we estimate

$$\begin{aligned}\left|\iint_\Omega \{u_n\phi_t + f(u_n)\phi_x\}\,dt\,dx - \int\int_\Omega \{u\phi_t + f(u)\phi_x\}\,dt\,dx\right| \\ \leq \iint_\Omega \{|u_n - u|\left|\phi_t\right| + \left|f(u_n) - f(u)\right|\left|\phi_x\right|\}\,dt\,dx \\ \leq \|u_n - u\|_{supp\phi,1}\left\|\phi_t\right\|_\infty + \left\|f(u_n) - f(u)\right\|_{supp\phi,1}\left\|\phi_x\right\|_\infty \to 0, \text{ as } n \to \infty\end{aligned} \tag{4.30}$$

2. The second part follows from the first part once we verify that $f(u_n) \to f(u)$ in L^1_{loc} under the assumption that all the functions u_n take values in a fixed compact subset K of Ω. As f is a smooth vector field, f is uniformly bounded on compact subsets. As the values of u_n stay inside K, it follows that $f(u_n)$ is uniformly bounded, say $\left\|f(u_n)\right\| \leq M$. Then $\left\|f(u_n(x)) - f(u(x))\right\| \leq 2M$ for all x in the support of ϕ where the constant function $2M$ is integrable over the support of ϕ.

   By dropping down to a subsequence if necessary, from the fact that $u_n \to u$ in L^1_{loc} we can also assume that $u_n \to u$ pointwise on the support of ϕ and hence also $f(u_n) \to f(u)$ on the support of ϕ. We now can use the *Lebesgue Dominated Convergence Theorem* (see [Roy88] or [DiB05]) to see that $f(u_n) \to f(u)$ in $L^1_{supp\phi}$) (or, more generally, in L^1_{loc})Ůas required.

$\square$

4.3.4 Weak Solution Property

There is a very important property that weak solutions possess but general distributional solutions don't. The property is the continuity of the solution with respect to the initial data. For problems with boundary data, similar continuity is required for the data given on the boundary. Generalized solutions are insensitive to data on a set of measure zero, but weak solutions have the continuity property that does not allow that. To understand the difference let us study a domain that is shown in Figure 4.8(see [Bre05]). The domain is

$$\Omega = \left\{(t,x);\ t \in [t_1,\ t_2], \gamma_1(t) \le x \le \gamma_2(t)\right\} \tag{4.31}$$

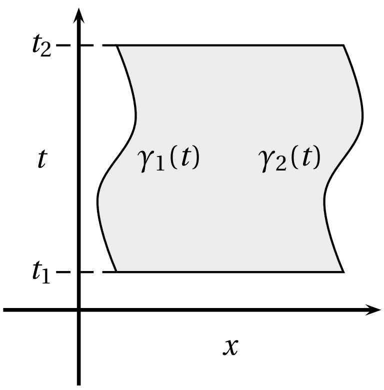

Figure 4.8: *Domain to Illustrate Trace Property*

A smooth solution, by the application of the *divergence theorem* (see [MT03]) should satisfy

$$0 = \iint_\Omega (u_t + f(u)_x)\,dt\,dx = \int_{\gamma_1(t_2)}^{\gamma_2(t_2)} f(u(t_2,x))\,dx - \int_{\gamma_1(t_1)}^{\gamma_2(t_1)} f(u(t_1,x))\,dx$$
$$+ \int_{t_1}^{t_2} [u(t,\gamma_1(t)) + f(u(t_1,\gamma_1(t)))\dot{\gamma}_1(t)]\,dt - \int_{t_1}^{t_2} [u(t_1,\gamma_2(t)) + f(u(t_1,\gamma_2(t)))\dot{\gamma}_2(t)]\,dt \tag{4.32}$$

Arbitrary generalized solutions will not satisfy equation (4.32) because the curves have measure zero and the values of the solution on these curves can be chosen arbitrarily. However, weak solutions will satisfy this equation due to continuity from $t \mapsto u(t,\cdot)$ when we consider point values satisfying $u(t,x) = u(t,x^+)$.

To see this, consider a smooth real-valued non-decreasing function β : $R \to [0,1]$, such that $\beta(r) = 0$ for $r \leq 0$ and $\beta = 1$ for $r \geq 1$. A scaled version of this function is $\beta^\epsilon(r) = \beta(r/\epsilon)$. We can use this scaled version to define a region surrounding Ω where the following function is non-zero and is equal to zero on its boundary.

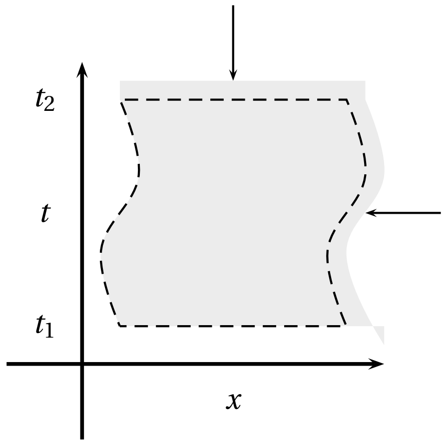

Figure 4.9: *Domain with* ϕ

$$\phi^\epsilon(t,x) = \left[\beta^\epsilon(x-\gamma_1(t)) - \beta^\epsilon(x-\gamma_2(t))\right]\left[\beta^\epsilon(t-t_1) - \beta^\epsilon(t-t_2)\right] \tag{4.33}$$

We obtain result from equation (4.32) here if we take the limit $\epsilon \to 0$ in

$$\int_0^\infty \int_{-\infty}^\infty \left\{u\phi_t^\epsilon + f(u)\phi_x^\epsilon\right\} dx\,dt = 0 \tag{4.34}$$

4.3.5 Trace Operator for Functions of Bounded Variation

The conservation law solutions are obtained using the sequential compactness property of functions of bounded variation (BV) shown by Helly's theorem (see [Nat55] and theorem 4.7) which is also implied by Alaoglu theorem (see [DiB05]). The trace property of BV functions is very useful in fixing initial and boundary conditions for conservation laws.

For a scalar conservation law on $(0,T)\times\Omega$ where Ω is a bounded subset of R^n with a piecewise regular boundary Γ, we present the following lemma from [CBN79] about the trace operator on BV functions.

Lemma 4.6. *For $u \in BV((0,T)\times\Omega)$, a trace γu in L^∞ for $t=0$ and in $L^\infty((0,T)\times\Omega)$ exists which is reached through L^1 convergence. Specifically, there is a bounded operator $\sigma : BV((0,T)\times\Omega) \to L^\infty(\Omega)$ such that $\sigma\phi = \phi(0,\cdot)$ for ϕ smooth on $L^\infty((0,T)\times\Omega)$.* ⊡

The proof depends on the fact that BV functions have right and left limits and the fact that for Ω bounded, a.e. convergence for the dominated sequence implies L^1 convergence (see [AMBT96] and [Bar95]). The proof is given in [CBN79].

The relationship between different modes of *dominated* convergence is shown in Figure 4.10. Specifically these relationships are valid when there exists $g \in L^1$ such that $|f_n| \le g$ for all n.

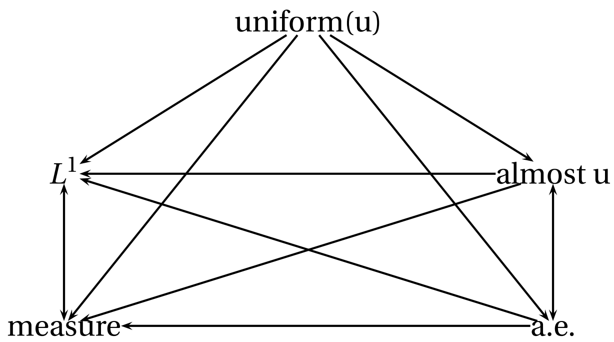

Figure 4.10: *Dominated Convergence Relationships*

In general, however, the relationships shown in Figure 4.11 are the ones that are valid. Here uniform convergence is shown as uniform(u) and almost uniform as almost u.

In case of finite measure space, the relationships shown in Figure 4.12 are the ones that are valid.

One very important theorem that we need for convergence deals with the sequential compactess property of sequence of functions of bounded variations (BV).

Theorem 4.7. (***Helly***) *Consider a sequence of functions given by $f_n : R \to R^n$ such that*

$$\textit{Total Variation}\{f_n\} \le C,\ |f_n(x)| \le M \textit{ for all } n, x \tag{4.35}$$

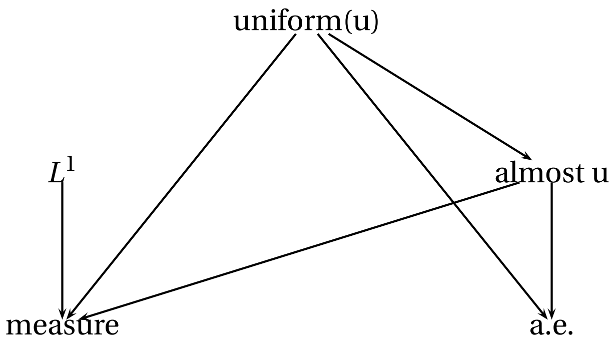

Figure 4.11: *General Convergence Relationships*

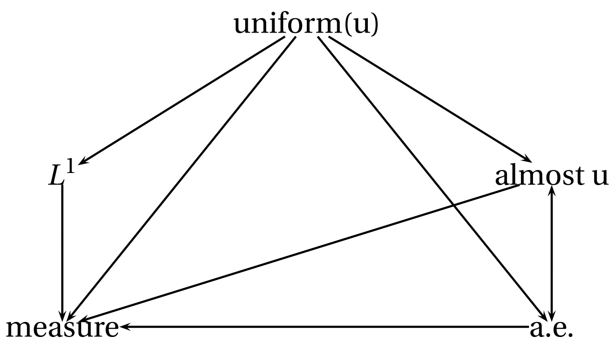

Figure 4.12: *Finite Measure Space Convergence Relationships*

for constants C, M. Then, there exists a function f and a subsequence f_{n_k} such that

$$\lim_{n_k \to \infty} f_{n_k}(x) = f(x) \textit{ for every } x \in R \tag{4.36}$$

$$\textit{Total Variation}\{f\} \le C, \ \left|f(x)\right| \le M \textit{ for all } x \tag{4.37}$$

$\boxdot$

4.4 Scalar Riemann Problem

Scalar Riemann problem is the Cauchy problem for the scalar conservation law where the initial data is a piecewise constant function with only two values. In both cases there will be two different values on both sides of $x = 0$ at time $t = 0$. In one case the left hand side value will be lower and in the other it will be higher than the right hand side value.

