Extremal Width

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February 13, 2026

Abstract

Given a family of paths Γ on a Riemann surface S, there is a natural conformal invariant associated to it defined in terms of measurable conformal metrics: the extremal width. The purpose of these notes is to rigorously define all of these notions, and present standard and well known results on them in an organized fashion.

Contents

1 Definitions

1.1 Measurable conformal metrics

Let S be a Riemann surface, possibly with boundary ∂S. We assume S is equipped with its Borel sigma-algebra $\mathcal{B}$. Though S does not come equipped with any prescribed measure, (Lebesgue) measure zero sets are well defined on S; this is because we may define measure zero sets on each holomorphic chart, measure zero sets are preserved under biholomorphisms, and S is second countable. Explicitly, we may say that a set $A \subseteq S$ has measure zero if, for all coordinate charts (U, φ), $\varphi(A \cap U) \subseteq \mathbb{C}$ has measure zero. In particular the boundary ∂S, if non-empty, has measure zero. Moreover, we may actually consider the sigma-algebra $\mathcal{L}$ of Lebesgue measurable sets on S, as it is the completion of the Borel sigma-algebra with the Lebesgue measure zero sets. For the most part, we will work with $\mathcal{B}$.

Definition 1.1. A *measurable conformal metric μ on S* is an assignment of every holomorphic chart (U, φ) for S to a non-negative measurable function $\mu_\varphi : \varphi(U) \to [0, +\infty)$, such that, on two overlapping charts (U, φ) and (V, ψ),

$$\mu_\varphi(\varphi(p)) = \mu_\psi((\psi(p))|(\psi \circ \varphi^{-1})'(\varphi(p))|,$$

for $p \in U \cap V$. Equivalently, for $z \in \varphi(U \cap V)$,

$$\mu_\varphi(z) = \mu_\psi((\psi \circ \varphi^{-1})(z))|(\psi \circ \varphi^{-1})'(z)|.$$

Morally, this condition on the μ_φ guarantess that the expression $\mu(z)|dz|$ is invariant under change of coordinates, and non-negativity is consistent as the multiplicative factor by which we change coordinates is always non-negative. We will often write μ as $\mu = \mu(z)|dz|$, where it is implied that this is a local description for μ with respect to a local complex coordinate $z = z(p)$ for $p \in S$ (instead of $\varphi(p)$). Tentatively,

$$\mu(z)|dz| = \mu(z(w))|z'(w)||dw|.$$

Henceforth, by *metric* we will always mean a measurable conformal metric, unless stated otherwise. We also only care about the functions that define the metric up to sets of measure zero, and will overlook this nunace to avoid having to state that properties hold "almost everywhere".

Remark 1.2. In order to define a measurable conformal metric on S it is sufficient to have, for some holomorphic atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ for S, an assignment $(U_i, \varphi_i) \mapsto \mu_i$ to a non-negative measurable function on $\varphi_i(U_i)$, such that the transformation law holds when changing between charts. This is because on any compatible chart we will get a corresponding measurable function with via change of coordinates, being locally defined in terms of the μ_i.

A tensorial definition of a measurable conformal metric is as follows. It is a measurable function $\mu : TS \to \mathbb{C}$ on the tangent bundle of S such that, if v is a tangent vector at some point p and $\lambda \in \mathbb{C}$, then $\mu(\lambda v) = |\lambda|\mu(v)$. We will, however, not utilize this definition often for most purposes. We leave it to the reader to check that a "chart-wise" metric uniquely determines a "tensorial" metric, and vice-versa.

Metrics can be added and scaled by non-negative constants, and they form a partial order by defining $\mu_1 \leq \mu_2$ if, for all $p \in S$ and some chart (U, φ) around z, $\mu_1(p) \leq \mu_2(p)$ (and therfore for all charts around p). This allows us to consider a supremum or infimum of a collection of metrics, as long as that supremum is not $+\infty$ on a set of positive measure. This definition does not depend on the choice of partition or of coordinate charts.

1.2 Line integrals and areas

Let $\gamma : I \to S$ be a continuous curve, where $I \subseteq \mathbb{R}$ is an interval. It makes sense to say that a curve on $\mathbb{R}^n$ is (locally) rectifiable, having a well-defined arc-length. On S, we say that γ is locally rectifiable if there exists a countable partition of I into subintervals I_j such that each restriction $\gamma|_{I_j}$ has its image on a coordinate chart (U_j, φ_j), and $\varphi_j \circ \gamma|_{I_j} : I_j \to \mathbb{C}$ is rectifiable.

Definition 1.3. Let $\phi : S \to [0, +\infty)$ be a non-negative measurable function, and $\gamma : I \to S$ a locally rectifiable curve. Given a partition of I as above, we define the line integral of ϕ along γ_j with respect to μ as

$$\int_{\gamma_j} \phi\mu := \int_{I_j} \phi(\gamma(t))\mu_{\varphi_j}((\varphi_j \circ \gamma)(t))dt,$$

and define the line integral of ϕ along γ as the sum of the line integrals over each of the subintervals of the partition:

$$\int_{\gamma} \phi\mu := \sum_j \int_{\gamma_j} \phi\mu.$$

It is straighforward to check that this will not depend on the choice of partition and charts. In particular, we define the *μ-length* of γ:

$$l_\mu(\gamma) := \int_\gamma \mu.$$

In principle, the μ-length can be equal to $+\infty$. Henceforth, all curves are assumed to be locally rectifiable.

Definition 1.4. A metric μ on S defines a corresponding area element $\mu^2 = \mu(z)^2|dz|^2 = \mu(z)^2 dz\overline{dz}$, satisfying the appropriate transformation rules. If (U, φ) is a coordinate chart on which $\mu = \mu(z)|dz|$ and $\phi : S \to [0, +\infty)$ is a non-negative measurable function, we define

$$\iint_U \phi\mu^2 \coloneqq \iint_{\varphi(U)} (\phi \circ \varphi^{-1})(z)\mu^2(z)|dz|^2.$$

Now let $\mathcal{A} = \{(U_i, \varphi_i)\}_{i\in I}$ be an atlas for S and $\{\rho_i\}_{i\in I}$ a partition of unity subordinate to $\mathcal{A}$. We define the (area) integral of ϕ over S with respect to μ as

$$\iint_S \phi\mu^2 \coloneqq \sum_{i\in I} \iint_{U_i} (\rho_i\phi)\mu^2,$$

potentially being equal to $+\infty$. In particular, $A_\mu(S) \coloneqq \iint_S \mu^2$ is the *μ-area of S*. Analogously to line integrals, the area integrals will not depend on the choice of charts and partition of unity.

Remark 1.5. We intially require non-negativity of the measurable functions that we integrate in order to not run into the problem of subtracting infinities. As soon as the (line or area) integral of $|\phi|$ with respect to μ or μ^2 is finite, we may define the integral of ϕ as a subtraction of its postive part with its negative part. We will be mostly concerned with metrics whose area is finite throughout these notes.

