Problem Refinement Critiques Report

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July 2, 2025

Summary Statistics

Critique Type	Pass	Fail	Removed	Pass Rate
Self Containment	66	6	0	91.7%
Difficulty	65	7	0	90.3%
Refinement Success	72	0	0	100.0%

Problem Details

Problem 1 (Paper: 2506.18641v1, Index: 0)

Original Problem Statement

Background: Consider a complex network represented by an unweighted, undirected graph with an adjacency matrix $\mathbf{A}$ and a diagonal matrix of node degrees $\mathbf{D}$. The network's structure can be analyzed using its Laplacian matrix, defined as $\mathbf{L} = \mathbf{D} - \mathbf{A}$. In analogy with statistical mechanics, a partition function Z_τ can be defined to characterize information flow dynamics on the network. This function is given by $Z_\tau = \mathrm{Tr}(e^{-\tau \mathbf{L}})$, where τ is a diffusion scale parameter analogous to inverse temperature. From this partition function, other key thermodynamic quantities can be defined. The average energy $\langle E \rangle_\tau$ is the expectation value of the Laplacian $\mathbf{L}$ and is given by the relation $\langle E \rangle_\tau = -(\partial/\partial\tau) \log Z_\tau$. The spectral entropy S_τ is related to the partition function and average energy by the standard thermodynamic formula $S_\tau = \log Z_\tau + \tau \langle E \rangle_\tau$. The free energy F_τ is defined in terms of the average energy and entropy as $F_\tau = \langle E \rangle_\tau - (1/\tau) S_\tau$.

Task: Using the provided relationships between the partition function Z_τ, average energy $\langle E \rangle_\tau$, and entropy S_τ, derive the simplified expression for the free energy F_τ as a function of only Z_τ and τ.

Original Solution

$$F_\tau = -\frac{\log Z_\tau}{\tau}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Trivial**

The problem involves a straightforward substitution and algebraic manipulation of given thermodynamic definitions without requiring synthesis of multiple complex concepts or multi-step reasoning.

Refined Problem

Refined Problem Statement: Background: Consider a complex network represented by an unweighted, undirected graph with an adjacency matrix $\mathbf{A}$ and a diagonal matrix of node degrees $\mathbf{D}$. The network's structure can be analyzed using its Laplacian matrix, defined as $\mathbf{L} = \mathbf{D} - \mathbf{A}$. In analogy with statistical mechanics, a partition function Z_τ can be defined to characterize information flow dynamics on the network. This function is given by $Z_\tau = \mathrm{Tr}(e^{-\tau \mathbf{L}})$, where τ is a diffusion scale parameter analogous to inverse temperature. From this partition function, other key thermodynamic quantities can be defined. The average energy $\langle E \rangle_\tau$ is the expectation value of the Laplacian $\mathbf{L}$ and is given by the relation $\langle E \rangle_\tau = -(\partial/\partial\tau) \log Z_\tau$. The spectral entropy S_τ is related to the partition function and average energy by the standard thermodynamic formula $S_\tau = \log Z_\tau + \tau \langle E \rangle_\tau$. The free energy F_τ is defined in terms of the average energy and entropy as $F_\tau = \langle E \rangle_\tau - (1/\tau) S_\tau$.

Task: Using the provided relationships between the partition function Z_τ, average energy $\langle E \rangle_\tau$, and entropy S_τ, derive the simplified expression for the free energy F_τ as a function of only Z_τ and τ.

Refined Solution:

$$F_\tau = -\frac{\log Z_\tau}{\tau}$$

Problem 2 (Paper: 2506.18641v1, Index: 1)

Original Problem Statement

Background: In the study of network reduction, a key challenge is to ensure that a reduced subgraph preserves the dynamical properties of the original large-scale network. Consider an original network G_0 and a subgraph G_l obtained through a reduction process. We wish to quantify the similarity of their epidemic spreading dynamics as described by the Susceptible-Infected-Recovered (SIR) model. A crucial observable in this model is the final fraction of the population that has recovered, denoted ρ_r, which is a function of the infection rate, β. Let the functions for the original network and the subgraph be $\rho_r^0(\beta)$ and $\rho_r^l(\beta)$, respectively.

A robust similarity metric, $f_{overlap}$, is required to compare the two dynamical response curves, $\rho_r^0(\beta)$ and $\rho_r^l(\beta)$, over a continuous range of the infection rate, for instance $\beta \in [\beta_{min}, \beta_{max}]$. This metric must satisfy the following criteria: 1. It should be functionally dependent on the total integrated absolute deviation, S_Δ, between the two curves over the specified range of β. 2. It must be normalized, mapping the non-negative deviation $S_\Delta \geq 0$ to a value $f_{overlap} \in (0, 1]$. 3. For identical dynamics, where $\rho_r^0(\beta) = \rho_r^l(\beta)$ for all β in the range, the deviation S_Δ is zero and the metric must yield its maximum value, $f_{overlap} = 1$. 4. The metric should be a simple, monotonically decreasing rational function of S_Δ.

Task: Based on the principles outlined, derive the complete mathematical expression for the similarity metric $f_{overlap}$ in terms of the functions $\rho_r^0(\beta)$ and $\rho_r^l(\beta)$ and the integration limits β_{min} and β_{max}.

Original Solution

$$f_{overlap} = \frac{1}{1 + \int_{\beta_{min}}^{\beta_{max}} |\rho_r^0(\beta) - \rho_r^l(\beta)| d\beta}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Trivial**

The problem essentially asks for the derivation of a simple normalized similarity metric directly from the integral of the absolute difference, which is a straightforward and standard construction without requiring multi-step or advanced reasoning.

Refined Problem

Refined Problem Statement: Background: In the study of network reduction, a key challenge is to ensure that a reduced subgraph preserves the dynamical properties of the original large-scale network. Consider an original network G_0 and a subgraph G_l obtained through a reduction process. We wish to quantify the similarity of their epidemic spreading dynamics as described by the Susceptible-Infected-Recovered (SIR) model. A crucial observable in this model is the final fraction of the population that has recovered, denoted ρ_r, which is a function of the infection rate, β. Let the functions for the original network and the subgraph be $\rho_r^0(\beta)$ and $\rho_r^l(\beta)$, respectively.

A robust similarity metric, $f_{overlap}$, is required to compare the two dynamical response curves, $\rho_r^0(\beta)$ and $\rho_r^l(\beta)$, over a continuous range of the infection rate, for instance $\beta \in [\beta_{min}, \beta_{max}]$. This metric must satisfy the following criteria: 1. It should be functionally dependent on the total integrated absolute deviation, S_Δ, between the two curves over the specified range of β. 2. It must be normalized, mapping the non-negative deviation $S_\Delta \geq 0$ to a value $f_{overlap} \in (0, 1]$. 3. For identical dynamics, where $\rho_r^0(\beta) = \rho_r^l(\beta)$ for all β in the range, the deviation S_Δ is zero and the metric must yield its maximum value, $f_{overlap} = 1$. 4. The metric should be a simple, monotonically decreasing rational function of S_Δ.

Task: Based on the principles outlined, derive the complete mathematical expression for the similarity metric $f_{overlap}$ in terms of the functions $\rho_r^0(\beta)$ and $\rho_r^l(\beta)$ and the integration limits β_{min} and β_{max}.

Refined Solution:

$$f_{overlap} = \frac{1}{1 + \int_{\beta_{min}}^{\beta_{max}} |\rho_r^0(\beta) - \rho_r^l(\beta)| d\beta}$$

Problem 3 (Paper: 2506.20163v1, Index: 0)

Original Problem Statement

Background: Consider a dynamical system defined on a 1-simplicial complex (a network) composed of N_n nodes (0-simplices) and N_l links (1-simplices). The state of the system is described by a set of variables on the nodes, called a 0-cochain and denoted by the vector $\vec{u} \in \mathbb{R}^{N_n}$, and a set of variables on the links, called a 1-cochain and denoted by the vector $\vec{v} \in \mathbb{R}^{N_l}$. The complete state of the system is given by the topological spinor $\vec{w} = (\vec{u}^\top, \vec{v}^\top)^\top \in \mathbb{R}^{N_n+N_l}$. The interactions between variables on nodes and links are mediated by the $N_n \times N_l$ boundary matrix $\boldsymbol{B}_1$ and its transpose, the coboundary matrix $\boldsymbol{B}_1^\top$. The dynamics of the system, which we refer to as a Dirac-Bianconi driven oscillator, are governed by a specific FitzHugh-Nagumo-like model given by the equations:

$$\begin{aligned} \dot{u}_i &= u_i - u_i^3 - (\boldsymbol{B}_1\vec{v})_i + I_i, \quad \text{for } i = 1, \dots, N_n \\ \dot{v}_j &= \delta_j \left((\boldsymbol{B}_1^\top \vec{u})_j - b_j v_j + \alpha_j a_j \right), \quad \text{for } j = 1, \dots, N_l \end{aligned}$$

where $(\boldsymbol{B}_1\vec{v})_i = \sum_{k=1}^{N_l} (\boldsymbol{B}_1)_{ik} v_k$ and $(\boldsymbol{B}_1^\top \vec{u})_j = \sum_{k=1}^{N_n} (\boldsymbol{B}_1^\top)_{jk} u_k$. The parameters $I_i, \delta_j, b_j, a_j, \alpha_j$ are real constants. The entire system can be written in the compact form $\dot{\vec{w}} = \vec{F}(\vec{w})$.

Task: Derive the Jacobian matrix $J(\vec{w}) = \frac{\partial \vec{F}}{\partial \vec{w}}$ for this system. Express your result as an $(N_n + N_l) \times (N_n + N_l)$ block matrix, with blocks corresponding to the partial derivatives with respect to $\vec{u}$ and $\vec{v}$. Define any necessary diagonal matrices in terms of the system parameters.

Original Solution

$$J(\vec{w}) = \begin{pmatrix} \operatorname{diag}(1 - 3u_i^2)_{i=1}^{N_n} & -\boldsymbol{B}_1 \\ \boldsymbol{\Delta}\boldsymbol{B}_1^\top & -\boldsymbol{\Delta}\boldsymbol{b} \end{pmatrix} \quad \text{where} \quad \boldsymbol{\Delta} = \operatorname{diag}(\delta_j)_{j=1}^{N_l} \quad \text{and} \quad \boldsymbol{b} = \operatorname{diag}(b_j)_{j=1}^{N_l}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing the nonlinear node dynamics with the network topology encoded by the boundary and coboundary matrices, and carefully computing the Jacobian as a block matrix involving diagonal parameter matrices and the incidence matrices, which is a multi-step, non-trivial derivation.

Refined Problem

Refined Problem Statement: Background: Consider a dynamical system defined on a 1-simplicial complex (a network) composed of N_n nodes (0-simplices) and N_l links (1-simplices). The state of the system is described by a set of variables on the nodes, called a 0-cochain and denoted by the vector $\vec{u} \in \mathbb{R}^{N_n}$, and a set of variables on the links, called a 1-cochain and denoted by the vector $\vec{v} \in \mathbb{R}^{N_l}$. The complete state of the system is given by the topological spinor $\vec{w} = (\vec{u}^\top, \vec{v}^\top)^\top \in \mathbb{R}^{N_n+N_l}$. The interactions between variables on nodes and links are mediated by the $N_n \times N_l$ boundary matrix $\boldsymbol{B}_1$ and its transpose, the coboundary matrix $\boldsymbol{B}_1^\top$. The dynamics of the system, which we refer to as a Dirac-Bianconi driven oscillator, are governed by a specific FitzHugh-Nagumo-like model given by the equations:

$$\begin{aligned} \dot{u}_i &= u_i - u_i^3 - (\boldsymbol{B}_1\vec{v})_i + I_i, \quad \text{for } i = 1, \dots, N_n \\ \dot{v}_j &= \delta_j \left((\boldsymbol{B}_1^\top \vec{u})_j - b_j v_j + \alpha_j a_j \right), \quad \text{for } j = 1, \dots, N_l \end{aligned}$$

where $(\boldsymbol{B}_1\vec{v})_i = \sum_{k=1}^{N_l} (\boldsymbol{B}_1)_{ik} v_k$ and $(\boldsymbol{B}_1^\top \vec{u})_j = \sum_{k=1}^{N_n} (\boldsymbol{B}_1^\top)_{jk} u_k$. The parameters $I_i, \delta_j, b_j, a_j, \alpha_j$ are real constants. The entire system can be written in the compact form $\dot{\vec{w}} = \vec{F}(\vec{w})$.

Task: Derive the Jacobian matrix $J(\vec{w}) = \frac{\partial \vec{F}}{\partial \vec{w}}$ for this system. Express your result as an $(N_n + N_l) \times (N_n + N_l)$ block matrix, with blocks corresponding to the partial derivatives with respect to $\vec{u}$ and $\vec{v}$. Define any necessary diagonal matrices in terms of the system parameters.

Refined Solution:

$$J(\vec{w}) = \begin{pmatrix} \mathrm{diag}(1-3u_i^2)_{i=1}^{N_n} & -\boldsymbol{B}_1 \\ \boldsymbol{\Delta}\boldsymbol{B}_1^\top & -\boldsymbol{\Delta}\boldsymbol{b} \end{pmatrix} \quad \text{where} \quad \boldsymbol{\Delta} = \mathrm{diag}(\delta_j)_{j=1}^{N_l} \quad \text{and} \quad \boldsymbol{b} = \mathrm{diag}(b_j)_{j=1}^{N_l}$$

Problem 4 (Paper: 2506.20163v1, Index: 1)

Original Problem Statement

Background: Consider a system called a Dirac-Bianconi driven oscillator, defined on a simplicial complex consisting of N_n nodes (0-simplices) and N_l links (1-simplices). The state of the system is described by a topological spinor $\vec{w} = (\vec{u}^\top, \vec{v}^\top)^\top$, where $\vec{u} \in \mathbb{R}^{N_n}$ is a 0-cochain (variables on nodes) and $\vec{v} \in \mathbb{R}^{N_l}$ is a 1-cochain (variables on links). The dynamics are given by $\dot{\vec{w}} = \vec{F}(\vec{w})$. Assume this system possesses a stable limit cycle solution $\vec{w}_c(t)$ with frequency ω. Using phase reduction theory, the state on the limit cycle can be described by a single phase variable $\vartheta(t)$ such that $\dot{\vartheta} = \omega$ and $\vec{w}_c(t) = \vec{w}_c(\vartheta(t))$. The response of the phase to a small perturbation $\epsilon\vec{P}$ is governed by the phase sensitivity function (PSF) $\vec{Z}(\vartheta) = \nabla_{\vec{w}}\vartheta|_{\vec{w}_c(\vartheta)}$, which can be partitioned according to the node and link variables as $\vec{Z}(\vartheta) = (\vec{Z}_u(\vartheta)^\top, \vec{Z}_v(\vartheta)^\top)^\top$.

Now, consider two such oscillators, indexed by $k = 1, 2$, with state vectors $\vec{w}_k = (\vec{u}_k^\top, \vec{v}_k^\top)^\top$ and natural frequencies ω_k. They are weakly coupled with strength $\epsilon \ll 1$. The dynamics of oscillator 1 are given by $\dot{\vec{w}}_1 = \vec{F}_1(\vec{w}_1) + \epsilon\vec{P}_1(\vec{w}_1, \vec{w}_2)$. To first order in ϵ, its phase dynamics are $\dot{\vartheta}_1 = \omega_1 + \epsilon\Gamma_1(\vartheta_1, \vartheta_2)$, where the phase coupling function is $\Gamma_1(\vartheta_1, \vartheta_2) = \vec{Z}_1(\vartheta_1) \cdot \vec{P}_1(\vec{w}_1(\vartheta_1), \vec{w}_2(\vartheta_2))$.

The coupling perturbation $\vec{P}_1$ has components acting on the node variables, $\vec{p}_{1,u}$, and link variables, $\vec{p}_{1,v}$. The coupling is a combination of a diffusive term and a Dirac-Bianconi term. 1. The diffusive coupling acts only between corresponding nodes: $\vec{p}_{1,u}^{\text{diff}} = C_u(\vec{u}_2 - \vec{u}_1)$, where C_u is an $N_n \times N_n$ matrix specifying which nodes are coupled. 2. The Dirac-Bianconi coupling is analogous to the internal coupling structure. It couples the link variables of oscillator 2 to the node variables of oscillator 1, and the node variables of oscillator 2 to the link variables of oscillator 1. This is mediated by the $N_n \times N_l$ boundary operator $\boldsymbol{B}_1$. The specific forms are: - Perturbation on $\vec{u}_1$: $\vec{p}_{1,u}^{\text{DB}} = -\boldsymbol{B}_1\vec{v}_2$ - Perturbation on $\vec{v}_1$: $\vec{p}_{1,v}^{\text{DB}} = \boldsymbol{\Delta}_1\boldsymbol{B}_1^\top\vec{u}_2$, where $\boldsymbol{\Delta}_1$ is a diagonal $N_l \times N_l$ matrix of internal system parameters $\delta_{1,j}$. The total perturbation on oscillator 1 is $\vec{P}_1 = (\vec{p}_{1,u}^{\text{diff}} + \vec{p}_{1,u}^{\text{DB}}, \vec{p}_{1,v}^{\text{DB}})^\top$.

Task: Derive the phase coupling function $\Gamma_1(\vartheta_1, \vartheta_2)$ for oscillator 1. Express your answer in terms of the partitioned PSFs $\vec{Z}_{u,1}(\vartheta_1)$ and $\vec{Z}_{v,1}(\vartheta_1)$, the limit cycle trajectories $\vec{u}_k(\vartheta_k)$ and $\vec{v}_k(\vartheta_k)$, the boundary operator $\boldsymbol{B}_1$, the coupling matrix C_u, and the parameter matrix $\boldsymbol{\Delta}_1$.

Original Solution

$$\Gamma_1(\vartheta_1, \vartheta_2) = \vec{Z}_{u,1}(\vartheta_1)^\top \left[C_u(\vec{u}_2(\vartheta_2) - \vec{u}_1(\vartheta_1)) - \boldsymbol{B}_1\vec{v}_2(\vartheta_2)\right] + \vec{Z}_{v,1}(\vartheta_1)^\top \left(\boldsymbol{\Delta}_1\boldsymbol{B}_1^\top\vec{u}_2(\vartheta_2)\right)$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing phase reduction theory, the structure of the Dirac-Bianconi oscillator on simplicial complexes, and the coupling perturbations involving boundary operators and internal parameters, demanding a multi-step derivation of the phase coupling function.

Refined Problem

Refined Problem Statement: Background: Consider a system called a Dirac-Bianconi driven oscillator, defined on a simplicial complex consisting of N_n nodes (0-simplices) and N_l links (1-simplices). The state of the system is described by a topological spinor $\vec{w} = (\vec{u}^\top, \vec{v}^\top)^\top$, where $\vec{u} \in \mathbb{R}^{N_n}$ is a 0-cochain (variables on nodes) and $\vec{v} \in \mathbb{R}^{N_l}$ is a 1-cochain (variables on links). The dynamics are given by $\dot{\vec{w}} = \vec{F}(\vec{w})$. Assume this system possesses a stable limit cycle solution $\vec{w}_c(t)$ with frequency ω. Using phase reduction theory, the state on the limit cycle can be described by a single phase variable $\vartheta(t)$ such that $\dot{\vartheta} = \omega$ and $\vec{w}_c(t) = \vec{w}_c(\vartheta(t))$. The response of the phase to a small perturbation $\epsilon\vec{P}$ is governed by the phase sensitivity function (PSF) $\vec{Z}(\vartheta) = \nabla_{\vec{w}}\vartheta\big|_{\vec{w}_c(\vartheta)}$, which can be partitioned according to the node and link variables as $\vec{Z}(\vartheta) = (\vec{Z}_u(\vartheta)^\top, \vec{Z}_v(\vartheta)^\top)^\top$.

Now, consider two such oscillators, indexed by $k = 1, 2$, with state vectors $\vec{w}_k = (\vec{u}_k^\top, \vec{v}_k^\top)^\top$ and natural frequencies ω_k. They are weakly coupled with strength $\epsilon \ll 1$. The dynamics of oscillator 1 are given by $\dot{\vec{w}}_1 = \vec{F}_1(\vec{w}_1) + \epsilon\vec{P}_1(\vec{w}_1, \vec{w}_2)$. To first order in ϵ, its phase dynamics are $\dot{\vartheta}_1 = \omega_1 + \epsilon\Gamma_1(\vartheta_1, \vartheta_2)$, where the phase coupling function is $\Gamma_1(\vartheta_1, \vartheta_2) = \vec{Z}_1(\vartheta_1) \cdot \vec{P}_1(\vec{w}_1(\vartheta_1), \vec{w}_2(\vartheta_2))$.