4.4.1 Shock Solution

Let us consider the following scalar Riemann problem for the traffic problem.

$$\frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x}v_f\rho(1-\frac{\rho}{\rho_m}) = 0 \tag{4.38}$$

with data $\rho(0,x) = \rho_\ell$ for $x < 0$ and $\rho(0,x) = \rho_r$ for $x \geq 0$, such that $\rho_\ell < \rho_r$. The characteristic speed for $t = 0$ and $x < 0$ is

$$\lambda(\rho_\ell) = f'(\rho_\ell) = v_f(1-2\frac{\rho_\ell}{\rho_m}) \tag{4.39}$$

The characteristic speed for $t = 0$ and $x \geq 0$ is

$$\lambda(\rho_r) = f'(\rho_r) = v_f(1-2\frac{\rho_r}{\rho_m}) \tag{4.40}$$

We see that the characteristic speed on the left is higher than that on the right and therefore the characteristic curves (straight lines) catch up with those on the right. This produces a shock curve with speed λ. This speed is given by *Rankine-Hugoniot* condition (see [Smo94]). The shock wave is shown in Figure 4.13.

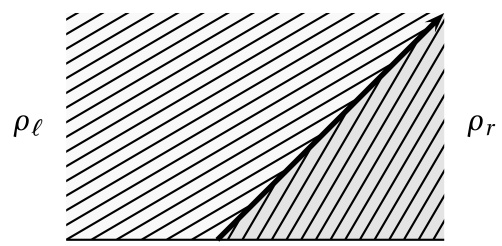

Figure 4.13: *Shockwave Solution to Riemann Problem*

The speed of the shockwave will satisfy the following Rankine-Hugoniot condition.

$$\lambda(\rho_r - \rho_\ell) = f(\rho_r) - f(\rho_\ell) \tag{4.41}$$

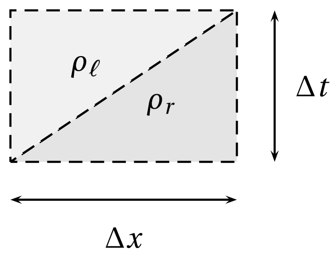

Figure 4.14: *Shockwave Speed Derivation*

For derivation, consider Figure 4.14.

We see that in time Δt the region of length Δx has changed its density completely from ρ_r to ρ_ℓ. Therefore, the mass conservation principle enforces that the change in mass should be equal to the change through the flux at the boundaries during the same time. Hence

$$\Delta x(\rho_r - \rho_\ell) = \Delta t[f(\rho_r) - f(\rho_\ell)] \tag{4.42}$$

Dividing both sides by Δt and then taking limits produces equation (4.41).

4.4.2 Rarefaction Solution

Let us consider the scalar Riemann problem of equation (4.38) with data $\rho(0, x) = \rho_\ell$ for $x < 0$ and $\rho(0, x) = \rho_r$ for $x \geq 0$, such that $\rho_\ell > \rho_r$.

We see that the characteristic speed on the left is lower than that on the right and this produces a gap in the characteristic lines that needs to be filled with some solution. This condition is shown in Figure 4.15.

There are many solutions possible that will fill up the gap and also be weak solutions. One possible solution is shown in Figure 4.16. However, we reject this solution since it is not stable to perturbation to initial data. In this rejected solution characteristics come out of the proposed shock line. In the correct shock solutions, characteristics can only impinge on the shock curve, not emanate from it.

There are many other solutions possible. We need to pick a solution that is stable. To accomplish this, many admissibility conditions have been proposed such as entropy, viscosity, and Lax condition. The correct solution which will also satisfy these conditions is a symmetry solution shown in Figure 4.17.

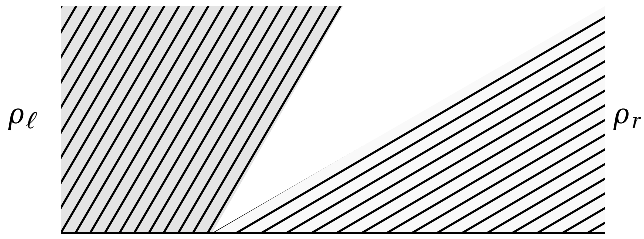

Figure 4.15: *Blank Region in $x - t$ Space*

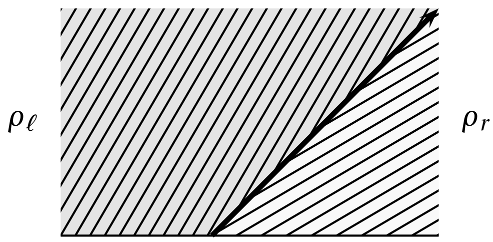

Figure 4.16: *Entropy Violating (Rejected) Solution*

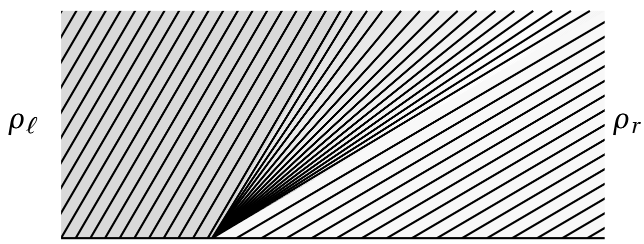

Figure 4.17: *Rarefaction Solution*

The symmetry rarefaction wave solution is given by

$$\rho(t,x) = \begin{cases} \rho_\ell & \text{if } \frac{x}{t} \leq \lambda(\rho_\ell) \\ \omega(\frac{x}{t}) & \text{if } \lambda(\rho_\ell) \leq \frac{x}{t} \leq \lambda(\rho_r) \\ \rho_r & \text{if } \frac{x}{t} \geq \lambda(\rho_r) \end{cases} \tag{4.43}$$

where

$$\omega(\frac{x}{t}) = \frac{\lambda_r - \lambda_\ell}{\rho_r - \rho_\ell}(x/t) \tag{4.44}$$

4.5 Admissibility Conditions

There are three admissibility conditions that help choose the physically relevant and stable solution out of the multiple ones possible. These are stated in this section.

4.5.1 Vanishing Viscosity Solution

This enables *viscous regularization* of the conservation law of equation (5.40) (see [HR02]). A weak solution u of (5.40) is admissible if there exists a sequence of smooth solutions u^ϵ to the following viscous conservation law such that $\lim_{\epsilon\to 0} \| u^\epsilon - u\|_{L^1_{loc}} = 0$.

$$u^\epsilon_t + f(u^\epsilon)_x = \epsilon u^\epsilon_{xx} \tag{4.45}$$

4.5.2 Entropy Admissible Solution

A weak solution u of (5.40) is entropy admissible if for all non-negative smooth functions ϕ with compact support

$$\iint [\eta(u)\phi_t + q(u)\phi_x]\, dx dt \geq 0 \tag{4.46}$$

$\forall(\eta, q)$ where η, an *entropy function* is a convex continuously differentiable function $\eta : R^n \to R$ with *entropy flux*, such that[1]

$$D\eta(u) \cdot Df(u) = Dq(u) \quad u \in R^n \tag{4.47}$$

The integral entropy condition (4.46) can be equivalently written in a differential form where the solution is implied in the distributional sense of

[1] In fact, it suffices to require η and q to be only Locally Lipschitz such that equation (4.47) is satisfied a.e.

$$\eta(u)_t + q(u)_x \leq 0 \tag{4.48}$$

To understand the relationship of the entropy condition with the viscosity solution, consider the scalar conservation law

$$u_t + f(u)_x = 0 \tag{4.49}$$

We have seen that there can be multiple solutions to this PDE. Hence we need to add some constraints such that only one solution remains. One way to do this is to add another variable that should satisfy a conservation law of its own. Let us take the new variable to be $\eta(u)$ and its corresponding flux to be $q(u)$. Hence, the conservation law for this additional variable is (see [JMR96])

$$n_t + q(u)_x = 0 \tag{4.50}$$

Let us assume smooth $\eta(u)$, $q(u)$ as well as smooth solution to (4.49). Then, multiplying (4.49) with $\eta'(u)$ gives

$$\eta'(u)u_t + \eta'(u)f(u)_x = 0 \tag{4.51}$$

Using the chain rule changes this equation to

$$\frac{\partial \eta(u)}{\partial t} + \eta'(u)f'(u)\frac{\partial u}{\partial x} = 0 \tag{4.52}$$

Comparing this equation with (4.50) we see that the following compatibility condition should be satisfied.

$$\eta'(u)f'(u) = q'(u) \tag{4.53}$$

Now, let us relax our assumption on u so that we allow piecewise C^1 solution to (4.49) in the weak sense. Now according to the Rankine-Hugoniot condition, the speed of shock wave for u will be

$$\lambda = \frac{[f(\rho)]}{[\rho(u)]} \tag{4.54}$$

where

$$[f(\rho)] = f(\rho_r) - f(\rho_\ell) \text{ and } [\rho(u)] = \rho_r - \rho_\ell \tag{4.55}$$

The same shock speed must also satisfy

$$\lambda = \frac{[q(\rho)]}{[\eta(u)]} \tag{4.56}$$

Satisfying (4.54) and (4.56) simultaneously is generally too restrictive as is clear from applying this to the Cauchy problem for non-viscous Burger's equation.