1.3 Measures

The area element μ^2 defines a measure on $(S, \mathcal{B})$. If $E \in \mathcal{B}$ (or $\mathcal{L}$), then it is not hard (though tiresome) to verify that

$$A_\mu(E) = \iint_E \mu^2 \coloneqq \iint_S \mathbf{1}_E\mu^2$$

defines a measure $A_\mu : \mathcal{B} \to [0, +\infty]$. Integration with respect to the measure A_μ coincides with the area integrals defined previously. For all charts (U, φ), the pushforward measure φ_*A_μ on $\varphi(U)$ (or more precisely the pushforward of the restriction of $A_\mu|_U$) is given by integration with respect to the non-negative measurable function $\mu^2_\varphi(z)$. This implies that φ_*A_μ is σ-finite and absolutely continuous with respect to the Lebesgue measure m on $\mathbb{C}$. As S is second countable, A_μ is σ-finite.

Conversely, let $\lambda : \mathcal{B} \to [0, +\infty]$ be a σ-finite measure on S such that for all charts (U, φ), $\varphi_* \lambda$ is absolutely continuous with respect to the Lebesgue measure. Then, by the Radon-Nikodym theorem [Sal16], there are measurable functions $g_\varphi : \varphi(U) \to [0, +\infty)$, uniquely defined up to measure zero, such that

$$\varphi_* \lambda(E) = \iint_E g_\varphi dm = \iint_E g_\varphi(z) |dz|^2.$$

We may recover a measurable conformal metric μ_λ from λ by defining $(\mu_\lambda)_\varphi := \sqrt{g_\varphi}$, and transformation rules for integrals will give the required transformation rules for μ. Additionally, $A_{\mu_\lambda} = \lambda$. Summarizing these ideas, we have a correspondence:

$$\begin{array}{c} \text{Measurable} \\ \text{conformal} \\ \text{metrics on } S \end{array} \longleftrightarrow \begin{array}{c} \sigma\text{-finite measures on } S \\ \text{absolutely continuous} \\ \text{with respect to Lebesgue on charts} \end{array}$$

1.4 Path families and extremal width

A path family Γ on S will mean a collection of curves, each assumed to be locally rectifiable. Its support in the closure of the union of the images of the curves. Let

$$l_\mu(\Gamma) = \inf_{\gamma \in \Gamma} l_\mu(\gamma).$$

Definition 1.6. The *extremal length* $\mathcal{L}(\Gamma)$ of Γ is defined as

$$\mathcal{L}(\Gamma) := \sup_\mu \frac{l_\mu(\Gamma)^2}{A_\mu(S)},$$

where the supremum ranges over all metrics on S whose area is not 0 or ∞.

Note that metrics that are scalar multiples of each other give the same ratio on the right, so that we could take the supremum over all metrics of area 1 (or some other homogeneous normalization). One could therefore talk about *projective measurable conformal metrics*, or equivalently absolutely continuous (with respect to the Lebesgue measure on each chart) probability measures on S.

Definition 1.7. The *extremal width* $\mathcal{W}(\Gamma)$ is defined as

$$\mathcal{W}(\Gamma) := \inf_\mu \{A_\mu(S) \mid \forall \gamma \in \Gamma,\ l_\mu(\gamma) \geq 1\}.$$

Similarly, the infimum above could be taken over all metrics μ such that $l_\mu(\Gamma) \geq 1$, but in practice the definition above will be the most useful to us. It is straightforward to check that $\mathcal{W}(\Gamma) = \mathcal{L}(\Gamma)^{-1}$, and that these are conformal invariants; that is, if $f : S \to T$ is a biholomorphism, then $\mathcal{W}(f(\Gamma)) = \mathcal{W}(\Gamma)$.

Example 1.8. Consider the rectangle $R = [0, u] \times [0, 1]$ as a subset of $\mathbb{C}$ with the induced conformal structure, and let Γ be the family of paths connecting the bottom side $[0, w] \times \{0\}$ to the top side $[0, w] \times \{1\}$. By taking the standard euclidean metric on (the interior of) R, we get from the infimum definition that $\mathcal{W}(\Gamma) \leq w$. Now let μ be an arbitrary metric on R such that $l_\mu(\gamma) \geq 1$ for $\gamma \in \Gamma$. In particular, for all vertical paths we have

$$\int_0^1 \mu(x_0, y)dy \geq 1,$$

and integrating over all vertical paths we get

$$\int_0^w \int_0^1 \mu(x, y)dydx = \iint_R \mu dxdy \geq w.$$

By Cauchy-Schwarz:

$$w \leq \iint_R (\mu \cdot 1)dxdy \leq \left(\iint_R \mu^2\right)^{1/2} \left(\iint_R 1\right)^{1/2} = A_\mu(R)^{1/2} w^{1/2},$$

which reduces to $w \leq A_\mu(R)$. As μ was arbitrary, $w \leq \mathcal{W}(\Gamma)$, and so $\mathcal{W}(\Gamma) = w$. Note that the extremal width would be the same if we restricted to the family of only vertical paths of R, and that in either case the measurable conformal metric that realizes the infimum is the standard euclidean metric.

Example 1.9. Let A be the standard round annulus of inner radius r and outer radius R centered at the origin, and Γ be the collection of paths connecting the inner boundary to the outer one. Consider the metric $|dz/z|$ on A; its area is

$$\iint_A \frac{|dz|^2}{|z|^2} = \int_0^{2\pi} \int_r^R \frac{1}{\rho^2} \rho d\rho d\theta = 2\pi \log(R/r),$$

and the length of a path connecting the boundaries is at least the length of a radial path, which is

$$\int_r^R \frac{1}{\rho} d\rho = \log(R/r).$$

Rescaling the metric by $\log(R/r)^{-1}$ so that $l_\mu(\Gamma) = 1$, we get

$$\mathcal{W}(\Gamma) \leq \frac{2\pi}{\log(R/r)}.$$

A very similar argument using Cauchy-Schwarz's inequality gives

$$\mathcal{W}(\Gamma) = \frac{2\pi}{\log(R/r)} = \frac{1}{\operatorname{mod}(A)},$$

where $\operatorname{mod}(A)$ is the usual modulus of an annulus. Another point of view for this example is that by slitting the annulus along some radial line, the exponential map takes the rectangle $[\log r, \log R] \times [0, 2\pi]$ conformally onto the slit annulus, but paths connecting the boundaries of A correspond to paths connecting the left and right sides of the rectangle. It is not hard to check that extremal width will be the inverse of that of the path family connecting the top and bottom sides.

2 Parallel and series laws

Given distinct path families satisfying certain properties, one may compare their extremal lengths and widths. If Γ and Γ' are such that every curve $\eta \in \Gamma'$ contains some $\gamma \in \Gamma$ as a segment, we say that Γ' *overflows* Γ, and denote this by $\Gamma \preceq \Gamma'$. A straightforward consequence is that $\mathcal{L}(\Gamma) \leq \mathcal{L}(\Gamma')$, or equivalently $\mathcal{W}(\Gamma) \geq \mathcal{W}(\Gamma')$. This is because every metric μ such that $l_\mu(\gamma) \geq 1$ for all $\gamma \in \Gamma$ also satisfies $l_\mu(\eta) \geq 1$ for all $\eta \in \Gamma'$, so that in the formula for $\mathcal{W}(\Gamma')$ we are taking an infimum over a bigger set of metrics, being therefore smaller. Intuitively, the Γ' are "longer and fewer" (and so more metrics satisfy the condition that their μ-lengths are large).