The coupling perturbation $\vec{P}_1$ has components acting on the node variables, $\vec{p}_{1,u}$, and link variables, $\vec{p}_{1,v}$. The coupling is a combination of a diffusive term and a Dirac-Bianconi term. 1. The diffusive coupling acts only between corresponding nodes: $\vec{p}_{1,u}^{\,\text{diff}} = C_u(\vec{u}_2 - \vec{u}_1)$, where C_u is an $N_n \times N_n$ matrix specifying which nodes are coupled. 2. The Dirac-Bianconi coupling is analogous to the internal coupling structure. It couples the link variables of oscillator 2 to the node variables of oscillator 1, and the node variables of oscillator 2 to the link variables of oscillator 1. This is mediated by the $N_n \times N_l$ boundary operator $\boldsymbol{B}_1$. The specific forms are: - Perturbation on $\vec{u}_1$: $\vec{p}_{1,u}^{\,\text{DB}} = -\boldsymbol{B}_1\vec{v}_2$ - Perturbation on $\vec{v}_1$: $\vec{p}_{1,v}^{\,\text{DB}} = \boldsymbol{\Delta}_1\boldsymbol{B}_1^\top\vec{u}_2$, where $\boldsymbol{\Delta}_1$ is a diagonal $N_l \times N_l$ matrix of internal system parameters $\delta_{1,j}$. The total perturbation on oscillator 1 is $\vec{P}_1 = (\vec{p}_{1,u}^{\,\text{diff}} + \vec{p}_{1,u}^{\,\text{DB}}, \vec{p}_{1,v}^{\,\text{DB}})^\top$.

Task: Derive the phase coupling function $\Gamma_1(\vartheta_1, \vartheta_2)$ for oscillator 1. Express your answer in terms of the partitioned PSFs $\vec{Z}_{u,1}(\vartheta_1)$ and $\vec{Z}_{v,1}(\vartheta_1)$, the limit cycle trajectories $\vec{u}_k(\vartheta_k)$ and $\vec{v}_k(\vartheta_k)$, the boundary operator $\boldsymbol{B}_1$, the coupling matrix C_u, and the parameter matrix $\boldsymbol{\Delta}_1$.

Refined Solution:

$$\Gamma_1(\vartheta_1, \vartheta_2) = \vec{Z}_{u,1}(\vartheta_1)^\top \left[C_u(\vec{u}_2(\vartheta_2) - \vec{u}_1(\vartheta_1)) - \boldsymbol{B}_1\vec{v}_2(\vartheta_2)\right] + \vec{Z}_{v,1}(\vartheta_1)^\top \left(\boldsymbol{\Delta}_1\boldsymbol{B}_1^\top\vec{u}_2(\vartheta_2)\right)$$

Problem 5 (Paper: 2506.20572v1, Index: 0)

Original Problem Statement

Background: Consider a network of channels represented by an undirected graph $\mathcal{G} = (\mathcal{V}, E)$, where $\mathcal{V}$ is the set of nodes and E is the set of edges. An incompressible fluid flows through this network. For each edge $(i, j) \in E$ connecting nodes i and j, we define its length L_{ij}, the fluid flux through it Q_{ij}, and its conductivity D_{ij}. The total power dissipated by the viscous flow in the network is given by $\mathcal{P} = \sum_{(i,j)\in E} \frac{Q_{ij}^2 L_{ij}}{D_{ij}}$. The channels are elastic, and the volume of a single channel (i, j) is related to its conductivity by $V_{ij} = \beta L_{ij}\sqrt{D_{ij}}$, where β is a constant. The total volume of fluid in the network, $V = \sum_{(i,j)\in E} V_{ij}$, is conserved. The conductivities of the channels adapt over time according to the differential equation:

$$\frac{d}{dt}\sqrt{D_{ij}} = \alpha \frac{g(Q_{ij})}{\sum_{(k,m)\in E} L_{km}\, g(Q_{km})} - \sqrt{D_{ij}}$$

where α is a constant related to the total volume, and $g(Q_{ij})$ is a function that describes the channel's elastic response to the local fluid flux. The functional form of $g(Q_{ij})$ is determined by the principle of energy minimization: at steady state ($d/dt = 0$), the network's configuration of conductivities $\{D_{ij}\}$ is such that it minimizes the total power dissipation $\mathcal{P}$ for a given, fixed set of fluxes $\{Q_{ij}\}$ and a constant total network volume V.

Task: Using the principle of minimum power dissipation under the constraint of constant total volume, derive the functional form of $g(Q_{ij})$. Provide the simplest possible expression for $g(Q_{ij})$ in terms of Q_{ij}.

Original Solution

$$g(Q_{ij}) = Q_{ij}^{2/3}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires applying variational calculus to minimize power dissipation under a nonlinear constraint, synthesizing multiple concepts including fluid dynamics, network theory, and elasticity, making it a sophisticated multi-step derivation rather than a direct formula recall.

Refined Problem

Refined Problem Statement: Background: Consider a network of channels represented by an undirected graph $\mathcal{G} = (\mathcal{V}, E)$, where $\mathcal{V}$ is the set of nodes and E is the set of edges. An incompressible fluid flows through this network. For each edge $(i, j) \in E$ connecting nodes i and j, we define its length L_{ij}, the fluid flux through it Q_{ij}, and its conductivity D_{ij}. The total power dissipated by the viscous flow in the network is given by $\mathcal{P} = \sum_{(i,j)\in E} \frac{Q_{ij}^2 L_{ij}}{D_{ij}}$. The channels are elastic, and the volume of a single channel (i, j) is related to its conductivity by $V_{ij} = \beta L_{ij}\sqrt{D_{ij}}$, where β is a constant. The total volume of fluid in the network, $V = \sum_{(i,j)\in E} V_{ij}$, is conserved. The conductivities of the channels adapt over time according to the differential equation:

$$\frac{d}{dt}\sqrt{D_{ij}} = \alpha \frac{g(Q_{ij})}{\sum_{(k,m)\in E} L_{km}\, g(Q_{km})} - \sqrt{D_{ij}}$$

where α is a constant related to the total volume, and $g(Q_{ij})$ is a function that describes the channel's elastic response to the local fluid flux. The functional form of $g(Q_{ij})$ is determined by the principle of energy minimization: at steady state ($d/dt = 0$), the network's configuration of conductivities $\{D_{ij}\}$ is such that it minimizes the total power dissipation $\mathcal{P}$ for a given, fixed set of fluxes $\{Q_{ij}\}$ and a constant total network volume V.

Task: Using the principle of minimum power dissipation under the constraint of constant total volume, derive the functional form of $g(Q_{ij})$. Provide the simplest possible expression for $g(Q_{ij})$ in terms of Q_{ij}.

Refined Solution:

$$g(Q_{ij}) = Q_{ij}^{2/3}$$

Problem 6 (Paper: 2506.20572v1, Index: 1)

Original Problem Statement

Background: Consider a network of channels, represented by a graph with edges (i, j), through which an incompressible fluid of dynamic viscosity η flows. The flow in each channel is governed by the Hagen-Poiseuille law. Each channel (i, j) has a fixed length L_{ij} and a variable radius r_{ij}. The conductivity of the channel is given by $D_{ij} = \pi r_{ij}^4/(8\eta)$, and the fluid flux Q_{ij} is related to the pressure difference $(p_i - p_j)$ by $Q_{ij} = D_{ij}(p_i - p_j)/L_{ij}$. The volume of a channel is $V_{ij} = \pi r_{ij}^2 L_{ij}$. The total volume of the network, $V = \sum_{(i,j)} V_{ij}$, is conserved.

The conductivities of the channels adapt over time according to the differential equation:

$$\frac{d}{dt}\sqrt{D_{ij}} = \mathcal{K} \cdot g(Q_{ij}) - \sqrt{D_{ij}}$$

where $\mathcal{K}$ is a normalization factor that is treated as constant for a given state of the network, and $g(\cdot)$ is a function that models the response of the channel walls to the flow.

The functional form of $g(Q_{ij})$ is determined by a physical optimization principle: for a given, fixed distribution of fluxes $\{Q_{ij}\}$ throughout the network, the conductivities $\{D_{ij}\}$ arrange themselves to minimize the total power dissipated by viscosity, $\mathcal{P} = \sum_{(i,j)} \frac{Q_{ij}^2 L_{ij}}{D_{ij}}$, subject to the constraint that the total network volume V is constant. The steady-state solution of the adaptation equation ($\frac{d}{dt}\sqrt{D_{ij}} = 0$) implies that $\sqrt{D_{ij}}$ is directly proportional to $g(Q_{ij})$.

Task: Using the principle of minimum power dissipation under the constraint of constant total volume, determine the functional form of $g(Q_{ij})$. Express your result in terms of the flux Q_{ij}, ignoring any arbitrary proportionality constants.

Original Solution

$$g(Q_{ij}) = Q_{ij}^{2/3}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires applying variational calculus to minimize power dissipation under a volume constraint, linking flow, conductivity, and geometry, thus demanding a multi-step derivation and synthesis of physical principles.

Refined Problem

Refined Problem Statement: Background: Consider a network of channels, represented by a graph with edges (i, j), through which an incompressible fluid of dynamic viscosity η flows. The flow in each channel is governed by the Hagen-Poiseuille law. Each channel (i, j) has a fixed length L_{ij} and a variable radius r_{ij}. The conductivity of the channel is given by $D_{ij} = \pi r_{ij}^4/(8\eta)$, and the fluid flux Q_{ij} is related to the pressure difference $(p_i - p_j)$ by $Q_{ij} = D_{ij}(p_i - p_j)/L_{ij}$. The volume of a channel is $V_{ij} = \pi r_{ij}^2 L_{ij}$. The total volume of the network, $V = \sum_{(i,j)} V_{ij}$, is conserved.

The conductivities of the channels adapt over time according to the differential equation:

$$\frac{d}{dt}\sqrt{D_{ij}} = \mathcal{K} \cdot g(Q_{ij}) - \sqrt{D_{ij}}$$

where $\mathcal{K}$ is a normalization factor that is treated as constant for a given state of the network, and $g(\cdot)$ is a function that models the response of the channel walls to the flow.

The functional form of $g(Q_{ij})$ is determined by a physical optimization principle: for a given, fixed distribution of fluxes $\{Q_{ij}\}$ throughout the network, the conductivities $\{D_{ij}\}$ arrange themselves to minimize the total power dissipated by viscosity, $\mathcal{P} = \sum_{(i,j)} \frac{Q_{ij}^2 L_{ij}}{D_{ij}}$, subject to the constraint that the total network volume V is constant. The steady-state solution of the adaptation equation ($\frac{d}{dt}\sqrt{D_{ij}} = 0$) implies that $\sqrt{D_{ij}}$ is directly proportional to $g(Q_{ij})$.

Task: Using the principle of minimum power dissipation under the constraint of constant total volume, determine the functional form of $g(Q_{ij})$. Express your result in terms of the flux Q_{ij}, ignoring any arbitrary proportionality constants.

Refined Solution:

$$g(Q_{ij}) = Q_{ij}^{2/3}$$

Problem 7 (Paper: 2506.21226v1, Index: 0)

Original Problem Statement

Background: Consider a network automaton model defined on a graph with N nodes. Each node v_i can be in one of two states, $s_i \in \{0, 1\}$. The state of each node is updated synchronously at discrete time steps. The update rule ϕ for a node v_i is uniform across the network and is of an outer-totalistic nature. It depends on the node's own state s_i at time t and the density of active neighbors $\rho_i^t = q/k_i$, where k_i is the degree of node v_i and $q = \sum_j A_{ij} s_j^t$ is the number of its neighbors in state 1. The new state is given by $s_i^{t+1} = \phi(s_i^t, \rho_i^t)$.

We analyze the propagation of a small perturbation through the network. Consider two state configurations of the network at time t, denoted by $\{s_i^t\}$ and $\{s_i'^t\}$. The defect at node v_i is defined as $d_i^t = s_i^t \oplus s_i'^t$, where $\oplus$ is the XOR operation. The global state of the system is characterized by two quantities: the average state density of the unperturbed configuration, $\rho^t = \frac{1}{N}\sum_i s_i^t$, and the normalized Hamming distance (or defect density) between the two configurations, $\delta^t = \frac{1}{N}\sum_i d_i^t$.

We adopt a mean-field approximation, ignoring local topological correlations. Under this assumption, the state of any node and its neighbors can be described by statistical distributions dependent only on the global quantities ρ^t and δ^t. For a single node with degree k, we define the following quantities and their probabilities: - $s \in \{0, 1\}$: the state of the central node in the unperturbed configuration. Its probability is $P(s|\rho^t) = (\rho^t)^s(1-\rho^t)^{1-s}$. - $q \in \{0, \dots, k\}$: the number of neighbors in state 1. Its probability is $P(q|k, \rho^t) = \binom{k}{q}(\rho^t)^q(1-\rho^t)^{k-q}$. - $c \in \{0, 1\}$: the defect state of the central node ($c = 1$ if $s \neq s'$). Its probability is $P(c|\delta^t) = (\delta^t)^c(1-\delta^t)^{1-c}$. - $d \in \{0, \dots, k\}$: the number of neighbors with a defect. Its probability is $P(d|k, \delta^t) = \binom{k}{d}(\delta^t)^d(1-\delta^t)^{k-d}$. - $\tau \in \{\max(0, d+q-k), \dots, \min(d, q)\}$: the number of defective neighbors that flip from state 1 to 0 (a "killer toggle"). Its probability, conditioned on k, d, q, is given by the hypergeometric distribution $P(\tau|k, d, q) = \binom{q}{\tau}\binom{k-q}{d-\tau}/\binom{k}{d}$.

The defect at the central node at time $t+1$, denoted $\mathcal{D}$, depends on these variables. The state of the perturbed node is $s' = s \oplus c$. The number of its active neighbors in the perturbed configuration is $q' = q - \tau + (d - \tau) = q - 2\tau + d$. The defect is thus $\mathcal{D}(k, s, q, c, d, \tau) = \phi(s, q/k) \oplus \phi(s \oplus c, (q - 2\tau + d)/k)$. The overall expected defect density at time $t+1$, $\langle\delta^{t+1}\rangle$, is found by averaging the single-node expected defect over the network's degree distribution, $P(k)$.

Task: Derive the analytical expression for the expected normalized Hamming distance at time $t+1$, $\langle\delta^{t+1}\rangle$, as a function of the normalized Hamming distance δ^t and the average state density ρ^t at time t. Your expression should be an average over the degree distribution $P(k)$ and should involve sums over all relevant probabilistic variables (s, q, c, d, τ).

Original Solution

$$\langle\delta^{t+1}\rangle = \sum_k P(k) \sum_{s\in\{0,1\}} P(s|\rho^t) \sum_{q=0}^{k} P(q|k, \rho^t) \sum_{c\in\{0,1\}} P(c|\delta^t) \sum_{d=0}^{k} P(d|k, \delta^t) \sum_{\tau} P(\tau|k, d, q)\, \mathcal{D}(k, s, q, c, d, \tau)$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing multiple probabilistic distributions, combinatorial reasoning, and the nonlinear update rule to derive a complex averaged expression for defect propagation, going well beyond a simple formula lookup or direct application of a known result.

Refined Problem

Refined Problem Statement: Background: Consider a network automaton model defined on a graph with N nodes. Each node v_i can be in one of two states, $s_i \in \{0, 1\}$. The state of each node is updated synchronously at discrete time steps. The update rule ϕ for a node v_i is uniform across the network and is of an outer-totalistic nature. It depends on the node's own state s_i at time t and the density of active neighbors $\rho_i^t = q/k_i$, where k_i is the degree of node v_i and $q = \sum_j A_{ij} s_j^t$ is the number of its neighbors in state 1. The new state is given by $s_i^{t+1} = \phi(s_i^t, \rho_i^t)$.

We analyze the propagation of a small perturbation through the network. Consider two state configurations of the network at time t, denoted by $\{s_i^t\}$ and $\{s_i'^t\}$. The defect at node v_i is defined as

$d_i^t = s_i^t \oplus s_i'^t$, where $\oplus$ is the XOR operation. The global state of the system is characterized by two quantities: the average state density of the unperturbed configuration, $\rho^t = \frac{1}{N}\sum_i s_i^t$, and the normalized Hamming distance (or defect density) between the two configurations, $\delta^t = \frac{1}{N}\sum_i d_i^t$.

We adopt a mean-field approximation, ignoring local topological correlations. Under this assumption, the state of any node and its neighbors can be described by statistical distributions dependent only on the global quantities ρ^t and δ^t. For a single node with degree k, we define the following quantities and their probabilities: - $s \in \{0,1\}$: the state of the central node in the unperturbed configuration. Its probability is $P(s|\rho^t) = (\rho^t)^s(1-\rho^t)^{1-s}$. - $q \in \{0,\dots,k\}$: the number of neighbors in state 1. Its probability is $P(q|k,\rho^t) = \binom{k}{q}(\rho^t)^q(1-\rho^t)^{k-q}$. - $c \in \{0,1\}$: the defect state of the central node ($c=1$ if $s \neq s'$). Its probability is $P(c|\delta^t) = (\delta^t)^c(1-\delta^t)^{1-c}$. - $d \in \{0,\dots,k\}$: the number of neighbors with a defect. Its probability is $P(d|k,\delta^t) = \binom{k}{d}(\delta^t)^d(1-\delta^t)^{k-d}$. - $\tau \in \{\max(0,d+q-k),\dots,\min(d,q)\}$: the number of defective neighbors that flip from state 1 to 0 (a ̈killer toggle ̈). Its probability, conditioned on k,d,q, is given by the hypergeometric distribution $P(\tau|k,d,q) = \binom{q}{\tau}\binom{k-q}{d-\tau}/\binom{k}{d}$.

The defect at the central node at time $t+1$, denoted $\mathcal{D}$, depends on these variables. The state of the perturbed node is $s' = s \oplus c$. The number of its active neighbors in the perturbed configuration is $q' = q-\tau+(d-\tau) = q-2\tau+d$. The defect is thus $\mathcal{D}(k,s,q,c,d,\tau) = \phi(s,q/k) \oplus \phi(s\oplus c,(q-2\tau+d)/k)$. The overall expected defect density at time $t+1$, $\langle\delta^{t+1}\rangle$, is found by averaging the single-node expected defect over the network's degree distribution, $P(k)$.

Task: Derive the analytical expression for the expected normalized Hamming distance at time $t+1$, $\langle\delta^{t+1}\rangle$, as a function of the normalized Hamming distance δ^t and the average state density ρ^t at time t. Your expression should be an average over the degree distribution $P(k)$ and should involve sums over all relevant probabilistic variables (s,q,c,d,τ).

Solution:

Refined Solution:

$$\langle\delta^{t+1}\rangle = \sum_k P(k) \sum_{s\in\{0,1\}} P(s|\rho^t) \sum_{q=0}^{k} P(q|k,\rho^t) \sum_{c\in\{0,1\}} P(c|\delta^t) \sum_{d=0}^{k} P(d|k,\delta^t) \sum_{\tau} P(\tau|k,d,q)\, \mathcal{D}(k,s,q,c,d,\tau)$$

Problem 8 (Paper: 2506.21226v1, Index: 1)

Original Problem Statement

Background: Consider a network automaton model where each node v_i in a network of N nodes can be in one of two states, $s_i \in \{0, 1\}$. The state of all nodes is updated synchronously at discrete time steps. The update rule for any node v_i is a uniform outer-totalistic function $\phi(s_i, \rho_i)$, which depends on the node's own state s_i and the density of its active neighbors, $\rho_i = \frac{1}{k_i} \sum_j A_{ij} s_j$, where k_i is the degree of node v_i and A_{ij} are the elements of the adjacency matrix.

We wish to study the propagation of a small perturbation through the network. Consider two configurations of the system at time t, $\{s_i(t)\}$ and $\{s'_i(t)\}$, which are nearly identical. The bitwise difference between these configurations is the defect configuration $\{d_i(t)\}$, where $d_i(t) = s_i(t) \oplus s'_i(t)$ and $\oplus$ is the XOR operator. The normalized Hamming distance between the two configurations is $\delta^t = \frac{1}{N} \sum_{i=1}^{N} d_i(t)$. We are interested in the expected normalized Hamming distance at the next time step, $\langle \delta^{t+1} \rangle$.