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)=0$$
$$u(0,x)=\begin{cases} u_\ell\,, & x<0\\ u_r\,, & x>0\end{cases} \tag{4.57}$$

We can use the following entropy pair for this problem.

$$\eta(u)=u^k \text{ and } q(u)=\frac{k}{k+1}u^{k+1} \text{ for } k\neq 1 \tag{4.58}$$

Then

$$\lambda=\frac{[f(\rho)]}{[\rho(u)]}=\frac{1}{2} \tag{4.59}$$

However

$$\lambda=\frac{[q(\rho)]}{[\eta(u)]}=\frac{k}{k+1} \tag{4.60}$$

This shows that the entropy equality condition is too restrictive. However, if we use the inequality, then it is just right. To see this, let us consider the viscous perturbation of (4.49)

$$u_t^\epsilon+f(u^\epsilon)_x=\epsilon\Delta u^\epsilon \tag{4.61}$$

Multiplying (4.61) with $\eta'(u^\epsilon)$ gives

$$\eta'(u^\epsilon)u_t^\epsilon+\eta'(u^\epsilon)f(u^\epsilon)_x=\epsilon\eta'(u^\epsilon)\Delta u^\epsilon \tag{4.62}$$

Using the chain rule changes this equation to

$$\frac{\partial\eta(u^\epsilon)}{\partial t}+\eta'(u^\epsilon)f'(u^\epsilon)\frac{\partial u^\epsilon}{\partial x}=\epsilon\eta'(u^\epsilon)\Delta u^\epsilon \tag{4.63}$$

Using the compatibility condition

$$\eta'(u^\epsilon)f'(u^\epsilon)=q'(u^\epsilon) \tag{4.64}$$

we obtain

$$\frac{\partial\eta(u^\epsilon)}{\partial t}+\frac{\partial q(u^\epsilon)}{\partial x}=\epsilon\Delta\eta(u^\epsilon)-\epsilon\eta''(u^\epsilon)\left|\nabla u^\epsilon\right|^2 \tag{4.65}$$

Taking a convex η, we obtain the inequality

$$\frac{\partial \eta(u^\epsilon)}{\partial t} + \frac{\partial q(u^\epsilon)}{\partial x} \le \epsilon \Delta \eta(u^\epsilon) \tag{4.66}$$

This equation can be viewed as the viscous perturbation of the inequality

$$\frac{\partial \eta(u^\epsilon)}{\partial t} + \frac{\partial q(u^\epsilon)}{\partial x} \le 0 \tag{4.67}$$

and it can be shown that equation (4.66) converges to (4.67) (see [JMR96]).

4.5.3 Lax Admissibility Condition

A weak solution u of 5.40 is Lax admissible if at every point of approximate discontinuity, the left state u_ℓ, the right state u_r and the shock speed λ are related as

$$\lambda(u_\ell) \ge \lambda \ge \lambda(u_r) \tag{4.68}$$

4.6 Kruzkov's Entropy Function

For a scalar balance law

$$u_t + f(x,t,u)_x = g(x,t,u) \tag{4.69}$$

with initial condition

$$u(x,0) = u_0(x). \tag{4.70}$$

To obtain entropy enabled generalized solution to the problem (4.69) with (4.70) we can use entropy function proposed by Kruzkov ([Kru70]).

Let $\Pi_T = R \times [0,T]$. Let $u_0(x)$ be a bounded measurable function satisfying $|u_0(x)| \le M_0 \ \forall x \in R$ on R.

Definition 4.8. A bounded measurable function $u(x,t)$ is called a generalized solution of problem (4.69) and (4.70) Π_T if:

i) for any constant k and any smooth function $\phi(x,t) \ge 0$ finite in Π_T ($supp(\phi) \subset \Pi_T$ strictly), if the following inequality holds,

$$\int\int_{\Pi_T} \{|u(x,t)-k|\phi_t + \mathrm{sign}(u(x,t)-k)[f(x,t,u(x,t)) - f(x,t,k)]\phi_x -$$

$$-\mathrm{sign}(u(x,t)-k)[f_x(x,t,u(x,t)) - g(x,t,u(x,t))]\}dxdt \ge 0 \tag{4.71}$$

ii) there exists a set E of zero measure on $[0,T]$, such that for $t \in [0,T]\backslash E$, the function $u(x,t)$ is defined almost everywhere in R, and for any ball $K_r = \{|x| \leq r\}$

$$\lim_{t\to 0}\int_{K_r} |u(x,t) - u_0(x)|dx = 0.$$

□

Inequality (4.82) is equivalent to condition E in [Ole63], if (u_-, u_+) is a discontinuity of u and v is any number between u_- and u_+, then

$$\frac{f(x,t,v) - f(x,t,u_-)}{v - u_-} \geq \frac{f(x,t,u_+) - f(x,t,u_-)}{u_+ - u_-} \tag{4.72}$$

The Kruzkov condition comes from using the following entopy flux pair.

$$\eta(u) = |u-k| \text{ and } q(u) = sign(u-k)\cdot(f(u) - f(k)) \tag{4.73}$$

4.7 Well-posedness

There are many methods to prove well-posedness of conservation laws (one and multi-dimensional Cauchy scalar case, one dimensional Cauchy systems case, boundary-initial value problems ([CBN79] and [Ott96]), for balance laws, and relaxation systems). These methods include vanishing viscosity method ([Lax72]), Glimm scheme ([Gli65]), front tracking method ([DiP76], [Bre05], [HR02]) and evolutionary integral equation regularization ([BCL96]).

4.7.1 Solution Properties for Scalar Cauchy Problem

Here we summarize the properties of the solution to the scalar conservation law from [HR02].

Theorem 4.9. *Given the initial data $u_0 \in BV \cap L^1$ and the corresponding flux $f(u) \in C_{Lip}$,, then the unique weak entropy solution $u(t,x)$ to the Cauchy problem*

$$u_t + f(u)_x = 0\,,\;\; u(0,x) = u_0(x) \tag{4.74}$$

satisfies the following properties for $t \in R^+$:

1. Maximum Principle*:*

$$\|u(t,\cdot)\|_\infty \leq \|u_0\|_\infty$$

2. Total variation diminishing*:*

$$TV(u(t,\cdot)) \leq TV(u_0)$$

3. L^1 Contractive*: If v_0 and $v(t,x)$ is another pair of admissible initial data and the corresponding solution, then*

$$\|u(t,\cdot) - v(t,\cdot)\| \leq \|u_0 - v_0\|$$

4. Monotonicity Preserving*:*

$$u_0 \textit{ monotone} \Rightarrow u(t,\cdot) \textit{ monotone}$$

5. Monotonicity*: If v_0 and $v(t,x)$ is another pair of admissible initial data and the corresponding solution, then*

$$u_0 \leq v_0 \Rightarrow u(t,\cdot) \leq v(t,\cdot)$$

6. Lipschitz Continuity in time*:*

$$\|u(t,\cdot) - u(s,\cdot)\|_1 \leq \left\|f\right\|_{Lip} TV(u_0)\,|t-s|$$

$\forall s,\, t \in R^+$

$\boxdot$

4.8 Oleinik Entropy Condition

We present here an alternate definition of the entropy admissible solution ([Eva98]) that uses Oleinik entropy condition ([Ole63]).

Definition 4.10. A function $u \in L^\infty(R \times (0,\infty))$ is an *(Oleinik)* entropy solution of the Cauchy problem

$$\begin{cases} u_t + f(u)_x = 0 & \text{in } R \times (0,\infty) \\ u = u_0 & \text{on } R \times \{t = 0\} \end{cases} \tag{4.75}$$

if for all test functions $\phi : R \times [0,\infty) \to R$ with compact support

$$\int_0^\infty \int_{-\infty}^\infty u\phi_t + f(u)\phi_x dx dt + \int_{-\infty}^\infty u_0 \phi dx\,|_{t=0} = 0 \tag{4.76}$$

and or some constant $C \geq 0$ and a.e. $x, z \in R$, $t > 0$, and $z > 0$ the following Oleinik entropy condition is satisfed.

$$u(x+z,t) - u(x,t) \leq C\left(1 + \frac{1}{t}\right) z \tag{4.77}$$

$\square$

4.8.1 Sup-norm Decay of the Solution

Assume that flux f is smooth, uniformly convex, satisfies $f(0) = 0$ and that the initial data u_0 is bounded and summable (integrable with a finite integral), then

Theorem 4.11. *The solution of $u(t,x)$ satisfies the following bound*

$$|u(t,x)| \le \frac{C}{t^{1/2}} \tag{4.78}$$

$\boxdot$

$\forall x \in R,\ t > 0.$

This theorem shows that the L^∞ norm of the solution u goes to zero as $t \to \infty$. It can be shown that ([Eva98]) the solution converges in the L^1 norm to an N-wave.

4.9 Scalar Initial-Boundary Problem

For a scalar conservation law

$$u_t + f(t,x,u)_x = 0 \tag{4.79}$$

with initial condition

$$u(0,x) = u_0(x), \tag{4.80}$$

and boundary conditions

$$u(t,a) = u_a(t) \text{ and } u(t,b) = u_b(t), \tag{4.81}$$

the definition of the generalized solutions of problem (5.44) with (5.45) is presented here. The boundary conditions cannot be prescribed point-wise, since characteristics from inside the domain might be traveling outside at the boundary. If there are any data at the boundary for that time, that has to be discarded. Moreover, the data also must satisfy entropy condition at the boundary so as to render the problem well-posed. This is shown in Figure 4.18 where for some time boundary data on the left can be prescribed when characteristics from the boundary can be *pushed in* (see [SB06]). However when the characteristics are coming from inside, the boundary data can not be prescribed.

Let $\Pi_T = [0,T] \times [a,b]$. Let $u_0(x)$ be a bounded measurable function satisfying $|u_0(x)| \le M_0\ \forall x \in [a,b]$ on R.

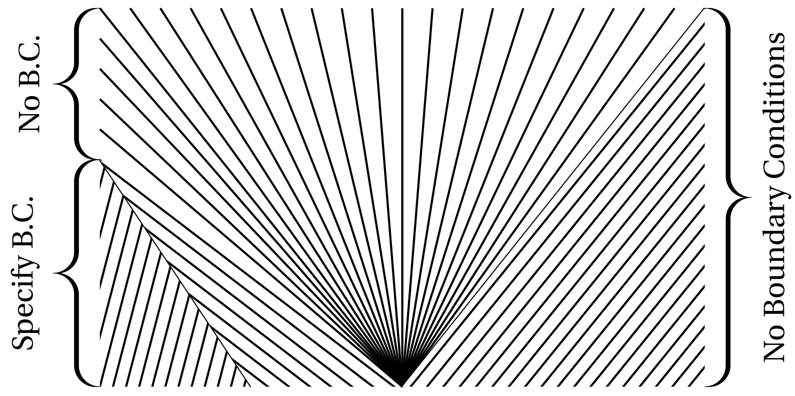

Figure 4.18: *Boundary Data*

Definition A bounded measurable function $u(t,x)$ is called a generalized solution of problem (5.44) with (5.45) in Π_T if:

i) for any constant k and any smooth function $\phi(t,x) \geq 0$ finite in Π_T ($supp(\phi) \subset \Pi_T$ strictly), if the following inequality holds,

$$\int\int_{\Pi_T} \{|u(t,x)-k|\phi_t + \text{sign}(u(t,x)-k)[f(t,x,u(t,x)) - f(t,x,k)]\phi_x -$$

