Thurston iteration on the Rees space

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June 22, 2025

1 Background

Let $f : S^2 \to S^2$ be a degree $d \geq 2$ topological branched covering of a 2-sphere. An important question is whether such a map has "the same dynamics" as that of a rational map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Let C_f be the set of critical points of f and $P_f = \bigcup_{n>0} f^n(C_f)$ its *postcritical set*; in the particular case where P_f is finite, a possible formalization of this notion of equivalence to a rational map was given by Thurston. For simplicity, hereafter we call a postcritically finite topological branched cover a *Thurston map*. Two Thurston maps f and g are said to be *Thurston equivalent* if there exists homeomorphisms $\theta, \theta' : (S^2, P_f) \to (S^2, P_g)$ such that the diagram below commutes, and θ is isotopic to θ' relative to P_f:

$$\begin{tikzcd}
(S^2, P_f) \arrow[r, "\theta'"] \arrow[d, "f"'] & (S^2, P_g) \arrow[d, "g"] \\
(S^2, P_f) \arrow[r, "\theta"] & (S^2, P_g)
\end{tikzcd}$$

This is a weaker notion of equivalence than that of topological conjugacy, and is akin to "combinatorial equivalence", or conjugacy up to isotopy. Any homeomorphism isotopic to θ rel P_f induces another Thurston equivalence, since we may lift this isotopy starting from θ'. We remark that there may be several non-isotopic Thurston equivalences between f and g.

Thurston gave a necessary and sufficient topological condition for a Thurston map f to be equivalent to a rational map on $\hat{\mathbb{C}}$, with a later proof appearing in print by Douady and Hubbard [4]. For completeness, we transcribe the definitions here. A *multicurve* Γ on $S^2 \setminus P_f$ is a collection of disjoint, simple, closed essential curves $\{\gamma_1, \ldots, \gamma_n\}$ on $S^2 \setminus P_f$, where an essential curve is non-nullhomotopic and not homotopic to a puncture (non-peripheral). As $f : S^2 \setminus f^{-1}(P_f) \to S^2 \setminus P_f$ is an unbranched degree d covering and $S^2 \setminus f^{-1}(P_f) \subseteq S^2 \setminus P_f$, for each $\gamma \in \Gamma$ the preimage $f^{-1}(\gamma)$ consists of disjoint simple closed curves on $S^2 \setminus P_f$, which we call the pullbacks of γ. We say that Γ is f*-invariant* if each essential pullback $\tilde{\gamma}$ of γ is homotopic to some curve $\gamma' \in \Gamma$, that is, $f^{-1}(\Gamma) \subseteq \Gamma$ up to isotopy.

Given an f-invariant multicurve Γ, we may define the *Thurston linear transformation* $f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ on the $\mathbb{R}$-vector space whose basis elements are curves $\gamma \in \Gamma$. Explicitly,

$$f_\Gamma(\gamma_j) = \sum_i \sum_k \frac{1}{d_{i,j,k}} \gamma_i,$$

where i ranges over $\gamma_i \in \Gamma$ and k ranges over all pullbacks γ_k of γ_j that are homotopic to γ_i, and $d_{i,j,k}$ is the degree of the map $f|_{\gamma_k} : \gamma_k \to \gamma_j$. Since the matrix of f_Γ has non-negative entries with respect to this basis, there exists a largest real non-negative eigenvalue $\lambda(f, \Gamma)$. We say that Γ is a *Thurston obstruction* if it is f-invariant and $\lambda(f, \Gamma) \geq 1$. Thurton's criterion is therefore the following:

Theorem 1.1. *If f is a Thurston map with hyperbolic orbifold, then f is equivalent to a rational map if and only if f has no Thurston obstructions.*

The first step of the proof is to transform the problem of finding such an equivalence into finding a fixed point for a certain iteration on the Teichmüller space $\mathcal{T}_f := \mathcal{T}_{P_f}$ of the punctured or marked sphere (S^2, P_f), which is non-canonically homeomorphic to $\mathcal{T}_{0,p}$, where $p = |P_f|$.

Given a finite set $P \subset S^2$, We may describe elements of $\mathcal{T}_P$ by diffeomorphisms $\varphi : (S^2, P) \to \hat{\mathbb{C}}$, where φ and η are equivalent if there is an conformal isomorphism $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\eta \circ \varphi^{-1}$ is isotopic to h relative to $\varphi(P)$, and $\eta|_P = (h \circ \varphi)|_P$:

$$\begin{tikzcd} & \hat{\mathbb{C}} \arrow[dd, "h"] \\ (S^2, P) \arrow[ur, "\varphi"] \arrow[dr, "\eta"'] & \\ & \hat{\mathbb{C}} \end{tikzcd}$$

In other words, the diagram commutes on P and commutes up to isotopy rel P. Given a Thurston map f, the Thurston pullback map $\sigma_f : \mathcal{T}_f \to \mathcal{T}_f$ is defined as follows. If $\tau \in \mathcal{T}_f$ is represented by a diffeomorphism $\varphi : (S^2, P_f) \to \hat{\mathbb{C}}$ mapping three given points $\{p_1, p_2, p_3\} \subseteq P_f$ to $\{0, 1, \infty\}$ in order, then there exists a diffeomorphism $\varphi' : (S^2, P_f) \to \hat{\mathbb{C}}$ mapping (p_1, p_2, p_3) to $(0, 1, \infty)$ and a holomorphic map f_τ such that the diagram below commutes:

$$\begin{tikzcd} (S^2, P_f) \arrow[r, "\varphi'"] \arrow[d, "f"'] & \hat{\mathbb{C}} \arrow[d, "f_\tau"] \\ (S^2, P_f) \arrow[r, "\varphi"'] & \hat{\mathbb{C}} \end{tikzcd}$$

We remark that the class of φ' in $\mathcal{T}_f$ depends only on τ, and f_τ depends only on τ and the choice of points $\{p_1, p_2, p_3\} \subseteq P_f$. We then define $\sigma_f(\tau)$ to be the element of $\mathcal{T}_f$ represented by φ'. It is important to recognize that f_τ is not to be considered as a dynamical system representative of f, since it will map $\varphi'(P_f)$ to $\varphi(P_f)$, and these sets can be identified differently through some diffeomorphism of $\hat{\mathbb{C}}$. It is, however, a rational map in the Hurwitz equivalence class of f, a fact we elaborate on later.

We may also describe $\mathcal{T}_P$ as the space of almost-complex structures μ on S^2, where $[\mu] = [\nu]$ if there is a diffeomorphism $h : S^2 \to S^2$ such that $h^*\nu = \mu$, $h|_P = \mathrm{id}$ and h is isotopic to the identity relative to P. Then $\sigma_f([\mu])$ corresponds to the pullback $[f^*\mu]$, which is always well-defined for topological branched covers and gives us a holomorphic map between Riemann surfaces $f : (S^2, f^*\mu) \to (S^2, \mu)$. Uniformizing these Riemann surfaces to the Riemann sphere is what gives us the holomorphic map f_τ. If θ is a Thurston equivalence between f and g, the induced pullback $\theta^* : \mathcal{T}_g \to \mathcal{T}_f$ on the Teichmuller spaces is a conjugacy between the iterations: $\sigma_f \circ \theta^* = \theta^* \circ \sigma_g$, as can be seen from the isotopy $\theta \sim \theta'$ rel P_f.

We define the moduli space $\mathcal{M}_P := (\iota : P \to \hat{\mathbb{C}})/\operatorname{Aut}(\hat{\mathbb{C}})$, represented by injections of P into $\hat{\mathbb{C}}$ up to postcomposition with Möbius transformations. There is a well-defined map $\pi : \mathcal{T}_P \to \mathcal{M}_P$ given by the restriction $\pi(\varphi) = \varphi|_P$, which is in fact a universal covering map. More precisely, $\mathcal{M}_P$ is the quotient of $\mathcal{T}_P$ by the free discrete right action of the pure mapping class group $\mathrm{PMCG}(S^2, P)$, acting by taking pullbacks of complex structures: for $\gamma \in \mathrm{PMCG}(S^2, P)$, we have $\gamma \cdot [\mu] = [\gamma^* \mu]$. Equivalently, $\mathrm{PMCG}(S^2, P)$ acts on $[\varphi] \in \mathcal{T}_P$ by precomposition: $\gamma \cdot [\varphi] = [\varphi \circ \gamma]$. Similarly, we define $\mathcal{M}_f := \mathcal{M}_{P_f}$. Throughout these notes, we abuse notation by often identifying a representative of a mapping class with the mapping class itself.