To derive an analytical expression for this quantity, we employ a mean-field approximation, assuming a large, random network where local topological correlations are negligible. The overall system is characterized by the global state average $\rho^t = \frac{1}{N} \sum_i s_i(t)$ and the normalized Hamming distance δ^t. The probability of any given node being in state 1 is $P(s = 1|\rho^t) = \rho^t$, and the probability of it having a defect is $P(c = 1|\delta^t) = \delta^t$, where c is the defect state of the central node. For a node of degree k, the number of its neighbors in state 1, denoted by q, follows a binomial distribution $P(q|k, \rho^t)$. Similarly, the number of its neighbors with a defect, denoted by d, follows a binomial distribution $P(d|k, \delta^t)$. Among the d defects in the neighborhood, let τ be the number of "killer toggles" (defects that change a neighbor's state from 1 to 0). The probability distribution for τ, given k, d, q, is hypergeometric. The defect at a node v_i at time $t + 1$ is given by $d_i^{t+1} = \phi(s_i(t), \rho_i(t)) \oplus \phi(s'_i(t), \rho'_i(t))$.

Task: Under the mean-field approximation, derive the full analytical expression for the expected normalized Hamming distance $\langle \delta^{t+1} \rangle$. Your final expression should be an average over the degree distribution $P(k)$ and should involve summations over all possible states and defect configurations of a representative node and its neighborhood (s, q, c, d, τ). The probabilities for these configurations are given by $P(s|\rho^t) = (\rho^t)^s(1 - \rho^t)^{1-s}$, $P(q|k, \rho^t) = \binom{k}{q}(\rho^t)^q(1 - \rho^t)^{k-q}$, $P(c|\delta^t) = (\delta^t)^c(1 - \delta^t)^{1-c}$, $P(d|k, \delta^t) = \binom{k}{d}(\delta^t)^d(1 - \delta^t)^{k-d}$, and $P(\tau|k, d, q) = \binom{q}{\tau}\binom{k-q}{d-\tau}/\binom{k}{d}$.

Original Solution

$$\langle \delta^{t+1} \rangle = \sum_k P(k) \sum_{s \in \{0,1\}} P(s|\rho^t) \sum_{q=0}^{k} P(q|k, \rho^t) \sum_{c \in \{0,1\}} P(c|\delta^t) \sum_{d=0}^{k} P(d|k, \delta^t) \sum_{\tau} P(\tau|k, d, q) \left(\phi\left(s, \frac{q}{k}\right) \oplus \phi\left(s \oplus c, \frac{q - 2\tau + d}{k}\right) \right)$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing multiple probability distributions, combinatorial reasoning, and the nonlinear update function to derive a complex mean-field expression for the expected Hamming distance, going well beyond a simple formula lookup or direct application.

Refined Problem

Refined Problem Statement: Background: Consider a network automaton model where each node v_i in a network of N nodes can be in one of two states, $s_i \in \{0, 1\}$. The state of all nodes is updated synchronously at discrete time steps. The update rule for any node v_i is a uniform outer-totalistic function $\phi(s_i, \rho_i)$, which depends on the node's own state s_i and the density of its active neighbors, $\rho_i = \frac{1}{k_i} \sum_j A_{ij} s_j$, where k_i is the degree of node v_i and A_{ij} are the elements of the adjacency matrix.

We wish to study the propagation of a small perturbation through the network. Consider two configurations of the system at time t, $\{s_i(t)\}$ and $\{s'_i(t)\}$, which are nearly identical. The bitwise difference between these configurations is the defect configuration $\{d_i(t)\}$, where $d_i(t) = s_i(t) \oplus s'_i(t)$ and $\oplus$ is the XOR operator. The normalized Hamming distance between the two configurations is $\delta^t = \frac{1}{N} \sum_{i=1}^{N} d_i(t)$. We are interested in the expected normalized Hamming distance at the next time step, $\langle \delta^{t+1} \rangle$.

To derive an analytical expression for this quantity, we employ a mean-field approximation, assuming a large, random network where local topological correlations are negligible. The overall system is characterized by the global state average $\rho^t = \frac{1}{N}\sum_i s_i(t)$ and the normalized Hamming distance δ^t. The probability of any given node being in state 1 is $P(s = 1|\rho^t) = \rho^t$, and the probability of it having a defect is $P(c = 1|\delta^t) = \delta^t$, where c is the defect state of the central node. For a node of degree k, the number of its neighbors in state 1, denoted by q, follows a binomial distribution $P(q|k, \rho^t)$. Similarly, the number of its neighbors with a defect, denoted by d, follows a binomial distribution $P(d|k, \delta^t)$. Among the d defects in the neighborhood, let τ be the number of "killer toggles" (defects that change a neighbor's state from 1 to 0). The probability distribution for τ, given k, d, q, is hypergeometric. The defect at a node v_i at time $t + 1$ is given by $d_i^{t+1} = \phi(s_i(t), \rho_i(t)) \oplus \phi(s_i'(t), \rho_i'(t))$.

Task: Under the mean-field approximation, derive the full analytical expression for the expected normalized Hamming distance $\langle\delta^{t+1}\rangle$. Your final expression should be an average over the degree distribution $P(k)$ and should involve summations over all possible states and defect configurations of a representative node and its neighborhood (s, q, c, d, τ). The probabilities for these configurations are given by $P(s|\rho^t) = (\rho^t)^s(1 - \rho^t)^{1-s}$, $P(q|k, \rho^t) = \binom{k}{q}(\rho^t)^q(1 - \rho^t)^{k-q}$, $P(c|\delta^t) = (\delta^t)^c(1 - \delta^t)^{1-c}$, $P(d|k, \delta^t) = \binom{k}{d}(\delta^t)^d(1 - \delta^t)^{k-d}$, and $P(\tau|k, d, q) = \binom{q}{\tau}\binom{k-q}{d-\tau}/\binom{k}{d}$.

Refined Solution:

$$\langle\delta^{t+1}\rangle = \sum_k P(k) \sum_{s\in\{0,1\}} P(s|\rho^t) \sum_{q=0}^{k} P(q|k, \rho^t) \sum_{c\in\{0,1\}} P(c|\delta^t) \sum_{d=0}^{k} P(d|k, \delta^t) \sum_{\tau} P(\tau|k, d, q) \left(\phi\left(s, \frac{q}{k}\right) \oplus \phi\left(s \oplus c, \frac{q - 2\tau + d}{k}\right)\right.$$

Problem 9 (Paper: 2506.21498v1, Index: 0)

Original Problem Statement

Background: Consider a strategic interaction between two players, Player 1 and Player 2. Each player can choose one of two strategies, **C** or **D**. The payoffs are symmetric and given by the matrix below, where the entries represent the payoff for Player 1 (the row player).

	C	**D**
C	1	$-B$
D	$1-A$	0

The players adapt their strategies over time through a learning process known as introspection dynamics. In each time step, one of the two players is chosen at random with equal probability to revise their strategy. The chosen player, say player i, compares their current payoff, π, with the payoff they would have received, π', by unilaterally switching to the other strategy (while the co-player's strategy remains fixed). Player i then switches to the alternative strategy with a probability given by the Fermi function:

$$\phi_i = \frac{1}{1+\mathrm{e}^{-\beta_i\,(\pi'-\pi)}}$$

The parameter $\beta_i \geq 0$ is the selection strength of player i. This process defines a continuous-time Markov chain on the four possible strategy profiles: **CC**, **CD**, **DC**, and **DD**. The system eventually settles into a unique stationary distribution, $\mathbf{u} = (u_{\mathbf{CC}}, u_{\mathbf{CD}}, u_{\mathbf{DC}}, u_{\mathbf{DD}})$, which gives the long-term average frequencies of observing each profile. Consider the specific case where both players have the same selection strength, $\beta_1 = \beta_2 = \beta$.

Task: Derive the stationary distribution vector $\mathbf{u}(A, B, \beta)$ for this symmetric case. Express your final answer as a single vector with a common normalization factor.

Original Solution

$$\frac{1}{2+e^{A\beta}+e^{B\beta}}(e^{A\beta}, 1, 1, e^{B\beta})$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires deriving the stationary distribution of a Markov chain defined by a non-trivial stochastic process involving Fermi update dynamics, necessitating synthesis of game theory, stochastic processes, and nonlinear probability functions.

Refined Problem

Refined Problem Statement: Background: Consider a strategic interaction between two players, Player 1 and Player 2. Each player can choose one of two strategies, **C** or **D**. The payoffs are symmetric and given by the matrix below, where the entries represent the payoff for Player 1 (the row player).

	C	**D**
C	1	$-B$
D	$1-A$	0

The players adapt their strategies over time through a learning process known as introspection dynamics. In each time step, one of the two players is chosen at random with equal probability to revise their strategy. The chosen player, say player i, compares their current payoff, π, with the payoff they would have received, π', by unilaterally switching to the other strategy (while the co-player's strategy remains fixed). Player i then switches to the alternative strategy with a probability given by the Fermi function:

$$\phi_i = \frac{1}{1+\mathrm{e}^{-\beta_i\,(\pi'-\pi)}}$$

The parameter $\beta_i \geq 0$ is the selection strength of player i. This process defines a continuous-time Markov chain on the four possible strategy profiles: **CC**, **CD**, **DC**, and **DD**. The system eventually settles into a unique stationary distribution, $\mathbf{u} = (u_{\mathbf{CC}}, u_{\mathbf{CD}}, u_{\mathbf{DC}}, u_{\mathbf{DD}})$, which gives the long-term average frequencies of observing each profile. Consider the specific case where both players have the same selection strength, $\beta_1 = \beta_2 = \beta$.

Task: Derive the stationary distribution vector $\mathbf{u}(A, B, \beta)$ for this symmetric case. Express your final answer as a single vector with a common normalization factor.

Solution:

Refined Solution:

$$\frac{1}{2 + e^{A\beta} + e^{B\beta}}(e^{A\beta}, 1, 1, e^{B\beta})$$

Problem 10 (Paper: 2506.21498v1, Index: 1)

Original Problem Statement

Background: Consider a symmetric 2×2 game between two players, where each can choose between two strategies, **C** and **D**. The payoff for player 1 is given by the matrix:

	C	**D**
C	1	$-B$
D	$1-A$	0

Players adapt their strategies over time through a process of introspection. At each step, a player compares their current payoff π with the payoff π' they would have received by choosing the other strategy. The player switches to the alternative strategy with a probability given by the Fermi function $\phi = (1 + e^{-\beta(\pi'-\pi)})^{-1}$, where $\beta \geq 0$ is the player's selection strength. This learning process can be modeled as a Markov chain on the four outcome states $\{\mathbf{CC}, \mathbf{CD}, \mathbf{DC}, \mathbf{DD}\}$. For a game between player 1 with selection strength β_1 and player 2 with β_2, this process leads to a unique stationary distribution $\mathbf{u}(\beta_1, \beta_2) = (u_{\mathbf{CC}}, u_{\mathbf{CD}}, u_{\mathbf{DC}}, u_{\mathbf{DD}})$. The components of this distribution are given by:

$$
\begin{aligned}
u_{\mathbf{CC}} &= N^{-1}(1 + e^{-B\beta_1})(1 + e^{-B\beta_2}) \\
u_{\mathbf{CD}} &= N^{-1}(1 + e^{-B\beta_1})(1 + e^{A\beta_2}) \\
u_{\mathbf{DC}} &= N^{-1}(1 + e^{A\beta_1})(1 + e^{-B\beta_2}) \\
u_{\mathbf{DD}} &= N^{-1}(1 + e^{A\beta_1})(1 + e^{A\beta_2})
\end{aligned}
$$

where N is a normalization factor given by $N(\beta_1, \beta_2) = (1 + e^{A\beta_1} + e^{-B\beta_1})(1 + e^{A\beta_2} + e^{-B\beta_2}) + (e^{A\beta_1} - e^{-B\beta_1})(e^{A\beta_2} - e^{-B\beta_2})$. Player 1's average payoff is $\Pi_{\beta_2}(\beta_1) = 1 \cdot u_{\mathbf{CC}} - B \cdot u_{\mathbf{CD}} + (1 - A) \cdot u_{\mathbf{DC}}$.

Now, consider the long-term evolution of the selection strength β itself using the framework of adaptive dynamics. We assume a large, well-mixed population of resident individuals, all with selection strength x. A rare mutant with selection strength y appears. The mutant's ability to invade is determined by its invasion fitness, $s_x(y) = \Pi_x(y) - \Pi_x(x)$. The direction of evolution is determined by the selection gradient, $D(x) = \left[\frac{\partial s_x(y)}{\partial y}\right]_{y=x}$.

Task: Derive the full analytical expression for the selection gradient $D(x)$. Your final expression should be simplified as much as possible.

Original Solution

$$
D(x) = \frac{e^{(A-B)x}\left[(A+B)(1+e^{Ax})(1+e^{-Bx}) - (A-B)(1+e^{Ax}+e^{-Bx})\right]}{\left((1+e^{Ax}+e^{-Bx})^2 + (e^{Ax}-e^{-Bx})^2\right)^2}
$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires deriving the selection gradient by synthesizing the stationary distribution of a Markov chain, payoff calculations, and adaptive dynamics, involving multiple non-trivial steps and algebraic manipulations.

Refined Problem

Refined Problem Statement: Background: Consider a symmetric 2×2 game between two players, where each can choose between two strategies, **C** and **D**. The payoff for player 1 is given by the matrix:

	C	**D**
C	1	$-B$
D	$1-A$	0

Players adapt their strategies over time through a process of introspection. At each step, a player compares their current payoff π with the payoff π' they would have received by choosing the other

strategy. The player switches to the alternative strategy with a probability given by the Fermi function $\phi = (1 + e^{-\beta(\pi' - \pi)})^{-1}$, where $\beta \geq 0$ is the player's selection strength. This learning process can be modeled as a Markov chain on the four outcome states $\{\mathbf{CC}, \mathbf{CD}, \mathbf{DC}, \mathbf{DD}\}$. For a game between player 1 with selection strength β_1 and player 2 with β_2, this process leads to a unique stationary distribution $\mathbf{u}(\beta_1, \beta_2) = (u_{\mathbf{CC}}, u_{\mathbf{CD}}, u_{\mathbf{DC}}, u_{\mathbf{DD}})$. The components of this distribution are given by:

$$
\begin{aligned}
u_{\mathbf{CC}} &= N^{-1}(1 + e^{-B\beta_1})(1 + e^{-B\beta_2}) \\
u_{\mathbf{CD}} &= N^{-1}(1 + e^{-B\beta_1})(1 + e^{A\beta_2}) \\
u_{\mathbf{DC}} &= N^{-1}(1 + e^{A\beta_1})(1 + e^{-B\beta_2}) \\
u_{\mathbf{DD}} &= N^{-1}(1 + e^{A\beta_1})(1 + e^{A\beta_2})
\end{aligned}
$$

where N is a normalization factor given by $N(\beta_1, \beta_2) = (1 + e^{A\beta_1} + e^{-B\beta_1})(1 + e^{A\beta_2} + e^{-B\beta_2}) + (e^{A\beta_1} - e^{-B\beta_1})(e^{A\beta_2} - e^{-B\beta_2})$. Player 1's average payoff is $\Pi_{\beta_2}(\beta_1) = 1 \cdot u_{\mathbf{CC}} - B \cdot u_{\mathbf{CD}} + (1 - A) \cdot u_{\mathbf{DC}}$.

Now, consider the long-term evolution of the selection strength β itself using the framework of adaptive dynamics. We assume a large, well-mixed population of resident individuals, all with selection strength x. A rare mutant with selection strength y appears. The mutant's ability to invade is determined by its invasion fitness, $s_x(y) = \Pi_x(y) - \Pi_x(x)$. The direction of evolution is determined by the selection gradient, $D(x) = \left[\frac{\partial s_x(y)}{\partial y}\right]_{y=x}$.

Task: Derive the full analytical expression for the selection gradient $D(x)$. Your final expression should be simplified as much as possible.

Refined Solution:

$$
D(x) = \frac{e^{(A-B)x}\left[(A+B)(1+e^{Ax})(1+e^{-Bx}) - (A-B)(1+e^{Ax}+e^{-Bx})\right]}{\left((1+e^{Ax}+e^{-Bx})^2 + (e^{Ax}-e^{-Bx})^2\right)^2}
$$

Problem 11 (Paper: 2506.22120v1, Index: 0)

Original Problem Statement

Background: Consider a system of settlements modeled as a directed network $G = (V, E)$, where V is the set of $|V|$ settlements (nodes) and E is the set of connections (edges). Displacements in this system are driven by a set of distinct hazards, $\mathcal{H}$. For each settlement $v \in V$ and each hazard $h \in \mathcal{H}$, the quantity $D_h(v)$ represents the number of individuals displaced from v due to that specific hazard. The *dominant hazard* at a settlement v, denoted $h^*(v)$, is defined as the hazard type responsible for the largest number of displacements from that settlement, i.e., $h^*(v) = \arg\max_{h \in \mathcal{H}} D_h(v)$.

The movement of displaced populations is modeled via a large number of simulated stochastic paths. For each ordered pair of source-target settlements $(s, t) \in V \times V$, a set of M distinct paths is generated. The j-th path, for $j \in \{1, \ldots, M\}$, is represented by a sequence of settlements $\mathcal{P}^{(j)}_{s \to t} = (v^{(j)}_1, v^{(j)}_2, \ldots, v^{(j)}_{m_j})$, where $v^{(j)}_1 = s$ is the source and m_j is the total number of steps (length) of the path. The settlement at step i of this path is $v^{(j)}_i$.

We aim to characterize the overall likelihood of encountering a specific dominant hazard at a given step along these paths. Let's define the necessary components. The indicator function $\mathbb{I}[\cdot]$ is 1 if its argument is true, and 0 otherwise. First, for a fixed source settlement s, we define the conditional probability $P_i(h \mid s)$ as the empirical probability that the dominant hazard at step i is h, given that the path started at s. This is found by considering all simulated paths originating from s (to all possible targets t). The probability is the ratio of the number of paths from s that both reach step i and have h as the dominant hazard at $v^{(j)}_i$, to the total number of paths from s that reach at least step i.

Task: Using the definitions provided, construct the formal mathematical expression for the overall hazard likelihood at step i, $\bar{P}_i(h)$. This quantity is defined as the arithmetic mean of the conditional probabilities $P_i(h \mid s)$ taken over all possible source settlements $s \in V$. Your final expression should be in terms of the quantities defined in the background, such as $|V|$, M, m_j, $v^{(j)}_i$, h^*, and the indicator function $\mathbb{I}[\cdot]$.

Original Solution

$$\bar{P}_i(h) = \frac{1}{|V|} \sum_{s \in V} \left(\frac{\sum_{t \in V} \sum_{j=1}^{M} \mathbb{I}[i \le m_j] \cdot \mathbb{I}[h^*(v^{(j)}_i) = h]}{\sum_{t \in V} \sum_{j=1}^{M} \mathbb{I}[i \le m_j]} \right)$$

Critiques

Self-Containment Critique: **Status: Not self-contained**

The variable $\mathrm{m}_j is used without explicitly clarifying that it depends on the path index j and the source-target pair (s,t), whi

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing multiple definitions and summations over complex path structures to derive a formal expression for an averaged conditional probability, involving careful indexing and indicator functions.

Refined Problem

Refined Problem Statement: Background: Consider a system of settlements modeled as a directed network $G = (V, E)$, where V is the set of $|V|$ settlements (nodes) and E is the set of connections (edges). Displacements in this system are driven by a set of distinct hazards, $\mathcal{H}$. For each settlement $v \in V$ and each hazard $h \in \mathcal{H}$, the quantity $D_h(v)$ represents the number of individuals displaced from v due to that specific hazard. The *dominant hazard* at a settlement v, denoted $h^*(v)$, is defined as the hazard type responsible for the largest number of displacements from that settlement, i.e., $h^*(v) = \arg\max_{h \in \mathcal{H}} D_h(v)$.