$$-\text{sign}(u(t,x)-k) f_x(t,x,u(x,t))\} dx dt \geq 0; \qquad (4.82)$$

ii) there exists sets E, E_ℓ and E_r of zero measure on $[0,T]$, such that for $t \in [0,T] \backslash E$, the function $u(t,x)$ is defined almost everywhere in $[a,b]$, and for any ball $K_r = \{|x| \leq r\}$

$$\lim_{t \to 0} \int_{K_r} |u(t,x) - u_0(x)| dx = 0.$$

$$\lim_{x \to a, x \notin E_\ell} \int_0^T L(u(t,x), u_a(t))\phi(t) dt = 0.$$

$$\lim_{x \to b, x \notin E_r} \int_0^T R(u(t,x), u_b(t))\phi(t) dt = 0.$$

where

$$L(x,y) = \sup_{k \in I(x,y)} (sign(x-y)(f(x) - f(k)))$$

$$R(x,y) = \inf_{k \in I(x,y)} (sign(x-y)(f(x)-f(k)))$$

and

$$I(x,y) = [\inf(x,y), \sup(x,y)]$$

CHAPTER 5

Dynamic Traffic Assignment: A Survey of Mathematical Models and Techniques

This chapter presents a survey of the mathematical methods used for modeling and solutions for the traffic assignment problem. It covers the static (steady state) traffic assignment techniques as well as dynamic traffic assignment in lumped parameter and distributed parameter settings. Moreover, it also surveys simulation based solutions. The chapter shows the models for static assignment, variational inequality method, projection dynamics for dynamic travel routing, discrete time and continuous time dynamic traffic assignment and macroscopic Dynamic Traffic Assignment (DTA). The chapter then presents the macroscopic DTA in terms of the Wardrop principle and derives a partial differential equation for experienced travel time function that can be integrated with the macroscopic DTA framework.

5.1 Introduction

Traffic assignment is an integral part of the four-stage transportation planning process (see [Gaz74] and [PO72]) that includes:

1. **Trip Generation**: Trip generation models estimate the number of trips generated at origin nodes and/or the number of trips attracted to destination nodes based on factors such as household income, demograph-

ics, and land-use pattern. This data is obtained using surveys conducted periodically.

2. **Trip Distribution**: From the total number trips generated and attracted at each node, trip distribution algorithms generate an origin-destination (O-D) matrix, in which each cell entry indicates the number of trips from one specific origin to one specific destination. Hitchcock model [Hit41], opportunity model [Sto40], gravity model [Voo56], and entropy models [Wil67] have been used for trip distribution algorithms.
3. **Modal Split**: Modal split analysis takes each cell value in the O-D matrix and divides it among various alternate modes of travel. The models are built based on performing discrete choice analysis on survey data (see [BAL85]).
4. **Traffic Assignment**: This step assigns each O-D flow value onto various alternate paths from that specific origin to the destination node. Assignments is based on optimization, usually using either Wardrop's user-equilibrium ([War52], [She85]) or system optimum.

This four-step process comes from the traditional transportation planning area and is not designed for real-time operations, such as traffic responsive real-time incident management. However, a lot of research has taken place in the area of traffic assignment, especially dynamic traffic assignment that enables researchers to study transient traffic behavior, not just steady state one which the static assignment is designed for. A survey paper [PZ01] provides an excellent survey for the research work that has been performed in the area of dynamic traffic assignment. This chapter, in contrast to that survey work provides a survey of the mathematical framework that has been used in this area, and presents the results to enable the reader to grasp the various mathematical tools that have been used to study and analyze this problem. The models and approaches that have been used are varied, and this review chapter brings them together in order for the readers to see them in a somewhat linear fashion.

Outline The remainder of this chapter is organized as follows. Section 5.2 gives account of various mathematical programming based static traffic assignment models that have been used. This section presents the user-equilibrium and system optimal formulations of the assignment problems, followed by the numerical schemes that have been used to solve those problems. Section 5.3 presents the fundamentals of the variational inequality framework which subsumes the mathematical programming methodology. Dynamic extension of the variational inequality framework is presented in the section 5.4. Section 5.5 presents the dynamic traffic assignment in continuous time. The discrete time and continuous time versions of this are presented. Section 5.7 presents the macroscopic DTA model including the new formulation and a new travel time partial differential equation. Section 5.8 presents a brief summary of the main features of simulation based DTA. Finally, Section 5.10 gives the conclusions.

5.2 Mathematical Programming for Static Traffic Assignment

To build the mathematical framework for our chapter, we will start with terminology and framework used in [She85]. We illustrate a sample network that is also taken from [She85] and is shown in Figure 5.1. The digraph shows four nodes and four arcs. Nodes 1 and 2 are origin nodes and node 4 is the destination node. Hence there are two O-D pairs: $1-4$ and $2-4$.

Table 5.1: Network Notation

$\mathfrak{N}$	Set of Nodes
$\mathfrak{A}$	Set of Arcs
$\mathfrak{R}$	Set of Origin Nodes
$\mathfrak{S}$	Set of Destination Nodes
$\mathfrak{K}$	Set of Paths connecting O-D pair $r-s$, $r \in \mathfrak{R}$, $s \in \mathfrak{S}$
x_a	Flow on arc $a \in \mathfrak{A}$
t_a	Travel time on arc $a \in \mathfrak{A}$
f_k^{rs}	Flow on path $k \in \mathfrak{K}$ between O-D pair $r-s$
c_k^{rs}	Travel time on path $k \in \mathfrak{K}$ between O-D pair $r-s$
q_{rs}	O-D Trip rate between O-D pair $r-s$
$\delta_{a,k}^{rs}$	$\delta_{a,k}^{rs} = 1$, if a is in path k between r and s, otherwise 0

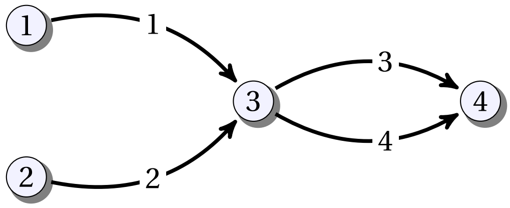

Figure 5.1: *Sample Network*

There are two main classical traffic assignment optimization problems considered. Those two are: user-equilibrium, and system optimum.

5.2.1 User-equilibrium

User-equilibrium problem is based on Wardrop's principle [War52] which is stated as:

The journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.

This equilibrium condition can be obtained as a solution of a mathematical programming problem presented below [She85].

Mathematical Programming Formulation

The user equilibrium problem is stated as the mathematical programming problem (see [She85], [DS69b]) shown in Equation 5.1.

$$\min z(x) = \sum_a \int_0^{x_a} t_a(\omega)\,d\omega \tag{5.1}$$

with the equality constraints:

$$\sum_k f_k^{rs} = q_{rs}\,\forall r,s \tag{5.2}$$

$$x_a = \sum_r \sum_s \sum_k f_k^{rs} \delta_{a,k}^{rs} \tag{5.3}$$

and the inequality constraint

$$f_k^{rs} \geq 0\,\forall r,s \tag{5.4}$$

The formulation given in Equation 5.1 is the Beckmann transformation [BMW55]. The link performance function $t_a(x_a)$ is a function of traffic flow on the link and the link capacity c_a. According to the Bureau of Public Roads (BPR) it is given by Equation 5.5

$$t_a(x_a) = v_f \left(1 + 0.15 \left(\frac{x_a}{c_a}\right)^4\right) \tag{5.5}$$

The plot of a BPR function is shown in Figure 5.2

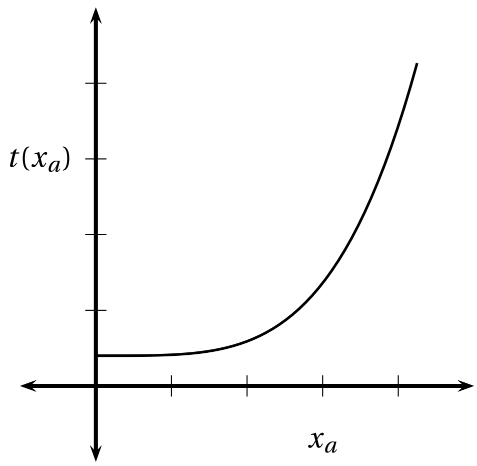

Figure 5.2: *BPR Link Performance Function*

Wellposedness The objective function is a smooth convex function ($\nabla^2(x)$ is positive definite), and the feasible region is convex, hence a unique solution exists.

Equivalence with Wardrop User-equilibrium Condition

The Kuhn-Tucker conditions for the mathematical programming problem given by Equation 5.1 can be obtained in terms of the Lagrangian given in Equation 5.6.

$$\mathfrak{L}(f,\lambda) = z[x(f)] + \sum_{rs} \lambda_{rs} \left(q_{rs} - \sum_k f_k^{rs}\right) \tag{5.6}$$

Here, λ_{rs} is the Lagrangian multiplier. The Kuhn-Tucker conditions $\forall k, r, s$ are:

$$
\begin{aligned}
& f_k^{rs} \frac{\partial \mathfrak{L}(f,\lambda)}{\partial f_k^{rs}} = 0 \\
& \frac{\partial \mathfrak{L}(f,\lambda)}{\partial f_k^{rs}} \geq 0 \\
& \frac{\partial \mathfrak{L}(f,\lambda)}{\partial \lambda^{rs}} = 0
\end{aligned}
\tag{5.7}
$$

Applying these necessary conditions 5.7 to the mathematical program 5.1 we obtain the Wardrop conditions $\forall k, r, s$ as:

$$
\begin{aligned}
& f_k^{rs}(c_k^{rs} - u_{rs}) = 0 \\
& c_k^{rs} - u_{rs} \geq 0 \\
& \sum_k f_k^{rs} = q_{rs} \\
& \sum_k f_k^{rs} \geq 0
\end{aligned}
\tag{5.8}
$$

5.2.2 System Optimal Solution

System optimal solution is a solution that provides the total minimum time for the entire network. This condition can be obtained as a solution of a mathematical programming problem presented below [She85].

Mathematical Programming Formulation

The system optimal problem is stated as the mathematical programming problem (see [She85], [DS69b]) shown in Equation 5.9.

$$
\min z(x) = \sum_a x_a t_a(x_a) \tag{5.9}
$$

with the equality constraints:

$$
\sum_k f_k^{rs} = q_{rs} \,\forall r,s \tag{5.10}
$$

$$
x_a = \sum_r \sum_s \sum_k f_k^{rs} \delta_{a,k}^{rs} \tag{5.11}
$$

and the inequality constraint

$$
f_k^{rs} \geq 0 \,\forall r,s \tag{5.12}
$$

Wellposedness The objective function is a smooth convex function ($\nabla^2(x)$ is positive definite), and the feasible region is convex, hence a unique solution exists.

