Exemplo 01

EXAMPLE 2 Let $f(x, y) = y - x$ and let $\mathbf{x}\colon [0, 3] \to \mathbf{R}^2$ be the planar path

$$\mathbf{x}(t) = \begin{cases} (2t, t) & \text{if } 0 \le t \le 1 \\ (t+1, 5-4t) & \text{if } 1 < t \le 3 \end{cases}.$$

Hence, $\mathbf{x}$ is piecewise C^1 (see Figure 6.2); the two path segments defined for t in $[0, 1]$ and for t in $[1, 3]$ are each of class C^1. Thus,

$$\int_{\mathbf{x}} f\,ds = \int_{\mathbf{x}_1} f\,ds + \int_{\mathbf{x}_2} f\,ds,$$

where $\mathbf{x}_1(t) = (2t, t)$ for $0 \le t \le 1$ and $\mathbf{x}_2(t) = (t+1, 5-4t)$ for $1 \le t \le 3$. Note that

$$\|\mathbf{x}_1'(t)\| = \sqrt{5} \quad \text{and} \quad \|\mathbf{x}_2'(t)\| = \sqrt{17}.$$

Consequently,

$$\int_{\mathbf{x}_1} f\,ds = \int_0^1 f(\mathbf{x}(t))\,\|\mathbf{x}'(t)\|\,dt = \int_0^1 (t - 2t)\cdot\sqrt{5}\,dt = -\frac{\sqrt{5}}{2}t^2\Big|_0^1 = -\frac{\sqrt{5}}{2}.$$

Also,

$$\begin{aligned}\int_{\mathbf{x}_2} f\,ds = \int_1^3 f(\mathbf{x}_2(t))\,\|\mathbf{x}_2'(t)\|\,dt &= \int_1^3 ((5-4t) - (t+1))\sqrt{17}\,dt \\ &= \sqrt{17}\left(4t - \tfrac{5}{2}t^2\right)\Big|_1^3 = -12\sqrt{17}.\end{aligned}$$

Hence,

$$\int_{\mathbf{x}} f\,ds = -\frac{\sqrt{5}}{2} - 12\sqrt{17}.$$

Exemplo 02

EXAMPLE 2.6 Calculating a Line Integral in Space

Compute $\int_C 4x\,dy + 2y\,dz$, where C consists of the line segment from $(0, 1, 0)$ to $(0, 1, 1)$, followed by the line segment from $(0, 1, 1)$ to $(2, 1, 1)$ and followed by the line segment from $(2, 1, 1)$ to $(2, 4, 1)$.

Solution We show a sketch of the curves in Figure 15.19. Parametric equations for the first segment C_1 are $x = 0$, $y = 1$ and $z = t$ with $0 \le t \le 1$. On this segment, we have $dy = 0\,dt$ and $dz = 1\,dt$. On the second segment C_2, parametric equations are $x = 2t$, $y = 1$ and $z = 1$ with $0 \le t \le 1$. On this segment, we have $dy = dz = 0\,dt$. On the third segment C_3, parametric equations are $x = 2$, $y = 3t + 1$ and $z = 1$ with $0 \le t \le 1$. On this segment, we have $dy = 3\,dt$ and $dz = 0\,dt$. Putting this all together, we have

$$\begin{aligned}\int_C 4x\,dy + 2y\,dz &= \int_{C_1} 4x\,dy + 2y\,dz + \int_{C_2} 4x\,dy + 2y\,dz + \int_{C_3} 4x\,dy + 2y\,dz \\ &= \int_0^1 [\underbrace{4(0)}_{4x}\underbrace{(0)}_{y'(t)} + \underbrace{2(1)}_{2y}\underbrace{(1)}_{z'(t)}]\,dt + \int_0^1 [\underbrace{4(2t)}_{4x}\underbrace{(0)}_{y'(t)} + \underbrace{2(1)}_{2y}\underbrace{(0)}_{z'(t)}]\,dt \\ &\quad + \int_0^1 [\underbrace{4(2)}_{4x}\underbrace{(3)}_{y'(t)} + \underbrace{2(3t+1)}_{2y}\underbrace{(0)}_{z'(t)}]\,dt \\ &= \int_0^1 26\,dt = 26.\end{aligned}$$

Exemplo 03

EXAMPLE 1 **Integrating along a Helix** Calculate

$$\int_C (x + y + z)\, ds$$

where C is the helix $\mathbf{c}(t) = (\cos t, \sin t, t)$ for $0 \le t \le \pi$ (Figure 3).

