Thurston critically periodic polynomial-like maps

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Let $f : S^2 \to S^2$ be a degree d branched covering map of a topological 2-sphere, and P_f be its postcritical set. We assume P_f is finite. An important question is whether such a map is *Thurston equivalent* to a postcritically finite rational map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, that is, there exists homeomorphisms $\theta, \theta' : S^2 \to \hat{\mathbb{C}}$ such that the diagram below commutes:

$$\begin{tikzcd}
(S^2, P_f) \arrow[r, "\theta"] \arrow[d, "f"'] & (\hat{\mathbb{C}}, P_g) \arrow[d, "g"] \\
(S^2, P_f) \arrow[r, "\theta'"] & (\hat{\mathbb{C}}, P_g)
\end{tikzcd}$$

and θ is isotopic to θ' relative to P_f. Thurston gave a necessary and suficcient topological condition on whether such an equivalence exists. Let Γ be a set of disjoint, non-peripheral, simple closed curves on $S^2 \setminus P_f$, no two homotopic to each other. For each $\gamma \in \Gamma$, the pullback $f^{-1}(\gamma)$ is a set of disjoint simple closed curves, mapping to γ with some degree. We say that Γ is *f-stable* if for all $\gamma \in \Gamma$, every component of $f^{-1}(\gamma)$ is homotopic rel P_f to some curve in Γ. If $\Gamma = \{\gamma_i\}_i$, by considering the vector space $\mathbb{R}^\Gamma$, we have an induced linear pullback map $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ given on each basis element γ_i by

$$f_\Gamma(\gamma_i) = \sum_j \sum_\alpha \frac{1}{d_{\alpha,j,i}} \gamma_j,$$

where α ranges over all components γ_α of $f^{-1}(\gamma_i)$ which are homotopic to γ_j rel P_f, and $d_{\alpha,j,i}$ is the mapping degree of γ_α onto γ_i. Note that in this calculation we disconsider those components of the pullback which are peripheral. Moreover, being a matrix with all entries non-negative, there exists a real eigenvalue $\lambda(f, \Gamma)$ of f_Γ of largest absolute value.

If f has hyperbolic orbifold (a condition visible from the portrait of f, that is, the information about $\Omega_f \cup P_f$ and the local degrees of the critical points), then f is Thurston-equivalent to a rational map if and only if $\lambda(f,\Gamma) < 1$. An f-stable multicurve with $\lambda(f,\Gamma) \geq 1$ is said to be an *obstruction*.

We restrict ourselves to polynomial-like mappings, that is, those for which there exists a fixed critical point of local degree d, so that f induces a proper degree d branched cover of the plane $\mathbb{R}^2$ onto itself. Moreover, we consider the additional hypothesis that all critical points of f are periodic, and we wish to prove the following:

Theorem 0.1. *Polynomial-like, critically periodic Thurston maps are unobstructed.*

Given two disjoint simple closed curves γ and η in $\mathbb{R}^2$, they are either separated or nested, as a consequence of the Jordan curve theorem. Equivalently, if D_γ denotes the open disk in $\mathbb{R}^2$ having γ as its boundary, we either have $D_\gamma \cap D_\eta = \emptyset$ or $D_\gamma \subset\subset D_\eta$ compactly contained (or the opposite inclusion). Given γ, we want to understand the compact set $f^{-1}(\overline{D_\gamma})$. It is the disjoint union

$$f^{-1}(\overline{D_\gamma}) = f^{-1}(\gamma) \sqcup f^{-1}(D_\gamma),$$

where $f^{-1}(D_\gamma)$ is open. Because f is open, we also have $f^{-1}(\overline{D_\gamma}) = \overline{f^{-1}(D_\gamma)}$, and as f is proper, both restricted maps

$$f|_{f^{-1}(\overline{D_\gamma})} : f^{-1}(\overline{D_\gamma}) \to \overline{D_\gamma}, \qquad f|_{f^{-1}(D_\gamma)} : f^{-1}(D_\gamma) \to D_\gamma$$

are also proper. Moreover, since f is a branched cover, $f^{-1}(D_\gamma)$ has finitely many open components. If U is one of these components, then $f|_U : U \to D_\gamma$ is also proper, which can be checked readily from the above facts. In particular, $\partial U \subseteq \partial f^{-1}(D_\gamma) = f^{-1}(\gamma)$, and from a general converse we obtain