Remark 2.1. If Γ and Γ' are two path families such that $\Gamma \subseteq \Gamma'$, then in fact $\Gamma \succeq \Gamma'$, and so $W(\Gamma) \leq W(\Gamma')$. This is evident in the example of a vertical foliation of a rectangle of width w and height 1 being a subset of the vertcial foliation of a wider rectangle of width $w' \geq w$ and height 1.

Proposition 2.2 (Parallel law)**.** *Suppose Γ_1 and Γ_2 are two path families. Then*

$$\mathcal{W}(\Gamma_1 \cup \Gamma_2) \leq \mathcal{W}(\Gamma_1) + \mathcal{W}(\Gamma_2).$$

If Γ_1 and Γ_2 have disjoint support, then

$$\mathcal{W}(\Gamma_1 \cup \Gamma_2) = \mathcal{W}(\Gamma_1) + \mathcal{W}(\Gamma_2).$$

Proof. Let μ_i be a conformal metric for which $l_{\mu_i}(\gamma) \geq 1$ for all $\gamma \in \Gamma_i$, for $i = 1, 2$. Letting $\mu = \max\{\mu_1, \mu_2\}$, we see that $l_\mu(\gamma) \geq 1$ for all $\gamma \in \Gamma_1 \cup \Gamma_2$, and so

$$\mathcal{W}(\Gamma_1 \cup \Gamma_2) \leq A_\mu(S) \leq A_{\mu_1}(S) + A_{\mu_2}(S).$$

as this holds for all μ_1 and μ_2 over which the infimum in the definition of extremal length is defined, we get the first inequality.

If the families are disjoint, then every metric μ such that $l_\mu(\gamma) \geq 1$ for all $\gamma \in \Gamma_1 \cup \Gamma_2$ can be written as a sum $\mu = \mu_1 + \mu_2$ with supports on Γ_1 and Γ_2 respectively, so that

$$\mathcal{W}(\Gamma_1) + \mathcal{W}(\Gamma_2) \leq A_\mu(S),$$

and since this holds for all μ, we get the opposite inequality. $\square$

For x, y positive real numbers, we denote their harmonic sum by

$$x \oplus y := \frac{1}{\frac{1}{x} + \frac{1}{y}}.$$

Proposition 2.3 (Series law)**.** *Suppose that Γ_1 and Γ_2 are disjoint path families and Γ overflows both Γ_1 and Γ_2. Then*

$$\mathcal{W}(\Gamma) \leq \mathcal{W}(\Gamma_1) \oplus \mathcal{W}(\Gamma_2).$$

Proof. We may equivalently show that $\mathcal{L}(\Gamma) \geq \mathcal{L}(\Gamma_1) + \mathcal{L}(\Gamma_2)$. Choosing a metric μ_i for Γ_i of finite non-zero area, we may scale it so that $l_\mu(\Gamma_i) = A_{\mu_i}(S)$. If $\mu = \mu_1 + \mu_2$, then $l_\mu(\Gamma) \geq l_{\mu_1}(\Gamma_1) + l_{\mu_2}(\Gamma_2)$, and

$$\mathcal{L}(\Gamma) \geq \frac{l_\mu(\Gamma)^2}{A_\mu(S)} \geq \frac{(l_{\mu_1}(\Gamma_1) + l_{\mu_2}(\Gamma_2))^2}{A_{\mu_1}(S) + A_{\mu_2}(S)} = l_{\mu_1}(\Gamma_1) + l_{\mu_2}(\Gamma_2) = \frac{l_{\mu_1}(\Gamma_1)^2}{A_{\mu_1}(S)} + \frac{l_{\mu_2}(\Gamma_2)^2}{A_{\mu_2}(S)}.$$

As this holds independently of the choice of μ_1 and μ_2, we get the desired inequality. $\square$

Remark 2.4. Both laws obviously generealize to finite collections of path families. More precisely, if $\Gamma_1, \ldots, \Gamma_n$ are path families and $\Gamma \subseteq \bigcup_{i=1}^n \Gamma_i$ (or, more weakly, every Γ_i overflows Γ), then

$$\mathcal{W}(\Gamma) \leq \sum_{i=1} \mathcal{W}(\Gamma_i),$$

and if Γ' simultaneously overflows all Γ_i, then

$$\mathcal{W}(\Gamma') \leq \bigoplus_{i=1}^{n} \mathcal{W}(\Gamma_i).$$

3 Changes under mappings

3.1 Pullbacks and pushforwards of metrics

In what follows, all holomorphic maps are assumed to be non-constant, unless stated otherwise.

Definition 3.1. Let $f : S \to T$ be a holomorphic map so that $f(z) = w$, and let $\nu = \nu(w)|dw|$ be a metric on T. The *pullback* $f^*\nu$ of ν is the metric on S locally given by

$$f^*\nu = \nu(f(z))|f'(z)||dz|.$$

More precisely, if (U, φ) is a chart around $p \in S$ and (V, ψ) is a chart around $q = f(p) \in T$, then

$$(f^*\nu)_\varphi(\varphi(p)) = \nu_\psi(\psi(f(p))|(\psi \circ f \circ \varphi^{-1})'(\varphi(p))|,$$

wherever the functions above are defined. Equivalently, for $z \in \varphi(U \cap f^{-1}(V))$,

$$(f^*\nu)_\varphi(z) = \nu_\psi((\psi \circ f \circ \varphi^{-1})(z))|(\psi \circ f \circ \varphi^{-1})'(z)|.$$

In the tangent bundle definition, $(f^*\nu)(v) = \nu(df(v))$, where v is a tangent vector and df is the differential.

By construction, the map f is a local isometry away from critical points. More explicitly, if $p \in S$ is not a critical point, it has a neighborhood N such that $f : (N, f^*\nu) \to (f(N), \nu)$ is an isometry. Though the metric is well defined even at critical points, having value 0, they are inconsequential, as they form a discrete set and therefore of measure zero. This implies that f preserves the lengths of curves, even if they pass through critical points: if γ is a curve in U, its image/pushforward $f_*\gamma = f \circ \eta$ satisfies

$$\int_{f_*\gamma} \nu ds = \int_\gamma (f^*\nu) ds,$$

or equivalently, $l_{f^*\nu}(\gamma) = l_\nu(f_*\gamma)$.

In the special case that f is a biholomorphism, we get the following transformation law for measurable functions $\phi : T \to [0, +\infty)$:

$$\iint_S (\phi \circ f)(f^*\nu)^2 = \iint_T \phi\nu^2.$$

In particular, biholomorphisms preserve areas.