The movement of displaced populations is modeled via a large number of simulated stochastic paths. For each ordered pair of source-target settlements $(s, t) \in V \times V$, a set of M distinct paths is generated. The j-th path (for $j \in \{1, \ldots, M\}$) connecting s to t is represented by a sequence of settlements $\mathcal{P}^{(j)}_{s \to t} = (v^{(j)}_1, v^{(j)}_2, \ldots, v^{(j)}_{m^{(s,t)}_j})$, where $v^{(j)}_1 = s$ is the source, $v^{(j)}_{m^{(s,t)}_j} = t$ is the target, and $m^{(s,t)}_j$ denotes the length (number of steps) of the j-th path from s to t. The settlement at step i of this path is $v^{(j)}_i$.

We aim to characterize the overall likelihood of encountering a specific dominant hazard at a given step along these paths. Let us define the necessary components. The indicator function $\mathbb{I}[\cdot]$ is 1 if its argument is true, and 0 otherwise. First, for a fixed source settlement s, we define the conditional probability $P_i(h \mid s)$ as the empirical probability that the dominant hazard at step i is h, given that the path started at s. This is found by considering all simulated paths originating from s (across all possible targets $t \in V$), where, for each pair (s,t), all M paths $\mathcal{P}_{s\to t}^{(j)}$ are considered. The probability is the ratio of the number of $s \to t$ paths that reach at least step i and encounter h as the dominant hazard at $v_i^{(j)}$, to the total number of $s \to t$ paths that reach at least step i.

Task: Using the definitions provided, construct the formal mathematical expression for the overall hazard likelihood at step i, $\bar{P}_i(h)$. This quantity is defined as the arithmetic mean of the conditional probabilities $P_i(h \mid s)$ taken over all possible source settlements $s \in V$. Your final expression should be in terms of the quantities defined in the background, such as $|V|$, M, $m_j^{(s,t)}$, $v_i^{(j)}$, h^*, and the indicator function $\mathbb{I}[\cdot]$. Be explicit in your notation regarding path indices and the dependence of path length $m_j^{(s,t)}$ on s, t, and j.

Refined Solution:

$$\bar{P}_i(h) = \frac{1}{|V|} \sum_{s \in V} \left(\frac{\sum_{t \in V} \sum_{j=1}^{M} \mathbb{I}[i \le m_j^{(s,t)}] \cdot \mathbb{I}[h^*(v_i^{(j)}) = h]}{\sum_{t \in V} \sum_{j=1}^{M} \mathbb{I}[i \le m_j^{(s,t)}]} \right)$$

Problem 12 (Paper: 2506.22120v1, Index: 1)

Original Problem Statement

Background: Consider a system of settlements modeled as a directed network $G = (V, E)$, where V is the set of settlements (nodes) and E is the set of displacement flows (edges). For each settlement $v \in V$, displacement can be triggered by a set of distinct hazards $\mathcal{H}$. The number of individuals displaced from settlement v due to a specific hazard $h \in \mathcal{H}$ is given by the count $D_h(v)$. For any settlement v, the single most influential hazard, termed the "dominant hazard" $h^*(v)$, is defined as the one corresponding to the maximum displacement count, i.e., $h^*(v) = \arg\max_{h \in \mathcal{H}} D_h(v)$.

To analyze potential displacement trajectories, a simulation is performed. For each ordered pair of source-target settlements (s, t), a set of M probabilistic paths are generated. A single simulated path, indexed by $j \in \{1, \ldots, M\}$, is a sequence of settlements $\mathcal{P}_{s \to t}^{(j)} = (v_1^{(j)}, v_2^{(j)}, \ldots, v_{m_j}^{(j)})$, where the first settlement is the source, $v_1^{(j)} = s$, and the path has a total length of m_j steps.

The analysis aims to find the likelihood of encountering a specific dominant hazard at a given step along these trajectories. Let $P_i(h \mid s)$ be the empirical probability of encountering the dominant hazard h at step i of a path, given that the path originates from source s. This probability is computed by considering all simulated paths starting from s to all possible targets $t \in V$. The normalization constant for this probability is $M_s^{(i)}$, defined as the total number of simulated paths originating from s that have a length of at least i steps. You may use the indicator function $\mathbb{I}[\cdot]$, which evaluates to 1 if its argument is true and 0 otherwise.

Task: Derive the expression for the overall hazard likelihood at step i, $\bar{P}_i(h)$. This quantity is defined as the average of the per-source likelihoods $P_i(h \mid s)$ over all possible source settlements $s \in V$. The final expression should be written in terms of the fundamental quantities defined above, such as $|V|$, M, $M_s^{(i)}$, h^*, and the indicator function $\mathbb{I}[\cdot]$.

Original Solution

$$\bar{P}_i(h) = \frac{1}{|V|} \sum_{s \in V} \left(\frac{1}{M_s^{(i)}} \sum_{t \in V} \sum_{j=1}^{M} \mathbb{I}[i \leq m_j] \cdot \mathbb{I}[h^*(v_i^{(j)}) = h] \right)$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing multiple definitions and summations over paths, sources, and steps to derive an aggregate probability expression, involving careful normalization and indicator functions, which is non-trivial and demands multi-step reasoning.

Refined Problem

Refined Problem Statement: Background: Consider a system of settlements modeled as a directed network $G = (V, E)$, where V is the set of settlements (nodes) and E is the set of displacement flows (edges). For each settlement $v \in V$, displacement can be triggered by a set of distinct hazards $\mathcal{H}$. The number of individuals displaced from settlement v due to a specific hazard $h \in \mathcal{H}$ is given by the count $D_h(v)$. For any settlement v, the single most influential hazard, termed the "dominant hazard" $h^*(v)$, is defined as the one corresponding to the maximum displacement count, i.e., $h^*(v) = \arg\max_{h \in \mathcal{H}} D_h(v)$.

To analyze potential displacement trajectories, a simulation is performed. For each ordered pair of source-target settlements (s, t), a set of M probabilistic paths are generated. A single simulated path, indexed by $j \in \{1, \ldots, M\}$, is a sequence of settlements $\mathcal{P}_{s \to t}^{(j)} = (v_1^{(j)}, v_2^{(j)}, \ldots, v_{m_j}^{(j)})$, where the first settlement is the source, $v_1^{(j)} = s$, and the path has a total length of m_j steps.

The analysis aims to find the likelihood of encountering a specific dominant hazard at a given step along these trajectories. Let $P_i(h \mid s)$ be the empirical probability of encountering the dominant hazard h at step i of a path, given that the path originates from source s. This probability is computed by considering all simulated paths starting from s to all possible targets $t \in V$. The normalization constant for this probability is $M_s^{(i)}$, defined as the total number of simulated paths originating from s that have

a length of at least i steps. You may use the indicator function $\mathbb{I}[\cdot]$, which evaluates to 1 if its argument is true and 0 otherwise.

Task: Derive the expression for the overall hazard likelihood at step i, $\bar{P}_i(h)$. This quantity is defined as the average of the per-source likelihoods $P_i(h \mid s)$ over all possible source settlements $s \in V$. The final expression should be written in terms of the fundamental quantities defined above, such as $|V|$, M, $M_s^{(i)}$, h^*, and the indicator function $\mathbb{I}[\cdot]$.

Refined Solution:

$$\bar{P}_i(h) = \frac{1}{|V|} \sum_{s \in V} \left(\frac{1}{M_s^{(i)}} \sum_{t \in V} \sum_{j=1}^{M} \mathbb{I}[i \leq m_j] \cdot \mathbb{I}[h^*(v_i^{(j)}) = h] \right)$$

Problem 13 (Paper: 2506.22160v1, Index: 0)

Original Problem Statement

Background: Consider a system of N globally coupled Kuramoto phase oscillators, where the dynamics of the j-th oscillator is given by

$$\frac{d\theta_j}{dt} = \omega_j + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_j).$$

Here, θ_j is the phase, ω_j is the natural frequency, and K is the uniform coupling strength. The collective behavior of the system can be described by the complex order parameter $s(t) = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$, which allows the dynamics to be rewritten as $\frac{d\theta_j}{dt} = \omega_j + K\,\mathrm{Im}(se^{-i\theta_j})$. The natural frequencies ω_j are randomly drawn from a stationary Lorentzian distribution with zero mean and unit half-width, given by the probability density function $g(\omega) = \frac{1}{\pi}\frac{1}{\omega^2+1}$.

For a finite number of oscillators N, the system exhibits fluctuations even for subcritical coupling ($K < 2$), where an infinite system would be asynchronous ($s = 0$). These fluctuations can be treated as a form of "shot noise". In the absence of coupling ($K = 0$), the system's order parameter is a "free shot noise" $\chi_0(t)$, whose power spectrum is given by $W_0(\omega) = \frac{2\pi}{N}g(\omega)$.

For non-zero subcritical coupling, the fluctuations of the finite system provide feedback, acting as a driving force on the population. This feedback can be analyzed using a self-consistent approach. We model the response of the finite population by considering an auxiliary infinite-size population driven by the order parameter $s(t)$ of the finite system. The order parameter of this auxiliary infinite population, denoted by $r(t)$, evolves according to the mean-field equation derived from the Ott-Antonsen ansatz:

$$\frac{dr}{dt} = -r + \frac{K}{2}s - \frac{K}{2}s^* r^2.$$

The total order parameter of the finite system $s(t)$ is then approximated as the sum of the free shot noise $\chi_0(t)$ and the response $r(t)$ of the auxiliary system: $s(t) = \chi_0(t) + r(t)$. This establishes a self-consistency condition, as the response $r(t)$ is driven by the total signal $s(t)$ which itself contains $r(t)$.

Task: Assuming the fluctuations are small (allowing for linearization) and using the self-consistency condition, determine the power spectrum $W(\omega)$ of the collective oscillations $s(t)$ in the finite-size system with subcritical coupling K. Express your result in terms of N, K, ω, and the distribution $g(\omega)$.

Original Solution

$$W(\omega) = \frac{2\pi}{N}\left|\frac{1+i\omega}{1+i\omega-K/2}\right|^2 g(\omega)$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing the Kuramoto model dynamics, the Ott-Antonsen reduction, and a self-consistent linear response analysis to derive the finite-size power spectrum, involving multiple advanced steps and concepts.

Refined Problem

Refined Problem Statement: Background: Consider a system of N globally coupled Kuramoto phase oscillators, where the dynamics of the j-th oscillator is given by

$$\frac{d\theta_j}{dt} = \omega_j + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_j).$$

Here, θ_j is the phase, ω_j is the natural frequency, and K is the uniform coupling strength. The collective behavior of the system can be described by the complex order parameter $s(t) = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$, which allows the dynamics to be rewritten as $\frac{d\theta_j}{dt} = \omega_j + K\,\mathrm{Im}(se^{-i\theta_j})$. The natural frequencies ω_j are randomly drawn

from a stationary Lorentzian distribution with zero mean and unit half-width, given by the probability density function $g(\omega) = \frac{1}{\pi}\frac{1}{\omega^2+1}$.

For a finite number of oscillators N, the system exhibits fluctuations even for subcritical coupling ($K < 2$), where an infinite system would be asynchronous ($s = 0$). These fluctuations can be treated as a form of "shot noise". In the absence of coupling ($K = 0$), the system's order parameter is a "free shot noise" $\chi_0(t)$, whose power spectrum is given by $W_0(\omega) = \frac{2\pi}{N}g(\omega)$.

For non-zero subcritical coupling, the fluctuations of the finite system provide feedback, acting as a driving force on the population. This feedback can be analyzed using a self-consistent approach. We model the response of the finite population by considering an auxiliary infinite-size population driven by the order parameter $s(t)$ of the finite system. The order parameter of this auxiliary infinite population, denoted by $r(t)$, evolves according to the mean-field equation derived from the Ott-Antonsen ansatz:

$$\frac{dr}{dt} = -r + \frac{K}{2}s - \frac{K}{2}s^* r^2.$$

The total order parameter of the finite system $s(t)$ is then approximated as the sum of the free shot noise $\chi_0(t)$ and the response $r(t)$ of the auxiliary system: $s(t) = \chi_0(t) + r(t)$. This establishes a self-consistency condition, as the response $r(t)$ is driven by the total signal $s(t)$ which itself contains $r(t)$.

Task: Assuming the fluctuations are small (allowing for linearization) and using the self-consistency condition, determine the power spectrum $W(\omega)$ of the collective oscillations $s(t)$ in the finite-size system with subcritical coupling K. Express your result in terms of N, K, ω, and the distribution $g(\omega)$.

Refined Solution:

$$W(\omega) = \frac{2\pi}{N}\left|\frac{1 + i\omega}{1 + i\omega - K/2}\right|^2 g(\omega)$$

Problem 14 (Paper: 2506.22160v1, Index: 1)

Original Problem Statement

Background: Consider a system of N globally coupled Kuramoto oscillators, where the phase θ_j of each oscillator evolves according to the equation:

$$\frac{d\theta_j}{dt} = \omega_j + K\,\mathrm{Im}(se^{-i\theta_j})$$

Here, $s = \frac{1}{N}\sum_{k=1}^{N} e^{i\theta_k}$ is the complex order parameter of the finite-size system, K is the uniform coupling strength, and ω_j are the natural frequencies. The frequencies are independent random variables drawn from a Lorentzian distribution with zero mean and unit half-width, given by the probability density function $g(\omega) = \frac{1}{\pi(1+\omega^2)}$.

In the subcritical regime ($0 < K < 2$), the mean-field theory for an infinite system ($N \to \infty$) predicts a stationary asynchronous state with zero order parameter. For a finite system, however, the order parameter $s(t)$ exhibits persistent fluctuations due to the finite number of oscillators, a phenomenon known as shot noise. The total signal $s(t)$ can be conceptually decomposed into two parts: $s(t) = \chi_0(t) + r(t)$, where $\chi_0(t)$ is the "free" shot noise that would exist for uncoupled oscillators ($K = 0$), and $r(t)$ is the induced collective response that arises from the coupling. The power spectrum of the free shot noise is given by $W_0(\omega) = \frac{2\pi}{N} g(\omega)$.

The induced response $r(t)$ can be modeled as the order parameter of an auxiliary infinite-size population driven by the total signal $s(t)$. For the given Lorentzian frequency distribution, the dynamics of this induced response can be described by the following equation, derived from the Ott-Antonsen ansatz:

$$\frac{dr}{dt} = -r + \frac{K}{2}s - \frac{K}{2}s^* r^2$$

For the purpose of analyzing small fluctuations, this equation can be linearized around the state $r = 0$. The relationship between the Fourier transforms of the signals, $s_\omega = \mathcal{F}[s(t)]$, $w_{0,\omega} = \mathcal{F}[\chi_0(t)]$, and $r_\omega = \mathcal{F}[r(t)]$, is then given by the self-consistency condition $s_\omega = w_{0,\omega} + r_\omega$.

Task: Using this theoretical framework, derive an expression for the total variance of the order parameter, $D = \langle |s(t)|^2 \rangle$, for the finite-size system with subcritical coupling ($0 < K < 2$). The variance is related to the two-sided power spectral density $W(\omega)$ of the signal $s(t)$ by Parseval's theorem, $D = \frac{1}{2\pi}\int_{-\infty}^{\infty} W(\omega)d\omega$.

Original Solution

$$D = \frac{1}{N}\frac{1}{1 - K/2}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing the stochastic finite-size effects, linearizing nonlinear dynamics from the Ott-Antonsen ansatz, applying Fourier transforms, and integrating the resulting power spectral density to derive the variance, involving multiple advanced steps beyond direct formula application.

Refined Problem

Refined Problem Statement: Background: Consider a system of N globally coupled Kuramoto oscillators, where the phase θ_j of each oscillator evolves according to the equation:

$$\frac{d\theta_j}{dt} = \omega_j + K\,\mathrm{Im}(se^{-i\theta_j})$$

Here, $s = \frac{1}{N}\sum_{k=1}^{N} e^{i\theta_k}$ is the complex order parameter of the finite-size system, K is the uniform coupling strength, and ω_j are the natural frequencies. The frequencies are independent random variables drawn from a Lorentzian distribution with zero mean and unit half-width, given by the probability density function $g(\omega) = \frac{1}{\pi(1+\omega^2)}$.

In the subcritical regime ($0 < K < 2$), the mean-field theory for an infinite system ($N \to \infty$) predicts a stationary asynchronous state with zero order parameter. For a finite system, however, the order parameter $s(t)$ exhibits persistent fluctuations due to the finite number of oscillators, a phenomenon known as shot noise. The total signal $s(t)$ can be conceptually decomposed into two parts: $s(t) = \chi_0(t) + r(t)$, where $\chi_0(t)$ is the "free" shot noise that would exist for uncoupled oscillators ($K = 0$), and $r(t)$ is the induced collective response that arises from the coupling. The power spectrum of the free shot noise is given by $W_0(\omega) = \frac{2\pi}{N} g(\omega)$.

The induced response $r(t)$ can be modeled as the order parameter of an auxiliary infinite-size population driven by the total signal $s(t)$. For the given Lorentzian frequency distribution, the dynamics of this induced response can be described by the following equation, derived from the Ott-Antonsen ansatz:

$$\frac{dr}{dt} = -r + \frac{K}{2}s - \frac{K}{2}s^* r^2$$

For the purpose of analyzing small fluctuations, this equation can be linearized around the state $r = 0$. The relationship between the Fourier transforms of the signals, $s_\omega = \mathcal{F}[s(t)]$, $w_{0,\omega} = \mathcal{F}[\chi_0(t)]$, and $r_\omega = \mathcal{F}[r(t)]$, is then given by the self-consistency condition $s_\omega = w_{0,\omega} + r_\omega$.

Task: Using this theoretical framework, derive an expression for the total variance of the order parameter, $D = \langle |s(t)|^2 \rangle$, for the finite-size system with subcritical coupling ($0 < K < 2$). The variance is related to the two-sided power spectral density $W(\omega)$ of the signal $s(t)$ by Parseval's theorem, $D = \frac{1}{2\pi} \int_{-\infty}^{\infty} W(\omega) d\omega$.

Refined Solution:

$$D = \frac{1}{N} \frac{1}{1 - K/2}$$

Problem 15 (Paper: 2506.22695v1, Index: 0)

Original Problem Statement

Background: In a non-equilibrium statistical mechanics model of protein dynamics, the state of a single protein is described by its position $X \in \mathbb{R}^d$ and its internal temperature $\theta_P \in \mathbb{R}$. The irreversible dynamics, which encompass both deterministic drift and dissipation, are governed by a symmetric, positive-semidefinite operator $K^{(1)}$. This operator couples the mechanical and thermal degrees of freedom and is given by the $(d+1) \times (d+1)$ block matrix:

$$K^{(1)} = \begin{pmatrix} \theta_P M_{XX} & -\frac{\theta_P M_{XX}(\nabla_X \mathcal{E})}{c_P} \\ -\frac{(\nabla_X \mathcal{E})^T \theta_P M_{XX}}{c_P} & \frac{(\nabla_X \mathcal{E})^T \theta_P M_{XX}(\nabla_X \mathcal{E})}{c_P^2} \end{pmatrix}$$

Here, M_{XX} is the symmetric and positive-definite $d \times d$ mobility tensor of the protein, c_P is the protein's heat capacity, and $\nabla_X \mathcal{E}$ is the conservative force (a $d \times 1$ column vector) derived from the system's total energy $\mathcal{E}$. To generate the corresponding thermal fluctuations in simulations, one must compute a matrix factor $R^{(1)}$ such that $K^{(1)} = R^{(1)} R^{(1),T}$. This factor $R^{(1)}$ is a $(d+1) \times d$ matrix.

Task: Given a matrix factorization of the mobility tensor $M_{XX} = R_M R_M^T$, where R_M is a $d \times d$ matrix, derive the expression for the factor $R^{(1)}$.

Original Solution

$$R^{(1)} = \begin{bmatrix} \sqrt{\theta_P} R_M \\ -\frac{\sqrt{\theta_P}(\nabla_X \mathcal{E})^T R_M}{c_P} \end{bmatrix}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing matrix factorization, block matrix algebra, and physical interpretation of coupled mechanical and thermal degrees of freedom to derive a nontrivial matrix square root factorization.

