Some Drawings of Annuli for the Yoccoz Puzzle

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Following Milnor's exposition [2] on the Yoccoz puzzle, we present some drawings for the proofs of various facts about annuli and their children, hoping to make the geometry more explicity from the combinatorics of the tableau.

We repeat the basic definitions of the objects involved, but defer to [2] for their use in proving local connectivity of the Julia set of non-renormalizable quadratic maps.

1 Preliminaries

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial. Recall the following well-known results [1]:

Theorem 1.1 (**Landing Criterion**). *For a polynomial connected Julia set J (that is, such that all of its critical orbits are bounded) with filled Julia set K, J is locally connected if and only if the Böttcher map $\phi : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K$ extends to the boundary as a continuous surjective map $\varphi : \mathbb{R}/\mathbb{Z} \to J$. The values in J correspond to the landing points of external rays.*

The proof follows from the theory of Carathéodory ends.

Theorem 1.2 (**Locally connected Julia sets**). *If the polynomial Julia set J is connected and locally connected, then every periodic point in J is either repelling or parabolic. Moreover, every cycle of Siegel disks will contain a critical point on its boundary.*

In other words, if f is a polynomial, J is connected, and has either a Cremer periodic point or a cycle of Siegel disks without boundary critical point, then J is not locally connected.

The proof follows from the Snail lemma, and compactness arguments on the set of external rays landing at the periodic (which can be assumed fixed) point. As for the Siegel disks, assuming local connectedness one may extend the linearization to the boundary homeomorphically, and repeat the compactness arguments for the rays landing on this boundary.

Theorem 1.3 (**Rational Rays Land**). *If the polynomial f has connected Julia set, every periodic external ray lands at a repelling or parabolic periodic point. If the angle is preperiodic, then it lands at a preperiodic point.*

The proof involves hyperbolic geometry in parametrizing the external ray, that rays landing at points must behave well with respect to images and preimages, and that the set of rays landing at a specific point must preserve the cyclic ordering.

Theorem 1.4 (**Repelling and Parabolic points are landing points**). *If the polynomial f has connected Julia set, and if $z_0 \in J$ is a repelling periodic point, then at least one rational ray lands on z_0. Moreover, finitely may rays land on z_0, all with the same period.*

Suppose z_0 is a parabolic fixed point whose multiplier is a primite q-th root of unity, then every repelling petal of z_0 has at least one ray landing through it. All rays landing at z_0 are periodic with period q. We may extend the result to periodic parabolic points.

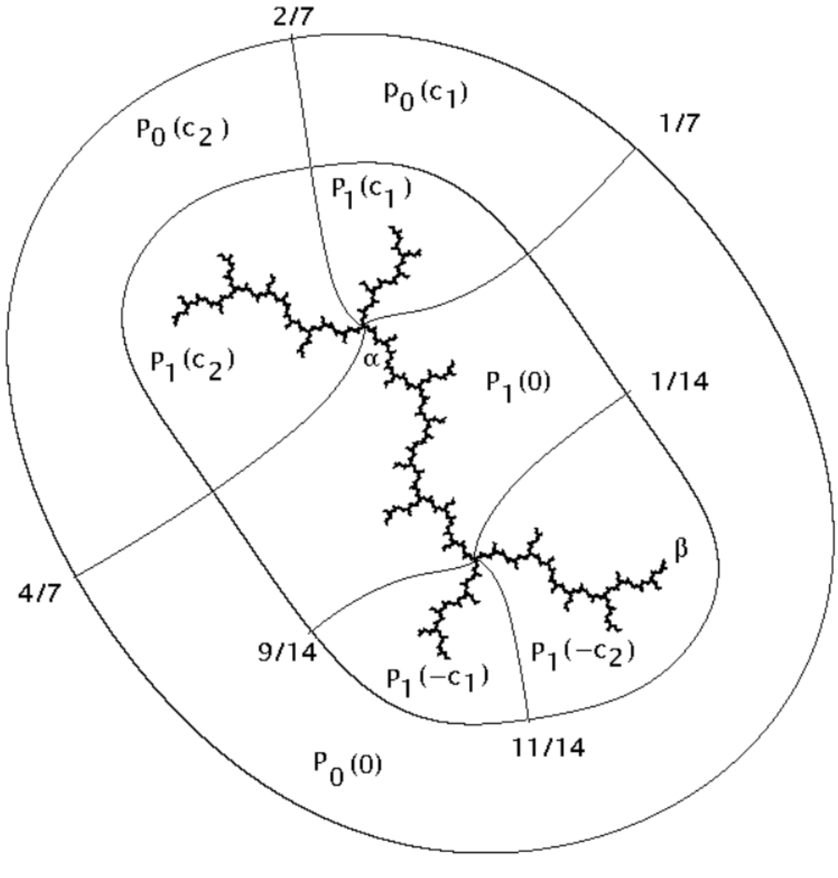
The main idea of the proof is to consider the "linearized Julia set" at the point, as a subset of a suitable shift space with respect to the repelling behavior near it. The "linearized Fatou components" will be coverings of the basin at infinity and must be eventually mapped onto themselves.