Remark 3.2. The above transformation may also be suggestively written as

$$\iint_S \phi(f(z))\nu(f(z))^2|f'(z)|^2|dz|^2 = \iint_T \phi(w)\nu(w)^2|dw|^2,$$

where one has to be careful with the fact that the expressions inside the integrals only represent local forms, since z and w may not be global coordinates.

Proposition 3.3. *If $f : S \to T$ is a holomorphic map that is at most d to 1 and ν is a metric on T, then*

$$A_\nu(f(S)) \leq A_{f^*\nu}(S) \leq d \cdot A_\nu(T).$$

If f is proper of degree d, then $A_{f^*\nu}(S) = d \cdot A_\nu(T)$.

Proof. As the set of critical points of f is discrete, we may assume without loss of generality that f has no critical points, and since $A_\nu(T) \geq A_\nu(f(S))$, we may also assume that f is surjective. Let $(U_i)_{i\in I}$ be an open cover of S such that for each $i \in I$, $f|_{U_i} : U_i \to f(U_i)$ is a biholomorphism, and let $(\rho_i)_{i\in I}$ be a partition of unity subordinate to this open cover. Then

$$A_{f^*\nu}(S) = \sum_{i\in I} \iint_S \rho_i(f^*\nu)^2 = \sum_{i\in I} \iint_{U_i} \rho_i(f^*\nu)^2.$$

Define $\widetilde{\rho}_i : f(U_i) \to \mathbb{R}$ by $\widetilde{\rho}_i = \rho_i \circ (f|_{U_i})^{-1}$, so that $\rho_i = \widetilde{\rho}_i \circ f$ on U_i. Hence

$$A_{f^*\nu}(S) = \sum_{i\in I} \iint_{U_i} (\widetilde{\rho}_i \circ f)(f^*\nu)^2 = \sum_{i\in I} \iint_{f(U_i)} \widetilde{\rho}_i\nu^2.$$

The collection $\{\widetilde{\rho}_i\}_{i\in I}$ is locally finite: if $q \in T$ and $f^{-1}(q) = \{p_1, \ldots, p_k\}$, each p_i has a neighborhood N_j that intersects finitely many of the supports $\operatorname{supp}\rho_i$. Hence $\widetilde{N} = f(N_1)\cap\ldots\cap f(N_k)$ is a neighborhood of q that intersects finitely many of the supports $\operatorname{supp}\widetilde{\rho}_i$. This implies that the non-negative function $\widetilde{\rho} = \sum_{i\in I}\widetilde{\rho}_i$ is well defined (and in fact positive, with the surjectiveness assumption) on T, and so the functions $\widehat{\rho}_i = \widetilde{\rho}_i/\widetilde{\rho}$ form a partition of unity subordinate to the $f(U_i)$. So

$$A_{f^*\nu}(S) = \sum_{i\in I} \iint_{f(U_i)} \widetilde{\rho}_i\nu^2 = \sum_{i\in I} \iint_{f(U_i)} (\widehat{\rho}_i\widetilde{\rho})\nu^2 = \iint_T \widetilde{\rho}\nu^2.$$

We prove that $1 \leq \widetilde{\rho} \leq d$. For $q \in T$, again we have $f^{-1}(q) = \{p_1, \ldots, p_k\}$, where $1 \leq k \leq d$. Given the previously defined neighborhoods N_j and $\widetilde{N}$, we may assume that if N_j intersects $\operatorname{supp}\rho_i$, then $N_j \subseteq U_i$. Let I_q be the set of indices for which $\widetilde{N}$ intersects $\operatorname{supp}\widetilde{\rho}_i$, so that for $i \in I_q$ we have $\widetilde{N} \subseteq f(U_i)$. There is a unique preimage p_j such that $p_j \in U_i$; this implies that we can partition I_w into $I_{p_1} \sqcup \cdots \sqcup I_{p_j}$ depending on which preimage U_i contains. Therefore

$$\widetilde{\rho} = \sum_{j=1}^{k} \sum_{i \in I_{p_j}} \widetilde{\rho}_i(q) = \sum_{j=1}^{k} \sum_{i \in I_{p_j}} \rho(p_j) = \sum_{j=1}^{k} 1 = k.$$

This immediately concludes the desired inequalities from the integral expression for $A_{f^*\nu}(S)$ in terms of $\widetilde{\rho}$. $\square$

Corollary 3.4. *If $f : S \to T$ is exactly d to 1, then $A_{f^*\nu}(S) = d \cdot A_\nu(T)$.*

Definition 3.5. Let $f : S \to T$ be a holomorphic map, and μ a metric on S. The *pushforward metric* $f_*\mu$ on T is the metric corresponding to the pushforward measure f_*A_μ, that is, such that $A_{f_*\mu} = f_*A_\mu$.

This implies the transformation formula

$$\iint_T \phi(f_*\mu)^2 = \iint_S (\phi \circ f)\mu^2,$$

where $\phi : T \to [0, +\infty)$ is any non-negative measurable function. In particular, $A_{f_*\mu}(T) = A_\mu(S)$; note that this does not require that f be a biholomorphism, in contrast to pullbacks of metrics. In fact, if $f : S \to T$ is a biholomorphism, pullbacks and pushforwards of metrics are inverse operations.

We want to understand what the local form for the pushforward metric is. Suppose that f is finite to one, and that $q \in T$ is a point that admits a chart (V, ψ) that contains no critical values and no asymptotic values. If $f^{-1}(q) = \{p_1, \dots, p_k\}$, by shrinking the neighborhoods we may assume that there are charts (U_j, φ_j) around p_j for $j = 1, \dots, k$ such that $f|_{U_i} : U_i \to V$ is a biholomorphism. Given any non-negative measurable function ϕ on V, we have

$$\iint_V \phi(f_*\mu)^2 = \iint_{f^{-1}(V)} (\phi \circ f)\mu^2 = \sum_{j=1}^{k} \iint_{U_i} (\phi \circ f)\mu^2.$$

If $g_j = (f|_{U_j})^{-1} : V \to U_j$ denotes the local branch of the inverse mapping to the j-th preimage, by the transformation rule with respect to pullbacks we get

$$\sum_{j=1}^{k} \iint_{U_i} (\phi \circ f)\mu^2 = \sum_{j=1}^{k} \iint_V (\phi \circ f \circ g_i)(g_i^*\mu^2) = \iint_V \phi \left(\sum_{j=1}^{k} (g_j^*\mu)^2 \right).$$

As this holds for all ϕ measurable and non-negative, we get the equality

$$(f_*\mu)^2 = \sum_{j=1}^{k} (g_j^*\mu)^2$$

on V. For each pullback $(g_j^*\mu)^2$, we have

$$(g_j^*\mu)_\psi(\psi(q)) = \mu_{\varphi_j}(\varphi_j(g_j(q)))|(\varphi_j \circ g_j \circ \psi^{-1})'(\psi(q))|,$$

and so

$$(f_*\mu)_\psi(\psi(q))^2 = \sum_{j=1}^{k} \frac{\mu_{\varphi_j}(\varphi_j(p_j))^2}{|(\psi \circ f \circ \varphi_j^{-1})'(\varphi_j(p_j))|^2},$$

or equivalently for $w \in \psi(V)$,

$$(f_*\mu)_\psi(w)^2 = \sum_{j=1}^{k} \frac{\mu_{\varphi_j}(z_j)^2}{|(\psi \circ f \circ \varphi_j^{-1})'(z)|^2}.$$