Solution

***Step 1.* Compute ds.**

$$\mathbf{c}'(t) = \langle -\sin t, \cos t, 1\rangle$$

$$\|\mathbf{c}'(t)\| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2}$$

$$ds = \|\mathbf{c}'(t)\| dt = \sqrt{2}\, dt$$

***Step 2.* Write out the integrand and evaluate.**

We have $f(x, y, z) = x + y + z$, and so

$$f(\mathbf{c}(t)) = f(\cos t, \sin t, t) = \cos t + \sin t + t$$

$$f(x, y, z)\, ds = f(\mathbf{c}(t))\, \|\mathbf{c}'(t)\|\, dt = (\cos t + \sin t + t)\sqrt{2}\, dt$$

By Eq. (4),

$$\int_C f(x, y, z)\, ds = \int_0^\pi f(\mathbf{c}(t))\, \|\mathbf{c}'(t)\|\, dt = \int_0^\pi (\cos t + \sin t + t)\sqrt{2}\, dt$$

$$= \sqrt{2}\left(\sin t - \cos t + \frac{1}{2}t^2\right)\Big|_0^\pi$$

$$= \sqrt{2}\left(0 + 1 + \frac{1}{2}\pi^2\right) - \sqrt{2}(0 - 1 + 0) = 2\sqrt{2} + \frac{\sqrt{2}}{2}\pi^2$$

Exemplo 04

Exemplo 4: Calcular $\int_C xy\, ds$, onde C é a intersecção das superfícies $x^2 + y^2 = 4$ e $y + z = 8$.

Solução: A Figura 9.4 mostra um esboço da curva C. Para parametrizá-la, observamos que x e y devem satisfazer a equação da circunferência $x^2 + y^2 = 4$, que é a projeção de C sobre o plano xy. Fazemos, então,

$$x = 2\cos t;\ y = 2 \operatorname{sen} t;\ t \in [0, 2\pi].$$

Substituindo o valor de y na equação $y + z = 8$, obtemos

$$z = 8 - 2 \operatorname{sen} t.$$

Portanto,

$$\vec{r}(t) = 2\cos t\,\vec{i} + 2\operatorname{sen} t\,\vec{j} + (8 - 2\operatorname{sen} t)\vec{k},\ t \in [0, 2\pi].$$

Usando (3), temos

$$\int_C xy\, ds = \int_0^{2\pi} 2\cos t \cdot 2\operatorname{sen} t \cdot 2\sqrt{1 + \cos^2 t}\, dt$$

$$= 4\int_0^{2\pi} (1 + \cos^2 t)^{1/2} \cdot 2\cos t \operatorname{sen} t\, dt$$

$$= -4 \cdot \frac{2}{3}(1 + \cos^2 t)^{3/2}\Big|_0^{2\pi} = 0.$$

Exemplo 05

EXAMPLE 1 Evaluate $\int_C x^2 y\, ds$, where C is determined by the parametric equations $x = 3\cos t$, $y = 3\sin t$, $0 \le t \le \pi/2$. Also show that the parametrization $x = \sqrt{9 - y^2}$, $y = y$, $0 \le y \le 3$, gives the same value.

SOLUTION Using the first parametrization, we obtain

$$\begin{aligned}\int_C x^2 y\, ds &= \int_0^{\pi/2} (3\cos t)^2 (3\sin t)\sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt \\ &= 81 \int_0^{\pi/2} \cos^2 t \sin t\, dt \\ &= \left[-\frac{81}{3}\cos^3 t\right]_0^{\pi/2} = 27\end{aligned}$$

For the second parametrization, we use another formula for ds as given in Section 5.4. This gives

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \sqrt{1 + \frac{y^2}{9 - y^2}}\, dy = \frac{3}{\sqrt{9 - y^2}}\, dy$$

and

$$\begin{aligned}\int_C x^2 y\, ds &= \int_0^3 (9 - y^2) y \frac{3}{\sqrt{9 - y^2}}\, dy = 3\int_0^3 \sqrt{9 - y^2}\, y\, dy \\ &= -\left[(9 - y^2)^{3/2}\right]_0^3 = 27\end{aligned}$$

■

Exemplo 06

▶ **Example 4** Find the area of the surface extending upward from the circle $x^2 + y^2 = 1$ in the xy-plane to the parabolic cylinder $z = 1 - x^2$ (Figure 15.2.9).