We may ask if there is a notion of iteration on the moduli space of f. If a map $\rho_f : \mathcal{M}_f \to \mathcal{M}_f$ were to satisfy $\rho_f \circ \pi = \pi \circ \sigma_f$, then it is because the mapping class group would act equivariantly with respect to σ_f:

$$\gamma \cdot \sigma_f(\tau) = \sigma_f(\gamma \cdot \tau), \quad \forall \gamma \in \mathrm{PMCG}(S^2, P_f).$$

If γ is a Thurston equivalence between f and itself, the above equation holds, but we cannot expect it to be true for all $\gamma \in \mathrm{PMCG}(S^2, P_f)$ in general. Bartholdi and Nekrashevych [2] produced a counterexample coming from the Douady rabbit polynomial, that is, $f(z) = z^2 + c$, where the critical point at 0 is periodic of period 3 and c has positive imaginary part. They show that if $\tau \in \mathcal{T}_f$ and $w_0 = \pi(\sigma_f(\tau))$, $w_1 = \pi(\tau)$, then

$$w_1 = 1 - \frac{1}{w_0^2},$$

where the moduli space for f is identified with the punctured sphere $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$. Such a phenomenon, where an inverse of the Thurston iteration can be defined on $\mathcal{M}_f$, has been studied in depth by Koch [6]. In a later section, we study more closely the failure of the iteration to descend to the moduli space, in the form of the “centralizer” subgroup $C(\sigma_f)$. This subgroup has also been discussed at length in [13].

2 The Rees space

Given a Thurston map $f : S^2 \to S^2$, its portrait consists of the dynamical data of its postcritical set, along with the branching degrees of its critical points. Let C and P be abstract, finite, not necessarily disjoint sets, equipped with a map $F : C \cup P \to P$ and a function $\deg : C \to \{2, 3, \ldots\}$. This defines a *portrait*, which we refer to simply by F.

Elements of the *Rees space* $\mathcal{R}_F$ of the portrait F consist of the data of a pairs of injections $e, e' : C \cup P \to \hat{\mathbb{C}}$, a holomorphic map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, and a homeomorphism $q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, such that the following holds:

(i) $g \circ e' = e \circ F$ on $C \cup P$;

(ii) $\deg g_{e'(c)} = \deg(c)$, for all $c \in C$;

(iii) $q \circ e' = e$ on P.

$$\begin{tikzcd} C \cup P \arrow[r, "e'"] \arrow[d, "F"'] & \hat{\mathbb{C}} \arrow[d, "g"] \\ P \arrow[r, "e|_P"] & \hat{\mathbb{C}} \end{tikzcd} \qquad \begin{tikzcd} & \hat{\mathbb{C}} \arrow[d, "q"] \\ P \arrow[ur, "e'|_P"] \arrow[r, "e|_P"] & \hat{\mathbb{C}} \end{tikzcd}$$

One should think of the two copies of $\hat{\mathbb{C}}$ as different Riemann spheres, which are identified by q up to isotopy relative to $e'(P)$. Then $g \circ q^{-1} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a topological branched cover such that $(g \circ q^{-1}) \circ e|_P = e \circ F|_P$, so that $g \circ q^{-1}$ is a genuine topological dynamical system which realizes on $\hat{\mathbb{C}}$ the dynamics of F on its postcritical set.

Two tuples (e_1, e_1', g_1, q_1) and (e_2, e_2', g_2, q_2) are equivalent if there are isomorphisms between the diagrams making them commute in the appropriate categories. Explicitly, an isomorphism is a pair of conformal maps $h, h' : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that the diagram on the left below commutes, and the one on the right commutes up to isotopy rel $e_1'(P)$.

$$\begin{tikzcd} C \cup P \arrow[r, "e_1'"] \arrow[d, "F"'] \arrow[rr, bend left, "e_2'"] & \hat{\mathbb{C}} \arrow[r, "h'"] \arrow[d, "g_1"] & \hat{\mathbb{C}} \arrow[d, "g_2"] \\ P \arrow[r, "e_1|_P"] \arrow[rr, bend right, "e_2|_P"'] & \hat{\mathbb{C}} \arrow[r, "h"] & \hat{\mathbb{C}} \end{tikzcd} \qquad \begin{tikzcd} & \hat{\mathbb{C}} \arrow[r, "h'"] \arrow[d, "q_1"] & \hat{\mathbb{C}} \arrow[d, "q_2"] \\ P \arrow[ur, "e_1'|_P"] \arrow[urr, bend left, "e_2'|_P"] \arrow[r, "e_1|_P"] \arrow[rr, bend right, "e_2|_P"'] & \hat{\mathbb{C}} \arrow[r, "h"] & \hat{\mathbb{C}} \end{tikzcd}$$

This emphasizes that q is only really defined up to isotopy, and that the data that represents a point in the Rees space can be seen as $(e, e', g, [q])$, where $[q]$ is its isotopy class rel $e'(P)$. We could even assume, and we do so later on, that q is a Teichmüller map from $\hat{\mathbb{C}} \setminus e'(P)$ to $\hat{\mathbb{C}} \setminus e(P)$.

Alternatively, an element of $\mathcal{R}_F$ may be represented by a tuple (e, e', g, q, μ), where $e, e' : C \cup P \to S^2$ are injections, μ is an almost complex structure on S^2, $g : S^2 \to S^2$ is such that $g^*\mu = \mu$, $q : S^2 \to S^2$ is a homeomorphism and properties (i), (ii) and (iii) are satisfied. Equivalence of data is analogous, where we require $h : (S^2, \mu_1) \to (S^2, \mu_2)$ to be conformal, that is, $h^*\mu_2 = \mu_1$:

$$\begin{tikzcd} C \cup P \arrow[r, "e_1'"] \arrow[d, "F"'] \arrow[rr, bend left, "e_2'"] & (S^2, g_1^*\mu_1) \arrow[r, "h'"] \arrow[d, "g_1"] & (S^2, g_2^*\mu_2) \arrow[d, "g_2"] \\ P \arrow[r, "e_1|_P"] \arrow[rr, bend right, "e_2|_P"'] & (S^2, \mu_1) \arrow[r, "h"] & (S^2, \mu_2) \end{tikzcd} \qquad \begin{tikzcd} & (S^2, \mu_1) \arrow[r, "h'"] \arrow[d, "q_1"] & (S^2, \mu_2) \arrow[d, "q_2"] \\ P \arrow[ur, "e_1'|_P"] \arrow[urr, bend left, "e_2'|_P"] \arrow[r, "e_1|_P"] \arrow[rr, bend right, "e_2|_P"'] & (S^2, \mu_1) \arrow[r, "h"] & (S^2, \mu_2) \end{tikzcd}$$

By uniformization, every tuple is equivalent to another one for which $\mu = \mu_0$, the canonical complex structure on $\hat{\mathbb{C}}$. Keeping track of the choice of almost complex structure, however, is advantageous is certain situations, and specifically for when we later define a self-map of the Rees space.

A second definition of $\mathcal{R}_F$ is as follows. An element consists of the data of an injection $e : C \cup P \to \hat{\mathbb{C}}$ along with a topological branched cover $G : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that:

(i') $G \circ e = e \circ F$ on $C \cup P$;

(ii') $\deg G|_{e(c)} = \deg(c)$ for $c \in C$.

$$\begin{tikzcd} C \cup P \arrow[r, "e"] \arrow[d, "F"'] & \hat{\mathbb{C}} \arrow[d, "G"] \\ P \arrow[r, "e|_P"] & \hat{\mathbb{C}} \end{tikzcd}$$

Two pairs (e_1, G_1) and (e_2, G_2) are equivalent if there is a conformal map $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and a homeomorphism $\tilde{h} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $h \circ e_1|_P = e_2|_P$, $h \circ G_1 = G_2 \circ \tilde{h}$, and h is isotopic to $\tilde{h}$ relative to $e(P)$:

$$\begin{tikzcd} \hat{\mathbb{C}} \arrow[r, "\tilde{h}"] \arrow[d, "G_1"'] & \hat{\mathbb{C}} \arrow[d, "G_2"] \\ \hat{\mathbb{C}} \arrow[r, "h"'] & \hat{\mathbb{C}} \end{tikzcd}$$

In essence, this means that G_1 and G_2 are Thurston equivalent, and that the Thurston equivalence h is conformal (though the lifted map $\tilde{h}$ in general is not).

Similarly, we may want to keep track of an almost complex structure on S^2 by representing elements of $\mathcal{R}_f$ as tuples (e, G, μ), where $e : C \cup P \to S^2$ is an injection, μ is an almost complex structure on S^2, and $G : (S^2, \mu) \to (S^2, \mu)$ is a topological branched cover such that (i') and (i") hold. Equivalence of data is defined analogously, requiring one complex structure to be pulled back to the other by the Thurston equivalence h.