Suggestively, we write

$$(f_*\mu)(w)|dw| = \left(\sum_{f(z)=w} \frac{\mu(z)^2}{|f'(z)|^2} \right)^{\frac{1}{2}} |dw|.$$

More generally, we may forgo the assumption that f is finite to one and that q has a neighborhood with no critical and asymptotic values. Suppose that q is not a critical value and $p_1, \ldots, p_k$ are points in $f^{-1}(q)$, still with biholomorphisms $f|_{U_i} : U_i \to V$. We get that $f_*\mu$ at q is larger than or equal to the local form given by pullling back these k local lifts, since the preimage $f^{-1}(V)$ will in general contain more components (either corresponding to other preimages of q, or corresponding to asymptotic curves in S).

Even when f is proper, the local form implies that the length of a curve on T is not necessarily the sum of the lengths of all its lifts on S, contrary to what one might expect. We do however, get some important inequalities:

Proposition 3.6. *Let $f : S \to T$ be a holomorphic map, and μ a metric on S.*

a) If γ is a curve on S, then

$$l_{f_*\mu}(f \circ \gamma) \geq l_\mu(\gamma).$$

b) If η is a curve on T and $\widetilde{\eta}_1, \ldots, \widetilde{\eta}_k$ are lifts of η, then

$$l_{f_*\mu}(\eta) \geq \frac{1}{\sqrt{k}} \sum_{j=1}^{k} l_\mu(\widetilde{\eta}_j).$$

Proof. It is sufficient to compare lengths of curves contained in holomorphic charts, as we may divide the curves into segments, each contained in a chart, and add up the lengths. The expression for the local form for $f_*\mu$ guarantees that the length of a curve on S is greater than or equal to the length of any of its local lifts, and the second inequality follows from

$$\sqrt{\frac{a_1^2 + \cdots + a_k^2}{k}} \geq \frac{a_1 + \cdots + a_k}{k}.$$

□

Remark 3.7. It is possible that η has "partial lifts" on S, for example if the image of η is not fully contained in $f(S)$, and another source of complications is if the critical values or asymptotic values are not discrete on S. But as long as η has some "full" lifts, the inequality above holds. These oddities never happen, for instance, if f is proper, so that it is a branched cover of some degree d.

We also have statements describing how pushforwards and pullbacks interact with each other:

Proposition 3.8. *a) If $f : S \to T$ is a d to 1 holomorphic map and ν is a metric on T, then*

$$f_* f^* \nu = \sqrt{d} \cdot \nu.$$

b) If $f : S \to T$ is a holomorphic map and μ a metric on S, then

$$f^* f_* \mu \geq \mu.$$

Proof. For the first statement, we may adapt the proof of Propostion 3.3 to be with respect to the integral of some non-measurable function ϕ, so that the following holds:

$$\iint_T \phi (f_* f^* \nu)^2 = \iint_S (\phi \circ f)(f^* \nu)^2 = \iint_T \phi \widetilde{\rho} \nu^2 = d \iint_T \phi \nu^2.$$

As this is true for all non-negative measurable ϕ, we get the desired equality. The second statement is true by virtue of the local form for f_μ, being greater than or equal than any one of its "lifts". □

3.2 Pullbacks and pushforwards of path families

If Γ is a path family on S and $f : S \to T$ is holomorphic, then the *pushforward* $f_*\Gamma := f(\Gamma) = \{f \circ \gamma \mid \gamma \in \Gamma\}$ is naturally a path family in T.

Proposition 3.9. *If $f : S \to T$ is holomorphic map that is at most d to 1 and Γ is a path family on S, then*

$$\frac{1}{d}\mathcal{W}(\Gamma) \leq \mathcal{W}(f(\Gamma)) \leq \mathcal{W}(\Gamma).$$

Proof. Let μ be a metric on S such that $l_\mu(\gamma) \geq 1$ for all $\gamma \in \Gamma$ and $f_*\mu$ its pushforward. Then $l_{f_*\mu}(f \circ \gamma) \geq l_\mu(\gamma) \geq 1$ for all $f \circ \gamma \in f(\Gamma)$, and so

$$\mathcal{W}(f(\Gamma)) \leq A_{f_*\mu}(T) = A_\mu(S).$$

By taking infima, we have $\mathcal{W}(f(\Gamma)) \leq \mathcal{W}(\Gamma)$.

Now let ν be a metric on T such that $l_\nu(f \circ \gamma) \geq 1$ for all $\gamma \in \Gamma$, and $f^*\nu$ be its pullback. Then $l_{f^*\nu}(\gamma) = l_\mu(f \circ \gamma) \geq 1$, and $A_{f^*\nu}(S) \leq d \cdot A_\nu(T)$, and so $\mathcal{W}(\Gamma) \leq d\,\mathcal{W}(f(\Gamma))$. $\square$

Remark 3.10. A particular case of the proposition above is if we have an inclusion $U \hookrightarrow V$ of two Riemann surfaces, and Γ is a path family on U. The width of Γ will be the same when seen as a family on U or on V, so that the conformal invariant is associated essentially to the family and its support (or any neighborhood of its support).

If Γ is a path family on T, we may define its *pullback* $f^*\Gamma$ as the collection of lifts of curves in Γ, that is, the curves $\widetilde{\gamma}$ in S such that $f \circ \widetilde{\gamma} = \gamma \in \Gamma$. This definition is most useful when f is a proper, so that all curves avoiding the critical values of f are liftable.

Proposition 3.11. *if $f : S \to T$ is a proper map of degree d and Γ is a path family on T, then*

$$\mathcal{W}(f^*\Gamma) = d\,\mathcal{W}(\Gamma).$$

Proof. Let ν be a metric in T such that $l_\nu(\gamma) \geq 1$ for all $\gamma \in \Gamma$ and $f^*\nu$ its pullback. Then $l_{f^*\nu}(\widetilde{\gamma}) = l_\nu(f \circ \widetilde{\gamma}) = l_\mu(\gamma)$, and $A_{f^*\nu}(S) = d \cdot A_\nu(T)$, so that $\mathcal{W}(f^*\Gamma) \leq d\,\mathcal{W}(\Gamma)$.