Equivalence with Marginal User-equilibrium Condition

Applying Kuhn-Tucker conditions in this case we get $\forall k, r, s$:

$$\begin{aligned}
&f_k^{rs}(\tilde{c}_k^{rs} - \tilde{u}_{rs}) = 0 \\
&\tilde{c}_k^{rs} - \tilde{u}_{rs} \geq 0 \\
&\sum_k f_k^{rs} = q_{rs} \\
&\sum_k f_k^{rs} \geq 0
\end{aligned} \tag{5.13}$$

Here, we have

$$\tilde{c}_k^{rs} = \sum_a \delta_{a,k}^{rs} \tilde{t}_a \tag{5.14}$$

where

$$\tilde{t}_a(x_a) = t_a(x_a) + x_a \frac{d t_a(x_a)}{x_a} \tag{5.15}$$

5.2.3 Numerical Schemes

The numerical scheme for solving user-equilibrium is based on the Frank-Wolfe algorithm that obtains the feasible direction and the maximum step-size for each iteration in one step. In fact, for the static traffic assignment problem, this amounts to simply applying all or nothing assignment to the shortest path for each O-D pair. The next step for each iteration involves finding the step size in the direction of the link flow solution of the all-or-nothing assignment step. Appropriate stopping criterion can be applied using some convergence principle. Details of this are provided in section 5.2, pages 116 – 122 of [She85].

There are heuristic numerical methods available to perform the assignment to achieve user-equilibrium. Two of the common heuristic techniques are:

FHWA (modified capacity restraint) method In this method at each iteration an all-or-nothing assignment of the entire OD flow is performed on a single path. Travel times are updated by performing a weighted average of the travel time obtained by the latest assignment and the previous one. A convergence criterion is used to stop the iteration steps (for instance when the maximum difference between two iterative steps of link flows is less than some ϵ). The final link flows assigned to the network are obtained by averaging the values from the last four iterative steps.

Incremental Assignment In incremental assignment, the OD values are divided into n parts, and then each part is assigned to the network using all or nothing assignment based on the previous travel time values.

Dafermos ([DS69a]) applied the Frank Wolfe method to traffic assignment problem. This method also results in an all or nothing assignment, followed by a line search step in each iteration. The details can be obtained from [She85].

5.3 Variational Inequality based Static Traffic Assignment Model

Variational inequality formulation for traffic equilibrium has been used as it generalizes the framework of mathematical programming even when the travel time function on one link depends on the conditions on other links as well ([Daf80]). Once the variational inequality model has been formulated, it can be solved using some appropriate numerical scheme, such as the ones based on projection method, linear approximation, relaxation method, or the more general iterative scheme of Dafermos ([Daf83]).

The variational inequality problem is stated as:

VI Problem: Given a continuous function $f : \mathbb{K} \to \mathbb{R}^n$, where $\mathbb{K}$ is a given closed and convex subset of $\mathbb{R}^n$, $\langle \cdot, \cdot \rangle$ denotes the inner product, find $x \in \mathbb{K}$, such that

$$\langle f(x), y - x \rangle \geq 0, \forall y \in \mathbb{K} \tag{5.16}$$

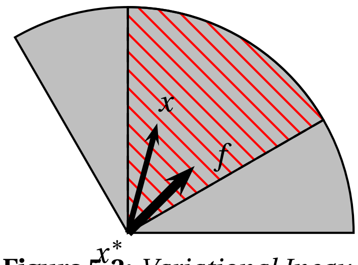

Figure 5.3: *Variational Inequality*

Figure 5.3 shows a convex set and the variational inequality condition at a corner. The relationship between variational inequalities and optimization problems is given by the following two theorems ([KS00]).

Theorem 5.1. $x \in \mathbb{K}$ *s.t.* $f(x) = \min_{y \in \mathbb{K}} f(y) \implies \langle \nabla f(x), y - x \rangle \geq 0, \forall y \in \mathbb{K}$. $\boxdot$

Theorem 5.2. *Convex* f *s.t.* $\langle \nabla f(x), y - x \rangle \geq 0, \forall y \in \mathbb{K} \implies f(x) = \min_{y \in \mathbb{K}} f(y)$. $\boxdot$

To understand the constrained optimization problem and its interplay with variational inequalities, we present two figures (Figure 5.4 and Figure 5.5). The first quadrant in the $x - y$ plane is the constrained region of search where we have assumed that $h(x, y) \geq 0$ is satisfied. The function to be minimized is given by $f(x, y)$. Figure 5.4 shows the case when the minimizing point (on the $x - y$ plane) for a given smooth cost function $f(x, y)$ is contained in the interior of the region $\mathbb{K}$ given by $h(x, y) \geq 0$. For the local minimum to exist, it is necessary that the gradient of the function is zero. Figure 5.5 shows the case when the minimizing point (on the $x - y$ plane) for a given smooth cost function $f(x, y)$ is contained at the boundary of the region $\mathbb{K}$ given by $h(x, y) = 0$. For the given point to be the minimizer, any movement from this point in any feasible direction, i.e. in the direction of increasing $h(x, y)$, should increase the value of $f(x, y)$. This is the variational inequality statement. Moreover, in this case (when certain regularity conditions are satisfied ([Avr03])), since, the boundary is given by $h(x, y) = 0$, the directional derivative of $f(x, y)$ in the direction of the tangent to the boundary should be zero. Moreover, the gradient of $h(x, y)$ as well as that of $f(x, y)$ should be pointing in the same direction. Kuhn-Tucker conditions (and Lagrangian method) state the condition on the relationship between the gradient of the cost function and that of the constraint functions. However, those are necessary conditions only if the problem satisfies certain regularity conditions (see [Avr03], [BSS06], and [Man94]).

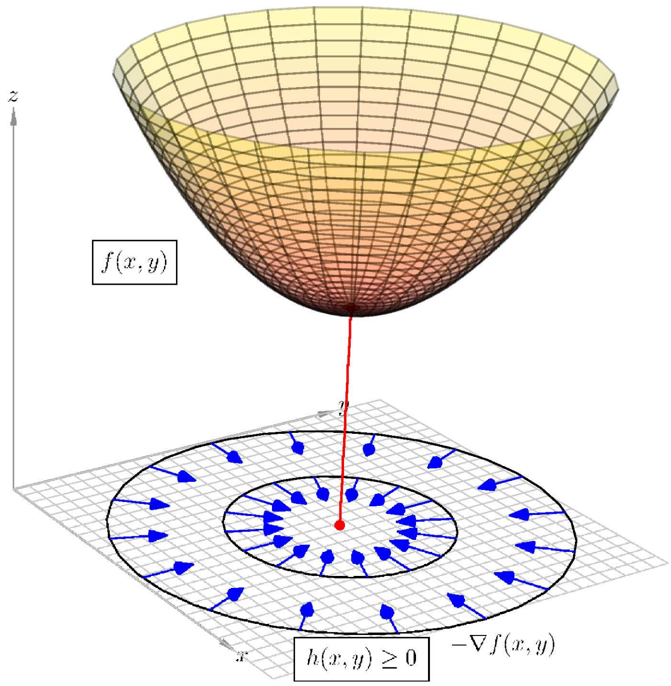

Figure 5.4: *Minimizer in the Interior*

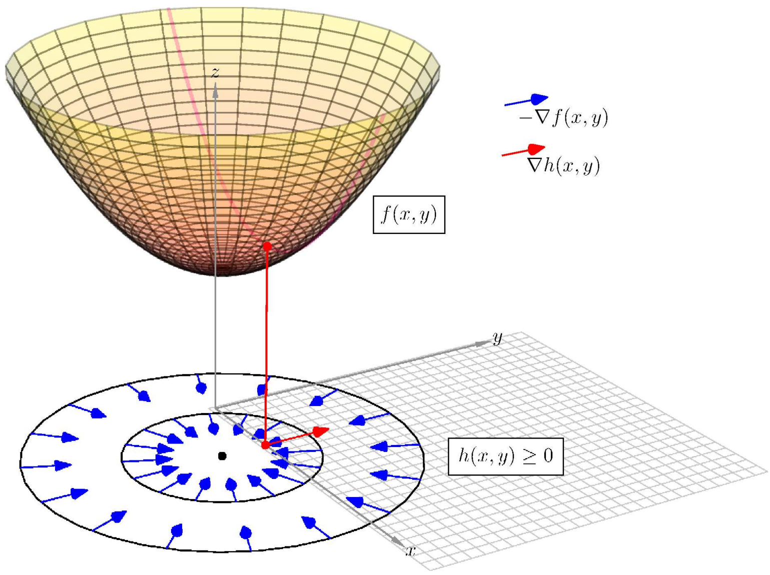

Figure 5.5: *Minimizer on the Boundary*

The theorems 5.1 and 5.2 demonstrate that variational inequality framework is more general than the mathematical programming framework. The variational inequality formulations of the traffic equilibrium (user) problems are stated below.

Theorem 5.3. *$x \in \mathbb{K}$ is a solution to the user equilibrium problem if and only*

$$\sum_{w \in W} \sum_{p \in P_w} C_p(x)(y - x) \geq 0, \forall x \in \mathbb{K}$$

$\boxdot$

Here, C_p is the travel time for the path p from the OD pair P_w from the set of OD pairs W. This variational inequality can also be written in terms of traffic flows instead of link flows ([Nug00]).

To understand how the variational inequality formulation is more general than the optimization problem, consider the variationa inequality formulation again.

$$\langle f(x), y - x \rangle \geq 0, \forall y \in \mathbb{K} \tag{5.17}$$

Now, if $f(x) = \nabla\theta(x)$, then the condition

$$\langle \nabla\theta(x), y - x \rangle \geq 0, \forall y \in \mathbb{K} \tag{5.18}$$

is the necessary condition for the optimization problem

$$\text{minimize}\, \theta(x),\ x \in \mathbb{K} \tag{5.19}$$

The variational inequality has a corresponding gradient relationship based on the following theorem that is about the symmetry of second partial derivatives ([FP03]).