We explicitly show that the two definitions of $\mathcal{R}_F$ are equivalent. Consider the data $E_1 = (e_1, e_1', g_1, q_1)$ and $E_2 = (e_2, e_2', g_2, q_2) \sim E_1$, and $G_i = g_i \circ q_i^{-1}$ for $i = 1, 2$. If h and h' are conformal maps realizing the equivalence between E_1 and E_2, the Thurston equivalence of G_1 and G_2 is given below, where h is conformal, the diagram commutes and h is isotopic to $q_2 \circ h' \circ q_1^{-1}$ relative to $e_1(P)$:

$$\begin{tikzcd} \hat{\mathbb{C}} \arrow[r, "q_2 h' q_1^{-1}"] \arrow[d, "G_1"'] & \hat{\mathbb{C}} \arrow[d, "G_2"] \\ \hat{\mathbb{C}} \arrow[r, "h"] & \hat{\mathbb{C}} \end{tikzcd}$$

so that (e_1, G_1) and (e_2, G_2) are equivalent. Conversely, suppose we are given the data (e, G), which gives us the holomorphic map $G : (S^2, G^*\mu_0) \to (S^2, \mu_0)$. The identity $\mathrm{id} : (S^2, G^*\mu_0) \to (S^2, \mu_0)$ is a diffeomorphism; we obtain the diagram

$$\begin{tikzcd} (S^2, G^*\mu_0) \arrow[rr, bend left, "G"] \arrow[rr, bend right, "\mathrm{id}"'] & & (S^2, \mu_0) \\ & C \cup P \arrow[ul, "e"] \arrow[ur, "e"'] & \end{tikzcd}$$

corresponding to the data (e, e, G, id). If (e_1, G_1) and (e_2, G_2) are equivalent and $(h, \tilde{h})$

realize the equivalence, then $(e_1, e_1, G_1, \mathrm{id}_1)$ and $(e_2, e_2, G_2, \mathrm{id}_2)$ must also be equivalent:

$$\begin{tikzcd}
C \cup P \arrow[r, "e_1'"] \arrow[d, "F"'] \arrow[rr, bend left, "e_2'"] & (S^2, G_1^*\mu_0) \arrow[r, "\tilde{h}"] \arrow[d, "G_1"'] & (S^2, G_2^*\mu_0) \arrow[d, "G_2"] \\
P \arrow[r, "e_1|_P"] \arrow[rr, bend right, "e_2|_P"'] & (S^2, \mu_0) \arrow[r, "h"] & (S^2, \mu_0)
\end{tikzcd}$$

where $\tilde{h} : (S^2, G_1^*\mu_0) \to (S^2, G_2^*\mu_0)$ will be a conformal isomorphism between the pulled back complex structures. Given that h and $\tilde{h}$ are isotopic rel $e_1(P)$, the commutativity up to homotopy of the diagram with $q_1 = \mathrm{id}_1$ and $q_2 = \mathrm{id}_2$ follows. It is straightforward to verify that the correspodence between the two definitions are inverses of each other.

There is a well defined map $\overline{\pi} : \mathcal{R}_F \to \mathcal{M}_P$ that sends $[e, G]$ to $[e|_P]$. Moreover, given a Thurston map f which realizes a portrait F, there is a well defined map $\hat{\pi} : \mathcal{T}_f \to \mathcal{R}_F$, which we describe in multiple ways. If $\tau = [\varphi]$, recall that from the iterate $\sigma_f(\tau)$ we get a commutative diagram:

$$\begin{tikzcd}
(S^2, P_f) \arrow[r, "\varphi'"] \arrow[d, "f"'] & \hat{\mathbb{C}} \arrow[d, "f_\tau"] \\
(S^2, P_f) \arrow[r, "\varphi"'] & \hat{\mathbb{C}}
\end{tikzcd}$$

Hence

$$\hat{\pi}([\varphi]) = [\varphi|_{C_f \cup P_f}, \varphi'|_{C_f \cup P_f}, f_\tau, \varphi \circ (\varphi')^{-1}].$$

This is well defined, because if $[\eta] = [\varphi]$, then $\eta \circ \varphi^{-1}$ is isotopic to a conformal map h rel $\varphi(P_f)$, which is lifted to a homotopy relative to $\varphi'(P_f)$ between $\eta' \circ (\varphi')^{-1}$ and a conformal map h', so that the diagram below commutes, and the conformal maps (h, h') realize the equivalence.

$$\begin{tikzcd}
\hat{\mathbb{C}} \arrow[r, "h' \sim \eta'(\varphi')^{-1}"] \arrow[d, "f_\tau"'] & \hat{\mathbb{C}} \arrow[d, "f_\tau"] \\
\hat{\mathbb{C}} \arrow[r, "h \sim \eta\varphi^{-1}"'] & \hat{\mathbb{C}}
\end{tikzcd}$$

Considering elements of $\mathcal{R}_f$ in the form $[e, G]$, we may also view $\hat{\pi} : \mathcal{T}_f \to \mathcal{R}_F$ as $[\varphi] \mapsto [\varphi|_{C_f \cup P_f}, \varphi \circ f \circ \varphi^{-1}]$. In this case, the equivalence of the data as above will be realized by the conformal maps h and $\tilde{h} = \eta \circ (\eta')^{-1} \circ h' \circ \varphi' \circ \varphi^{-1}$:

$$\begin{tikzcd}
\hat{\mathbb{C}} \arrow[r, "\eta(\eta')^{-1}h'\varphi'\varphi^{-1}"] \arrow[d, "\varphi f \varphi^{-1}"'] & \hat{\mathbb{C}} \arrow[d, "\eta f \eta^{-1}"] \\
\hat{\mathbb{C}} \arrow[r, "h"'] & \hat{\mathbb{C}}
\end{tikzcd}$$

From the point of view of elements of $\mathcal{R}_F$ as (e, G, μ), we have $[\varphi] \mapsto (\mathrm{id}\,|_{C_f \cup P_f}, f, \varphi^* \mu_0)$, since φ is conformal in the diagram below:

$$\begin{tikzcd}
(S^2, \varphi^*\mu_0) \arrow[r, "\varphi"] \arrow[d, "f"'] & (S^2, \mu_0) \arrow[d, "\varphi f \varphi^{-1}"] \\
(S^2, \varphi^*\mu_0) \arrow[r, "\varphi"'] & (S^2, \mu_0)
\end{tikzcd}$$

and if we represent $\tau \in \mathcal{T}_f$ by a complex structure $[\mu]$, we have $\tilde{\pi}([\mu]) = [\mathrm{id}\,|_{C_f \cup P_f}, f, \mu]$.

Elements of $\mathcal{R}_F$ are distributed into disjoint equivalence classes, where two elements $\alpha_1 = [e_1, G_1, \mu_1]$ and $\alpha_2 = [e_2, G_2, \mu_2]$ are in the same class if G_1 and G_2 are Thurston equivalent. By construction, $\mathcal{T}_f$ maps surjectively onto a subset $\mathcal{R}_f \subseteq \mathcal{R}_F$, corresponding to those maps Thurston equivalent to f.

2.1 The Rees Space as a quotient

A third definition of $\mathcal{R}_f$ is possible, which refers back to how we in general cannot define a corresponding iteration on $\mathcal{M}_f$. Consider the subgroup of the pure mapping class group $\mathrm{PMCG}(S^2, P_f)$ given by the *special liftable mapping classes* $\mathrm{LS}(f)$, that is, those pure mapping classes represented by homeomorphisms h such that h is a Thurston equivalence of f with itself:

$$\begin{tikzcd}
(S^2, P_f) \arrow[r, "h'"] \arrow[d, "f"'] & (S^2, P_f) \arrow[d, "f"] \\
(S^2, P_f) \arrow[r, "h"] & (S^2, P_f)
\end{tikzcd}$$

where $h \sim h'$ rel P_f. We remark again that, as a pure mapping class, h is the identity on P_f. The following is immediate from the observations made in the introduction:

Lemma 2.1. $\mathrm{LS}(f)$ *is equivariant with respect to* σ_f. *In other words, for all* $h \in \mathrm{LS}(f)$,

$$h \cdot \sigma_f(\tau) = \sigma_f(h \cdot \tau), \quad \forall \tau \in \mathcal{T}_f\,.$$

We consider the quotient $\mathcal{T}_f/\mathrm{LS}(f)$, being an intermediary cover between $\mathcal{T}_f$ and $\mathcal{M}_f = \mathcal{T}_f/\mathrm{PMCG}(S^2, P_f)$. The proposition below guarantees that $\mathcal{R}_f$ is this quotient, being an intermediary covering space of $\pi : \mathcal{T}_f \to \mathcal{M}_f$, and coming equipped with a natural topology and complex manifold structure.