2 The Yoccoz Puzzle

Let $f : \mathbb{C} \to \mathbb{C}$ be a quadratic polynomial with both fixed points repelling and bounded critical orbit $c_i = f^i(c_0)$, so that the Julia set J is connected. We also assume that f is not post-critically finite, as local connectivity of J would already follow from hyperbolicity of f on a neighborhood of J with respect to an orbifold metric. Let β be the fixed landing point of the external ray R_0, and α the other fixed point. We assume that there are $q > 1$ external rays landing at α, being permuted by the doubling map on the angles. We also consider $V_1 = \{G(z) \leq 1\}$ as the simply connected neighborhood of the filled Julia set K bounded by the equipotential $\{G(z) = 1\}$.

V_1 is partitioned into q simply connected sectors, bounded by the external rays and the equipotential, and meeting at the landing point α. These will be the **depth 0** puzzle pieces of the Yoccoz puzzle. Inductively, if at depth d we have the puzzle pieces $P_d^{(1)}, \ldots, P_d^{(m)}$, the puzzle pieces of depth $d+1$ consist of the connected components of the preimages $f^{-1}(P_d^{(j)})$. This means that $f : P_{d+1} \to P_d$ is a homeomorphism unless $c_0 \in P_{d+1}$, in which case it is a 2 to 1 branched covering map, conjugate to the squaring map on the disk. In particular, all puzzle pieces are simply connected. The puzzle pieces of depth d partition the set $V_{2^{-d}} = \{G(z) \leq 2^{-d}\}$, have disjoint interiors, and if P_{d+1} intersects P_d, then $P_{d+1} \subset P_d$.

Below is a depiction of depth 0 and depth 1 puzzle pieces for the quadratic map $z^2 + i$ with $q = 3$, taken from [2]:



Given $z \in J$, let $P_d(z)$ be the puzzle piece of depth d which contains z. If z is a preimage of α, $P_d(z)$ is not uniquely identified, but we forego this complication for the moment. The depth 0 pieces above are labeled according to the critical orbit. The **critical pieces** are $P_d(c_0)$. The main purpose of puzzle pieces is to prove that the nested intersection

$$P_0(z) \supset P_1(z) \supset \cdots P_d(z) \supset \cdots$$

reduces to the singleton $\{z\}$, thereby proving local connectivity of J at this point (where the intersection of J with each puzzle piece is connected). For this, we consider the *annuli* $A_d(z) \coloneqq P_d(z) \setminus P_{d+1}(z)$, where by the Branner-Hubbard criterion, if the sum of the moduli $\sum_{d\geq 1} A_d(z)$ diverges to ∞, then the intersection reduces to $\{z\}$.