Theorem 5.4. *Given $f : \mathbb{K} \to \mathbb{R}^n$, a continuously differentiable function on the open convex set $\mathbb{K} \subset \mathbb{R}^n$, then the following three conditions are equivalent.*

1. *$\exists\theta$, s.t. $f(x) = \nabla\theta(x)$*
2. *$\nabla f(x) = [\nabla f(x)]^T\ \forall x \in \mathbb{K}$*
3. *f is integrable on $\mathbb{K}$*

$\boxdot$

Theorem 5.4 shows that if the function f has a symmetric Jacobian then there is a corresponding optimization problem associated with it. However, if the Jacobian is asymmetric, for instance, when the user-equilibrium cost is asymmetric with respect to traffic flows, then the Wardrop solution (variational inequality) is the framework without a corresponding mathematical programming problem.

On a cautionary note, Kuhn-Tucker conditions (and Lagrangian method) state the condition on the relationship between the gradient of the cost function and that of the constraint functions. However, those are necessary conditions only if the problem satisfies certain regularity conditions (see [Avr03], [BSS06], and [Man94]). For instance Figure 5.6 shows a function $f(x,y) = -x$ to be minimized which at the minimum point $(x,y) = (1,0)$ does not satisfy the Kuhn-Tucker conditions for the region constrained by the first quadrant and the curve $y = 1 - x^3$.

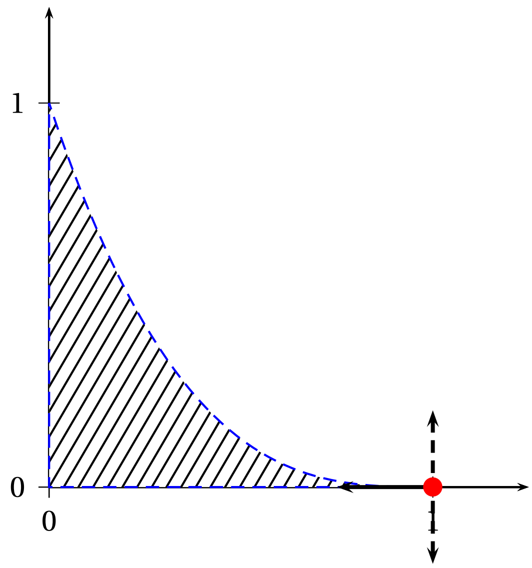

Figure 5.6: *Violation of Kuhn-Tucker Condition*

5.4 Projected Dynamical Systems: Dynamic Variational Equation Model

Dynamics of route switching has been analyzed using dynamic variational inequality by Nagurney and Zhang ([NZ96], [ZN95], [NZ97], [ZN96], and [Daf88]). They developed the theory for projected dynamical systems in [ZN95], and applied the theory to traffic assignment in [ZN96] and [NZ97]. The paper by Dupuis and Nagurney ([DN93]) shows the main results in the theory and applications of projected dynamical systems including its relationship to the Skorokhod problem ([Sko61]) for the study of its wellposedness.

Since variational inequality is related to the solution of a fixed point problem, we can related the variational inequality solution to be the equilibrium point of a dynamic system. The stability of the equilibrium point can be studied within the framework of this dynamic system, and then those dynamics can be used to model a time varying route assignment problem. This is precisely what Nagurney and Zhang do in their various papers. We summarize the technical results here.

5.4.1 Dynamic Route Choice

The dynamics of route choice adjustment are given by ([NZ96]):

$$\dot{x} = \Pi_{\mathbb{K}}(x, -C(x)) \tag{5.20}$$

where

$$\Pi_{\mathbb{K}}(x, v) = \lim_{\epsilon \to 0} \frac{P_{\mathbb{K}}(+\epsilon v) - x}{\epsilon} \tag{5.21}$$

and

$$P_{\mathbb{K}}(x) = Argmin_{x \in \mathbb{K}} \|x - z\| \tag{5.22}$$

Figure 5.7 shows the convex region inside which the vector field of the dynamics are shown. The equilibrium point as well as the solution of the variational inequality is at $(0,0)$.

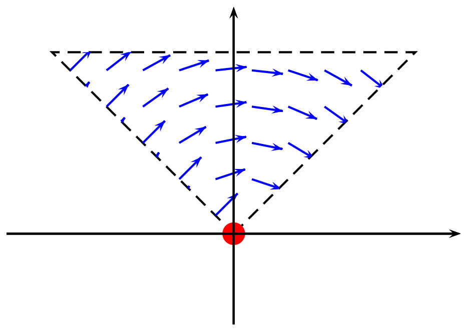

Figure 5.7: *The vector field*

The path flow vector $x^* \in \mathbb{K}$ is the solution of

$$0 = \Pi_{\mathbb{K}}(x^*, -C(x^*)) \tag{5.23}$$

if and only if it satisfies

$$\langle C(x^*), x - x^* \rangle \geq 0, \forall y \in \mathbb{K} \tag{5.24}$$

The following theorem from [NZ96] gives the condition for asymptotic stability of the equilibrium point of the projected dynamics related to the route adjustment process.

Theorem 5.5. *If the link cost is a strictly monotonic continuous function of link flows, then the equilibrium point for dynamics shown in Equation 5.20 is asymptotically stable.* ⊡

The major result from [NZ96] for applying the discrete algorithm for the dynamic route choice problem is the following.

Theorem 5.6. *The Euler method given by*

$$x^{\tau+1} = P_{\mathbb{K}}(x^\tau - a_\tau C(x^\tau)) \tag{5.25}$$

when

$$\lim_{\tau \to \infty} a_\tau = 0 \tag{5.26}$$

and

$$\sum_{\tau=1}^{\infty} a_\tau = \infty \tag{5.27}$$

for $\mathbb{K}$ *being the positive orthant converges to some traffic network equilibrium path flow.* ⊡

5.5 Dynamic Traffic Assignment

There are some nice reviews that provide summary of the models and work that has been performed in the area of Dynamic Traffic Assignment (DTA), such as [CBM$^+$09], [PZ01], [RB96], and [Fri01]. Our review will focus on the mathematical aspects of these developments.

5.5.1 Dynamic Traffic Assignment: Discrete Time

Merchant and Nemhauser ([MN78a] and [MN78b]) were the first to present a dynamic traffic assignment problem where time varying O-D flows are considered. Their formulation uses a state difference equation to represent the link dynamics, a conservation equation at the nodes of the digraph, and a cost function to minimize which leads to the following mathematical programming problem.

$$\min z(x) = \sum_{i=1}^{I} \sum_{j=1}^{a} t_{ij}(x_{ij}) \tag{5.28}$$

with the link discrete time dynamics as equality constraints:

$$x_j[i+1] = x_j[i] - g_j(x_j[i]) + d_j[i], i = 0, 1, \cdots I-1, \forall j \in \mathfrak{A} \tag{5.29}$$

the node conservation equation as

$$\sum_{j \in A(q)} d_j[i] = F_q[i] + \sum_{j \in B(q)} g_j(x_j[i]),\ i = 0, 1, \cdots I-1, \forall q \in \mathfrak{N} \tag{5.30}$$

and the inequality constraints

$$x_j[i] \geq 0\ i = 0, 1, \cdots I-1,\ \forall j \in \mathfrak{A} \tag{5.31}$$

$$d_j[i] \geq 0\ i = 0, 1, \cdots I-1,\ \forall j \in \mathfrak{A} \tag{5.32}$$

$$x_j[0] = x_0[j]\ \forall j \in \mathfrak{A} \tag{5.33}$$

Here, $x_j[i]$ is the number of vehicles at the beginning of time period i in link j, $g_j(x_j[i])$ is the number of vehicles exiting the link in the unit time as a function of $x_j[i]$, and $d_j[i]$ is the number of vehicles entering the link j. This problem formulation is a single destination network model. $F_q[i]$ show the inflow rates as the time varyting O-D flows. This can be extended to a multi orgin multi destination formulation.

5.5.2 Dynamic Traffic Assignment: Continuous Time

Now we present a continuous time formulation of the DTA problem ([BLR01]) where a dynamic variational inequality is used. The traffic dynamics utilize ordinary differential equations instead of finite difference equation as was the case for the discrete time formulation. There are other models that use dynamic continuous time models in optimal control or variational setting such as [FLTW89], [FBS$^+$93], and [Che99].

The time dependent Wardrop condition for the DTA are

$$\begin{aligned}
&f_k^{rs}(t)(c_k^{rs}(t) - u_{rs}(t)) = 0 \\
&c_k^{rs}(t) - u_{rs}(t) \geq 0 \\
&\sum_k f_k^{rs}(t) = q_{rs}(t) \\
&\sum_k f_k^{rs}(t) \geq 0
\end{aligned} \tag{5.34}$$

The traffic dynamics for this DTA problem are the continuous version of the difference equation for the Merchant Nemhauser model, and are given by the following conservation ordinary differential equation.

$$\dot{x}_{ak}^{rs}(t) = u_{ak}^{rs}(t) - g_{ak}^{rs}(x_a(t)) \tag{5.35}$$

Here, $u_{ak}^{rs}(t)$ is the time varying inflow to link a on path k from origin r to destination s, and $g_{ak}^{rs}(x^{rs}(t))$ is the corresponding time varying outflow which is the exit function which depends on the link density $x_a(t)$.

We have the following equality among matching constraints for various flows and links ([BLR01]).

$$\sum_r \sum_s \sum_k x_{ak}^{rs}(t)\delta_{a,k}^{rs} = x_a(t) \tag{5.36}$$

Numerical techniques are available to solve this variational inequality (see [BLR01]). Optimal control formulation for this problem can also be obtained which can be solved by calculus of variations or dynamic programming methods.

5.6 Travel Time and FIFO Issue

One major issue in dynamic traffic assignment problem is that of First In First Out (FIFO) constraint as discussed in [Car92]. According to FIFO if $x_{t\tau a} > 0$ where $x_{t\tau a}$ is the traffic flow that enters link a at time t and exits at time τ, then any flow that enters before time t can not exit after time τ at an average. This condition is shown to be nonconvex in [Car92] and is presented in Equation 5.37.

$$(x_{t\tau a} > 0) \Rightarrow \left(\sum_{t'\tau' a} x_{t'\tau' a} | t' < t, \tau' > \tau \right) = 0 \tag{5.37}$$

A violation of this condition is shown in Figure 5.8. The violation essentially occurs because of the nature of the exit function and also the time and space discretization of the traffic link and dynamics. Both of these issues get resolved by a proper choice of space and time discretization that are chosen after the original modeling is performed in a hydrodynamic setting using the dynamic distributed parameter traffic flow theory. This theory allows for a proper development of a travel time function as well as a travel time vector field. This development is the main original technical contribution of this chapter.