Proposition 2.2. $\mathcal{T}_f/\mathrm{LS}(f)$ *is canonically identified with* $\mathcal{R}_f$.

Proof. The map $\tilde{\pi} : \mathcal{T}_f \to \mathcal{R}_f$ descends to this quotient, because if $h \in \mathrm{LS}(f)$, then

$$[\mathrm{id}\,|_{C_f \cup P_f}, f, h^*\mu] = [\mathrm{id}\,|_{C_f \cup P_f}, f, \mu],$$

since

$$\begin{tikzcd}
(S^2, h^*\mu) \arrow[r, "h'"] \arrow[d, "f"'] & (S^2, \mu) \arrow[d, "f"] \\
(S^2, h^*\mu) \arrow[r, "h"'] & (S^2, \mu)
\end{tikzcd}$$

is a Thurston equivalence with h conformal. Conversely, if $\hat{\pi}([\varphi]) = \hat{\pi}([\eta])$, we have an equivalence

$$\begin{array}{ccc} \hat{\mathbb{C}} & \xrightarrow{h'} & \hat{\mathbb{C}} \\ {\scriptstyle \varphi f \varphi^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle \eta f \eta^{-1}} \\ \hat{\mathbb{C}} & \xrightarrow{h} & \hat{\mathbb{C}} \end{array}$$

where h is conformal, $h \circ \varphi|_{P_f} = \eta|_{P_f}$ and $h' \sim h$ rel $\varphi(P_f)$. This gives us the self Thurston equivalence

$$\begin{array}{ccc} S^2 & \xrightarrow{\eta^{-1}h'\varphi} & S^2 \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ S^2 & \xrightarrow{\eta^{-1}h\varphi} & S^2 \end{array}$$

since $\eta^{-1}h\varphi|_{P_f} = \mathrm{id}\,|_{P_f}$ and $\eta^{-1}h\varphi \sim \eta^{-1}h'\varphi$ rel P_f. Moreover,

$$(\eta^{-1}h\varphi) \cdot [\eta] = [\eta \circ \eta^{-1}h\varphi] = [h \circ \varphi] = [\varphi],$$

since h is conformal. $\square$

Remark. Considering the notion of *Hurwitz equivalence* defined in [6], the self liftable mapping classes $\mathrm{LS}(f)$ form a subgroup of the *liftable mapping classes* H_f, that is, those mapping classes represented by homeomorphisms h such that there exists another homeomorphism $\tilde{h}$ such that the diagram below commutes. Note that we do not stipulate that $h \sim \tilde{h}$ rel P_f.

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{\tilde{h}} & (S^2, P_f) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ (S^2, P_f) & \xrightarrow{h} & (S^2, P_f) \end{array}$$

Consequently, the Rees space forms a cover of the *Hurwitz space* $\mathcal{W}_f$ (see [6]). It is known that H_f is a subgroup of finite index of $\mathrm{PMCG}(S^2, P_f)$, so that $\mathcal{W}_f \to \mathcal{M}_f$ is a finite cover.

This in fact provides another point of view of the Rees space, building it "from the ground up" as a cover of $\mathcal{M}_{P_f}$ by keeping track of more combinatorial and topological information. As we recall, elements of M_f are injections $e : P_f \to \hat{\mathbb{C}}$ up to conjugacy by Möbius transformations. The fibers of W_f over $[e] \in M_f$ correspond to the possible choices (e', g) of a rational map g and an injection e', up to the appropriate equivalences, such that there exists homeomorphisms $\varphi, \varphi' : (S^2, P_f) \to \hat{\mathbb{C}}$ such that the diagram below commutes, and $\varphi|_{P_f} = e$, $\varphi'|_{P_f} = e'$:

$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{\varphi'} & \hat{\mathbb{C}} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ (S^2, P_f) & \xrightarrow{\varphi} & \hat{\mathbb{C}} \end{array}$$

This is exactly the diagram that defines the Thurston pullback of $[\varphi]$. As there are finitely many non-isomorphic degree d coverings of $\hat{\mathbb{C}} \setminus e(P_f)$ and finitely many injections of P_f into $g^{-1}(e(P_f))$, we recover that $W_f \to \mathcal{M}_f$ is a finite cover.

We now decide to keep track of the topological information of the isotopy cass of $\varphi \circ (\varphi')^{-1}$, which identifies the two Riemann spheres. The fibers of $\mathcal{R}_f$ over $[e, e', g] \in \mathcal{W}_f$ corresponds to the possible isotopy classes of $\varphi \circ (\varphi')^{-1}$ coming from diagrams as above. Any two such diagrams will "differ" by a self-liftable mapping class, justifying the equivalence and relating back to the two previous definitions.

With this, a final definition of the Rees space R_f is possible. It consists of a choice of element $[e, e', g]$ of W_f, and an isotopy class of paths $\gamma : [0,1] \to \mathcal{M}_f$ such that $\gamma(0) = [e]$, $\gamma(1) = [e']$. This is because any such isotopy class of paths corresponds to an isotopy class of homeomorphisms $h^{-1} : (\hat{\mathbb{C}}, e(P_f)) \to (\hat{\mathbb{C}}, e'(P_f))$, going back to the first definition given.

Remark. The definition of the Rees was introduced in more generality in [5] based on work by Mary Rees [12], while we emphasize again that other authors have previously studied the self-liftable mapping classes [13]. In [5], the authors prove several properties of $\mathcal{R}_f$ in the context of studying deformation spaces for one-dimensional families of quadratic rational maps.

3 Iteration on the Rees Space

For the Rees space $\mathcal{R}_F$ of a portrait F, we may define $\psi : \mathcal{R}_F \to \mathcal{R}_F$ by

$$\psi([e, G, \mu]) = [e, G, G^*\mu].$$

This is well defined because, given equivalent data (e_i, G_i, μ_i) for $i = 1, 2$, we have the equivalence

$$\begin{tikzcd} (S^2, \mu_1) \arrow[r, "h'"] \arrow[d, "G_1"'] & (S^2, \mu_2) \arrow[d, "G_2"] \\ (S^2, \mu_1) \arrow[r, "h"'] & (S^2, \mu_2) \end{tikzcd}$$

where $h^*\mu_2 = \mu_1$ and $h \sim h'$ rel $e_1(P)$. Therefore $h'^* G_2^* \mu_2 = G_1^* h^* \mu_2 = G_1^* \mu_1$, and $h' : (S^2, G_1^*\mu_1) \to (S^2, G_2^*\mu_2)$ is conformal. We may lift the isotopy $h \sim h'$ rel $e_1(P)$ to an isotopy $h' \sim h''$ rel $e_1(P)$ so that we have the diagram

$$\begin{tikzcd} (S^2, G_1^*\mu_1) \arrow[r, "h''"] \arrow[d, "G_1"'] & (S^2, G_2^*\mu_2) \arrow[d, "G_2"] \\ (S^2, G_1^*\mu_1) \arrow[r, "h'"'] & (S^2, G_2^*\mu_2) \end{tikzcd}$$

witnessing the equivalence of the images. This map commutes with $\hat{\pi} : \mathcal{T}_f \to \mathcal{R}_f$ since

$$[\mathrm{id}|_{C_f \cup P_f}, f, \mu] \mapsto [\mathrm{id}|_{C_f \cup P_f}, f, f^*\mu].$$

The iteration $\psi : \mathcal{R}_f \to \mathcal{R}_f$ can also be obtained from the fact that, because $\mathrm{LS}(f)$ is equivariant with respect to σ_f, it descends to the quotient. This is, in fact, one of the motivations for defining the Rees space, which keeps track of enough dynamical information of the pullback but still has some "finiteness" properties over $\mathcal{M}_f$, a fact later made precise in theorem 3.4.

A fixed point for σ_f corresponds to a Thurston equivalence to a rational map: so if f is Thurston equivalent to a rational map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, then $\alpha = [e, g, \mu_0] \in \mathcal{R}_f$ is a fixed point for ψ_f. However, if $\alpha \in \mathcal{R}_f$ is a fixed point for ψ_f, we can only guarantee the existence of some $h \in \mathrm{LS}(f)$ and $\tau \in \mathcal{T}_f$ such that $\sigma_f(\tau) = h^*\tau$, so that $f \circ h^{-1}$ is Thurston equivalent to a rational map. Equivalently, since $\tilde{h}^*\tau = h^*\tau$ and $f \circ \tilde{h} = h \circ f$, also $h^{-1} \circ f$ is equivalent to a rational map. We remedy this situation by showing that the translation distance on $\mathcal{T}_f$ with respect to σ_f descends to $\mathcal{R}_f$, and consider when this function is zero.