It could be the case that the set $P_d(z) \setminus P_{d+1}(z)$ is not topologically an annulus, for instance $P_0(c_1)$ in the picture above. This is so if $P_d(z)$ and $P_{d+1}(z)$ intersect on the boundaries. We will still call $A_d(z)$ a *degenerate annulus*, of modulus 0. One can check that $A_d(z)$ has positive modulus if and only if $A_{d-1}(f(z))$ also has.

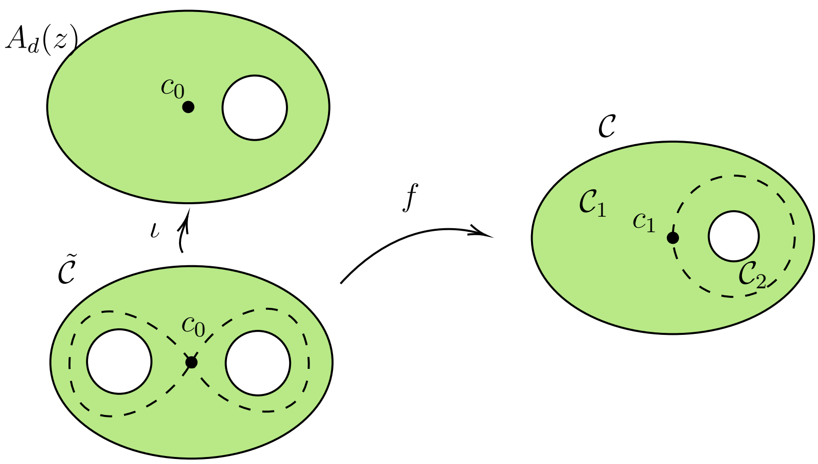
The annuli are further classified into three possibilities:

- *Critical*, if $c_0 \in P_{d+1}(z)$ (or equivalently $P_{d+1}(z)$ is the critical piece). In this case, $f : A_d(z) \to A_{d-1}(f(z))$ is a 2 to 1 unramified covering, and $\operatorname{mod} A_d(z) = \frac{1}{2} \operatorname{mod} A_{d-1}(f(z))$.

- *Off-critical*, if $c_0 \notin P_d(z)$. In this case, $f : A_d(z) \to A_{d-1}(f(z))$ is a conformal isomorphism, and $\operatorname{mod} A_d(z) = \operatorname{mod} A_{d-1}(f(z))$.

- *Semi-critical*, if $c_0 \in A_d(z)$.

In the semi-critical case, $f^{-1}(A_{d-1}(f(z)))$ is a two-holed disk in $P_d(z)$, one being $P_{d+1}(z)$, and f mapping each hole conformally onto $P_d(f(z))$. We consider a conformal isomorphism between $A_{d-1}(f(z))$ and a straight cylinder $\mathcal{C}$, and cut up $\mathcal{C}$ into two straight subcylinders $\mathcal{C}_1$ and $\mathcal{C}_2$ at the critical value $c_1 = f(c_0)$. Grötzch's inequality applied to the inclusion of the preimages of these cylinders into $A_d(z)$ shows that $\operatorname{mod} A_d(z) > \frac{1}{2} \operatorname{mod} A_{d-1}(f(z))$.