Solution. It follows from (7) that the area A of the surface can be expressed as the line integral

$$A = \int_C (1 - x^2)\, ds \tag{15}$$

where C is the circle $x^2 + y^2 = 1$. This circle can be parametrized in terms of arc length as

$$x = \cos s, \quad y = \sin s \qquad (0 \le s \le 2\pi)$$

Thus, it follows from (13) and (15) that

$$\begin{aligned}A &= \int_C (1 - x^2)\, ds = \int_0^{2\pi} (1 - \cos^2 s)\, ds \\ &= \int_0^{2\pi} \sin^2 s\, ds = \frac{1}{2}\int_0^{2\pi} (1 - \cos 2s)\, ds = \pi\end{aligned}$$

◀

Exemplo 07

Sobre el límite del escenario se va a levantar una cortina hasta el techo del teatro. Se quiere saber qué área se cubrirá con la cortina. Aplicando matemáticas a este problema, vemos que se trata de integrar la función $f(x, y) = \frac{b}{a}(a - x - y)$ sobre el camino $\lambda\colon [0, \pi/2] \to \mathbb{R}^2$. $\lambda(t) = (r\cos t, r\,\mathrm{sen}\, t)$. Tenemos entonces que el área A procurada es

$$\begin{aligned}A &= \int_\lambda f\, ds = \int_0^{\pi/2} \frac{b}{a}(a - r\cos t - r\,\mathrm{sen}\, t) r\, dt \\ &= \frac{br}{a}\left[at - r\,\mathrm{sen}\, t + r\cos t\right]_0^{\pi/2} = \frac{br}{2a}(\pi a - 4r) \text{ unidades}^2\end{aligned}$$

■

Exemplo 07

Example 3 Suppose that a semicircular wire has the equation $y = \sqrt{25 - x^2}$ and that its mass density is $\delta(x, y) = 15 - y$ (Figure 15.2.8). Physically, this means the wire has a maximum density of 15 units at the base ($y = 0$) and that the density of the wire decreases linearly with respect to y to a value of 10 units at the top ($y = 5$). Find the mass of the wire.

Solution. The mass M of the wire can be expressed as the line integral

$$M = \int_C \delta(x, y)\, ds = \int_C (15 - y)\, ds$$

along the semicircle C. To evaluate this integral we will express C parametrically as

$$x = 5\cos t, \quad y = 5\sin t \qquad (0 \le t \le \pi)$$

Thus, it follows from (11) that

$$\begin{aligned}
M &= \int_C (15 - y)\, ds = \int_0^\pi (15 - 5\sin t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^\pi (15 - 5\sin t)\sqrt{(-5\sin t)^2 + (5\cos t)^2}\, dt \\
&= 5\int_0^\pi (15 - 5\sin t)\, dt \\
&= 5\big[15t + 5\cos t\big]_0^\pi \\
&= 75\pi - 50 \approx 185.6 \text{ units of mass}
\end{aligned}$$

Exemplo 08

Example 3 Find the mass of a wire lying along the first octant part $\mathcal{C}$ of the curve of intersection of the elliptic paraboloid $z = 2 - x^2 - 2y^2$ and the parabolic cylinder $z = x^2$ between $(0, 1, 0)$ and $(1, 0, 1)$ (see Figure 15.7) if the density of the wire at position (x, y, z) is $\delta(x, y, z) = xy$.