3.1 The translation distance

Let $\alpha \in \mathcal{R}_F$ be given by $[e, e', g, q]$, where q is a Teichmüller map. Define $D(\alpha) := \frac{1}{2} \log \mathrm{Dil}(q)$, where $\mathrm{Dil}(q)$ is the dilatation of q. Equivalently, if $\alpha = [e, G, \mu]$, we take $D(\alpha)$ to be half the logarithm of the minimal dilatation for a map $(S^2, G^*\mu) \to (S^2, \mu)$ isotopic to the identity rel $e(P)$. This quantity is well defined irrespective of the representative of α, and $D(\alpha) = 0$ if and only if α represents the data of a rational map. In fact, on $\mathcal{R}_f$ this quantity is just the translation distance $d(\tau, \sigma_f(\tau))$ on $\mathcal{T}_f$ with respect to the Teichmüller metric; $D : \mathcal{T}_f \to [0, +\infty)$ is such that, for all $h \in \mathrm{LS}(f)$, $D(h^*\tau) = D(\tau)$, so that this descends to a continuous map on $\mathcal{R}_f$. We remark that $D(\alpha)$ is **not** the the translation distance between α and $\psi(\alpha)$, as we have not defined a metric on $\mathcal{R}_f$.

Proposition 3.1. *For all $\alpha \in \mathcal{R}_f$, $D(\psi(\alpha)) \leq D(\alpha)$. Moreover, if f has hyperbolic orbifold, for all α either $D(\alpha) = 0$ or there exists $n \geq 1$ such that $D(\psi^n(\alpha)) < D(\alpha)$.*

Proof. If $\alpha = [\mathrm{id}|_{C_f \cup P_f}, f, \mu]$ and $q : (S^2, f^*\mu) \to (S^2, \mu)$ is the Teichmüller map isotopic to the identity rel P_f, so that $D(\alpha) = \frac{1}{2} \log \mathrm{Dil}(q)$, we may lift q to a map $\tilde{q} : (S^2, f^*f^*\mu) \to (S^2, f^*\mu)$ with the same dilatation, and isotopic to the identity. Hence the minimal dilatation of a map $(S^2, (f^*)^2\mu) \to (S^2, f^*\mu)$ isotopic to the identity will be no greater than $\mathrm{Dil}(\tilde{q}) = \mathrm{Dil}(q)$.

If $D(\psi(\alpha)) = D(\alpha)$, then, by Teichmüller's uniqueness theorem, $\tilde{q}$ is the Teichmüller map between the corresponding Riemann surfaces. Let s and s' be the holomorphic integrable quadratic differentials on (S^2, μ) and $(S^2, f^*\mu)$ for q respectively, with pullbacks f^*s and f^*s'. We recall that, if Z_s and P_s are the number of zeros and poles of s counting multiplicity, then $Z_s - P_s = -4$. In order for s to be a holomorphic integrable quadratic differential on $S^2 \setminus P_f$, the set of poles of s must be contained in P_f. Moreover, if w is a zero of order $m \in \{-1, 0, \ldots\}$ for s, and z is such that it maps to w with local degree k, then z is a zero of order $k(m + 2) - 2$ for f^*s. This is because a zero of order m corresponds to a $(m+2)$-pronged singularity, which is pulled back to a $k(m+2)$-pronged singularity by f.

We want to show that there exists some n for which $(f^*)^n s$ has a pole outside of P_f, so that the corresponding lift of q cannot be the Teichmüller map in its isotopy class. Consequently, the dilatation strictly decreases. Since poles are always simple for integrable holomorphic quadratic differentials, it is sufficient to show that the number of zeros eventually increases under iterated pullbacks. If w is a zero of order $m \geq 1$ for s, then each $z \in f^{-1}(w)$ is a zero of order $k_z(m + 2) - 2 \geq k_z m$. Hence f^*s doesn't have more zeros than s only when s itself has no zeros, and only 4 simple poles p_1, p_2, p_3, p_4 contained in P_f.

If f^*s has no zeros, then all points mapping to one of the p_i must have local degree $k = 1$ or 2, and if w is not a pole and $z \in f^{-1}(w)$, then z must map to w with local degree 1. In particular, P_f is exactly the set of 4 poles. If some pole is also a critical point, it would be a regular point for f^*s. But as f^*s has at least 4 poles, there would some pole for f^*s outside of P_f. Therefore $C_f \cap P_f = \emptyset$, and the orbifold is euclidean of type $(2, 2, 2, 2)$, contradicting our assumptions. □

A more refined analysis can show that we may take $n = 2$ for all $\alpha \in \mathcal{R}_F$ [4], but the above result is sufficient for our purposes.

The strategy of the proof above shows that, if f is (Thurston equivalent to) a rational map, then the Rees space $\mathcal{R}_f$ is just the Teichmüller space itself:

Corollary 3.2. *If $f : (\hat{\mathbb{C}}, P_f) \to (\hat{\mathbb{C}}, P_f)$ is a postcritically finite rational map with hyperbolic orbifold, then $\mathrm{LS}(f) = \{\mathrm{id}_{\hat{\mathbb{C}}}\}$. In particular, $\mathcal{R}_f = \mathcal{T}_f$.*

Proof. Let $h \in \mathrm{LS}(f)$ be such that $h \sim \tilde{h}$ rel P_f in the diagram below:

$$\begin{tikzcd} (\hat{\mathbb{C}}, P_f) \arrow[r, "\tilde{h}"] \arrow[d, "f"'] & (\hat{\mathbb{C}}, P_f) \arrow[d, "f"] \\ (\hat{\mathbb{C}}, P_f) \arrow[r, "h"] & (\hat{\mathbb{C}}, P_f) \end{tikzcd}$$

We may also assume that h is the Teichmüller map in its isotopy class rel P_f. Since $\mathrm{Dil}(\tilde{h}) \leq \mathrm{Dil}(h)$, by uniqueness we have $\tilde{h} = h$. If we lift h by f a sufficient number of times and h is not holomorphic, eventually the dilatation strictly decreases, contradicting minimality. Hence h is holomorphic, and because it is the identity on P_f, it must be the identity on $\hat{\mathbb{C}}$. □

3.2 Self-liftable mapping classes

The statements of this section also have proofs in [5]. The previous corollary states that if f is unobstructed (which, by Thurston's criterion, is equivalent to being Thurston equivalent to a rational map), then $\mathcal{R}_f$ is simply the Teichmuller space $\mathcal{T}_f$. We show that, for certain Thurston maps, the Rees space is genuinely distinct from $\mathcal{T}_f$, and in fact $\hat{\pi} : \mathcal{T}_f \to \mathcal{R}_f$ will be an infinite cover.

Let A be an annulus with boundary, $f_d : A \to A$ be a degree d unbranched covering, and $h : A \to A$ a Dehn twist around the core curve of A. Since h preserves this core curve, The Dehn twist lifts to a map $\tilde{h} : A \to A$ as in the diagram below:

$$\begin{tikzcd} A \arrow[r, "\tilde{h}"] \arrow[d, "f_d"'] & A \arrow[d, "f_d"] \\ A \arrow[r, "h"] & A \end{tikzcd}$$

As h is the identity on the two boundary components of A, $\tilde{h}$ is a lift of the identity on each component. But even by choosing an appropriate lift, $\tilde{h}$ will in general not be the identity on ∂A, since one boundary component will be "offset" from the other by a "rotation" of

$2\pi/d$. The d-th power h^d of the Dehn twist, however, does lift a homeomorphism isotopic to h relative to ∂A.