Refined Problem

Refined Problem Statement: Background: In a non-equilibrium statistical mechanics model of protein dynamics, the state of a single protein is described by its position $X \in \mathbb{R}^d$ and its internal temperature $\theta_P \in \mathbb{R}$. The irreversible dynamics, which encompass both deterministic drift and dissipation, are governed by a symmetric, positive-semidefinite operator $K^{(1)}$. This operator couples the mechanical and thermal degrees of freedom and is given by the $(d+1) \times (d+1)$ block matrix:

$$K^{(1)} = \begin{pmatrix} \theta_P M_{XX} & -\frac{\theta_P M_{XX}(\nabla_X \mathcal{E})}{c_P} \\ -\frac{(\nabla_X \mathcal{E})^T \theta_P M_{XX}}{c_P} & \frac{(\nabla_X \mathcal{E})^T \theta_P M_{XX}(\nabla_X \mathcal{E})}{c_P^2} \end{pmatrix}$$

Here, M_{XX} is the symmetric and positive-definite $d \times d$ mobility tensor of the protein, c_P is the protein's heat capacity, and $\nabla_X \mathcal{E}$ is the conservative force (a $d \times 1$ column vector) derived from the system's total energy $\mathcal{E}$. To generate the corresponding thermal fluctuations in simulations, one must compute a matrix factor $R^{(1)}$ such that $K^{(1)} = R^{(1)} R^{(1),T}$. This factor $R^{(1)}$ is a $(d+1) \times d$ matrix.

Task: Given a matrix factorization of the mobility tensor $M_{XX} = R_M R_M^T$, where R_M is a $d \times d$ matrix, derive the expression for the factor $R^{(1)}$.

Refined Solution:

$$R^{(1)} = \begin{bmatrix} \sqrt{\theta_P} R_M \\ -\frac{\sqrt{\theta_P}(\nabla_X \mathcal{E})^T R_M}{c_P} \end{bmatrix}$$

Problem 16 (Paper: 2506.22695v1, Index: 1)

Original Problem Statement

Background: In the non-equilibrium statistical mechanics of a membrane-protein system, thermal exchanges between different components give rise to irreversible dynamics and fluctuations. Consider the coupling between a single protein's thermal interface (temperature θ_I, heat capacity c_I) and the surrounding membrane. The membrane is a continuum field with temperature $\theta_C(x)$ and specific heat per unit volume c_C. In a discrete picture where the membrane is composed of cells of volume , the total heat capacity of a cell at position x is c_C. The thermal exchange between the interface and the membrane cell at x is described by a local dissipative operator, which we denote $K_{loc}(x)$. This operator is a 2×2 matrix acting on the degrees of freedom $(\theta_C(x), \theta_I)$ and is given by:

$$K_{loc}(x) = \kappa_{CI}(x)\theta_I\theta_C(x) \begin{pmatrix} \frac{1}{(c_C)^2} & -\frac{1}{(c_C)c_I} \\ -\frac{1}{c_I(c_C)} & \frac{1}{c_I^2} \end{pmatrix}$$

where $\kappa_{CI}(x)$ is the local heat conduction coefficient density. To generate stochastic thermal fluctuations in a simulation, one must find an operator $R_2(x)$ such that $K_{loc}(x) = R_2(x)R_2(x)^T$. This operator $R_2(x)$ is a vector in the state space spanned by the basis vectors $\mathbf{e}_{\theta_C(x)}$ and $\mathbf{e}_{\theta_I}$.

Task: Derive the operator $R_2(x)$ that satisfies the factorization $K_{loc}(x) = R_2(x)R_2(x)^T$.

Original Solution

$$R_2(x) = \sqrt{\kappa_{CI}(x)\theta_I\theta_C(x)} \left(\frac{1}{c_C}\mathbf{e}_{\theta_C(x)} - \frac{1}{c_I}\mathbf{e}_{\theta_I} \right)$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires deriving a matrix factorization of a non-trivial dissipative operator involving coupled thermal degrees of freedom, demanding synthesis of linear algebra and physical interpretation beyond direct formula application.

Refined Problem

Refined Problem Statement: Background: In the non-equilibrium statistical mechanics of a membrane-protein system, thermal exchanges between different components give rise to irreversible dynamics and fluctuations. Consider the coupling between a single protein's thermal interface (temperature θ_I, heat capacity c_I) and the surrounding membrane. The membrane is a continuum field with temperature $\theta_C(x)$ and specific heat per unit volume c_C. In a discrete picture where the membrane is composed of cells of volume , the total heat capacity of a cell at position x is c_C. The thermal exchange between the interface and the membrane cell at x is described by a local dissipative operator, which we denote $K_{loc}(x)$. This operator is a 2×2 matrix acting on the degrees of freedom $(\theta_C(x), \theta_I)$ and is given by:

$$K_{loc}(x) = \kappa_{CI}(x)\theta_I\theta_C(x) \begin{pmatrix} \frac{1}{(c_C)^2} & -\frac{1}{(c_C)c_I} \\ -\frac{1}{c_I(c_C)} & \frac{1}{c_I^2} \end{pmatrix}$$

where $\kappa_{CI}(x)$ is the local heat conduction coefficient density. To generate stochastic thermal fluctuations in a simulation, one must find an operator $R_2(x)$ such that $K_{loc}(x) = R_2(x)R_2(x)^T$. This operator $R_2(x)$ is a vector in the state space spanned by the basis vectors $\mathbf{e}_{\theta_C(x)}$ and $\mathbf{e}_{\theta_I}$.

Task: Derive the operator $R_2(x)$ that satisfies the factorization $K_{loc}(x) = R_2(x)R_2(x)^T$.

Refined Solution:

$$R_2(x) = \sqrt{\kappa_{CI}(x)\theta_I\theta_C(x)} \left(\frac{1}{c_C}\mathbf{e}_{\theta_C(x)} - \frac{1}{c_I}\mathbf{e}_{\theta_I} \right)$$

Problem 17 (Paper: 2506.22909v1, Index: 0)

Original Problem Statement

Background: Consider an ensemble of N globally coupled FitzHugh-Nagumo oscillators, where the dynamics of the i-th oscillator ($i = 1, 2, ..., N$) are described by the following set of stochastic differential equations:

$$\varepsilon \frac{dx_i}{dt} = x_i - \frac{x_i^3}{3} - y_i + f_i(x_1, x_2, ..., x_N)$$
$$\frac{dy_i}{dt} = x_i + a + \sqrt{2D} n_i(t)$$

Here, x_i and y_i are the fast activator and slow inhibitor variables, respectively. The parameter $\varepsilon \ll 1$ ensures time-scale separation, a is a threshold parameter, and D is the intensity of the statistically independent Gaussian white noise sources $n_i(t)$, which satisfy $\langle n_i(t) \rangle = 0$ and $\langle n_i(t) n_j(t') \rangle = \delta_{ij} \delta(t - t')$. The global coupling is introduced through the term f_i, defined as:

$$f_i(x_1, x_2, ..., x_N) = \frac{\sigma}{N} \sum_{j=1}^{N} (x_j - x_i)$$

where σ is the coupling strength. We can define the instantaneous mean fields as $\bar{x}(t) = \frac{1}{N} \sum_{j=1}^{N} x_j(t)$ and $\bar{y}(t) = \frac{1}{N} \sum_{j=1}^{N} y_j(t)$. In the thermodynamic limit ($N \to \infty$), the noise term in the equation for the mean field $\bar{y}(t)$ averages to zero, and the dynamics of the mean fields can be approximated by a deterministic system. The state of each oscillator can be decomposed into a mean-field component and a fluctuation component: $x_i(t) = \bar{x}(t) + \delta x_i(t)$ and $y_i(t) = \bar{y}(t) + \delta y_i(t)$. Assume that the fluctuations δx_i and δy_i are small, which allows for a linearization of the system's dynamics around the mean-field trajectory $(\bar{x}(t), \bar{y}(t))$.

Task: In the limit of large N and small fluctuations, derive the linearized equation of motion for the fluctuation of the activator variable, $\delta x_i(t)$. Your final result should be a single expression for $\varepsilon \frac{d(\delta x_i)}{dt}$ in terms of δx_i, δy_i, the coupling strength σ, and the mean-field variable $\bar{x}(t)$.

Original Solution

$$\varepsilon \frac{d(\delta x_i)}{dt} = (1 - \sigma - \bar{x}(t)^2) \delta x_i - \delta y_i$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing the original nonlinear stochastic system, performing a mean-field decomposition, applying linearization around the mean-field trajectory, and carefully handling the coupling term to derive the fluctuation dynamics, which involves multiple conceptual and mathematical steps.

Refined Problem

Refined Problem Statement: Background: Consider an ensemble of N globally coupled FitzHugh-Nagumo oscillators, where the dynamics of the i-th oscillator ($i = 1, 2, \ldots, N$) are described by the following set of stochastic differential equations:

$$\varepsilon \frac{dx_i}{dt} = x_i - \frac{x_i^3}{3} - y_i + f_i(x_1, x_2, \ldots, x_N)$$
$$\frac{dy_i}{dt} = x_i + a + \sqrt{2D} n_i(t)$$

Here, x_i and y_i are the fast activator and slow inhibitor variables, respectively. The parameter $\varepsilon \ll 1$ ensures time-scale separation, a is a threshold parameter, and D is the intensity of the statistically

independent Gaussian white noise sources $n_i(t)$, which satisfy $\langle n_i(t)\rangle = 0$ and $\langle n_i(t)n_j(t')\rangle = \delta_{ij}\delta(t-t')$. The global coupling is introduced through the term f_i, defined as:

$$f_i(x_1, x_2, \ldots, x_N) = \frac{\sigma}{N}\sum_{j=1}^{N}(x_j - x_i)$$

where σ is the coupling strength. We can define the instantaneous mean fields as $\bar{x}(t) = \frac{1}{N}\sum_{j=1}^{N} x_j(t)$ and $\bar{y}(t) = \frac{1}{N}\sum_{j=1}^{N} y_j(t)$. In the thermodynamic limit ($N \to \infty$), the noise term in the equation for the mean field $\bar{y}(t)$ averages to zero, and the dynamics of the mean fields can be approximated by a deterministic system. The state of each oscillator can be decomposed into a mean-field component and a fluctuation component: $x_i(t) = \bar{x}(t) + \delta x_i(t)$ and $y_i(t) = \bar{y}(t) + \delta y_i(t)$. Assume that the fluctuations δx_i and δy_i are small, which allows for a linearization of the system's dynamics around the mean-field trajectory $(\bar{x}(t), \bar{y}(t))$.

Task: In the limit of large N and small fluctuations, derive the linearized equation of motion for the fluctuation of the activator variable, $\delta x_i(t)$. Your final result should be a single expression for $\varepsilon\frac{d(\delta x_i)}{dt}$ in terms of δx_i, δy_i, the coupling strength σ, and the mean-field variable $\bar{x}(t)$.

Refined Solution:

$$\varepsilon\frac{d(\delta x_i)}{dt} = (1 - \sigma - \bar{x}(t)^2)\delta x_i - \delta y_i$$

Problem 18 (Paper: 2506.22909v1, Index: 1)

Original Problem Statement

Background: An ensemble of N coupled excitable systems is modeled by the FitzHugh-Nagumo equations. The elements are arranged in a one-dimensional ring with periodic boundary conditions, such that the distance between oscillators i and j is given by $|i-j|_N = \min(|i-j|, N-|i-j|)$. The coupling between elements is nonlocal and diffusive, described by the term f_i for the i-th oscillator: $f_i = \frac{\sigma}{2R} \sum_{k=i-R}^{i+R} (x_k - x_i)$ where x_k is the activator variable of the k-th oscillator, σ is the coupling strength, R is the coupling radius, and the indices in the sum are interpreted modulo N. This nonlocal coupling scheme bridges two important limit cases: local coupling, corresponding to the smallest radius $R = 1$, and global coupling, corresponding to the largest possible radius. For comparison, a standard definition for global coupling is given by: $f_i^{\text{global}} = \frac{\sigma}{N} \sum_{k=1}^{N} (x_k - x_i)$ The transition between these topologies by varying the radius R is a key mechanism for controlling the collective dynamics of the ensemble. To analyze this transition, one can examine the form of the nonlocal coupling term in its limits.

Task: Consider an ensemble with an odd number of oscillators, $N = 2k + 1$ for some integer $k \geq 1$. In this configuration, the nonlocal coupling becomes fully global when the radius is maximal, i.e., when $R = k = (N - 1)/2$, as this radius ensures each oscillator is coupled to every other oscillator in the ring. Derive the expression for the nonlocal coupling term f_i for this specific maximal radius. By comparing your derived expression for f_i with the standard global coupling form f_i^{global}, determine the analytical expression for the ratio $\frac{f_i}{f_i^{\text{global}}}$.

Original Solution

$$\frac{N}{N-1}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Trivial**

The problem involves a straightforward algebraic manipulation and comparison of two coupling terms that are explicitly defined, requiring no deeper synthesis beyond recognizing the maximal radius case.

Refined Problem

Refined Problem Statement: Background: An ensemble of N coupled excitable systems is modeled by the FitzHugh-Nagumo equations. The elements are arranged in a one-dimensional ring with periodic boundary conditions, such that the distance between oscillators i and j is given by $|i - j|_N = \min(|i - j|, N - |i - j|)$. The coupling between elements is nonlocal and diffusive, described by the term f_i for the i-th oscillator:

$$f_i = \frac{\sigma}{2R} \sum_{k=i-R}^{i+R} (x_k - x_i)$$

where x_k is the activator variable of the k-th oscillator, σ is the coupling strength, R is the coupling radius, and the indices in the sum are interpreted modulo N. This nonlocal coupling scheme bridges two important limit cases: local coupling, corresponding to the smallest radius $R = 1$, and global coupling, corresponding to the largest possible radius. For comparison, a standard definition for global coupling is given by:

$$f_i^{\text{global}} = \frac{\sigma}{N} \sum_{k=1}^{N} (x_k - x_i)$$

The transition between these topologies by varying the radius R is a key mechanism for controlling the collective dynamics of the ensemble. To analyze this transition, one can examine the form of the nonlocal coupling term in its limits.

Task: Consider an ensemble with an odd number of oscillators, $N = 2k + 1$ for some integer $k \geq 1$. In this configuration, the nonlocal coupling becomes fully global when the radius is maximal, i.e., when $R = k = (N - 1)/2$, as this radius ensures each oscillator is coupled to every other oscillator in the ring. Derive the expression for the nonlocal coupling term f_i for this specific maximal radius. By comparing your derived expression for f_i with the standard global coupling form f_i^{global}, determine the analytical expression for the ratio $\frac{f_i}{f_i^{\text{global}}}$.

Refined Solution:

$$\frac{N}{N-1}$$

Problem 19 (Paper: 2506.23181v1, Index: 0)

Original Problem Statement

Background: Consider a system of N globally coupled phase oscillators in the continuum limit ($N \to \infty$). The dynamics of each oscillator are governed by the overdamped Kuramoto model with triadic interactions. In the mean-field representation, the equation of motion for an oscillator with phase θ and intrinsic frequency ω is given by:

$$\dot{\theta} = \omega + K_2 r_2 r_1 \sin(\psi_2 - \psi_1 - \theta)$$

Here, K_2 is the coupling strength. The macroscopic state of the system is described by the complex order parameters $z_p = r_p e^{i\psi_p} = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} e^{ip\theta_j}$ for $p = 1, 2$, where r_p is the magnitude and ψ_p is the phase of the p-th order parameter. The intrinsic frequencies ω are drawn from a Lorentzian distribution $g(\omega) = \frac{\Delta}{\pi[(\omega-\omega_0)^2+\Delta^2]}$ with mean frequency $\omega_0 = 0$ and width $\Delta = 1$. In the continuum limit, the system's state is described by a probability density function $\rho(\theta, \omega, t)$ that satisfies the continuity equation $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta}(\rho \dot{\theta}) = 0$. The dynamics can be analyzed using the Ott-Antonsen (OA) ansatz, which posits that the density function takes the form $\rho(\theta, \omega, t) = \frac{g(\omega)}{2\pi}\left[1 + \sum_{n=1}^{\infty}(\alpha(\omega, t)e^{i\theta})^n + \text{c.c.}\right]$, where $\alpha(\omega, t)$ is a complex function analytic in the lower half of the complex ω-plane. For this system, the OA ansatz leads to the following relations between the order parameters and the function $\alpha(\omega, t)$:

$$z_p^* = \int_{-\infty}^{\infty} g(\omega)[\alpha(\omega, t)]^p d\omega$$

Evaluating this integral for the given Lorentzian distribution via Cauchy's residue theorem yields $z_p^* = [\alpha(\omega_0 - i\Delta, t)]^p$. A direct consequence for $p = 1, 2$ is the simplification $z_2 = (z_1)^2$, which implies $r_2 = r_1^2$ and $\psi_2 = 2\psi_1$.

Task: Using the provided framework, derive the differential equation governing the temporal evolution of the global order parameter magnitude, $r_1(t)$. Your final answer should be an expression for $\dot{r}_1$ in terms of r_1 and K_2.

Original Solution

$$\dot{r}_1 = -r_1 + \frac{K_2}{2}(r_1^3 - r_1^5)$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires applying the Ott-Antonsen ansatz, performing contour integration with the Lorentzian distribution, using the relation between order parameters, and deriving a nonlinear differential equation for the order parameter magnitude, involving multiple advanced steps and synthesis of concepts.

Refined Problem

Refined Problem Statement: Background: Consider a system of N globally coupled phase oscillators in the continuum limit ($N \to \infty$). The dynamics of each oscillator are governed by the overdamped Kuramoto model with triadic interactions. In the mean-field representation, the equation of motion for an oscillator with phase θ and intrinsic frequency ω is given by:

$$\dot{\theta} = \omega + K_2 r_2 r_1 \sin(\psi_2 - \psi_1 - \theta)$$

Here, K_2 is the coupling strength. The macroscopic state of the system is described by the complex order parameters $z_p = r_p e^{i\psi_p} = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} e^{ip\theta_j}$ for $p = 1, 2$, where r_p is the magnitude and ψ_p is the phase of the p-th order parameter. The intrinsic frequencies ω are drawn from a Lorentzian distribution $g(\omega) = \frac{\Delta}{\pi[(\omega-\omega_0)^2+\Delta^2]}$ with mean frequency $\omega_0 = 0$ and width $\Delta = 1$. In the continuum limit, the system's state is described by a probability density function $\rho(\theta, \omega, t)$ that satisfies the continuity equation $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta}(\rho \dot{\theta}) = 0$. The dynamics can be analyzed using the Ott-Antonsen (OA) ansatz, which posits that the density function takes the form $\rho(\theta, \omega, t) = \frac{g(\omega)}{2\pi}\left[1 + \sum_{n=1}^{\infty}(\alpha(\omega, t)e^{i\theta})^n + \text{c.c.}\right]$, where

$\alpha(\omega, t)$ is a complex function analytic in the lower half of the complex ω-plane. For this system, the OA ansatz leads to the following relations between the order parameters and the function $\alpha(\omega, t)$:

$$z_p^* = \int_{-\infty}^{\infty} g(\omega)[\alpha(\omega, t)]^p d\omega$$

Evaluating this integral for the given Lorentzian distribution via Cauchy's residue theorem yields $z_p^* = [\alpha(\omega_0 - i\Delta, t)]^p$. A direct consequence for $p = 1, 2$ is the simplification $z_2 = (z_1)^2$, which implies $r_2 = r_1^2$ and $\psi_2 = 2\psi_1$.

Task: Using the provided framework, derive the differential equation governing the temporal evolution of the global order parameter magnitude, $r_1(t)$. Your final answer should be an expression for $\dot{r}_1$ in terms of r_1 and K_2.

Refined Solution:

$$\dot{r}_1 = -r_1 + \frac{K_2}{2}(r_1^3 - r_1^5)$$

Problem 20 (Paper: 2506.23181v1, Index: 1)

Original Problem Statement

Background: Consider a single dynamical unit whose phase evolution $\theta(t)$ is governed by the equation for a Kuramoto oscillator with inertia under a mean-field approximation. In a suitable rotating frame, the equation of motion is given by: $m\ddot{\theta} + \dot{\theta} = \omega - q\sin(\theta)$ where m is the inertia, ω is the intrinsic frequency, and q is an effective coupling strength. The dots denote derivatives with respect to time t. By rescaling time as $\tau = \sqrt{\frac{q}{m}}t$, the dynamics can be described by the dimensionless equation:

$$\frac{d^2\theta}{d\tau^2} = -\alpha\frac{d\theta}{d\tau} + \beta - \sin(\theta)$$

Here, $\alpha = \sqrt{\frac{1}{qm}}$ and $\beta = \frac{\omega}{q}$ are dimensionless parameters representing damping and driving, respectively. We are interested in the "drifting" state of the oscillator, which occurs when $|\beta| > 1$. In this regime, the system settles into a stable limit cycle in the $(\theta, d\theta/d\tau)$ phase space, rather than a fixed point. The time average of a periodic function $f(\theta)$ over one period of this limit cycle is denoted by $\langle f(\theta)\rangle$.