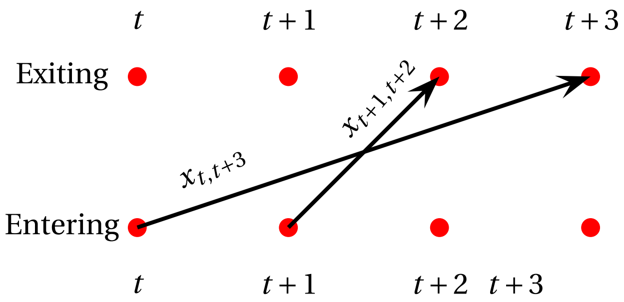

Figure 5.8: *FIFO Violation*

5.7 Macroscopic Model for DTA

We propose to use a hydrodynamic traffic model in the framework of the DTA problem. The Lighthill-Whitham-Richards (LWR) model, named after the authors in [LW55] and [Ric56], is a macroscopic one-dimensional traffic model. The conservation law for traffic in one dimension is given by

$$\frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x}f(\rho(t,x)) = 0 \tag{5.38}$$

In this equation ρ is the traffic density (vehicles or pedestrians) and f is the flux which is the product of traffic density and the traffic speed v, i.e. $f = \rho v$. There are many models researchers have proposed for how the flux should be dependent on traffic conditions. This relationship is given by the *fundamental diagram.*

5.7.1 Greenshield's Model

Greenshield's model (see [Gre35]) uses a linear relationship between traffic density and traffic speed.

$$v(\rho) = v_f(1 - \frac{\rho}{\rho_m}) \tag{5.39}$$

where v_f is the free flow speed and ρ_m is the maximum density. Free flow speed is the speed of traffic when the density is zero. This is the maximum speed. The maximum density is the density at which there is a traffic jam and the speed is equal to zero. The flux function is concave as can be confirmed by noting the negative sign of the second derivative of flow with respect to density, i.e. $\partial^2 f/\partial\rho^2 < 0$. The fundamental diagram refers to the relationship that the traffic density ρ, traffic speed v and traffic flow f have with each other. These relationships are shown in Figure 5.9.

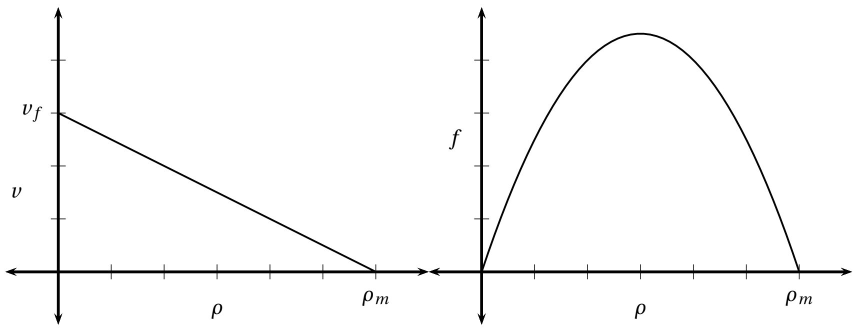

Figure 5.9: *Fundamental Diagram using Greenshield Model*

5.7.2 Generalized/Weak Solution for the LWR Model

The hyperbolic Partial Differential Equation (PDE) for the LWR model given by Equation 5.38 can be solved by using the method of characteristics ([LeV94]). Figure 5.10 shows a $x-t$ plot for traffic density $\rho(t,x)$. Initially the traffic density is constant at ρ_0. At time $t=0$, there is a traffic light at $x=0$ that turns red. We see the shockwaves travelling backward so that there is a discontinuity between traffic density being ρ_0 to the left of teh shock line and being ρ_m to the right of it. On the right there is another shockwave travelling to the right between zero traffic density and ρ_0. At time $t=t_c$, the light turns green and we see raraefaction of traffic starting at $x=0$. Corresponding to time $t=t_u$ we see the plot of traffic density $\rho(t_u,x)$ that shows to the two shock waves as well as rarefaction of the traffic density. This shows that the traffic solution has discontinuities and a weak solution of the LWR model is required that allows for these discontinuous solutions.

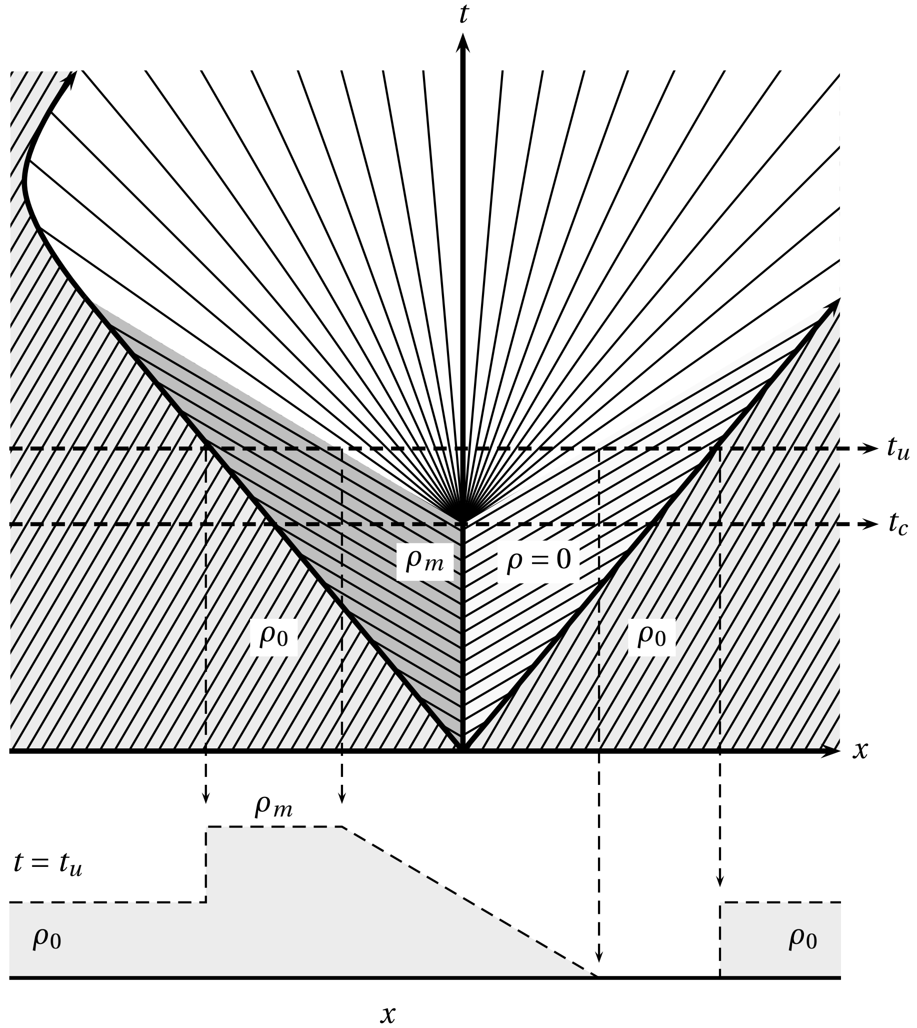

Figure 5.10: *Traffic Characteristics*

Generalized Solutions

For a conservation law

$$\rho_t + f(\rho)_x = 0 \tag{5.40}$$

with initial condition

$$\rho(x, 0) = \rho_0(x), \tag{5.41}$$

where $u_0(x) \in L^1_{loc}(R; R^n)$, solution in the distributional sense is defined below for smooth vector field $f : R^n \to R^n$ (see [Bre05]).

Definition 5.7. A measurable locally integrable function $\rho(t,x)$ is a solution in the distributional sense of the Cauchy problem 5.40 if for every $\phi \in C_0^\infty(R^+ \times R) \mapsto R^n$

$$\iint_{R^+\times R} [\rho(t,x)\,\phi_t(t,x) + f(\rho(t,x))\,\phi_x(t,x)]\,dx\,dt + \int_R u_0(x)\,\phi(x,0)\,dx = 0 \tag{5.42}$$

□

Weak Solutions

A measurable locally integrable function $u(t,x)$ is a weak solution in the distributional sense of the Cauchy problem (5.40) if it is a distributional solution in the open strip $(0,T)\times\mathbb{R}$, satisfies the initial condition 5.41 and if u is continuous as a function from $[0,T]$ into L^1_{loc}. We require $u(t,x) = u(t,x^+)$ and

$$\lim_{t\to 0}\int_R |u(t,x) - u_0(x)|\,dx = 0 \tag{5.43}$$

5.7.3 Scalar Initial-Boundary Problem

Consider the scalar conservation law given here.

$$u_t + f(t,x,u)_x = 0 \tag{5.44}$$

with initial condition

$$u(0,x) = u_0(x), \tag{5.45}$$

and boundary conditions

$$u(t,a) = u_a(t) \text{ and } u(t,b) = u_b(t), \tag{5.46}$$

The boundary conditions cannot be prescribed point-wise, since characteristics from inside the domain might be traveling outside at the boundary. If there are any data at the boundary for that time, that has to be discarded. Moreover, the data also must satisfy entropy condition at the boundary so as to render the problem well-posed. This is shown in Figure 5.11 where for some time boundary data on the left can be prescribed when characteristics from the boundary can be *pushed in* (see [SB06]). However when the characteristics are coming from inside, the boundary data can not be prescribed.

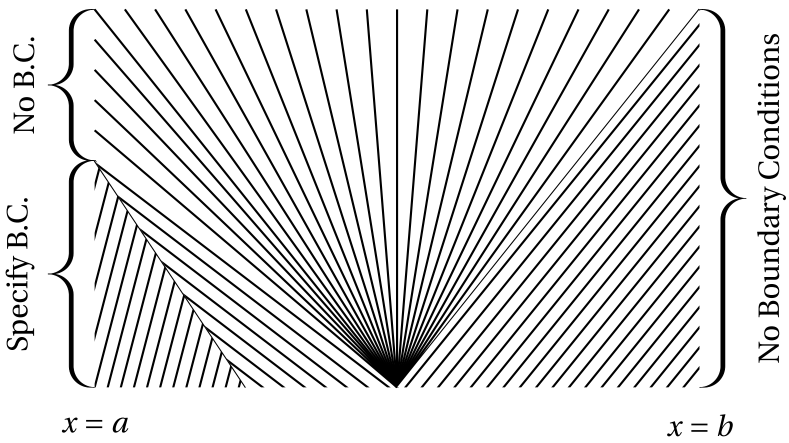

Figure 5.11: *Boundary Data*

5.7.4 Macroscopic (PDE) Traffic Network

The network problem for traffic flow has been studied by researchers ([GP06], [HR95], [Leb96] and [CP02]). They consider a traffic node with incoming n junctions and outgoing m junctions as shown in Figure 5.12.