Consider now an f-invariant multicurve Γ for a Thurston map $f : S^2 \to S^2$, and $D_\Gamma \cong \mathbb{Z}^\Gamma$ the abelian group generated by Dehn twists h_γ around each of the curves $\gamma \in \Gamma$. We want to decide when an element $h \in D_\Gamma$ is liftable, that is, there is some homeomorphism $\tilde{h}$ such that the diagram bellow commutes:

$$\begin{tikzcd} (S^2, P_f) \arrow[r, "\tilde{h}"] \arrow[d, "f"'] & (S^2, P_f) \arrow[d, "f"] \\ (S^2, P_f) \arrow[r, "h"] & (S^2, P_f) \end{tikzcd}$$

If η is a non-essential curve on $S^2 \setminus P_f$, the Dehn twist around η is isotopic to the identity relative to the boundary, by a version of the Alexander trick. This shows that the only obstruction to liftability of $\gamma \in \Gamma$ comes from its essential pullbacks. If $\tilde{\gamma}_1, \dots, \tilde{\gamma}_m$ are the pullbacks of γ in $S^2 \setminus f^{-1}(P_f)$, each mapping to γ with degree $d_1, \dots, d_m$, then $h_\gamma^{\mathrm{lcm}(d_1,\dots,d_m)}$ is liftable, and the lift corresponds to Dehn twists $h_{\tilde{\gamma}_i}^{\mathrm{lcm}(d_1,\dots,d_m)/d_i}$ around each pullback curve. Under the inclusion $S^2 \setminus f^{-1}(P_f) \hookrightarrow S^2 \setminus P_f$, pullbacks which are homotopic in $S^2 \setminus P_f$ "add up" their Dehn twists around the corresponding curve in Γ. This characterizes which basis elements of D_γ are liftable, knowing the combinatorics of Γ under pullbacks.

The lift $\tilde{h}$ of a liftable element in D_Γ corresponds exactly to applying the Thurston linear transformation f_Γ to $h \in D_\Gamma \cong \mathbb{Z}^\Gamma \subset \mathbb{R}^\Gamma$. This shows that the self-liftable elements are exactly the ones corresponding to integer-valued positive eigenvectors for f_Γ with eigenvalue 1, so that if f_Γ has an eigenvalue equal to 1, $\mathrm{LS}(f)$ contains at least an infinite cyclic subgroup. In summary:

Proposition 3.3. *If f is a Thurston map and Γ is an f-invariant multicurve such that f_Γ has an eigenvalue equal to 1, then $\mathrm{LS}(f)$ contains all multi-twists around Γ corresponding to positive integer-valued eigenvectors of eigenvalue 1. In particular, $\hat{\pi} : \mathcal{T}_f \to \mathcal{R}_f$ is an infinite cover.*

3.3 Properness

Given the non-increasing properties of the translation distance $D(\alpha)$, if we want to decide whether or not f is equivalent to a rational map, need to look for global minima of $D : \mathcal{R}_F \to [0, +\infty)$, so that if it exists, it will be a zero. The Rees space is not compact, but the result below shows that the set of $\alpha \in \mathcal{R}_F$ having uniformly bounded translation distance and bounded over $\mathcal{M}_P$ is compact:

Theorem 3.4. $(\overline{\pi}, D) : \mathcal{R}_F \to \mathcal{M}_P \times [0, +\infty)$ *is proper.*

Proof. We may assume $|P| \geq 3$, otherwise $\mathcal{T}_P$ consists of a single point. Let $K \subset \mathcal{M}_P$ be compact and $d \geq 0$. We want to show that $R_{K,d} := \{\alpha \in \mathcal{R}_F \mid \overline{\pi}(\alpha) \in K,\ D(\alpha) \leq d\}$ is compact. Let $(\alpha_i)_{i \geq 0}$ be a sequence in $\mathcal{R}_F$, where $\alpha_i = [e_i, e_i', g_i, q_i]$ and the q_i are Teichmüller maps. Given Möbius transformations M_i and N_i, we have

$$\alpha = [e_i, e_i', g_i, q_i] = [M_i e_i, N_i e_i', M_i g_i N_i^{-1}, M_i q_i N_i^{-1}],$$

so that we may assume there exists $p_1, p_2, p_3 \in P$ which are sent to $0, 1, \infty$ by all the e_i by choosing appropriate M_i, and that these points are fixed by q_i by choosing appropriate N_i. Under this normalization, and due to compactness in the moduli space, we may pass to a subsequence and assume that $e_i|_P$ converges to an injection $\iota : P \to \hat{\mathbb{C}}$. Moreover, since the q_i are normalized e^{2d}-quasiconformal maps, we may also take a subsequence and assume that $q_i \to q$ uniformly, and $e_i'|_P \to q^{-1}\iota$.

We consider diffeomorphisms $h_i : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $h_i \circ e_i|_P = \iota$, "pushing" the points in $e_i(P)$ towards their limit $\iota(P)$, where the h_i converge uniformly to the identity on $\hat{\mathbb{C}}$. In fact, we may assume that the h_i are quasiconformal and, for sufficiently large i, they are the identity outside of some small neighborhood of $\iota(P)$, the neighborhood shrinks with respect to i (in the spherical metric), and $\mathrm{Dil}(h_i) \to 1$. Let $V = F(C) \subseteq P$, representing the abstract critical values, and $V_i = e_i(V) = g_i(e_i'(C))$ the critical values of g_i. The composition $h_i g_i : \hat{\mathbb{C}} \setminus g^{-1}(V_i) \to \hat{\mathbb{C}} \setminus \iota(V)$ is a covering map of degree d, inducing a subgroup $H_i < \pi_1(\hat{\mathbb{C}} \setminus \iota(V))$ of index d. There are only finitely many subgroups of index d, so by passing to a subsequence we may assume all of the H_i are equal to some fixed $H < \pi_1(\hat{\mathbb{C}} \setminus \iota(V))$. There exists an associated covering space $g : E \to \hat{\mathbb{C}} \setminus \iota(V)$ and diffeomorphisms $k_i : \hat{\mathbb{C}} \setminus g_i^{-1}(V_i) \to E$ such that the diagram below commutes:

$$\begin{tikzcd}
\hat{\mathbb{C}} \setminus g_i^{-1}(V_i) \arrow[r, "k_i"] \arrow[d, "g_i"'] & E \arrow[d, "g"] \\
\hat{\mathbb{C}} \setminus V_i \arrow[r, "h_i"'] & \hat{\mathbb{C}} \setminus \iota(V)
\end{tikzcd}$$

Topologically, E is a 2-sphere with some finite number of punctures, and if we pullback the conformal structure of $\hat{\mathbb{C}} \setminus \iota(V)$ by g we may assume that $E = \hat{\mathbb{C}} \setminus L$. Consequently, the k_i are quasiconformal diffeomorphisms such that $\mathrm{Dil}(k_i) \le \mathrm{Dil}(h_i) \to 1$, and each extends to $g_i^{-1}(V_i)$, mapping it bijectively onto L.

$$\begin{tikzcd}
C \cup P \arrow[r, "e_i'"] \arrow[d, "F"'] & \hat{\mathbb{C}} \arrow[r, "k_i"] \arrow[d, "g_i"] & \hat{\mathbb{C}} \arrow[d, "g"] \\
P \arrow[r, "e_i|_P"] & \hat{\mathbb{C}} \arrow[r, "h_i"] & \hat{\mathbb{C}}
\end{tikzcd}$$

The maps k_i send $0, 1, \infty$ into points in $g^{-1}(\iota(F(\{p_1, p_2, p_3\}))$, so by passing to a subsequence we may assume that the k_i are normalized to send $0, 1, \infty$ into three specific points independently of i. Moreover, by taking an isomorphic covering, we may assume that these three points are themselves $0, 1, \infty$. With this normalization, k_i converges uniformly to the identity, and by continuity of composition under uniform convergence,

$$g_i = h_i^{-1}(h_i g_i) = h_i(g k_i) \to g.$$

For $p \in P$, we see that $g k_i e_i'(p) = \iota(p)$, so that by passing to a subsequence $k_i(e_i(p))$ is a fixed element $\widetilde{p}$ in $g^{-1}(\iota(p))$. Then

$$e_i'(p) = k_i^{-1}(\widetilde{p}) \to \widetilde{p},$$

but also $e_i'(p) \to q^{-1}\iota(p)$, so that $\widetilde{p} = q^{-1}\iota(p)$. In particular

$$\widetilde{p} = q^{-1}\iota(p) \implies g(\widetilde{p}) = \iota(p) = g q^{-1}(\iota(p)).$$

This shows that g is the desired rational map and q the quasiconformal map whose composition forms a postcritically finite branched cover. We also get the Thurston equivalences

$$\begin{tikzcd} \hat{\mathbb{C}} \arrow[r, "qk_iq_i^{-1}"] \arrow[d, "g_iq_i^{-1}"'] & \hat{\mathbb{C}} \arrow[d, "gq^{-1}"] \\ \hat{\mathbb{C}} \arrow[r, "h_i"] & \hat{\mathbb{C}} \end{tikzcd}$$

where $h_i \sim qk_iq_i^{-1}$ rel $e_i(P)$ since both are sufficiently close to the identity, and $\log \mathrm{Dil}(h_i) \to 0$, showing convergence of the $(\alpha_i)_i$. $\square$