Now let μ be a metric on S such that, for all lifts $\widetilde{\gamma} \in f^*\Gamma$, we have $l_\mu(\widetilde{\gamma}) \geq 1$. If $f_*\mu$ is the pushforward, then

$$l_{f_*\mu}(\gamma) \geq \frac{1}{\sqrt{d}} \sum_{j=1}^{d} l_\mu(\widetilde{\gamma}_j) \geq \sqrt{d}$$

since any curve will have d lifts, and $A_{f_*\mu}(T) = A_\mu(S)$. If we let $\overline{\mu} := \frac{1}{\sqrt{d}} f_*\mu$, then $l_{\overline{\mu}}(\gamma) \geq 1$ for all $\gamma \in \Gamma$ and $A_{\overline{\mu}}(T) = \frac{1}{d}A_\mu(S)$. Therefore $\mathcal{W}(\Gamma) \leq \frac{1}{d}\,\mathcal{W}(f^*\Gamma)$, concluding the desired equality. $\square$

4 Large width, small overlap

This section is a presentation of the "small overlap" principle for path families with large widths [KL09].

Definition 4.1. Given a Riemann surface S, a *quadrilateral* Q on S is a simply connected open subset Q of S whose boundary is a Jordan curve, along with four marked points p_1, p_2, p_3, p_4 on ∂Q in cyclic order with positive orientation. Letting I be the arc $[p_1, p_2]$ on ∂Q and J be the arc $[p_3, p_4]$, we may denote $Q = Q(p_1, p_2, p_3, p_4) = Q(I, J)$. A quadrilateral naturally defines the family of paths (in Q) connecting the arc I to J, whose extremal width we denote by $\mathcal{W}(Q) = \mathcal{W}(I, J)$.

The uniformization theorem and the theory of elliptic functions guarantee the existence of $w > 0$ and a unique homeomorphism $h : \overline{Q} \to R$, where R is the rectangle $[0, w] \times [0, 1]$, that maps I to $[0, w] \times \{0\}$ and J to $[0, w] \times \{1\}$ (preserving orientations), and is conformal on the interiors. The value w is uniquely determined, and is equal to the extremal width $\mathcal{W}(Q)$. This essentially says that all quadrilaterals form a moduli space of one positive real parameter. Moreover, the vertical foliation on the rectangle $[0, w] \times [0, 1]$ can be pulled back to a foliation on Q, which we call its *vertical foliation*, whose extremal width again is equal to w. More generally, a path family Λ on Q is said to be *vertical* if it is a (measurable) subset of the vertical foliation.

Letting m be the Lebesgue measure on R, its pushforward $\pi_* m$ under the projection to the x coordinate induces a *transverse measure* on the vertical foliation (identified with the interval $[0, w]$ itself). By pulling back the measures to S by h, we get a measure $h^* m$ on Q that realizes the extremal width, and a transverse measure ν on its vertical foliation (or, more generally, on any vertical path family). Note that, with this specific normalization of the rectangle, $\nu(\Lambda) = \mathcal{W}(\Lambda)$ for any vertical path family Λ on Q.

Given a path $\gamma \in S$, we may compute "how much" it intersects a vertical path family Λ on Q; that is, how many vertical leaves it crosses. This intersection is measured by the ratio

$$\varepsilon(\gamma, \Lambda) := \frac{\nu(\{\lambda \in \Lambda \mid \lambda \cap \gamma \neq \emptyset\})}{\nu(\Lambda)}.$$

Let S be a Riemann surface, possibly with boundary, Γ a path family on S, and Λ a vertical path family on a quadrilateral Q on S. The "small overlap" principle can be states as follows:

Lemma 4.2. *Suppose that, for all paths $\gamma \in \Gamma$, $\varepsilon(\gamma, \Lambda) \geq p$, where $0 < p \leq 1$. Then*

$$\mathcal{W}(\Gamma)\, \mathcal{W}(\Lambda) \leq \frac{1}{p^2}.$$

Proof. Invoking the notation utilized previously in this section, let $m|_{h(\Lambda)}$ be the Lebesgue metric on R restricted to $h(\Lambda)$, so that $A_{m|_{h(\Lambda)}}(R) = \nu(\Lambda) = \mathcal{W}(\Lambda)$. Let $\mu = h^*(m|_{h(\Lambda)})$ be its pullback to Q, extended to S so that it is equal to 0 outside of $\overline{Q}$. Since every path $\gamma \in \Gamma$ crosses definite p proportion of leaves of λ, we have $l_\mu(\gamma) \geq p\nu(\Lambda)$. Let $\overline{\mu}$ be the rescaling of μ by $(p\nu(\Lambda))^{-1}$, so that for all $\gamma \in \Gamma$, $l_{\overline{\mu}}(\gamma) \geq 1$. Moreover, $A_{\overline{\mu}}(S) = (p\nu(\Lambda))^{-2}\nu(\Lambda)$, so that

$$\mathcal{W}(\Gamma) \leq \frac{1}{p^2\nu(\Lambda)},$$

which reduces to the desired inequality. □

Corollary 4.3. *If $W > 0$ is such that $\mathcal{W}(\Gamma), \mathcal{W}(\Lambda) \geq W$, then there exists at least one path $\gamma \in \Gamma$ for which $\varepsilon(\gamma, \Lambda) < 1/W$. In particular, if $W > 1$, there is at least one path γ in Γ and one leaf $\lambda \in \Lambda$ such that $\lambda \cap \gamma = \emptyset$.*

Consider now two quadrilaterals Q_1 and Q_2 on S, with respective vertical path families Λ_1 and Λ_2 and transverse measures ν_1 and ν_2. We may consider $\varepsilon(\lambda, \Lambda_2)$ as a measurable function on Λ_1, and in fact,

$$\int_{\Lambda_1} \varepsilon(\lambda, \Lambda_2) d\nu_1 = \int_0^1 \nu_1(\{\lambda \in \Lambda_1 \mid \varepsilon(\lambda, \Lambda_2) \geq y\}) dy.$$

By the above, an upper bound for the integrand is $1/(y^2\nu(\Lambda_2))$, but the resulting function is not integrable. A better approach is to first estimate the average length of a curve with respect to a non-extremal metric:

Lemma 4.4. *Let Q be a quadrilateral on a Riemann surface S, and Λ a vertical path family on S. For all metrics ρ on S,*

$$\frac{1}{\nu(\Lambda)} \int_\Lambda l_\rho(\lambda) d\nu \leq \left(\frac{A_\rho(S)}{\mathcal{W}(\Lambda)}\right)^{\frac{1}{2}}.$$

The equality is achieved exactly when ρ is a scalar multiple of the extremal metric for Λ.