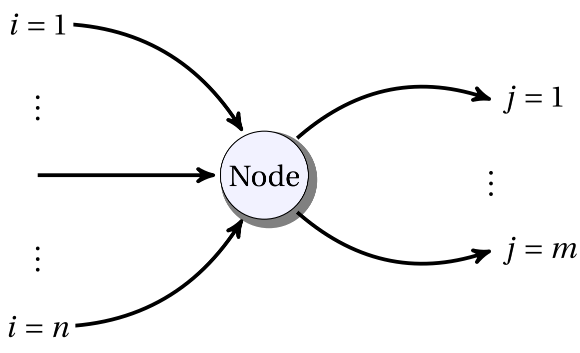

Figure 5.12: *Traffic Node with Incoming and Outgoing Links*

The traffic distribution at the junction is performed based on a traffic distribution matrix that must be provided for the node as well as using an entropy condition at the node that is equivalent to maximizing the flow at the node.

We present the summary of the Coclite/Piccoli model for the network ([CP02], [GP05] and [GP06]). That summary is also used in [GHKL05]. The formulation in terms of demand and supply is shown in the work by Lebacaque ([Leb96], [LK04], and [BLL96]). This formulation is equivalent to the Coclite/Piccoli formulation, and both then show numerical method using the Godunov scheme.

Each arc of the traffic network is an interval $[a_i, b_i]$ The model for the network is

$$\frac{\partial}{\partial t}\rho^i(t,x) + \frac{\partial}{\partial x} f(\rho^i(t,x)) = 0 \,\forall x \in [a_i, b_i], t \in [0,T] \tag{5.47}$$

$$\frac{\partial}{\partial t}\pi^i(t,x,k,r,s) + v^i(\rho^i(t,x))\frac{\partial}{\partial x}\pi^i(t,x,k,r,s) = 0 \,\forall x \in [a_i, b_i], t \in [0,T] \tag{5.48}$$

Here $\pi(t,x,k,r,s)$ is a function whose range is $[0,1]$ and gives the fraction of the traffic density on path k of the OD pair (r,s) on the arc i. Hence, we have

$$\rho^i(t,x,k,r,s) = \pi^i(t,x,k,r,s)\rho^i(t,x) \tag{5.49}$$

This ensures the FIFO condition automatically since vehicle speed is a function of traffic density, and hence vehicles don't cross each other in this model (unless we add lane modeling with lane change logic).

At any node the following flow conservation condition (Kirchoff's law) must be satisfied. This equation says that the total inflow to a node equals its outflow.

$$\sum_{i=1}^{n} f_i(\rho_i(b_i,t)) = \sum_{i=n+1}^{n+m} f_i(\rho_i(a_i,t)),\ \forall t \geq 0 \tag{5.50}$$

At the nodes, we have traffic splitting factor $\alpha_{j,i}$ that tell us what fraction of a given incoming arc i is going to an outgoing arc j of that node. The factors $\alpha_{j,i}$ have to be consistent with $\pi^i(t,x,k,r,s)$.

$$\alpha_{j,i} = \sum_r \sum_s \sum_k \pi^i(t, b_i-, k, r, s) \tag{5.51}$$

The weak solution of the traffic density at a node is given by a collection of functions ρ_i such that the following is satisfied.

$$\sum_i^{n+m} \int_0^\infty \int_{a_i}^{b_i} (\rho_i \frac{\partial \phi_i}{\partial t} + f(\rho_i)\frac{\partial \phi_i}{\partial x}) dx dt = 0 \tag{5.52}$$

All the details of this model can be obtained from [GP06]. The Wardrop condition for this macroscopic DTA model become the following.

$$
\begin{aligned}
&(\delta^{rs}_{a,k}\pi^i(t,a_i,k,r,s))(c^{rs}_k(t)-u_{rs}(t))=0\\
&c^{rs}_k(t)-u_{rs}(t)\geq 0\\
&\sum_k \delta^{rs}_{a,k}\pi^i(t,a_i,k,r,s)=q_{rs}(t)\\
&\sum_k \delta^{rs}_{a,k}\pi^i(t,a_i,k,r,s)\geq 0
\end{aligned}
\tag{5.53}
$$

Here, i in the expression $\pi^i(t,a_i,k,r,s)$ is the link connected to the source r for the particular k and s. The travel time $c^{rs}_k(t)$ is developed in the next section.

5.7.5 Travel Time Dynamics

This section provides a model for obtaining the experienced travel time function for the hydrodynamic model that can be used for the macroscopic DTA model.

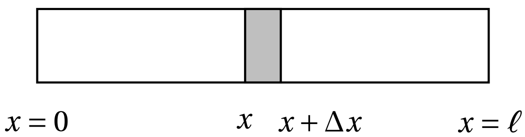

Figure 5.13: *Travel Time on a Link*

Consider a link as shown in the Figure 5.13. We want to develop a travel time function $T(t,x)$ that provides the travel time for a vehicle at position x and time t to reach $x=\ell$. It takes a vehicle time $\Delta x/v(t,x)$ to move from x to $x+\Delta x$. Hence, we have the following travel time condition.

$$T(t+\Delta t, x+\Delta x) = T(t,x) - \frac{\Delta x}{v(t,x)} \tag{5.54}$$

Taking the Taylor series first terms for $T(t,x)$ and simplifying, we obtain

$$\frac{\partial T(t,x)}{\partial t}\Delta t + \frac{\partial T(t,x)}{\partial x}\Delta x = -\frac{\Delta x}{v(t,x)} \tag{5.55}$$

Multiplying by $v(t,x)$, dividing by Δx, and then taking limits and simplifying we get the travel time partial differential equation.

$$\frac{\partial T(t,x)}{\partial t} + \frac{\partial T(t,x)}{\partial x} v(\rho(t,x)) + 1 = 0 \tag{5.56}$$

Hence, the one-way coupled PDE system for LWR and travel time for a link is given by

$$\frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x}[\rho(t,x)v(\rho(t,x))] = 0 \tag{5.57}$$

$$\frac{\partial T(t,x)}{\partial t} + \frac{\partial T(t,x)}{\partial x} v(\rho(t,x)) + 1 = 0 \tag{5.58}$$

$$v(\rho(t,x)) = v_f(1 - \frac{\rho}{\rho_m}) \tag{5.59}$$

5.8 Simulation based DTA

With the availability of faster processors and computers using simulation based DTA is becoming more and more popular ([PZ01], [MHA+98], and [BABKM98]). Summary of simualtion based DTA and the methodology is presented in [CBM+09] and [PZ01]. In principle, the simulation of the network can be accomplished using microscopic, mesoscopic, or macroscopic simulations. Microscopic simulation are based on car-following models and they model the vehcile dynamics for each indvidual vehicle. Macroscopic simulations are based on discretization and numerical solutions of the macroscopic models, such as LWR based models. Mesoscopic simulations use the fundamental relationship for obtaining vehicle speeds (macroscopic behavior), but also have individual vehicles (microscopic behavior) modeled with the tracking of their location and speeds. Since the mesoscopic modeling based DTA is more prevalent, we will focus on that in this section.

There are two main steps to prepare the simulation based DTA. A three stage iterative process to obtain user equilibrium behavior, as well as a field data based calibration process. Once these two processes have been successful, the software can be used for various studies.

5.8.1 Iterations for User Equilibrium

This equilibration process is performed in three steps ([CBM+09]). These three steps are iterated till the user equilibrium condition is obtained within some tolerance limit.

Network Loading This step is obtained by running the network simulation for a given time varying OD and traffic assignment to various paths between each OD pairs. The result is the set of travel times for each path.

Path Set Update The traffic loading obtained from the previous step is used to calculate the set of k-shortest paths between each OD pair.

Path Assignment Adjustment In this step the OD flows are assigned to new updated paths from the previous step.

5.8.2 Calibration from Field Data

Data obtained from field surveys and sensors can be used to calibrate the simulation based DTA models. Some parameters that can be tuned include the time varying OD values, road capacities, and vehicle speed density parameters. The calibration can be performed in order to maximize the match between the simulated outputs and the observed data. Various numerical optimization methods have been used such as gradient based methods, SPSA, etc. The general scheme is to find the parameter vector that will minimize the least squared error of the observations, where the observations are y_i, and the output from simulation is dependent on the parameters as $h_i(\theta)$.

$$\theta^* = Argmin_\theta \sum_i (y_i - h_i(\theta))^2 \tag{5.60}$$

A typical iterative scheme if it is gradient based to find the optimal parameters can be

$$\theta^*[k+1] = \theta^*[k] - \eta \nabla_\theta \sum_i (y_i - h_i(\theta))^2 \tag{5.61}$$

OD estimation has been performed (see [BABKM98]) using an auto-regressive model for OD variations from nominal values, and then applying Kalman filter techniques on it.

5.9 Traffic Operation Design and Feedback Control

Traffic assignment problem and its solutions have very strong roots in the transportation planning process, especially the four-stage process shown in Section 5.1. It is very important to keep this context in mind in order to ensure its proper use. DTA models can help in performing before and after studies for various transportation projects. They can also help in many other studies by enhancing its basic framework with additional features such as enviroment effects of congestion, costs etc.

For real-time traffic operations we must use and develop techniques specifically for real-time operations. For instance, if we have to design an isolated ramp control at one location, the entire OD matrix obtained and calibrated from field studies during some limited time is not relevant to that problem. Feedback control based methods are extremely suited for design of traffic control and real-time operations. The details of many specific feedback control designs for traffic operations such as real-time traffic routing and ramp metering are available in multiple publications ([KO99a], [KÖ03], [KÖ98], [KÖ06], and [KÖ05]).

5.10 Conclusions

This chapter provided a mathematical survey of the static and dynamic traffic asssignment problems. It presented the macroscopic DTA model using the LWR distributed parameter model as the basis. The chapter presented a new partial differential equation for travel time function for a link. It also provided a brief summary of simulation based DTA.

Appendices

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