$$f^{-1}(\gamma) = \bigcup_{i=1}^{m} \partial U_i.$$

Suppose that for some $i \neq j$ we had $x \in \partial U_i \cap \partial U_j$. We may take small neighborhoods V of $f(x)$ and W of x such that W maps to V homeomorphically, since $f(x) \notin P_f$. If we assume the curve γ to be C^1 in $\mathbb{R}$ so that, by taking a small enough V, the intersection $D_\gamma \cap V$ is connected, the corresponding intersection in W is also connected, so that there is a unique component U_i of $f^{-1}(D_\gamma)$ intersecting W and for which $x \in \partial U_i$. This shows that boundaries of distinct components cannot intersect, and due to connectedness of each curve in $f^{-1}(\gamma)$, each one will be equal to some ∂U_i.

Because each U_i is a precompact open set having some curve $\gamma' \subset f^{-1}(\gamma)$ as its boundary, we must have that $D_{\gamma'} = U_i$. This in particular shows the following:

Lemma 0.2. *The components of $f^{-1}(\gamma)$ are all separated.*

We also obtain the following results:

Lemma 0.3. *If γ and η are two separated curves, and γ' and η' are components of $f^{-1}(\gamma)$ and $f^{-1}(\eta)$ respectively, then γ' and η' are separated.*

Proof. Suppose on the contrary that $D_{\gamma'} \subset D_{\eta'}$. Since f maps $D_{\gamma'}$ onto D_γ and $D_{\eta'}$ onto D_η, this would imply that there $D_\gamma \subset D_\eta$, a contradiction. $\square$

Lemma 0.4. *Suppose that γ and η are two nested curves, with $D_\gamma \subset D_\eta$. Then for each component γ of $f^{-1}(\gamma)$, there is some component η' of $f^{-1}(\eta)$ such that γ' is nested inside of η'.*

Proof. Since $D_\gamma \subset D_\eta$, $f^{-1}(D_\eta)$ must intersect $D_{\gamma'}$, And by connectedness of $D_{\gamma'}$ it must be contained in some component of $f^{-1}(D_\eta)$. $\square$

Let p_γ be the number of postcritical points inside of D_γ, and q_γ to be number of postcritical points outside of γ (Here $p_\gamma + q_\gamma = p - 1$, where we are not considering the point at infinity). Note that this number is a homotopy invariant in $\mathbb{R}^2 \setminus P_f$, and in order for γ to be non-peripheral, we must have $p_\gamma \geq 2$, $q_\gamma \geq 1$. Recall our assumption that f is critically periodic; this implies that $\Omega_f \subseteq P_f$ and that $f|_{P_f} : P_f \to P_f$ is injective, since P_f is distributed into the cyclic orbits of the critical points.

If γ' is a component of $f^{-1}(\gamma)$, then

$$p_{\gamma'} \leq p_\gamma,$$

since each point in $P_f \cap D_{\gamma'}$ has to map injectively to a point in $P_f \cap D_\gamma$. In other words, the number of postcritical points inside a curve is non-increasing under pullback. Furthermore, since the components of the pullback $f^{-1}(\gamma)$ must be separated, in particular they cannot be homotopic in $\mathbb{R}^2 \setminus P_f$ unless they are peripheral.

It's important to note, however, that homotopy classes in $\mathbb{R}^2 \setminus P_f$ may collapse together under pullback. More precisely, if γ is nested inside η in $\mathbb{R}^2 \setminus P_f$, and γ' is a component of $f^{-1}(\gamma)$ and η' is the unique component of $f^{-1}(\eta)$ which nests γ', it could be the case that γ' and η' are homotopic in $\mathbb{R}^2 \setminus P_f$ even if γ and η are not homotopic; This is because our actual mapping is $\mathbb{R}^2 \setminus f^{-1}(P_f) \to \mathbb{R}^2 \setminus P_f$, so that if x is a postcritical point which is inside η but outside γ, its corresponding preimage x' inside of η' and outside of γ'

may not be a postcritical point. Another possible way to view this situation is through the two maps

$$\iota : \mathbb{R}^2 \setminus f^{-1}(P_f) \to \mathbb{R}^2 \setminus P_f, \qquad f : \mathbb{R}^2 \setminus f^{-1}(P_f) \to \mathbb{R}^2 \setminus P_f$$

which determine the dynamics.