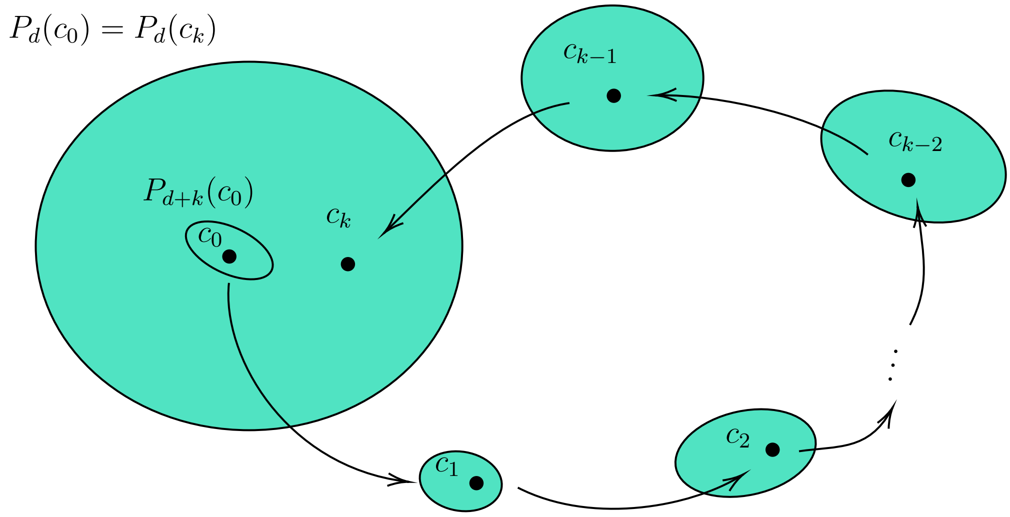


We are particularly interested in the case where the critical orbit $\{c_i\}$ passes through all critical pieces $P_d(c_0)$; we call f **critically combinatorially recurrent**. This is in principle weaker than f being critically recurrent, so that c_0 is an accumulation point of the critical orbit, since we do not know if $\bigcap P_d(c_0) = \{c_0\}$.

In this, we say that the critical annulus $A_{d+k}(c_0)$ is a **child** of the critical annulus $A_d(c_0)$ if $f^k : A_{d+k}(c_0) \to A_d(c_0)$ is an unramified 2-fold covering. From the above, we have that $\bmod A_{d+k}(c_0) = \frac{1}{2} \bmod A_d(c_0)$. The main idea is that if we prove that a given critical annulus has many children, as its children have many children and so on, we may prove divergence of the sum $\sum A_d(c_0)$, thereby showing local connectivity of J at the critical point.

In [2], the proofs of various facts are combinatorial, by translating the geometry of the annuli into a *tableau* that keeps track of their mappings. Here, we instead opt to keep the geometry explicit, at the expense of lengthier arguments, but with the benefit of being able to visualize them.

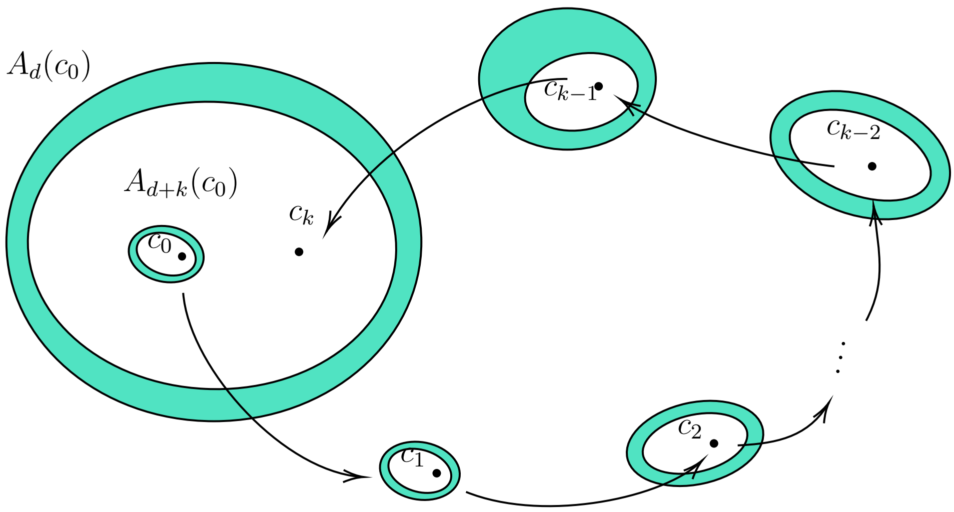
We give a preliminary argument to establish the main geometrical ideas involed. If f is critically combinatorially recurrent, then every critical piece has a "child". Given $P_d(c_0)$, let c_k be the first return of c_0 to the critical piece $P_d(c_0)$. We then "pullback" the piece $P_d(c_k) = P_d(c_0)$ along the orbit $\{c_0, c_1, \ldots, c_{k-1}\}$, obtaining the pieces $P_{d+1}(c_{k-1}), \ldots, P_{d+k-1}(c_1), P_{d+k}(c_0)$:



It is then easy to see that $f^k : P_{d+k}(c_0) \to P_d(c_0)$ is a two to one ramified covering. As a remark, the situation above illustrates that the first return c_k does not fall into $P_{d+k}(c_0)$, but this could have been the case.