Solution We need a convenient parametrization of $\mathcal{C}$. Since the curve $\mathcal{C}$ lies on the cylinder $z = x^2$ and x goes from 0 to 1, we can let $x = t$ and $z = t^2$. Thus, $2y^2 = 2 - x^2 - z = 2 - 2t^2$, so $y^2 = 1 - t^2$. Since $\mathcal{C}$ lies in the first octant, it can be parametrized by

$$x = t, \qquad y = \sqrt{1 - t^2}, \qquad z = t^2, \qquad (0 \le t \le 1).$$

Then $dx/dt = 1$, $dy/dt = -t/\sqrt{1 - t^2}$, and $dz/dt = 2t$, so

$$ds = \sqrt{1 + \frac{t^2}{1 - t^2} + 4t^2}\, dt = \frac{\sqrt{1 + 4t^2 - 4t^4}}{\sqrt{1 - t^2}}\, dt.$$

Hence, the mass of the wire is

$$\begin{aligned}
m &= \int_{\mathcal{C}} xy\, ds = \int_0^1 t\sqrt{1 - t^2}\frac{\sqrt{1 + 4t^2 - 4t^4}}{\sqrt{1 - t^2}}\, dt \\
&= \int_0^1 t\sqrt{1 + 4t^2 - 4t^4}\, dt \qquad \text{Let } u = t^2 \\
&= \frac{1}{2}\int_0^1 \sqrt{1 + 4u - 4u^2}\, du \\
&= \frac{1}{2}\int_0^1 \sqrt{2 - (2u - 1)^2}\, du \qquad \text{Let } v = 2u - 1 \\
&= \frac{1}{4}\int_{-1}^1 \sqrt{2 - v^2}\, dv = \frac{1}{2}\int_0^1 \sqrt{2 - v^2}\, dv \\
&= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) = \frac{\pi + 2}{8}.
\end{aligned}$$

(The final integral above was evaluated by interpreting it as the area of part of a circle. You are invited to supply the details. It can also be done by the substitution $v = \sqrt{2}\sin w$.)

Exemplo 10

■ **EXAMPLE 4** **Electric Potential** A charged semicircle of radius R centered at the origin in the xy-plane (Figure 5) has charge density

$$\rho(x, y, 0) = 10^{-8}\left(2 - \frac{x}{R}\right) \quad \text{C/m}$$

Find the electric potential at a point $P = (0, 0, a)$ if $R = 0.1$ m.

Solution Parametrize the semicircle by $\mathbf{c}(t) = (R\cos t, R\sin t, 0)$ for $-\pi/2 \le t \le \pi/2$:

$$\|\mathbf{c}'(t)\| = \|\langle -R\sin t, R\cos t, 0\rangle\| = \sqrt{R^2\sin^2 t + R^2\cos^2 t + 0} = R$$

$$ds = \|\mathbf{c}'(t)\|\,dt = R\,dt$$

$$\rho(\mathbf{c}(t)) = \rho(R\cos t, R\sin t, 0) = 10^{-8}\left(2 - \frac{R\cos t}{R}\right) = 10^{-8}(2 - \cos t)$$

In our case, the distance r_P from P to a point $(x, y, 0)$ on the semicircle has the constant value $r_P = \sqrt{R^2 + a^2}$ (Figure 5). Thus,

$$V(P) = k\int_C \frac{\rho(x, y, z)\,ds}{r_P} = k\int_C \frac{10^{-8}(2 - \cos t)\,R dt}{\sqrt{R^2 + a^2}}$$

$$= \frac{10^{-8}kR}{\sqrt{R^2 + a^2}}\int_{-\pi/2}^{\pi/2}(2 - \cos t)\,dt = \frac{10^{-8}kR}{\sqrt{R^2 + a^2}}(2\pi - 2)$$

With $R = 0.1$ m and $k = 8.99 \times 10^9$, we then obtain $10^{-8}kR(2\pi - 2) \approx 38.5$ and $V(P) \approx \dfrac{38.5}{\sqrt{0.01 + a^2}}$ volts. ■