Let Γ be an f-stable multicurve. For $\gamma_i, \gamma_j \in \Gamma$, there can be at most one component γ_α of $f^{-1}(\gamma_i)$ which is homotopic to γ_j in $\mathbb{R}^2 \setminus P_f$, given that the components of the pullback are separated; so the sum in $f_\Gamma(\gamma_i)$ has at most one term coming from the components α for each j. We define d_{ij} to be the degree of the mapping $f : \gamma_\alpha \to \gamma_i$, and $d_{ij} = 0$ if such γ_α homotopic to γ_j does not exist. Note that $\sum_j d_{ij} \leq d$, and that the non-zero entries of f_Γ are $1/d_{ij}$.

Fix some $\gamma_j \in \Gamma$, and suppose that γ_i and γ_k are separated curves in Γ such that there are components γ_α of $f^{-1}(\gamma_i)$ and γ_β of $f^{-1}(\gamma_k)$ where γ_α and γ_β are homotopic to γ_j (recall that this homotopy is in $\mathbb{R}^2 \setminus P_f$, and that these components are unique being homotopic to γ_j). This implies that γ_α and γ_β are nested, but this is only possible if γ_i and γ_k are nested, which is a contradiction; hence either $d_{ij} = 0$ or $d_{kj} = 0$. In other words, among separated curves $\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_m} \in \Gamma$, at most one can have $d_{i_l j} \neq 0$.

This suggests that we should organize the multicurve Γ into a structure which guarantees that some curves are separated. One way to do so is by the values p_γ; naturally curves in Γ having the same number of postcritical points enclosed inside them must be separated. Moreover, with the hypothesis that the critical orbits are all periodic, the number p_γ is non-increasing under pullback, which further dictates how the pullback acts on this structure of Γ.

We consider a different structure of Γ, that of nesting. Curves in Γ which are not nested inside of another one have depth 0; we inductively define the depth of nesting for the other curves. This depth n_γ is non-increasing under pullback, since separated curves stay separated and nested curves stay nested, but possibly collapsing to the same homotopy class in $\mathbb{R}^2 \setminus P_f$. Evidently curves having the same depth of nesting are separated.

If we order the curves in Γ by their depth of nesting, the matrix for f_Γ becomes block upper triangular, since a curve γ_i may be pulled back only to curves γ_j having depth of nesting $n_{\gamma_j} \leq n_{\gamma_i}$. Because the eigenvalues of a block upper triangular matrix are the collection of eigenvalues of each diagonal block, if Γ is an obstruction then for some nesting depth n the curves Γ_n having this depth can be extended to form an obstruction. By this reasoning, we may assume from the outset that our f-stable multicurve Γ which is an obstruction consists of only separated curves.

In this situation, every column of f_Γ has at most one non-zero entry. In other words, if γ_j is the pullback (or more precisely, homotopic to a pullback) of a curve in Γ, it is the pullback of a unique such curve $\gamma_i \in \Gamma$; we may represent this as a mapping $\gamma_j \to \gamma_i$. If γ_j is not the pullback of any curve, so that the column of γ_j is identically zero, then any non-zero eigenvalue of f_Γ has to come from the minor matrix obtained from f_Γ by excluding the j-th row and the j-th column. Hence we may assume that no column is identically zero. Similarly, if there was some curve $\gamma_i \in \Gamma$ whose pullbacks were entirely peripheral or homotopically trivial, its corresponding row would be identically zero; so we can also assume that this does not happen for our obstruction.

In other words, we have a well defined mapping $\hat{f} : \Gamma \to \Gamma$ of the curves as above, which must be surjective, and hence bijective. This implies that the curves are permuted among themselves according to the degrees of the mappings; we may assume that our obstruction Γ consists of a single cycle of the curves, and in order for there to be an eigenvalue greater than or equal to 1, all the mapping degrees must be 1. Such an obstruction is called a *Levy cycle*.

Proof of the theorem. Recall the hypothesis that all critical orbits are periodic. As each $\gamma_i \in \Gamma$ will enclose at least one critical point in its cycle, the degree of the mapping cannot be one, so that there can be no obstruction. $\square$

The proof strategy above works even under the weaker hypothesis that all cycles in P_f contain a critical point.