3.4 Reproving Thurston's criterion

We consider the following result, essentially proven in [4]:

Lemma 3.5. *Let $D \geq 0$. Suppose that f has no Thurston obstructions and that there exists some $\tau_0 \in \mathcal{T}_f$ such that $D(\tau_0) \leq D$. Then there is an integer $m \geq 1$ and a non-empty compact set $K \subset \mathcal{M}_f$ such that, if $\pi(\tau) \in K$ and $D(\tau) \leq D$, then $\pi(\sigma_f^m \tau) \in K$.*

Proof. For $\tau \in \mathcal{T}_f$, define

$$\omega(\tau) \coloneqq \sup_\gamma \{-\log l_\tau(\gamma)\},$$

where the supremum ranges over all non-trivial simple closed curves γ in $S^2 \setminus P_f$, and $l_\tau(\gamma)$ is the length of the unique geodesic of τ homotopic to γ. Proposition 7.3 in [4] states that ω is 2-Lipschitz and that, for all M, $\{\tau \in \mathcal{T}_f : \omega(\tau) \leq M\}$ is the preimage of a compact set K in $\mathcal{M}_f$, so that a sequence in $\mathcal{T}_f$ can only go to infinity in $\mathcal{M}_f$ if the length of the shortest geodesic is going to 0.

Given that f is unobstructed, Proposition 8.2 in [4] guarantees that there exists an integer $m \geq 1$, depending only on $d = \deg f$ and $p = |P_f|$, and $C > 0$, depending only on d, p and D, such that, if $\omega(\tau) > C$ and $D(\tau) \leq D$, then $\omega(\sigma_f^m \tau) < \omega(\tau)$. Suppose that $d(\tau) \leq D$ and $\omega(\tau) \leq C + 2mD$. There are two possibilites; if $\omega(\tau) \leq C$, then $\omega(\sigma_f^m \tau) \leq C + 2mD$, because ω is 2-Lipschitz. If $\omega(\tau) > C$, then by Proposition 8.2 we get that $\omega(\sigma_f^m \tau) < \omega(\tau) \leq C + 2mD$. This concludes that $K = \{\tau : \omega(\tau) \leq C + 2mD\}$ is our desired compact set. $\square$

Choose $D \geq 0$ for which there exists some $\tau_0 \in \mathcal{T}_f$ such that $D(\tau_0) \leq D$. Then, for K obtained from Lemma 3.5,

$$E_{K,D} = \{\tau \in \mathcal{T}_f \mid \pi(\tau) \in K \text{ and } D(\tau) \leq D\}$$

is σ_f^m-invariant. Projecting this set to $\mathcal{R}_f$, we get a ψ^m-invariant set $R_{K,D}$, which is compact by Theorem 3.4. If $\beta \in R_{K,D}$ is a minimum of D in this set and $D(\beta) \neq 0$, then for some n we would get $D((\psi_f)^{mn}\beta) < D(\beta)$, a contradiction. Hence:

Theorem 3.6. *If f has hyperbolic orbifold and is unobstructed, then f is Thurston equivalent to a rational map.*

4 The Cover to Moduli Space

Throughout this section we assume that f is obstructed. We prove the following:

Theorem 4.1. *If f has hyperbolic orbifold, then $\overline{\pi} : \mathcal{R}_f \to \mathcal{M}_f$ is an infinite cover.*

Consider the group

$$C(\sigma_f) \coloneqq \{[\theta] \in \mathrm{PMCG}(S^2, P_f) : \theta^* \sigma_f = \sigma_f \theta^*\}$$

of those pure mapping classes that commute with the Thurston pullback. We call this the *centralizer subgroup* of σ_f (or simply of f); here we maintain the abuse of notation of identifying a mapping class with its representative homeomorphisms. Note that $\mathrm{LS}(f) \subseteq C(\sigma_f)$, since the lift of $\theta \in \mathrm{LS}(f)$ is isotopic to θ itself, which has the same action on $\mathcal{T}_f$.

Lemma 4.2. $C(\sigma_f) \cap H_f = \mathrm{LS}(f)$.

Proof. One of the implications is immediate. In the other direction, if h lifts to h', then $(h')^* \sigma_f = \sigma_f h^* = h^* \sigma_f$, and so $(hh'^{-1})^* \sigma_f = \sigma_f$. As the action of $\mathrm{PMCG}(S^2, P_f)$ on $\mathcal{T}_f$ is free, necessarily $h \sim h'$ rel P_f. □

Lemma 4.3. *If f has hyperbolic orbifold, then $C(\sigma_f)$ does not contain pseudo-Anosov mapping classes.*

Proof. If $\theta \in C(\sigma_f)$ is pseudo-Anosov, the translation distance $d(\tau, \theta^*\tau)$ has a positive infimum and it is realized for some $\tau_0 \in \mathcal{T}_f$ [?]. Let $\tau_i \coloneqq \sigma_f^i(\tau_0)$ for $i > 0$. By Proposition 3.1, we have

$$d(\tau_2, \theta^*\tau_2) = d(\sigma_f^2(\tau_0), \sigma_f^2(\theta^*\tau_0)) < d(\tau_0, \theta^*\tau_0),$$

where the first equality is because θ^* commutes with σ_f. This is a contradiction with the fact that the translation distance of θ assumes its minimum at τ_0. □

Proof of Theorem 4.1. Suppose by way of contradiction that $\mathrm{LS}(f) \leqslant \mathrm{PMCG}(S^2, P_f)$ is a finite index subgroup. Let γ be an essential simple closed curve. From the discussion in section 3.2, there exists some $l > 0$ such that the Dehn twist h_γ^k is liftable. As $\mathrm{LS}(f) \subseteq H_f$ will also have finite index, some higher power h_γ^l for $l > 0$ will be self-liftable; this can be seen due to every finite index subgroup containing a finite index normal subgroup.

If $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m$ are the essential pullbacks of γ in $S^2 \setminus f^{-1}(P_f)$ with corresponding Dehn twists $h_1, \ldots, h_m$, the lift of h_γ^l in $S^2 \setminus f^{-1}(P_f)$ will be $h_1^{l/d_1} \ldots h_m^{l/d_m}$, where d_i is the degree of the map $\tilde{\gamma}_i \to \gamma$. As before, the inclusion $S^2 \setminus f^{-1}(P_f) \hookrightarrow S^2 \setminus P_f$ may collpase some of the pullbacks to the same isotopy class. A product of disjoint Dehn twists cannot be a single Dehn twist, so that if $h_1^{l/d_1} \ldots h_m^{l/d_m} \sim h_\gamma^l$, all pullbacks of γ in $S^2 \setminus f^{-1}(P_f)$ collapse to the isotopy class of γ and $h_\gamma^l \sim h_\gamma^{l \sum_i 1/d_i}$. As the subgroup generated by h_γ is cyclic, $\sum 1/d_i = 1$, and every essential curve forms a Levy cycle for f consisting of a single curve. In particular, every Dehn twist is liftable.

As Dehn twists generate $\mathrm{PMCG}(S^2, P_f)$, we obtain that $\mathrm{LS}(f) = \mathrm{PMCG}(S^2, P_f)$, so $\mathcal{R}_f = \mathcal{M}_f$ and every mapping class is self-liftable. But $\mathrm{LS}(f) \leqslant C(\sigma_f)$ cannot contain pseudo-Anosov elements by lemma 4.3, giving a contradiction. □

In fact, we obtain the stronger result:

Proposition 4.4. *If f has hyperbolic orbifold, then $C(\sigma_f)$ is not a finite index subgroup of* $\mathrm{PMCG}(S^2, P_f)$.

Proof. if $C(\sigma_f)$ were a finite index subgroup, then $H_f \cap C(\sigma_f) = \mathrm{LS}(f)$ would also be. $\square$

In particular, this shows that any intermediary cover $\mathcal{T}_f \to E \to \mathcal{M}_f$ such that σ_f descends continuously to a self-map of E must be an infinite cover of $\mathcal{M}_f$.