Proposition 2.1. *If f is critically combinatorially recurrent, then every critical annulus has a child.*

The proof is through the same picture as above, but in this case considering the first return of c_k to the interior region $P_{d+1}(c_0)$ of the annulus, and pulling back the annuli along the orbit. None of the annuli $A_{d+i}(c_{k-i})$ for $1 \leq i \leq k-1$ can be critical or semi-critical, otherwise $A_d(c_{k-i})$ would be critical and c_{k-i} would return to $P_{d+1}(c_0)$ sooner than the first return. The mapping $f^k : A_{d+k}(c_0) \to A_d(c_0)$ is then indeed an unramified double cover.

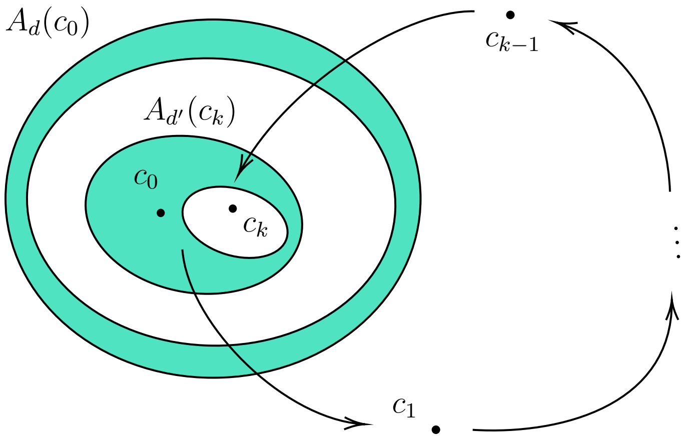


We need further assumptions on the map f to proceed, akin to non-renormalizability. We say that f is **combinatorially critically periodic** if there exists some c_k such that $c_k \in P_d(c_0)$ for all depths d. In [2], it is proved that if f is combinatorially critically periodic, then f is renormalizable. In fact, this condition is equivalent to f being *simply renormalizable*, something expanded upon in an errata to the notes. From now on, we will assume that f is combinatorially critically recurrent but not periodic (or more generally that f is not renormalizable).

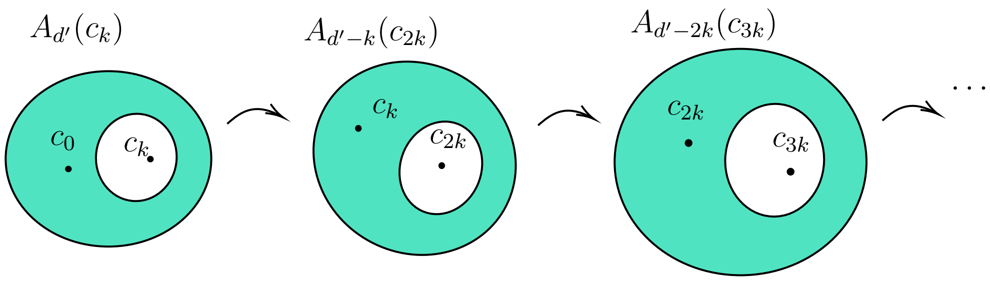
A critical annulus is said to be **excellent** if it contains no post-critical points.