Task: For an oscillator in the drifting regime ($|\beta| > 1$), derive an expression for the steady-state time-averaged value of $\cos(2\theta)$, denoted as $\langle\cos(2\theta)\rangle$. Your final answer should be expressed solely in terms of the parameters α and β.

Original Solution

$$\left[\frac{\beta^2-\alpha^4}{\beta^2+\alpha^4}\right]\left[\frac{2\beta(\beta^2+\alpha^4)}{\alpha^3}\left(\frac{\beta}{\alpha} - \sqrt{\frac{\beta^2}{\alpha^2} - \frac{\alpha^2}{\beta^2+\alpha^4}}\right) - 1\right]$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires deriving a nontrivial time-averaged quantity for a nonlinear oscillator in the drifting regime, involving rescaling, limit cycle analysis, and expressing the result purely in terms of dimensionless parameters, which demands a sophisticated multi-step reasoning beyond direct application of known formulas.

Refined Problem

Refined Problem Statement: Background: Consider a single dynamical unit whose phase evolution $\theta(t)$ is governed by the equation for a Kuramoto oscillator with inertia under a mean-field approximation. In a suitable rotating frame, the equation of motion is given by: $m\ddot{\theta} + \dot{\theta} = \omega - q\sin(\theta)$ where m is the inertia, ω is the intrinsic frequency, and q is an effective coupling strength. The dots denote derivatives with respect to time t. By rescaling time as $\tau = \sqrt{\frac{q}{m}}t$, the dynamics can be described by the dimensionless equation:

$$\frac{d^2\theta}{d\tau^2} = -\alpha\frac{d\theta}{d\tau} + \beta - \sin(\theta)$$

Here, $\alpha = \sqrt{\frac{1}{qm}}$ and $\beta = \frac{\omega}{q}$ are dimensionless parameters representing damping and driving, respectively. We are interested in the "drifting" state of the oscillator, which occurs when $|\beta| > 1$. In this regime, the system settles into a stable limit cycle in the $(\theta, d\theta/d\tau)$ phase space, rather than a fixed point. The time average of a periodic function $f(\theta)$ over one period of this limit cycle is denoted by $\langle f(\theta)\rangle$.

Task: For an oscillator in the drifting regime ($|\beta| > 1$), derive an expression for the steady-state time-averaged value of $\cos(2\theta)$, denoted as $\langle\cos(2\theta)\rangle$. Your final answer should be expressed solely in terms of the parameters α and β.

Solution:

Refined Solution:

$$\left[\frac{\beta^2-\alpha^4}{\beta^2+\alpha^4}\right]\left[\frac{2\beta(\beta^2+\alpha^4)}{\alpha^3}\left(\frac{\beta}{\alpha} - \sqrt{\frac{\beta^2}{\alpha^2} - \frac{\alpha^2}{\beta^2+\alpha^4}}\right) - 1\right]$$

Problem 21 (Paper: 2506.23385v1, Index: 0)

Original Problem Statement

Background: Consider a set of functions $\mathcal{D}_{\nu}^{(k)}(z)$ for a non-negative integer k, complex order ν, and complex argument z. For $\mathrm{Re}(\nu) < 0$, these functions obey the differentiation rule with respect to their order:

$$\frac{\partial \mathcal{D}_{\nu}^{(k)}(z)}{\partial \nu} = \phi_{\nu} \mathcal{D}_{\nu}^{(k)}(z) + \mathcal{D}_{\nu}^{(k+1)}(z)$$

where $\phi_{\nu} = \psi^{(0)}(-\nu) - \frac{\ln 2}{2}$ and $\psi^{(m)}(x)$ is the polygamma function. Let us define a set of polynomials $\mathcal{P}_{n,m}(\nu)$ through the n-th order derivative relation for a specific order dependence $-i\nu - 1$:

$$\frac{\partial^n \mathcal{D}_{-i\nu-1}^{(k)}(z)}{\partial \nu^n} = \sum_{m=0}^{n} \mathcal{P}_{n,m}(\nu)\, \mathcal{D}_{-i\nu-1}^{(k+m)}(z)$$

where ν is a real parameter. Consider the function $F_r(\nu, \tau) = \left| \mathcal{D}_{-i\nu-1}^{(r)}(-i\mu_0 \tau) \right|^2$, where r is a non-negative integer, τ is a real variable, and $\mu_0 = \sqrt{2} e^{-i\pi/4}$. You may use the property $(\mathcal{D}_{\nu}^{(k)}(z))^* = \mathcal{D}_{\nu^*}^{(k)}(z^*)$, where the star denotes complex conjugation.

Task: Derive a general expression for the n-th derivative $\frac{\partial^n F_r(\nu,\tau)}{\partial \nu^n}$ evaluated at $\nu = 0$. Express your answer as a triple summation involving the polynomials $\mathcal{P}_{n-j,k}(0)$, their complex conjugates $\mathcal{P}_{j,\ell}^*(0)$, binomial coefficients, and the functions $\mathcal{D}_{-1}^{(k')}(-i\mu_0\tau)$ and $\mathcal{D}_{-1}^{(k'')}(i\bar{\mu}_0\tau)$ for various integer k', k''.

Original Solution

$$\sum_{j=0}^{n} \sum_{k=0}^{n-j} \sum_{\ell=0}^{j} \binom{n}{j} \mathcal{P}_{n-j,k}(0)\, \mathcal{P}_{j,\ell}^*(0)\, \mathcal{D}_{-1}^{(r+k)}(-i\mu_0\tau)\, \mathcal{D}_{-1}^{(r+\ell)}(i\bar{\mu}_0\tau)$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing differentiation rules, complex conjugation properties, and polynomial expansions to derive a multi-index summation formula, involving advanced special functions and their parameter derivatives, which is a sophisticated multi-step reasoning task.

Refined Problem

Refined Problem Statement: Background: Consider a set of functions $\mathcal{D}_{\nu}^{(k)}(z)$ for a non-negative integer k, complex order ν, and complex argument z. For $\mathrm{Re}(\nu) < 0$, these functions obey the differentiation rule with respect to their order:

$$\frac{\partial \mathcal{D}_{\nu}^{(k)}(z)}{\partial \nu} = \phi_{\nu} \mathcal{D}_{\nu}^{(k)}(z) + \mathcal{D}_{\nu}^{(k+1)}(z)$$

where $\phi_{\nu} = \psi^{(0)}(-\nu) - \frac{\ln 2}{2}$ and $\psi^{(m)}(x)$ is the polygamma function. Let us define a set of polynomials $\mathcal{P}_{n,m}(\nu)$ through the n-th order derivative relation for a specific order dependence $-i\nu - 1$:

$$\frac{\partial^n \mathcal{D}_{-i\nu-1}^{(k)}(z)}{\partial \nu^n} = \sum_{m=0}^{n} \mathcal{P}_{n,m}(\nu)\, \mathcal{D}_{-i\nu-1}^{(k+m)}(z)$$

where ν is a real parameter. Consider the function $F_r(\nu, \tau) = \left| \mathcal{D}_{-i\nu-1}^{(r)}(-i\mu_0 \tau) \right|^2$, where r is a non-negative integer, τ is a real variable, and $\mu_0 = \sqrt{2} e^{-i\pi/4}$. You may use the property $(\mathcal{D}_{\nu}^{(k)}(z))^* = \mathcal{D}_{\nu^*}^{(k)}(z^*)$, where the star denotes complex conjugation.

Task: Derive a general expression for the n-th derivative $\frac{\partial^n F_r(\nu,\tau)}{\partial \nu^n}$ evaluated at $\nu = 0$. Express your answer as a triple summation involving the polynomials $\mathcal{P}_{n-j,k}(0)$, their complex conjugates $\mathcal{P}_{j,\ell}^*(0)$, binomial coefficients, and the functions $\mathcal{D}_{-1}^{(k')}(-i\mu_0\tau)$ and $\mathcal{D}_{-1}^{(k'')}(i\bar{\mu}_0\tau)$ for various integer k', k''.

Refined Solution:

$$\sum_{j=0}^{n}\sum_{k=0}^{n-j}\sum_{\ell=0}^{j}\binom{n}{j}\mathcal{P}_{n-j,k}\left(0\right)\mathcal{P}_{j,\ell}^{*}\left(0\right)\mathcal{D}_{-1}^{(r+k)}\left(-i\mu_{0}\tau\right)\mathcal{D}_{-1}^{(r+\ell)}\left(i\bar{\mu}_{0}\tau\right)$$

Problem 22 (Paper: 2506.23385v1, Index: 1)

Original Problem Statement

Background: Consider a class of highly oscillatory nested integrals given by

$$\mathcal{I}_k(\tau) = \int_{-\infty}^{\tau} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \cdots \int_{-\infty}^{\tau_{2k-1}} d\tau_{2k} \cos[\tau_1^2 - \tau_2^2] \times \cdots \times \cos[\tau_{2k-1}^2 - \tau_{2k}^2], \quad k \geq 1.$$

These integrals can be related to the finite-time Landau-Zener (LZ) transition probability, $P_{\mathrm{LZ}}(\tau, \nu)$, for a two-level quantum system. The relationship is given by the k-th derivative of the transition probability with respect to the LZ parameter ν at $\nu = 0$:

$$\mathcal{I}_k(\tau) = \frac{(-1)^{k+1}}{2^{3k-1} k!} \left(\frac{\partial^k P_{\mathrm{LZ}}(\tau, \nu)}{\partial \nu^k}\Big|_{\nu=0} \right).$$

The LZ transition probability is expressed in terms of the Parabolic Cylinder Function (PCF), $D_\nu(z)$, as

$$P_{\mathrm{LZ}}(\tau, \nu) = \nu e^{-\pi\nu/2} \left| D_{-i\nu-1}\left(-i\mu_0\tau\right) \right|^2,$$

where $\mu_0 = \sqrt{2}e^{-i\pi/4}$. To compute the derivatives with respect to the index ν, we introduce a set of modified PCFs, $\mathcal{D}_\nu^{(k)}(z)$, for non-negative integers k, with $\mathcal{D}_\nu^{(0)}(z) = D_\nu(z)$. The derivative of these functions with respect to their index ν follows the rule:

$$\frac{\partial \mathcal{D}_\nu^{(k)}(z)}{\partial \nu} = \phi_\nu \mathcal{D}_\nu^{(k)}(z) + \mathcal{D}_\nu^{(k+1)}(z),$$

where $\phi_\nu = \psi^{(0)}(-\nu) - \frac{\ln 2}{2}$, and $\psi^{(n)}(z)$ is the polygamma function of order n. You may use the following properties: 1. The complex conjugate of a modified PCF is given by $\overline{\mathcal{D}_\nu^{(k)}(z)} = \mathcal{D}_{\bar{\nu}}^{(k)}(\bar{z})$. For our purposes, this means $\overline{\mathcal{D}_{-i\nu-1}^{(k)}(-i\mu_0\tau)} = \mathcal{D}_{i\nu-1}^{(k)}(i\bar{\mu}_0\tau)$. 2. At $\nu = 0$, the function $\phi_{-i\nu-1}$ is real, i.e., $\mathrm{Im}(\phi_{-1}) = 0$. 3. The first derivative of $\phi_{-i\nu-1}$ with respect to ν at $\nu = 0$ is $\frac{\partial}{\partial\nu}\phi_{-i\nu-1}\Big|_{\nu=0} = i\psi^{(1)}(1) = i\frac{\pi^2}{6}$.

Task: Using the provided framework, derive the finite-time expression for the integral $\mathcal{I}_3(\tau)$. Your final answer should be expressed in terms of $D_{-1}(-i\mu_0\tau)$, $\mathcal{D}_{-1}^{(1)}(-i\mu_0\tau)$, $\mathcal{D}_{-1}^{(2)}(-i\mu_0\tau)$, and their complex conjugates.

Original Solution

$$\mathcal{I}_3(\tau) = \frac{1}{2^8 \cdot 3!} \left[\frac{7\pi^2}{4} \left| D_{-1}\left(-i\mu_0\tau\right) \right|^2 + 6 \left| \mathcal{D}_{-1}^{(1)}\left(-i\mu_0\tau\right) \right|^2 + 6\pi \mathrm{Im}\left[D_{-1}\left(-i\mu_0\tau\right) \mathcal{D}_{-1}^{(1)}\left(i\overline{\mu}_0\tau\right) \right] - 6\mathrm{Re}\left[D_{-1}\left(-i\mu_0\tau\right) \mathcal{D}_{-1}^{(2)}\left(i\overline{\mu}_0\right.\right.\right.$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing multiple advanced concepts including nested oscillatory integrals, derivatives of special functions with respect to their order, and complex conjugation properties, culminating in a multi-step derivation of a finite-time expression involving modified Parabolic Cylinder Functions.

Refined Problem

Refined Problem Statement: Background: Consider a class of highly oscillatory nested integrals given by

$$\mathcal{I}_k(\tau) = \int_{-\infty}^{\tau} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \cdots \int_{-\infty}^{\tau_{2k-1}} d\tau_{2k} \cos[\tau_1^2 - \tau_2^2] \times \cdots \times \cos[\tau_{2k-1}^2 - \tau_{2k}^2], \quad k \geq 1.$$

These integrals can be related to the finite-time Landau-Zener (LZ) transition probability, $P_{\mathrm{LZ}}(\tau,\nu)$, for a two-level quantum system. The relationship is given by the k-th derivative of the transition probability with respect to the LZ parameter ν at $\nu=0$:

$$\mathcal{I}_k(\tau)=\frac{(-1)^{k+1}}{2^{3k-1}k!}\left(\frac{\partial^k P_{\mathrm{LZ}}(\tau,\nu)}{\partial\nu^k}\Big|_{\nu=0}\right).$$

The LZ transition probability is expressed in terms of the Parabolic Cylinder Function (PCF), $D_\nu(z)$, as

$$P_{\mathrm{LZ}}(\tau,\nu)=\nu e^{-\pi\nu/2}\left|D_{-i\nu-1}\left(-i\mu_0\tau\right)\right|^2,$$

where $\mu_0=\sqrt{2}e^{-i\pi/4}$. To compute the derivatives with respect to the index ν, we introduce a set of modified PCFs, $\mathcal{D}_\nu^{(k)}(z)$, for non-negative integers k, with $\mathcal{D}_\nu^{(0)}(z)=D_\nu(z)$. The derivative of these functions with respect to their index ν follows the rule:

$$\frac{\partial\mathcal{D}_\nu^{(k)}(z)}{\partial\nu}=\phi_\nu\mathcal{D}_\nu^{(k)}(z)+\mathcal{D}_\nu^{(k+1)}(z),$$

where $\phi_\nu=\psi^{(0)}(-\nu)-\frac{\ln 2}{2}$, and $\psi^{(n)}(z)$ is the polygamma function of order n. You may use the following properties: 1. The complex conjugate of a modified PCF is given by $\overline{\mathcal{D}_\nu^{(k)}(z)}=\mathcal{D}_{\bar\nu}^{(k)}(\bar z)$. For our purposes, this means $\overline{\mathcal{D}_{-i\nu-1}^{(k)}(-i\mu_0\tau)}=\mathcal{D}_{i\nu-1}^{(k)}(i\bar\mu_0\tau)$. 2. At $\nu=0$, the function $\phi_{-i\nu-1}$ is real, i.e., $\mathrm{Im}(\phi_{-1})=0$. 3. The first derivative of $\phi_{-i\nu-1}$ with respect to ν at $\nu=0$ is $\frac{\partial}{\partial\nu}\phi_{-i\nu-1}\Big|_{\nu=0}=i\psi^{(1)}(1)=i\frac{\pi^2}{6}$.

Task: Using the provided framework, derive the finite-time expression for the integral $\mathcal{I}_3(\tau)$. Your final answer should be expressed in terms of $D_{-1}(-i\mu_0\tau)$, $\mathcal{D}_{-1}^{(1)}(-i\mu_0\tau)$, $\mathcal{D}_{-1}^{(2)}(-i\mu_0\tau)$, and their complex conjugates.

Refined Solution:

$$\mathcal{I}_3(\tau)=\frac{1}{2^8\cdot 3!}\left[\frac{7\pi^2}{4}\left|D_{-1}\left(-i\mu_0\tau\right)\right|^2+6\left|\mathcal{D}_{-1}^{(1)}\left(-i\mu_0\tau\right)\right|^2+6\pi\mathrm{Im}\left[D_{-1}\left(-i\mu_0\tau\right)\mathcal{D}_{-1}^{(1)}\left(i\overline{\mu}_0\tau\right)\right]-6\mathrm{Re}\left[D_{-1}\left(-i\mu_0\tau\right)\mathcal{D}_{-1}^{(2)}\left(i\overline{\mu}_0\right.\right.\right.$$

Problem 23 (Paper: 2506.23386v1, Index: 0)

Original Problem Statement

Background: Consider a hybrid quantum system composed of a two-level atom (qubit) and a single-mode quantized electromagnetic field, described by the Jaynes-Cummings (JC) model. The system is in the resonant regime, where the atomic transition frequency Ω equals the field frequency ω. The atom's Hilbert space is spanned by the excited state e and ground state g, while the field's Hilbert space is spanned by the Fock states n for $n = 0, 1, 2, \ldots$. At time $t = 0$, the system is prepared in a separable state where the atom is excited and the field is in a coherent state: $\psi(0) = e \otimes \alpha$. The coherent state is given in the Fock basis by $\alpha = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} n \equiv \sum_{n=0}^{\infty} C_n n$. The time-evolved density operator of the system, $\hat{\rho}(t)$, is given by

$$\begin{aligned}\hat{\rho}(t) = \sum_{n,m=0}^{\infty} C_n C_m^* e^{-it(E_n - E_m)} \Big[&\cos(gt\sqrt{n+1}) \cos(gt\sqrt{m+1}) e, ne, m \\ &+ i \cos(gt\sqrt{n+1}) \sin(gt\sqrt{m+1}) e, ng, m+1 \\ &- i \sin(gt\sqrt{n+1}) \cos(gt\sqrt{m+1}) g, n+1e, m \\ &+ \sin(gt\sqrt{n+1}) \sin(gt\sqrt{m+1}) g, n+1g, m+1 \Big],\end{aligned}$$

where g is the atom-field coupling strength, $E_n = \omega(n + 1/2)$, and $a, k \equiv a \otimes k$.

The state of this hybrid system can be described by a Wigner function $W_{\hat{\rho}}(\theta, \phi, \beta, \beta^*)$ on a phase space parameterized by spherical coordinates (θ, ϕ) for the qubit and complex coordinates (β, β^*) for the field. The Wigner function is defined by the Stratonovich-Weyl correspondence as $W_{\hat{\rho}} = [\hat{\rho}(\hat{\Delta}_q \otimes \hat{\Delta}_f)]$, where $\hat{\Delta}_q$ and $\hat{\Delta}_f$ are the Stratonovich-Weyl kernels for the qubit and field, respectively. In the $\{e, g\}$ basis, the qubit kernel is

$$\hat{\Delta}_q(\theta, \phi) = \frac{1}{2} \begin{pmatrix} 1 + \sqrt{3}\cos\theta & \sqrt{3} e^{-i\phi} \sin\theta \\ \sqrt{3} e^{i\phi} \sin\theta & 1 - \sqrt{3}\cos\theta \end{pmatrix}.$$

For the field, the trace of the operator nm with the kernel $\hat{\Delta}_f$ is given by

$${}_f[nm\hat{\Delta}_f(\beta, \beta^*)] = \frac{2}{\pi} \frac{e^{-2|\beta|^2}}{\sqrt{n!m!}} L_{n,m}(2\beta, 2\beta^*),$$

where $L_{n,m}(x, y)$ are the Laguerre 2D polynomials.

Task: Using the provided density operator $\hat{\rho}(t)$ and the Stratonovich-Weyl kernels, derive the full Wigner function $W_{\hat{\rho}}(\theta, \phi, \beta, \beta^*)$ for the Jaynes-Cummings model. Express your final answer as a sum over indices n and m.