Proof. Without loss of generality, we may assume S is a rectangle $R = [0, w] \times [0, 1]$. The projection of Λ to $[0, w]$ is a measurable set E; if μ is the restriction of the Lebesgue measure to $E \times [0,1]$, then $\nu(\Lambda) = A_\mu(E \times [0,1]) = \mathcal{W}(\Lambda)$, and it is the extremal metric for Λ. By Cauchy-Schwarz,

$$\begin{aligned}\frac{1}{\nu(\Lambda)} \int_\Lambda l_\rho(\lambda) d\nu &= \frac{1}{\mathcal{W}(\Lambda)} \int_E \int_0^1 \rho(x,y) dy dx \\ \leq \frac{1}{\mathcal{W}(\Lambda)} &\left(\iint_{E\times[0,1]} \rho^2 dy dx \right)^{\frac{1}{2}} \left(\iint_{E\times[0,1]} 1 dy dx \right)^{\frac{1}{2}} \\ &= \frac{1}{\mathcal{W}(\Lambda)} A_\rho(R)^{\frac{1}{2}} A_\mu(R)^{\frac{1}{2}} = \left(\frac{A_\rho(R)}{\mathcal{W}(\Lambda)} \right)^{\frac{1}{2}}.\end{aligned}$$

Equality in Cauchy-Schwarz occurs exacly when ρ is a scalar multiple of 1 on $E \times [0,1]$. $\square$

Another way to understand the lemma above is as follows. The extremal metric μ for the vertical path family is the one which maximizes the minimal ρ-length of leaves $\lambda \in \Lambda$, over all area 1 metrics ρ. The inequality above states that μ is also the metric that maximizes the *average* ρ-length of leaves $\lambda \in \Lambda$ over all area 1 metrics ρ.

Corollary 4.5. *Let Q_1 and Q_2 be two quadrilaterals on a Riemann surface, with respective vertical path families Λ_i, extremal metrics μ_i and transverse measures ν_i for $i = 1, 2$, normalized so that $\mathcal{W}(\Lambda_i) = \nu_i(\Lambda_i) = A_{\mu_i}(S)$. Then*

$$\frac{1}{\nu(\Lambda_1)} \int_{\Lambda_1} \varepsilon(\lambda, \Lambda_2) d\nu_1 \leq \frac{1}{\sqrt{\mathcal{W}(\Lambda_1)\,\mathcal{W}(\Lambda_2)}}.$$

Proof. This is immediate from the lemma above and the fact that, for any path $\gamma \in S$, its μ_2-length is no less than the proportion of leaves in Λ_2 it crosses, times the measure of Λ_2, so $l_{\mu_2}(\gamma) \geq \varepsilon(\gamma, \Lambda_2)\nu(\Lambda_2)$. $\square$

In other words, the average proportion of leaves of Λ_2 that a leaf of Λ_1 crosses is bounded above by $(\mathcal{W}(\Lambda_1)\,\mathcal{W}(\Lambda_2))^{1/2}$.

We can in fact establish that $\varepsilon(\lambda, \Lambda_2)$ is square integrable as a function of $\lambda \in \Lambda_1$. This comes from Jensen's inequality applied to

$$\frac{1}{\nu(\Lambda)} \int_\Lambda l_\rho(\lambda)^2 d\nu = \frac{1}{\nu(\Lambda)} \int_\Lambda \left(\int_\lambda \rho \right)^2 \leq \frac{1}{\nu(\Lambda)} \iint_\Lambda \rho^2 \leq \frac{A_\rho(S)}{\mathcal{W}(\Lambda)},$$

in the context of lemma 4.4, and so

$$\frac{1}{\nu(\Lambda_1)} \int_{\Lambda_1} \varepsilon(\lambda, \Lambda_2)^2 d\nu_1 \leq \frac{1}{\mathcal{W}(\Lambda_1)\,\mathcal{W}(\Lambda_2)}.$$

This also implies bounded variance on the proportion.

Remark 4.6. Though we defined vertical path families on quadrilaterals, we can do the same for annuli on Riemann surfaces. Every annulus in conformally equivalent to around annulus on $\mathbb{C}$ having a canonical vertical foliation, and we may pull it back to a foliation on the conformal annulus. The results above carry over verbatim.

5 A special family

This brief section is a retelling of some known facts contained in [KL09]. Let U be a Riemann surface of finite type with boundary ∂U, and $K \subseteq \operatorname{int} U$ a compact set. We consider the path family $\Gamma(U, K)$ of all paths connecting K to ∂U, and $\operatorname{mod}(U, K) := \mathcal{L}(\Gamma(U, K))$, $\mathcal{W}(U, K) := \mathcal{W}(\Gamma(U, K))$. We first note that the family of paths Γ' *contained in* $U \setminus K$ and connecting K to ∂U has the same width, because $\Gamma' \subseteq \Gamma$ and Γ overflows Γ'.

Proposition 5.1. *Let $f : U \to V$ be a proper holomorphic map of degree d between Riemann surfaces with boundary, $A \subseteq \operatorname{int} U$ compact and $B = f(A) \subseteq \operatorname{int} V$. Then*

$$\operatorname{mod}(U, A) \leq \operatorname{mod}(V, B) \leq d \cdot \operatorname{mod}(U, A).$$

Proof. If $\Gamma = \Gamma(U, A)$ and $\Gamma' = (V, B)$, then $f(\Gamma) = \Gamma'$ as f is proper. The inequality is an immediate consequence of 3.9. □

6 Extremal metrics and the Beurling criterion

A natural question to ask in terms of extremal length and width is if, given a path family Γ on a (possibly bordered) Riemann surface S, there exists a metric μ that realizes $\mathcal{W}(\Gamma)$, that is, such that $l_\mu(\gamma) \geq 1$ for all $\gamma \in \Gamma$ and $A_\mu(S) = \mathcal{W}(\Gamma)$. We say that such a metric μ is *extremal* for Γ, and so, for all other metrics μ' with the same normalization condition that $l_{\mu'}(\gamma) \geq 1$ for all $\gamma \in \Gamma$, we have $A_\mu(S) \leq A_{\mu'}(S)$. Naturaly μ also maximizes the ratio

$$\frac{l_{\mu'}(\Gamma)^2}{A_{\mu'}(S)}$$

over all metrics μ' of finite non-zero area, where we recall that $l_{\mu'}(\Gamma) = \inf_{\gamma \in \Gamma} l_{\mu'}(\gamma)$, and so μ realizes $\mathcal{L}(\Gamma)$. Rescalings of μ also attain $\mathcal{L}(\Gamma)$, but wouldn't follow the normalization $l_\mu(\Gamma) = 1$. (The infimum must be equal to 1, otherwise we could scale down μ by $l_\mu(\Gamma)$ to obtain a metric with smaller area, according the "normalized" definition of an extremal metric.)

Proposition 6.1. *If $\mathcal{W}(\Gamma) < +\infty$ and an extremal metric μ exists for Γ, then it is unique (up to a measure zero set).*

Proof. Suppose μ is an extremal metric such that $l_\mu(\gamma) \geq 1$ for all $\gamma \in \Gamma$ and $A_\mu(S) = \mathcal{W}(\Gamma)$, and that ν also has these properties. Then $\rho = \frac{1}{2}\mu + \frac{1}{2}\nu$ also satisfies $l_\rho(\gamma) \geq 1$ for all $\gamma \in \Gamma$, and by Minkowski's inequality for $p = 2$ adapted to metrics on Riemann surfaces,

$$A_\rho(S)^{1/2} \leq \frac{1}{2}A_\mu(S)^{1/2} + \frac{1}{2}A_\nu(S)^{1/2} = \mathcal{W}(\Gamma)^{1/2},$$

with equality if and only μ and ν are linearly dependent. But given the normalizations, we obtain equality almost everywhere. □

A conceptually similar problem is as follows: for any subfamily $\Gamma' \subseteq \Gamma$, we have $\mathcal{W}(\Gamma') \leq \mathcal{W}(\Gamma)$. Is there a smaller (or smallest) subfamily such that we still have $\mathcal{W}(\Gamma') = \mathcal{W}(\Gamma)$? We have seen that such a subfamily exists in the example of the paths connecting opposite sides of a rectangle, where a subfamily which has same extremal width is the vertical foliation.