Proposition 2.2. *Every excellent critical annulus has at least two children.*

Proof. First, note that if a and b belong to the same piece of depth d , then $f^i(a)$ and $f^i(b)$ belong to the same piece of depth $d-i$, for $i \leq d$. Letting $A_d(c_0)$ be an excellent critical annulus, suppose c_k is the first return of the critical orbit to $P_{d+1}(c_0)$, so that $f^k : A_{d+k}(c_0) \to A_d(c_0)$ is a 2 to 1 unramified double cover as before. Since f is not combinatorially critically periodic, there must be some depth $d' > d$ such that $c_k \in P_{d'}(c_0)$ but $c_k \notin P_{d'+1}(c_0)$, so that $c_k \in A_{d'}(c_0)$. Equivalently, the annulus $A_{d'}(c_k)$ is off-critical. (Milnor calls d' the *semi-critical depth* of c_k.)

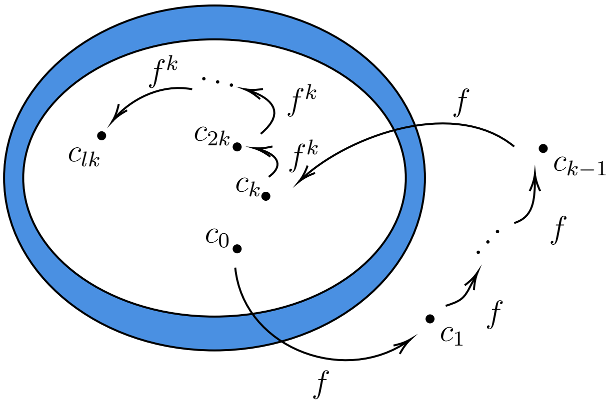


For simplicity, let's assume that d' is much bigger than d, but the arguments carry over in all cases. Applying the first return map f^k to the annulus $A_{d'}(c_k)$ iteratively, we consider the various annuli $A_{d'-k}(c_{2k})$, $A_{d'-2k}(c_{3k})$, succesively. Note that we do not strictly have a mapping $f^k : A_{d'}(c_k) \to A_{d'-k}(c_{2k})$ since $A_{d'}(c_k)$ is semi-critical, but we do have the double ramified cover $f^k : P_{d'}(c_k) \to P_{d'-k}(c_{2k})$.



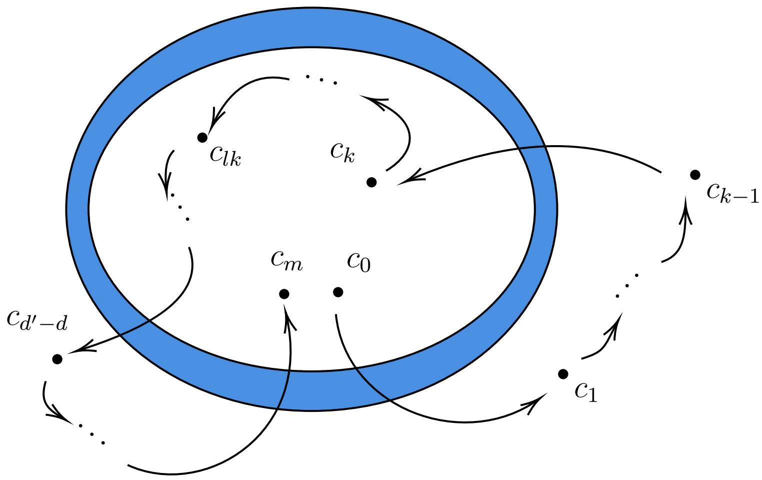
We now that c_0 and c_k are in the same piece of depth d', and in particular $c_k \in P_{d+1}(c_0)$ as the first return. Similarly, c_k and c_{2k} are in the same piece of depth $d' - k$, and if $d' - k \geq d$, they are in the same piece of depth d and $c_{2k} \in P_d(c_0)$. But as the annulus is excellent, $c_{2k} \in P_{d+1}(c_0)$. We also show that the annulus $A_{d'-k}(c_{2k})$ is semi-critical; this is because f^k maps $A_{d'}(c_0)$ onto $A_{d'-k}(c_0)$, where $c_k \in A_{d'}(c_0)$. Hence $c_{2k} \in A_{d'-k}(c_0)$, and $c_0 \in A_{d'-k}(c_{2k})$. Moreover, we cannot have that $d' - k = d$, otherwise $c_{2k} \in A_{d'-k}(c_0) = A_d(c_0)$, a contradiction with the annulus being excellent.