Original Solution

$$\begin{aligned}W_{\hat{\rho}}(\theta, \phi, \beta, \beta^*) = \sum_{n,m=0}^{\infty} \Big[&\frac{1}{2}(1 + \sqrt{3}\cos\theta) C_n {C_m}^* e^{-it(E_n - E_m)} \cos(tg\sqrt{n+1}) \cos(tg\sqrt{m+1}) \\ &+ i\frac{\sqrt{3}}{2} e^{i\phi} \sin\theta C_n C_{m-1}^* e^{-it(E_n - E_{m-1})} \cos(tg\sqrt{n+1}) \sin(tg\sqrt{m}) \\ &- i\frac{\sqrt{3}}{2} e^{-i\phi} \sin\theta C_{n-1} C_m^* e^{-it(E_{n-1} - E_m)} \sin(tg\sqrt{n}) \cos(tg\sqrt{m+1}) \\ &+ \frac{1}{2}(1 - \sqrt{3}\cos\theta) C_{n-1} C_{m-1}^* e^{-it(E_{n-1} - E_{m-1})} \sin(tg\sqrt{n}) \sin(tg\sqrt{m}) \Big] \\ &\times \frac{2}{\pi} \frac{e^{-2|\beta|^2}}{\sqrt{n!m!}} L_{n,m}(2\beta, 2\beta^*)\end{aligned}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing the Jaynes-Cummings time-evolved density operator with the Stratonovich-Weyl kernels for both qubit and field, involving nontrivial operator expansions, infinite sums, and special functions, thus demanding a multi-step derivation beyond direct formula application.

Refined Problem

Refined Problem Statement: Background: Consider a hybrid quantum system composed of a two-level atom (qubit) and a single-mode quantized electromagnetic field, described by the Jaynes-Cummings (JC) model. The system is in the resonant regime, where the atomic transition frequency Ω equals the field frequency ω. The atom's Hilbert space is spanned by the excited state e and ground state g, while the field's Hilbert space is spanned by the Fock states n for $n = 0, 1, 2, \ldots$. At time $t = 0$, the system is prepared in a separable state where the atom is excited and the field is in a coherent state: $\psi(0) = e \otimes \alpha$. The coherent state is given in the Fock basis by $\alpha = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} n \equiv \sum_{n=0}^{\infty} C_n n$. The time-evolved density operator of the system, $\hat{\rho}(t)$, is given by

$$\begin{aligned}\hat{\rho}(t) = \sum_{n,m=0}^{\infty} C_n C_m^* e^{-it(E_n - E_m)} \Big[& \cos(gt\sqrt{n+1}) \cos(gt\sqrt{m+1}) e, ne, m \\ & + i \cos(gt\sqrt{n+1}) \sin(gt\sqrt{m+1}) e, ng, m+1 \\ & - i \sin(gt\sqrt{n+1}) \cos(gt\sqrt{m+1}) g, n+1e, m \\ & + \sin(gt\sqrt{n+1}) \sin(gt\sqrt{m+1}) g, n+1g, m+1 \Big],\end{aligned}$$

where g is the atom-field coupling strength, $E_n = \omega(n + 1/2)$, and $a, k \equiv a \otimes k$.

The state of this hybrid system can be described by a Wigner function $W_{\hat{\rho}}(\theta, \phi, \beta, \beta^*)$ on a phase space parameterized by spherical coordinates (θ, ϕ) for the qubit and complex coordinates (β, β^*) for the field. The Wigner function is defined by the Stratonovich-Weyl correspondence as $W_{\hat{\rho}} = [\hat{\rho}(\hat{\Delta}_q \otimes \hat{\Delta}_f)]$, where $\hat{\Delta}_q$ and $\hat{\Delta}_f$ are the Stratonovich-Weyl kernels for the qubit and field, respectively. In the $\{e, g\}$ basis, the qubit kernel is

$$\hat{\Delta}_q(\theta, \phi) = \frac{1}{2} \begin{pmatrix} 1 + \sqrt{3} \cos\theta & \sqrt{3} e^{-i\phi} \sin\theta \\ \sqrt{3} e^{i\phi} \sin\theta & 1 - \sqrt{3} \cos\theta \end{pmatrix}.$$

For the field, the trace of the operator nm with the kernel $\hat{\Delta}_f$ is given by

$${}_f[nm\hat{\Delta}_f(\beta, \beta^*)] = \frac{2}{\pi} \frac{e^{-2|\beta|^2}}{\sqrt{n!m!}} L_{n,m}(2\beta, 2\beta^*),$$

where $L_{n,m}(x, y)$ are the Laguerre 2D polynomials.

Task: Using the provided density operator $\hat{\rho}(t)$ and the Stratonovich-Weyl kernels, derive the full Wigner function $W_{\hat{\rho}}(\theta, \phi, \beta, \beta^*)$ for the Jaynes-Cummings model. Express your final answer as a sum over indices n and m.

Refined Solution:

$$\begin{aligned}W_{\hat{\rho}}(\theta, \phi, \beta, \beta^*) = \sum_{n,m=0}^{\infty} \Big[& \frac{1}{2}(1 + \sqrt{3} \cos\theta) C_n {C_m}^* e^{-it(E_n - E_m)} \cos(tg\sqrt{n+1}) \cos(tg\sqrt{m+1}) \\ & + i \frac{\sqrt{3}}{2} e^{i\phi} \sin\theta C_n C_{m-1}^* e^{-it(E_n - E_{m-1})} \cos(tg\sqrt{n+1}) \sin(tg\sqrt{m}) \\ & - i \frac{\sqrt{3}}{2} e^{-i\phi} \sin\theta C_{n-1} C_m^* e^{-it(E_{n-1} - E_m)} \sin(tg\sqrt{n}) \cos(tg\sqrt{m+1}) \\ & + \frac{1}{2}(1 - \sqrt{3} \cos\theta) C_{n-1} C_{m-1}^* e^{-it(E_{n-1} - E_{m-1})} \sin(tg\sqrt{n}) \sin(tg\sqrt{m}) \Big] \\ & \times \frac{2}{\pi} \frac{e^{-2|\beta|^2}}{\sqrt{n!m!}} L_{n,m}(2\beta, 2\beta^*)\end{aligned}$$

Problem 24 (Paper: 2506.23386v1, Index: 1)

Original Problem Statement

Background: Consider a hybrid quantum system composed of a two-level atom (qubit) and a single-mode quantized electromagnetic field, evolving under the Jaynes-Cummings (JC) Hamiltonian. We assume the resonant condition, where the atomic transition frequency Ω equals the field frequency ω. The atom is initially in its excited state e, and the field is in a coherent state α, which can be expanded in the Fock basis as $\alpha = \sum_{n=0}^{\infty} C_n n$, with coefficients $C_n = e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}$. The state of the composite system at time t can be described by a Wigner function $W_{\hat{\rho}}(\theta, \phi, \beta, \beta^*)$ defined on the total phase space. The qubit's phase space is a two-sphere S^2 parametrized by spherical coordinates (θ, ϕ), while the field's phase space is the complex plane parametrized by (β, β^*). The full Wigner function for the time-evolved state is given by:

$$
\begin{aligned}
W_{\hat{\rho}}(\theta,\phi,\beta,\beta^*) = \sum_{n,m=0}^{\infty} \Big[& \frac{1}{2}(1+\sqrt{3}\cos\theta) C_n {C_m}^* e^{-it(E_n - E_m)} \cos(tg\sqrt{n+1}) \cos(tg\sqrt{m+1}) \\
& + i\frac{\sqrt{3}}{2} e^{i\phi} \sin\theta C_n C^*_{m-1} e^{-it(E_n - E_{m-1})} \cos(tg\sqrt{n+1}) \sin(tg\sqrt{m}) \\
& - i\frac{\sqrt{3}}{2} e^{-i\phi} \sin\theta C_{n-1} C^*_m e^{-it(E_{n-1} - E_m)} \sin(tg\sqrt{n}) \cos(tg\sqrt{m+1}) \\
& + \frac{1}{2}(1-\sqrt{3}\cos\theta) C_{n-1} C^*_{m-1} e^{-it(E_{n-1} - E_{m-1})} \sin(tg\sqrt{n}) \sin(tg\sqrt{m}) \Big] \\
& \times \frac{2}{\pi} \frac{e^{-2|\beta|^2}}{\sqrt{n!m!}} L_{n,m}(2\beta, 2\beta^*)
\end{aligned}
$$

Here, g is the atom-field coupling strength, $E_k = \omega(k + 1/2)$ are the free energy eigenvalues, and $L_{n,m}(\gamma, \gamma^*)$ are the 2D Laguerre polynomials. To analyze the properties of the field subsystem alone, one can compute its reduced Wigner function, $W_{\hat{\rho}_f}(\beta, \beta^*)$, by tracing out the qubit degrees of freedom. In the phase space formalism, this operation corresponds to integrating the full Wigner function over the qubit's phase space with the appropriate measure:

$$
W_{\hat{\rho}_f}(\beta,\beta^*) = \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin\theta \, W_{\hat{\rho}}(\theta,\phi,\beta,\beta^*)
$$

Task: By performing the integration over the qubit's phase space coordinates (θ, ϕ), derive the expression for the reduced Wigner function of the bosonic field, $W_{\hat{\rho}_f}(\beta, \beta^*)$.

Original Solution

$$
W_{\hat{\rho}_f}(\beta,\beta^*) = \sum_{n,m=0}^{\infty} \Big[C_n {C_m}^* e^{-it(E_n - E_m)} \cos(tg\sqrt{n+1}) \cos(tg\sqrt{m+1}) + C_{n-1} C^*_{m-1} e^{-it(E_{n-1} - E_{m-1})} \sin(tg\sqrt{n}) \sin(tg\sqrt{m}) \Big]
$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires performing a nontrivial integration over the qubit's spherical phase space, carefully handling angular dependencies and orthogonality relations, to reduce the full Wigner function to the field's reduced Wigner function, thus synthesizing knowledge of quantum phase space methods, special functions, and the Jaynes-Cummings model dynamics.

Refined Problem

Refined Problem Statement: Background: Consider a hybrid quantum system composed of a two-level atom (qubit) and a single-mode quantized electromagnetic field, evolving under the Jaynes-Cummings (JC) Hamiltonian. We assume the resonant condition, where the atomic transition frequency Ω equals the field frequency ω. The atom is initially in its excited state e, and the field is in a coherent state α, which can be expanded in the Fock basis as $\alpha = \sum_{n=0}^{\infty} C_n n$, with coefficients $C_n = e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}$.

The state of the composite system at time t can be described by a Wigner function $W_{\hat{\rho}}(\theta, \phi, \beta, \beta^*)$ defined on the total phase space. The qubit's phase space is a two-sphere S^2 parametrized by spherical coordinates (θ, ϕ), while the field's phase space is the complex plane parametrized by (β, β^*). The full Wigner function for the time-evolved state is given by:

$$\begin{aligned} W_{\hat{\rho}}(\theta, \phi, \beta, \beta^*) = \sum_{n,m=0}^{\infty} \Big[& \frac{1}{2}(1 + \sqrt{3}\cos\theta) C_n {C_m}^* e^{-it(E_n - E_m)} \cos(tg\sqrt{n+1}) \cos(tg\sqrt{m+1}) \\ & + i\frac{\sqrt{3}}{2} e^{i\phi} \sin\theta C_n C^*_{m-1} e^{-it(E_n - E_{m-1})} \cos(tg\sqrt{n+1}) \sin(tg\sqrt{m}) \\ & - i\frac{\sqrt{3}}{2} e^{-i\phi} \sin\theta C_{n-1} C^*_m e^{-it(E_{n-1} - E_m)} \sin(tg\sqrt{n}) \cos(tg\sqrt{m+1}) \\ & + \frac{1}{2}(1 - \sqrt{3}\cos\theta) C_{n-1} C^*_{m-1} e^{-it(E_{n-1} - E_{m-1})} \sin(tg\sqrt{n}) \sin(tg\sqrt{m}) \Big] \\ & \times \frac{2}{\pi} \frac{e^{-2|\beta|^2}}{\sqrt{n!m!}} L_{n,m}(2\beta, 2\beta^*) \end{aligned}$$

Here, g is the atom-field coupling strength, $E_k = \omega(k + 1/2)$ are the free energy eigenvalues, and $L_{n,m}(\gamma, \gamma^*)$ are the 2D Laguerre polynomials. To analyze the properties of the field subsystem alone, one can compute its reduced Wigner function, $W_{\hat{\rho}_f}(\beta, \beta^*)$, by tracing out the qubit degrees of freedom. In the phase space formalism, this operation corresponds to integrating the full Wigner function over the qubit's phase space with the appropriate measure:

$$W_{\hat{\rho}_f}(\beta, \beta^*) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin\theta \, W_{\hat{\rho}}(\theta, \phi, \beta, \beta^*)$$

Task: By performing the integration over the qubit's phase space coordinates (θ, ϕ), derive the expression for the reduced Wigner function of the bosonic field, $W_{\hat{\rho}_f}(\beta, \beta^*)$.

Refined Solution:

$$W_{\hat{\rho}_f}(\beta, \beta^*) = \sum_{n,m=0}^{\infty} \Big[C_n {C_m}^* e^{-it(E_n - E_m)} \cos(tg\sqrt{n+1}) \cos(tg\sqrt{m+1}) + C_{n-1} C^*_{m-1} e^{-it(E_{n-1} - E_{m-1})} \sin(tg\sqrt{n}) \sin(tg\sqrt{m})$$

Problem 25 (Paper: 2506.23447v1, Index: 0)

Original Problem Statement

Background: In a theoretical model of information encoding, the length of a codeword for a continuous variable x from the space $\mathcal{C} = (1, +\infty)$ is given by a function $\ell(x)$. The probability density of a codeword is $p(x)$. The optimal code length function $\ell(x)$ is one that minimizes the average codeword length, $\mathbb{E}[\ell] = \int_{\mathcal{C}} p(x)\ell(x)dx$, subject to the continuous form of the Kraft-McMillan inequality, $\int_{\mathcal{C}} \exp(-\ell(x))dx \leq 1$. This constrained optimization problem can be analyzed using the Lagrangian formalism with a multiplier $\lambda > 0$:

$$\mathcal{L}(\ell) = \int_{\mathcal{C}} p(x)\ell(x)dx - \lambda \left(1 - \int_{\mathcal{C}} \exp(-\ell(x))dx\right)$$

Furthermore, the system is assumed to possess a scale-invariance property. This property is captured by a renormalization rule which, for a fixed-point probability distribution $p(x)$, is expressed by the relation:

$$p(\exp(x)) = p(x) \cdot \exp(-x)$$

where $\exp(x)$ is the exponential function. The process of calculating the length is defined recursively. This recursion terminates, contributing nothing further to the total length, when any iterated argument to the length function becomes less than or equal to 1.

Task: First, find the optimal relationship between $p(x)$ and $\ell(x)$ by functionally minimizing the Lagrangian $\mathcal{L}(\ell)$. Then, use this relationship to translate the renormalization fixed-point condition on $p(x)$ into a recursive equation for the optimal, fixed-point code length function, which we denote $\ell_*(x)$. Finally, by recursively expanding this equation, determine the complete functional form of $\ell_*(x)$. Express your answer as a sum of iterated natural logarithms.

Original Solution

$$\ell_*(x) = \log x + \log(\log x) + \log\big(\log(\log x)\big) + \cdots$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires functional minimization under a continuous constraint, application of a scale-invariance renormalization condition, and recursive expansion to derive a nontrivial infinite sum expression for the optimal code length function.

Refined Problem

Refined Problem Statement: Background: In a theoretical model of information encoding, the length of a codeword for a continuous variable x from the space $\mathcal{C} = (1, +\infty)$ is given by a function $\ell(x)$. The probability density of a codeword is $p(x)$. The optimal code length function $\ell(x)$ is one that minimizes the average codeword length, $\mathbb{E}[\ell] = \int_{\mathcal{C}} p(x)\ell(x)dx$, subject to the continuous form of the Kraft-McMillan inequality, $\int_{\mathcal{C}} \exp(-\ell(x))dx \leq 1$. This constrained optimization problem can be analyzed using the Lagrangian formalism with a multiplier $\lambda > 0$:

$$\mathcal{L}(\ell) = \int_{\mathcal{C}} p(x)\ell(x)dx - \lambda \left(1 - \int_{\mathcal{C}} \exp(-\ell(x))dx\right)$$

Furthermore, the system is assumed to possess a scale-invariance property. This property is captured by a renormalization rule which, for a fixed-point probability distribution $p(x)$, is expressed by the relation:

$$p(\exp(x)) = p(x) \cdot \exp(-x)$$

where $\exp(x)$ is the exponential function. The process of calculating the length is defined recursively. This recursion terminates, contributing nothing further to the total length, when any iterated argument to the length function becomes less than or equal to 1.

Task: First, find the optimal relationship between $p(x)$ and $\ell(x)$ by functionally minimizing the Lagrangian $\mathcal{L}(\ell)$. Then, use this relationship to translate the renormalization fixed-point condition on $p(x)$ into a recursive equation for the optimal, fixed-point code length function, which we denote $\ell_*(x)$. Finally, by recursively expanding this equation, determine the complete functional form of $\ell_*(x)$. Express your answer as a sum of iterated natural logarithms.

Refined Solution:

$$\ell_*(x) = \log x + \log(\log x) + \log\big(\log(\log x)\big) + \cdots$$

Problem 26 (Paper: 2506.23447v1, Index: 1)

Original Problem Statement

Background: Consider a prefix code for the set of positive integers $\mathbb{N} = \{1, 2, 3, \ldots\}$. Let $\beta(n) = \lfloor \log_2 n \rfloor + 1$ denote the number of bits in the standard binary representation of an integer n. The length of the codeword for an integer n, denoted $L(n)$, is defined by the following recursive relationship:

- For $n = 1$, the code length is $L(1) = 1$.
- For $n > 1$, the code length is given by the recurrence $L(n) = L(\beta(n) - 1) + \beta(n)$.

The completeness of such a code can be investigated by examining its Kraft's sum, defined as $S = \sum_{n=1}^{\infty} 2^{-L(n)}$. To analyze this sum, we partition the set of positive integers $\mathbb{N}$ into disjoint subsets I_k for $k = 1, 2, 3, \ldots$, where each subset contains all integers that have the same binary length: $I_k = \{n \in \mathbb{N} \mid \beta(n) = k\}$. Note that this implies $I_k = \{n \in \mathbb{N} \mid 2^{k-1} \leq n \leq 2^k - 1\}$ for $k \geq 1$.

Task: By first partitioning the total Kraft's sum S into a sum over the sets I_k, and then evaluating the partial sums over each I_k, derive a linear algebraic equation that relates S to itself.

Original Solution

$$S = \frac{1 + S}{2}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires partitioning an infinite sum, applying a recursive definition of code lengths, and synthesizing these to derive a self-referential linear equation for the Kraft sum, involving multiple steps and concepts beyond direct formula application.

Refined Problem

Refined Problem Statement: Background: Consider a prefix code for the set of positive integers $\mathbb{N} = \{1, 2, 3, \ldots\}$. Let $\beta(n) = \lfloor \log_2 n \rfloor + 1$ denote the number of bits in the standard binary representation of an integer n. The length of the codeword for an integer n, denoted $L(n)$, is defined by the following recursive relationship:

- For $n = 1$, the code length is $L(1) = 1$.
- For $n > 1$, the code length is given by the recurrence $L(n) = L(\beta(n) - 1) + \beta(n)$.

The completeness of such a code can be investigated by examining its Kraft's sum, defined as $S = \sum_{n=1}^{\infty} 2^{-L(n)}$. To analyze this sum, we partition the set of positive integers $\mathbb{N}$ into disjoint subsets I_k for $k = 1, 2, 3, \ldots$, where each subset contains all integers that have the same binary length: $I_k = \{n \in \mathbb{N} \mid \beta(n) = k\}$. Note that this implies $I_k = \{n \in \mathbb{N} \mid 2^{k-1} \leq n \leq 2^k - 1\}$ for $k \geq 1$.

Task: By first partitioning the total Kraft's sum S into a sum over the sets I_k, and then evaluating the partial sums over each I_k, derive a linear algebraic equation that relates S to itself.