Remark 6.2. For the rest of the chapter, we assume that all curves in a path family Γ are one-to-one, or composed of finitely many one-to-one arcs.

In general, existence of an extremal metric is not guaranteed, but it always exists for a subfamily of Γ whose complement is "exceptional":

Theorem 6.3 (Fuglede's lemma [Fug57])**.** *If $\mathcal{W}(\Gamma) < +\infty$, then there exists a subfamily $\Gamma_e \subseteq \Gamma$ such that $\mathcal{W}(\Gamma_e) = 0$ and $\Gamma \setminus \Gamma_e$ admits an extremal metric.*

Beurling established an important sufficient criterion for when a conformal metric μ is extremal for a path family Γ. It is classicaly stated for when S is an open subset of the plane [Ahl73], so in order to adapt it to general Riemann surfaces, we introduce the following definition. A *signed measurable conformal metric* has the same definition as a measurable conformal metric, except that the corresponding measurable functions on holomorphic charts are not required to be non-negative. If ρ is a signed (measurable, conformal) metric, then $|\rho|$ is a metric. We say that ρ is L^2-integrable if $|\rho|$ has finite area.

Theorem 6.4. *Let Γ be a path family on S, and μ a metric on S of finite non-zero area. If there exists a subfamily $\Gamma_0 \subseteq \Gamma$ such that:*

(i) For all $\gamma \in \Gamma_0$, $l_\mu(\gamma) = l_\mu(\Gamma)$;

(ii) For every L^2-integrable signed metric ρ on S the following implication holds:

If $\int_\gamma \rho \geq 0$ for all $\gamma \in \Gamma_0$, then $\iint_S \rho\mu \geq 0$;

Then $\frac{1}{l_\mu(\Gamma)}\mu$ is extremal for Γ.

Note that because ρ is a signed metric, it makes sense to define line integrals of ρ and area integrals of $\rho\mu$.

Proof. Suppose that the hypothesis are true for a metric μ and subfamily $\Gamma_0 \subseteq \Gamma$, and we normalize it so that $l_\mu(\Gamma) = 1$. Let μ' be another metric, which we may assume has finite area, also normalized such that $l_{\mu'}(\Gamma) = 1$. Then for all $\gamma \in \Gamma_0$, we have

$$l_{\mu'}(\gamma) \geq 1 = l_\mu(\gamma),$$

since $\Gamma_0 \subseteq \Gamma$. Consider the L^2-integrable signed metric $\rho = \mu' - \mu$. As $\int_\gamma \rho \geq 0$ for all $\gamma \in \Gamma_0$, we get

$$\iint_S \rho\mu = \iint_S (\mu' - \mu)\mu \geq 0.$$

By Cauchy-Schwarz,

$$\iint_S \mu'\mu \leq \left(\iint_S \mu'^2\right)^{1/2} \left(\iint_S \mu^2\right)^{1/2},$$

which combined with the inequality above gives $A_{\mu'}(S) \geq A_\mu(S)$. Hence μ is extremal. $\square$

Remark 6.5. By the same proof as above, we may conclude that if μ is a metric for which such a family Γ_0 exists, then necessarily $\mathcal{W}(\Gamma_0) = \mathcal{W}(\Gamma)$, and (a rescaling of) μ is extremal for both Γ and Γ_0. Therefore, even though the converse to Beurling's criterion is not always true, given an extremal metric μ for Γ, a good candidate for a subfamily $\Gamma' \subseteq \Gamma$ such that $\mathcal{W}(\Gamma') = \mathcal{W}(\Gamma)$ is that of the paths whose μ-length is equal to the infimal μ-length (that is, the unit length paths, under the specified normalization).

A partial converse to the Beurling criterion is true, given a suitable extension of Corollary 1 in [Bad13] to Riemann surfaces:

Theorem 6.6. *A finite area metric μ is extremal for Γ if and only if there exists a family of curves Γ' such that the following conditions hold:*

(i) $\mathcal{W}(\Gamma \cup \Gamma') = \mathcal{W}(\Gamma')$;

(ii) *For all* $\gamma \in \Gamma'$, $l_\mu(\gamma) = l_\mu(\Gamma)$;

(iii) *For all* L^2*-integrable signed metrics* ρ, *if* $\int_\gamma \rho \geq 0$ *for all* $\gamma \in \Gamma'$, *then* $\iint_S \rho\mu \geq 0$.

Badger's proof shows that if μ is extremal, one may take the family Γ' to be the family of all curves in S (not just in Γ) such that $l_\mu(\gamma) = 1$.

6.1 Relationship with quadratic differentials

It turns out that, for many path families naturally defined in terms of homotopy classes of paths, an extremal metric exists, and it is given by a quadratic differential. For a treatment of quadratic differentials and their basic properties, we refer to [Str84].

Let S be a bordered Riemann surface of finite genus and with finitely many punctures. For simplicity, assume that the boundary components of S are all homeomorphic to S^1. Consider a set of $H_1 = \{[\gamma_i]\}_i$ of non-peripheral (that is, not null-homotopic and not homotopic to a puncture) free homotopy classes of simple closed curves on S, and $H_2 = \{[\eta_j]\}_j$ a set of non-trivial homotopy classes of arcs on S whose endpoints are on boundary components, and the homotopy allows the endpoints to move on their respective boundaries. Non-triviality here means that each arc is not homotopic to a point on ∂S. Assume that there exists a choice of representative curves and arcs $\{\gamma_i, \eta_j\}_{i,j}$ that are pairwise disjoint, and note that $H = H_1 \cup H_2$ is therefore necessarily finite.

We define Γ_H to be the set of all metrics μ on S such that, for all simple closed curves γ such that $\gamma \in [\gamma_i]$ for some homotopy class in H_1, we have $l_\mu(\gamma) \geq 1$, and for all arcs η whose endpoints are on ∂S and $\eta \in [\eta_j]$ for some homotopy class in H_2, also $l_\mu(\eta) \geq 1$. A specialization of a theorem of Jenkins [Jen57] guarantees the following:

Theorem 6.7. *There exists an extremal metric* μ *for* Γ_H, *which is given by a unique holomorphic integrable quadratic differential* q *on* S *having closed trajectories.*

We actually get a lot more information: The critical graph of q subdivides S into domains that correspond to quadrilaterals and annuli associated to the homotopy classes, though these may be degenerate. Moreover, this metric satisfies an extremal property with respect to a general choice of these domains and the sum of their moduli. We omit all the details for the sake of exposition.

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