Exemplo 11

Ejemplo 8. Calculemos la masa total de un alambre cuya forma es la de la imagen de la hélice $\boldsymbol{\lambda}: [0, 2\pi] \to \mathbb{R}^3$, $\boldsymbol{\lambda}(t) = (\cos t, \operatorname{sen} t, t)$, si la densidad en cada punto es proporcional al cuadrado de la distancia del punto al origen, valiendo 1 gr/cm en el punto inicial $\boldsymbol{\lambda}(0) = (1, 0, 0)$. Se tiene entonces la función densidad $\rho(x, y, z) = k(x^2 + y^2 + z^2)$, donde k es la constante de proporcionalidad, que se calcula usando que $\rho(1, 0, 0) = 1$, obteniéndose que $k = 1$. Entonces $\rho(x, y, z) = x^2 + y^2 + z^2$, y la masa total del alambre es

$$M = \int_{\boldsymbol{\lambda}} \rho\,ds = \int_0^{2\pi} \rho(\boldsymbol{\lambda}(t))\|\boldsymbol{\lambda}'(t)\|\,dt$$

$$= \int_0^{2\pi} (\cos^2 t + \operatorname{sen}^2 t + t^2)\sqrt{(-\operatorname{sen} t)^2 + (\cos t)^2 + 1}\,dt$$

$$= \int_0^{2\pi} (1 + t^2)\sqrt{2}\,dt = 2\sqrt{2}\pi(1 + \frac{4}{3}\pi^2)\text{ gr}$$ ■

Ejemplo 11. Calculemos el centro de masa del alambre del ejemplo 8, en forma de la hélice $\boldsymbol{\lambda}: [0, 2\pi] \to \mathbb{R}^3$, $\boldsymbol{\lambda}(t) = (\cos t, \operatorname{sen} t, t)$, cuya función densidad es $\rho(x, y, z) = x^2 + y^2 + z^2$. La masa total del alambre ya ha sido calculada y es $M = 2\sqrt{2}\pi(1 + \frac{4}{3}\pi^2)$ gr. Calculemos los momentos estáticos del alambre respecto de los planos coordenados

$$M_{xy} = \int_{\boldsymbol{\lambda}} z\rho(x, y, z)\,ds = \int_0^{2\pi} t(\cos^2 t + \operatorname{sen}^2 t + t^2)\sqrt{2}\,dt$$

$$= \sqrt{2}\int_0^{2\pi} t(1 + t^2)\,dt = 2\sqrt{2}\pi^2(1 + 2\pi^2)$$

$$M_{xz} = \int_{\boldsymbol{\lambda}} y\rho(x, y, z)\,ds = \int_0^{2\pi} \operatorname{sen} t(\cos^2 t + \operatorname{sen}^2 t + t^2)\sqrt{2}\,dt$$

$$= \sqrt{2}\int_0^{2\pi} \operatorname{sen} t(1 + t^2)\,dt = -4\sqrt{2}\pi^2$$

$$M_{yz} = \int_{\boldsymbol{\lambda}} x\rho(x, y, z)\,ds = \int_0^{2\pi} \cos t(\cos^2 t + \operatorname{sen}^2 t + t^2)\sqrt{2}\,dt$$

$$= \sqrt{2}\int_0^{2\pi} \cos t(1 + t^2)\,dt = 4\sqrt{2}\pi$$

Entonces el centro de masa del alambre está en el punto $(\bar{x}, \bar{y}, \bar{z})$ en donde

$$\bar{x} = \frac{M_{yz}}{M} = \frac{\int_\lambda x\rho(x, y, z)\, ds}{\int_\lambda \rho(x, y, z)\, ds} = \frac{4\sqrt{2}\pi}{2\sqrt{2}\pi(1 + \frac{4}{3}\pi^2)} = \frac{2}{1 + \frac{4}{3}\pi^2}$$

$$\bar{y} = \frac{M_{xz}}{M} = \frac{\int_\lambda y\rho(x, y, z)\, ds}{\int_\lambda \rho(x, y, z)\, ds} = \frac{-4\sqrt{2}\pi^2}{2\sqrt{2}\pi(1 + \frac{4}{3}\pi^2)} = \frac{-2\pi}{1 + \frac{4}{3}\pi^2}$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{\int_\lambda z\rho(x, y, z)\, ds}{\int_\lambda \rho(x, y, z)\, ds} = \frac{2\sqrt{2}\pi^2(1 + 2\pi^2)}{2\sqrt{2}\pi(1 + \frac{4}{3}\pi^2)} = \frac{\pi(1 + 2\pi^2)}{1 + \frac{4}{3}\pi^2}$$

que corresponde aproximadamente al punto (0.141248, −0.443744, 4.60145).