We can induct this reasoning to other iterates $c_{2k}, c_{3k}, \dots, c_{lk}$ of the critical orbit, as long as $d' - lk \geq d$. All the annuli $A_{d'-jk}(c_{(j+1)k})$ are semi-critical, the c_{jk} all belong to the critical piece $P_{d+1}(c_0)$, and $d' - d$ cannot be a multiple of k.



Moreover, the same ideas apply to show that the iterates $c_i, c_{i+k}, c_{i+2k}, \dots$ are in the same depth d piece for $0 < i < k$, as long as $i+jk > d$. In particular, all the iterates c_i where $i \neq jk$ and $i < d' - d$ are outside of the critical piece $P_{d+1}(c_0)$. and since the annulus is excellent, they are also outside of $P_d(c_0)$.

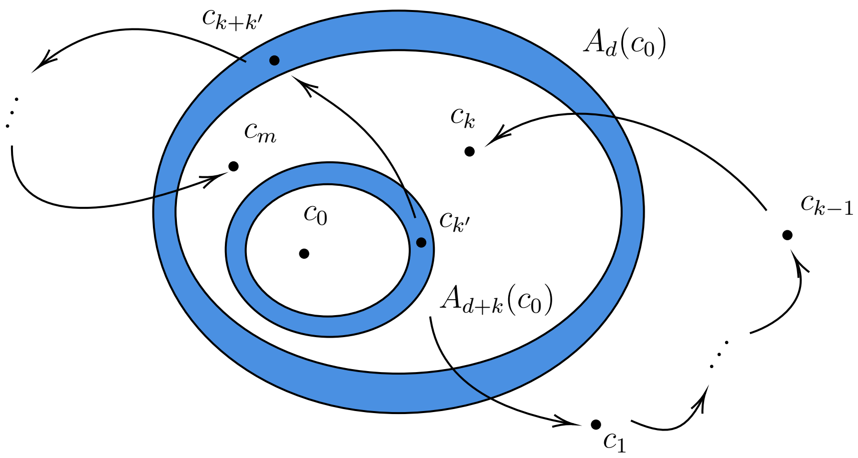
We then conclude that $c_{d'-d}$ must lie outside of the original critical piece $P_d(c_0)$. Let c_m be the next return of $c_{d'-d}$ to $P_{d+1}(c_0)$, since f is critically combinatorially recurrent, and we pullback the annulus $A_d(c_0)$ along f^m to a critical annulus $A_{d+m}(c_0)$. As $m > d' - d$, c_k is outside of $P_{d+m}(c_0)$, as will be all other iterates. Hence $f^m : A_{d+m}(c_0) \to A_d(c_0)$ is a 2 to 1 unramified covering, and $A_{d+m}(c_0)$ is the second child of $A_d(c_0)$.



□

Proposition 2.3. *Every child of an excellent parent is excellent, and an only child must be excellent.*

Proof. Suppose $A_{d+k}(c_0)$ is a child of $A_d(c_0)$, and assume that it is not excellent, so that there exists $c_{k'} \in A_{d+k}(c_0)$, where $k' \geq k$. Then $c_{k'}$ is mapped by f^k to $c_{k+k'}$ on the annulus $A_d(c_0)$, so that $A_d(c_0)$ is not excellent. Letting c_m be the first return of $c_{k+k'}$ to the piece $P_{d+1}(c_0)$, we then pullback the annulus $A_d(c_0)$ along f^m and obtain another child of $A_d(c_0)$, similarly to the previous proof.



$\square$

With these results about annuli and their children, [2] proceeds to show local connectivity of the Julia sets under the assumption of non-renormalizability, and various cases of critical recurrence or not.

References

[1] J. Milnor, *Dynamics in One Complex Variable*, Vieweg, 2000.

[2] ———, *Local connectivity of Julia sets: expository lectures* (1992).