5 Applications to matings

Let f_1 and f_2 be two degree d postcritically finite polynomials on $\mathbb{C}$, which have a natural meromorphic extensions to $\hat{\mathbb{C}}$ where ∞ is a superattracting fixed point. We take two copies $\mathbb{C}_1$ and $\mathbb{C}_2$ of the complex plane and adjoin a circle at infinity to each, naturally parametrized by its angle, to form closed disks $\overline{\mathbb{C}}_1$ and $\overline{\mathbb{C}}_2$. We glue the two disks along their circles at infinity, identifying $(\theta, \infty)_1$ with $(-\theta, \infty)_2$, to form a topological 2-sphere S^2. The polynomials, acting on each copy of $\mathbb{C}$, have a natural continuous extension to the circles at infinity, where they act by angle doubling. By gluing together these dynamics, we obtain the *formal mating* $f = f_1 \sqcup f_2 : S^2 \to S^2$, which is a Thurston map. A natural question to ask is whether f admits Thurston obstructions or not, and if so, when the obstructions are "degenerate" and not essential, in a precise sense. This has been answered completely in the degree 2 case by Tan Lei [7].

Other notions of matings of polynomials exist, such as the topological mating [11]. We define here the *slow mating* of two postcritically finite polynomials. Let φ_1, φ_2 be the Bottcher coordinates around ∞ for f_1 and f_2, with associated Green functions G_1 and G_2 for the filled Julia sets $K_1 := K_{f_1}$ and $K_2 := K_{f_2}$. For a fixed $h > 0$, let $E_i(h)$ be the equipotential $\{G_i(z) = h\}$ and $D_i(h)$ the closed disk $\{G_i(z) \leq h\}$, for $i = 1, 2$. We consider these disks as bordered Riemann surfaces. We may glue $D_1(h)$ and $D_2(h)$ along their boundary equipotentials, with the angle $(\theta, h)_1$ in Bottcher coordinates identified with $(-\theta, h)_2$, to obtain a topological 2-sphere S_h. This sphere inherits a conformal structure from the gluing of the bordered Riemann surfaces. Moreover, since we have holomorphic maps $f_i|_{D_i} : D_i(h) \to D_i(2h)$, we also obtain a glued holomorphic map $f_h : S_h \to S_{2h}$.

We also construct a 2-quasiconformal map $q_h : S_h \to S_{2h}$ as follows. The annulus $\operatorname{int} D_i(h) \setminus K_i$ has module $2h/2\pi$, and for $0 < h < H$, there exists a natural (H/h)-quasiconformal map $\operatorname{int} D_i(h) \setminus K_i \to \operatorname{int} D_i(H) \setminus K_i$ that stretches uniformly along the vertical foliations of the annuli. For $\varepsilon > 0$, we may consider a map $D_i(h) \to D_i(2h)$ which is the identity on $D_i(\varepsilon)$ and is the aforementioned quasiconformal map on the annuli $D_i(h) \setminus D_i(\varepsilon) \to D_i(2h) \setminus D_i(\varepsilon)$. This $(2h - \varepsilon)/(h - \varepsilon)$-quasiconformal map converges to a 2-quasiconformal map $D_i(h) \to D_i(2h)$ which is the identity on K_i as $\varepsilon > 0$, and by gluing together the disks, we obtain $q_h : S_h \to S_{2h}$ that identifies the filled Julia sets in S_h with the filled Julia sets in S_{2h}.

We keep track of the injections $e_h : C \cup P \to S_h$ which parametrize the critical and postcritical points of the polynomials, thus obtaining a genuine Thurston map $g_h = q_h^{-1} \circ f_h : S_h \to S_h$, and remark that

$$g_h = q_h^{-1} \circ f_h = f_{h/2} \circ q_{h/2}^{-1}.$$

A natural question to ask is if the "rational maps" $f_h : S_h \to S_{2h}$ converge in a meaningful sense on the moduli space of rational maps of degree d, and if so, under what conditions [8]. If $d = 2$, this can be rephrased by uniformizing each sphere $S_h \to \hat{\mathbb{C}}$ by mapping the two critical points to -1 and 1, and mapping the point at angle $t = 0$ on the equator (the identified equipotentials) to ∞, so that the normalized rational map $R_h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ has the form

$$w \mapsto \frac{1}{\lambda}\left(w + \frac{1}{w} + c\right),$$

where λ is the multiplier of the fixed point at ∞. Note that this question makes sense even in the case where f_1 and f_2 are not postcritically finite.

In the postcritically finite case, if μ_h is the almost complex structure on S^2 such that $S_h = (S^2, \mu_h)$, we have the data of an element $\alpha_h = [e_h, g_h, \mu_h]$ of the Rees space $\mathcal{R}_F$ for the portrait F of the critical and postcritical points of the glued polynomials. In fact, it is true that $\psi(\alpha_h) = \alpha_{h/2}$, given by the equivalence

$$\begin{tikzcd}
(S^2, g_h^* \mu_h) \arrow[r, "q_{h/2}^{-1}"] \arrow[d, "g_h"'] & (S^2, \mu_{h/2}) \arrow[d, "g_{h/2}"] \\
(S^2, g_h^* \mu_h) \arrow[r, "q_{h/2}^{-1}"] & (S^2, \mu_{h/2})
\end{tikzcd}$$

From Theorem 3.4, we obtain:

Proposition 5.1. *If the formal mating has hyperbolic orbifold and, for some $h > 0$, the marked spheres $(S^2, e_{2^{-i}h})$ for $i \in \mathbb{N}$ are bounded in $\mathcal{M}_P$, then the slow mating of f_1 and f_2 exists.*

Proof. Let $\alpha_i := \alpha_{2^{-i}h}$, and consider the set $\Omega := \bigcap_{k \geq 0} \overline{\{\alpha_i : i \geq k\}} \subseteq \mathcal{R}_f$ of all limits of subsequences of $(\alpha_i)_{i \geq 0}$. Since $D(\alpha_i) \leq 2$ for all i, and by the hypothesis of boundedness in $\mathcal{M}_P$, Ω is compact. Hence D attains a minimimum on Ω at some $\alpha \in \Omega$, and because f has hyperbolic orbifold and Ω is ψ-invariant, necessarily $D(\alpha) = 0$. There is some subsequence $\alpha_{i_k} \to \alpha$, so that $D(\alpha_{i_k}) \to 0$, and since the sequence $D(\alpha_i)$ is already non-increasing, this implies that $D(\alpha_i) \to 0$.

Again by compactness of Ω, any subsequence α_{i_j} has a further subsequence $\alpha_{i_{j_l}}$ that converges to some $\alpha' \in \Omega$, and $D(\alpha') = 0$. If $\hat{\pi}(\tau) = \alpha$ and $\hat{\pi}(\tau') = \alpha'$, then both τ and τ' are fixed points for σ_f, so that $\tau = \tau'$ and $\alpha = \alpha'$. This concludes that the sequence α_i itself converges to α with $D(\alpha) = 0$. □

Corollary 5.2. *If the formal mating has hyperbolic orbifold and the marked spheres (S^2, e_h) are bounded in $\mathcal{M}_P$ for $h > 0$, then the slow mating of f_1 and f_2 exists.*

Proof. Consider the continuous [?] curve $\gamma : [0, +\infty) \to \mathcal{R}_F$ given by $\gamma(t) = \alpha_{2^{-t}}$, so that $\psi(\gamma(t)) = \gamma(t+1)$, and $D(\gamma(t)) \leq 2$ for all t. By hypothesis, the image of γ is contained in a compact subset of $\mathcal{R}_F$. By the assumption of hyperbolic orbifold, there exists $0 \leq L < 1$ such that, for all α in the image of this curve,

$$D(\psi^2(\alpha)) \leq LD(\alpha),$$

where we use the strengthened form of Proposition 3.1. Define $d(t) \coloneqq D(\gamma(2t))$, so that $d(t+1) \leq Ld(t)$, for all $t \geq 0$. Then

$$d(t) \leq L^{\lfloor t \rfloor} d(t - \lfloor t \rfloor) \leq L^t M,$$

where M is the maximum of $d(t)$ for $t \in [0,1]$, and this implies that $\lim_{t\to\infty} d(t) = 0$. Let Ω be the omega limit of the curve γ in $\mathcal{R}_F$, being non-empty, connected, compact ψ-invariant. By continuity of d, every $\beta \in \Omega$ is such that $D(\beta) = 0$. If $\beta_1 = \hat{\pi}(\tau_1)$ and $\beta_2 = \hat{\pi}(\tau_2)$ are both in Ω, then $d(\tau_1, \sigma_f(\tau_1)) = d(\tau_2, \sigma_f(\tau_2)) = 0$, so that $\sigma_f(\tau_1) = \tau_1$ and $\sigma_f(\tau_2) = \tau_2$. But if σ_f has a fixed point, it must be unique [4], so that $\tau_1 = \tau_2$ and therefore $\beta_1 = \beta_2$. This shows that Ω is a singleton $\{\alpha\}$, and $\gamma(t) \to \alpha$. $\square$

6 Path lifting

7 Examples

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