Refined Solution:

$$S = \frac{1 + S}{2}$$

Problem 27 (Paper: 2506.23451v1, Index: 0)

Original Problem Statement

Background: Consider a one-dimensional continuous dynamical system for a variable $x(t)$ governed by the differential equation $\frac{dx}{dt} = F(x, \alpha)$, where the function F is given by $F(x, \alpha) = f(x, \alpha) - g(x, \alpha)$. The functions f and g are positive and smooth, and $\alpha = (\alpha_1, \alpha_2)$ is a two-component positive bifurcation parameter. This system can be discretized using the tropical discretization scheme, resulting in the discrete map $x_{n+1} = F_\tau(x_n, \alpha)$, where n is the iteration step and τ is the time interval. The map F_τ is defined as:

$$F_\tau(x, \alpha) = \frac{x + \tau f(x, \alpha)}{x + \tau g(x, \alpha)} x$$

Assume that the system exhibits a cusp bifurcation at a point $(\bar{x}, \bar{\alpha})$. At this nonhyperbolic fixed point, the following conditions for the continuous system are satisfied:

1. $F(\bar{x}, \bar{\alpha}) = 0$, which implies $f(\bar{x}, \bar{\alpha}) = g(\bar{x}, \bar{\alpha})$.
2. $\dfrac{\partial F(\bar{x}, \bar{\alpha})}{\partial x} = 0.$

For convenience, we define an auxiliary function $Z_\tau(x, \alpha) = \frac{\tau x}{x + \tau f(x, \alpha)}$.

Task: At the cusp bifurcation point $(\bar{x}, \bar{\alpha})$, the discrete map F_τ also satisfies certain conditions. One of these involves a combination of its partial derivatives. Derive the expression for the quantity $C(F_\tau) = \dfrac{\partial F_\tau(\bar{x}, \bar{\alpha})}{\partial \alpha_1} \dfrac{\partial^2 F_\tau(\bar{x}, \bar{\alpha})}{\partial x \partial \alpha_2} - \dfrac{\partial F_\tau(\bar{x}, \bar{\alpha})}{\partial \alpha_2} \dfrac{\partial^2 F_\tau(\bar{x}, \bar{\alpha})}{\partial x \partial \alpha_1}$ in terms of the function $Z_\tau(\bar{x}, \bar{\alpha})$ and the derivatives of the original function $F(x, \alpha)$ evaluated at the bifurcation point.

Original Solution

$$Z_\tau^2(\bar{x}, \bar{\alpha}) \left(\frac{\partial F(\bar{x}, \bar{\alpha})}{\partial \alpha_1} \frac{\partial^2 F(\bar{x}, \bar{\alpha})}{\partial x \partial \alpha_2} - \frac{\partial F(\bar{x}, \bar{\alpha})}{\partial \alpha_2} \frac{\partial^2 F(\bar{x}, \bar{\alpha})}{\partial x \partial \alpha_1} \right)$$

Critiques

Self-Containment Critique: **Status: Not self-contained**

The final solution expression contains a malformed LaTeX environment and an unmatched closing brace, which may confuse readers and obscure the intended formula.

Difficulty Critique: **Status: Non-trivial**

The problem requires deriving a complex expression involving mixed partial derivatives of a discretized map at a cusp bifurcation point, synthesizing the continuous system's properties with the tropical discretization scheme and auxiliary functions, thus demanding a multi-step, advanced chain of reasoning.

Refined Problem

Refined Problem Statement: Background: Consider a one-dimensional continuous dynamical system for a variable $x(t)$ governed by the differential equation $\frac{dx}{dt} = F(x, \alpha)$, where the function F is given by $F(x, \alpha) = f(x, \alpha) - g(x, \alpha)$. The functions f and g are positive and smooth, and $\alpha = (\alpha_1, \alpha_2)$ is a two-component positive bifurcation parameter. This system can be discretized using the tropical discretization scheme, resulting in the discrete map $x_{n+1} = F_\tau(x_n, \alpha)$, where n is the iteration step and τ is the time interval. The map F_τ is defined as:

$$F_\tau(x, \alpha) = \frac{x + \tau f(x, \alpha)}{x + \tau g(x, \alpha)} x$$

Assume that the system exhibits a cusp bifurcation at a point $(\bar{x}, \bar{\alpha})$. At this nonhyperbolic fixed point, the following conditions for the continuous system are satisfied:

1. $F(\bar{x}, \bar{\alpha}) = 0$, which implies $f(\bar{x}, \bar{\alpha}) = g(\bar{x}, \bar{\alpha})$.
2. $\dfrac{\partial F(\bar{x}, \bar{\alpha})}{\partial x} = 0.$

For convenience, we define an auxiliary function $Z_\tau(x,\alpha) = \frac{\tau x}{x+\tau f(x,\alpha)}$.

Task: At the cusp bifurcation point $(\bar{x},\bar{\alpha})$, the discrete map F_τ also satisfies certain conditions. One of these involves a combination of its partial derivatives. Derive the expression for the quantity

$$C(F_\tau) = \frac{\partial F_\tau(\bar{x},\bar{\alpha})}{\partial \alpha_1}\frac{\partial^2 F_\tau(\bar{x},\bar{\alpha})}{\partial x \partial \alpha_2} - \frac{\partial F_\tau(\bar{x},\bar{\alpha})}{\partial \alpha_2}\frac{\partial^2 F_\tau(\bar{x},\bar{\alpha})}{\partial x \partial \alpha_1}$$

in terms of the function $Z_\tau(\bar{x},\bar{\alpha})$ and the derivatives of the original function $F(x,\alpha)$ evaluated at the bifurcation point.

Refined Solution:

$$Z_\tau^2(\bar{x},\bar{\alpha})\left(\frac{\partial F(\bar{x},\bar{\alpha})}{\partial \alpha_1}\frac{\partial^2 F(\bar{x},\bar{\alpha})}{\partial x \partial \alpha_2} - \frac{\partial F(\bar{x},\bar{\alpha})}{\partial \alpha_2}\frac{\partial^2 F(\bar{x},\bar{\alpha})}{\partial x \partial \alpha_1}\right)$$

Problem 28 (Paper: 2506.23451v1, Index: 1)

Original Problem Statement

Background: Consider a one-dimensional biochemical model described by the differential equation $\frac{dx}{dt} = p - qx + \frac{x^2}{1+x^2}$, where the state variable $x \geq 0$ and the parameters p, q are positive. This continuous system can be related to a discrete-time map through a procedure known as tropical discretization. In the limit of an infinite time step, this procedure yields the following discrete map for the state variable x_n at step n: $x_{n+1} = \frac{p}{q} + \frac{x_n^2}{q(1+x_n^2)}$. This discrete map can be further transformed into a max-plus dynamical system via the process of ultradiscretization. This involves the variable substitutions $x_n = e^{X_n/\varepsilon}$, $p = e^{P/\varepsilon}$, and $q = e^{Q/\varepsilon}$, where ε is a small positive parameter. The ultradiscrete limit is then taken as $\varepsilon \to +0$. The fundamental identity for this limit is $\lim_{\varepsilon \to +0} \varepsilon \log(e^{A/\varepsilon} + e^{B/\varepsilon}) = \max(A, B)$ for any real numbers A and B. After deriving the max-plus equation for X_{n+1}, a final transformation of the state variable, $X_n \to -X_n$, is applied to the system to obtain a canonical form.

Task: Apply the ultradiscretization procedure to the given discrete-time map for x_{n+1}. First, substitute the exponential variables and take the ultradiscrete limit $\varepsilon \to +0$ to find the max-plus equation for X_{n+1}. Then, apply the transformation $X_n \to -X_n$ to this resulting max-plus equation. Determine the final expression for the transformed state variable X_{n+1} in terms of X_n, P, and Q.

Original Solution

$$X_{n+1} = Q - \max(P, -\max(0, 2X_n))$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires multiple non-trivial steps including variable substitution, taking a non-standard ultradiscrete limit, applying max-plus algebra, and performing a final variable transformation, thus demanding synthesis of several advanced concepts.

Refined Problem

Refined Problem Statement: Background: Consider a one-dimensional biochemical model described by the differential equation $\frac{dx}{dt} = p - qx + \frac{x^2}{1+x^2}$, where the state variable $x \geq 0$ and the parameters p, q are positive. This continuous system can be related to a discrete-time map through a procedure known as tropical discretization. In the limit of an infinite time step, this procedure yields the following discrete map for the state variable x_n at step n: $x_{n+1} = \frac{p}{q} + \frac{x_n^2}{q(1+x_n^2)}$. This discrete map can be further transformed into a max-plus dynamical system via the process of ultradiscretization. This involves the variable substitutions $x_n = \exp(X_n/\varepsilon)$, $p = \exp(P/\varepsilon)$, and $q = \exp(Q/\varepsilon)$, where ε is a small positive parameter. The ultradiscrete limit is then taken as $\varepsilon \to +0$. The fundamental identity for this limit is $\lim_{\varepsilon \to +0} \varepsilon \log(\exp(A/\varepsilon) + \exp(B/\varepsilon)) = \max(A, B)$ for any real numbers A and B. After deriving the max-plus equation for X_{n+1}, a final transformation of the state variable, $X_n \to -X_n$, is applied to the system to obtain a canonical form.

Task: Apply the ultradiscretization procedure to the given discrete-time map for x_{n+1}. First, substitute the exponential variables and take the ultradiscrete limit $\varepsilon \to +0$ to find the max-plus equation for X_{n+1}. Then, apply the transformation $X_n \to -X_n$ to this resulting max-plus equation. Determine the final expression for the transformed state variable X_{n+1} in terms of X_n, P, and Q.

Refined Solution:

$$X_{n+1} = Q - \max(P, -\max(0, 2X_n))$$

Problem 29 (Paper: 2506.23496v1, Index: 0)

Original Problem Statement

Background: Consider a reversible, detailed-balanced Chemical Reaction Network (CRN). The state of the system is described by a vector of species concentrations $q \in \mathbb{R}^S_{\geq 0}$. The network consists of a set of reactions R. For each reaction $r \in R$, the forward and reverse reaction fluxes are given by mass-action kinetics: $J_r(q) = k_r q^{r^-}$ and $J^*_r(q) = k^*_r q^{r^+}$, where r^- and r^+ are the stoichiometric vectors of reactants and products, respectively, and the notation q^v denotes $\prod_s q_s^{v_s}$. The system is detailed-balanced, meaning there exists an equilibrium concentration vector at which the forward and reverse fluxes for every reaction are equal: $k_r^{r^-} = k_r^{*r^+} =: \Phi_r$. The quantity Φ_r is the detailed-balanced flux for reaction r. A reaction r is said to be "blocked" if its net flux is constrained to be zero. The minimal log-improbability cost rate to maintain this blocked state is given by $\dot{\delta}(r, q) = (\sqrt{J_r(q)} - \sqrt{J^*_r(q)})^2$. We analyze the system in the linear response regime, where concentrations q are close to . In this regime, the net species current $I \in \mathbb{R}^S$ is related to a species potential vector $V(q) = (q/ - \mathbf{e})$ by a generalized Ohm's law: $V(q) = -RI$. Here, the division is element-wise, $\mathbf{e}$ is a vector of all ones, and R is the symmetric, positive semi-definite resistance matrix of the network. The net change in species due to a single firing of reaction r is given by its stoichiometric vector $\Delta r = r^+ - r^-$.

Task: For a detailed-balanced CRN in the linear response regime, derive an expression for the blocking cost rate $\dot{\delta}(r, q)$ of a single reaction r. Your final expression should be in terms of the species current I, the resistance matrix R, the detailed-balanced flux Φ_r, and the stoichiometric vector Δr.

Original Solution

$$\dot{\delta}(r, q) = I^T R(\Delta r)\frac{\Phi_r}{4}(\Delta r)^T RI$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing multiple concepts including detailed balance, mass-action kinetics, linear response theory, and matrix relations to derive a nontrivial expression for the blocking cost rate in terms of species currents and resistance matrices.

Refined Problem

Refined Problem Statement: Background: Consider a reversible, detailed-balanced Chemical Reaction Network (CRN). The state of the system is described by a vector of species concentrations $q \in \mathbb{R}^{\mathcal{S}}_{\geq 0}$. The network consists of a set of reactions $\mathcal{R}$. For each reaction $r \in \mathcal{R}$, the forward and reverse reaction fluxes are given by mass-action kinetics: $J_r(q) = k_r q^{r^-}$ and $J^*_r(q) = k^*_r q^{r^+}$, where r^- and r^+ are the stoichiometric vectors of reactants and products, respectively, and the notation q^v denotes $\prod_s q_s^{v_s}$. The system is detailed-balanced, meaning there exists an equilibrium concentration vector eq at which the forward and reverse fluxes for every reaction are equal: $k_r \text{eq}^{r^-} = k^*_r \text{eq}^{r^+} =: \Phi_r$. The quantity Φ_r is the detailed-balanced flux for reaction r. A reaction r is said to be "blocked" if its net flux is constrained to be zero. The minimal log-improbability cost rate to maintain this blocked state is given by $\dot{\delta}(r, q) = (\sqrt{J_r(q)} - \sqrt{J^*_r(q)})^2$. We analyze the system in the linear response regime, where concentrations q are close to eq. In this regime, the net species current $\mathcal{I} \in \mathbb{R}^{\mathcal{S}}$ is related to a species potential vector $\mathcal{V}(q) = (q/\text{eq} - \mathbf{e})$ by a generalized Ohm's law: $\mathcal{V}(q) = -\mathbb{R}\mathcal{I}$. Here, the division is element-wise, $\mathbf{e}$ is a vector of all ones, and $\mathbb{R}$ is the symmetric, positive semi-definite resistance matrix of the network. The net change in species due to a single firing of reaction r is given by its stoichiometric vector $\Delta r = r^+ - r^-$.

Task: For a detailed-balanced CRN in the linear response regime, derive an expression for the blocking cost rate $\dot{\delta}(r, q)$ of a single reaction r. Your final expression should be in terms of the species current $\mathcal{I}$, the resistance matrix $\mathbb{R}$, the detailed-balanced flux Φ_r, and the stoichiometric vector Δr.

Refined Solution:

$$\dot{\delta}(r, q) = \mathcal{I}^T \mathbb{R}(\Delta r)\frac{\Phi_r}{4}(\Delta r)^T \mathbb{R}\mathcal{I}$$

Problem 30 (Paper: 2506.23496v1, Index: 1)

Original Problem Statement

Background: Consider a detailed-balanced, reversible Chemical Reaction Network (CRN) denoted by the graph $G' = (S, R')$, which sustains a constant species throughput current I. The network operates in the linear response regime, where its non-equilibrium steady state (NESS) concentration vector q is close to the detailed-balanced equilibrium concentration vector . The system's behavior can be described by an analogy to Ohm's law. The species potential, defined as $V(q) = \frac{q}{-} - e$ (where e is the vector of all ones), is related to the current by $V(q) = -RI$. Here, R is the symmetric resistance matrix of the active network G', which is the Moore-Penrose pseudoinverse of the network's conductance matrix $C(G') = \frac{1}{2}\sum_{r\in R'} \Phi_r(\Delta r)(\Delta r)^T$. The term $\Phi_r = k_r^{r^-}$ is the detailed balanced flux for a reaction r, and $\Delta r = r^+ - r^-$ is its stoichiometric vector.

Now, consider an additional reversible reaction, r_b, which is not part of the active pathway G'. This reaction is kept "blocked," meaning its net flux is constrained to be zero. The thermodynamic cost associated with maintaining this block, known as the blocking cost, is evaluated at the NESS concentration q established by the active pathway G'. The blocking cost is given by the expression:

$$\dot{\delta}(r_b, q) = \Phi_{r_b}\left(\left(\frac{q}{-}\right)^{r_b^-/2} - \left(\frac{q}{-}\right)^{r_b^+/2}\right)^2$$

where Φ_{r_b} is the detailed balanced flux of reaction r_b, and r_b^- and r_b^+ are its reactant and product stoichiometry vectors, respectively.

Task: For the system described, derive the expression for the blocking cost $\dot{\delta}(r_b, q)$ in the linear response regime. Your final answer should be expressed in terms of the throughput current I, the resistance matrix of the active pathway R, the detailed balanced flux Φ_{r_b}, and the stoichiometric vector Δr_b of the blocked reaction.

Original Solution

$$I^T R(\Delta r)\frac{\Phi_r}{4}(\Delta r)^T RI$$

Critiques

Self-Containment Critique: **Status: Not self-contained**

The final solution uses variables Δr and Φ_r which are not explicitly defined for the blocked reaction r_b in the final expression, and the notation is inconsistent with the problem statement where the blocked reaction is denoted by r_b and its stoichiometric vector by Δr_b.

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing the network's linear response theory, the relation between potentials and currents, and the evaluation of the blocking cost expression in terms of the resistance matrix and stoichiometric vectors, involving multiple non-trivial steps and rederivations.

Refined Problem

Refined Problem Statement: Background: Consider a detailed-balanced, reversible Chemical Reaction Network (CRN) denoted by the graph $G' = (S, R')$, which sustains a constant species throughput current I. The network operates in the linear response regime, where its non-equilibrium steady state (NESS) concentration vector q is close to the detailed-balanced equilibrium concentration vector . The system's behavior can be described by an analogy to Ohm's law. The species potential, defined as $V(q) = \frac{q}{-} - e$ (where e is the vector of all ones), is related to the current by $V(q) = -RI$. Here, R is the symmetric resistance matrix of the active network G', which is the Moore-Penrose pseudoinverse of the network's conductance matrix $C(G') = \frac{1}{2}\sum_{r\in R'} \Phi_r(\Delta r)(\Delta r)^T$. The term $\Phi_r = k_r^{r^-}$ is the detailed balanced flux for a reaction r, and $\Delta r = r^+ - r^-$ is its stoichiometric vector.

Now, consider an additional reversible reaction, r_b, which is not part of the active pathway G'. This reaction is kept "blocked," meaning its net flux is constrained to be zero. The thermodynamic

cost associated with maintaining this block, known as the blocking cost, is evaluated at the NESS concentration q established by the active pathway G'. The blocking cost is given by the expression:

$$\dot{\delta}(r_b, q) = \Phi_{r_b} \left(\left(\frac{q}{-} \right)^{r_b^-/2} - \left(\frac{q}{-} \right)^{r_b^+/2} \right)^2$$

where Φ_{r_b} is the detailed balanced flux of reaction r_b, and r_b^- and r_b^+ are its reactant and product stoichiometry vectors, respectively.

Task: For the system described, derive the expression for the blocking cost $\dot{\delta}(r_b, q)$ in the linear response regime. Your final answer should be expressed in terms of the throughput current I, the resistance matrix of the active pathway R, the detailed balanced flux Φ_{r_b}, and the stoichiometric vector Δr_b of the blocked reaction.

Refined Solution:

$$I^T R(\Delta r_b) \frac{\Phi_{r_b}}{4} (\Delta r_b)^T R I$$

Problem 31 (Paper: 2506.23600v1, Index: 0)

Original Problem Statement

Background: A heavy point particle of mass m is confined by a harmonic potential with frequency ω and coupled to a thermal environment at temperature T. The coupling is characterized by a relaxation rate γ. The dynamics of the particle's reduced density matrix, $\hat{\rho}(T)$, is governed by the Caldeira-Leggett master equation (with $\hbar = 1$):

$$\partial_t \hat{\rho}(T) = -i[\hat{H}, \hat{\rho}] - i\gamma[\hat{x}, \{\hat{p}, \hat{\rho}\}] - 2\gamma m T[\hat{x}, [\hat{x}, \hat{\rho}]]$$

where $\hat{x}$ and $\hat{p}$ are the position and momentum operators satisfying $[\hat{x}, \hat{p}] = i$, and the Hamiltonian is $\hat{H} = b\hat{p}^2 + c\hat{x}^2$ with $b = 1/(2m)$ and $c = m\omega^2/2$. The Symmetric Logarithmic Derivative (SLD), $\hat{\Lambda}_T$, is an observable that quantifies the ultimate sensitivity to the parameter T and is implicitly defined by:

$$\partial_T \hat{\rho}(T) = \frac{1}{2}\left(\hat{\Lambda}_T \hat{\rho}(T) + \hat{\rho}(T)\hat{\Lambda}_T\right) - \hat{\rho}(T)\langle\hat{\Lambda}_T\rangle$$

The SLD can be found by enforcing the consistency condition $\partial_T(\partial_t\hat{\rho}) = \partial_t(\partial_T\hat{\rho})$. Consider an operator basis expansion for the SLD of the form $\hat{\Lambda}_T = \sum_i c_T^{(i)} \hat{A}_i$, where the basis operators are $\hat{A}_i \in \{\hat{x}, \hat{p}, \hat{x}^2, \hat{p}^2, \{\hat{x}, \hat{p}\}\}$. At late times, the system reaches a steady Gaussian state characterized by the following non-zero expectation values:

$$\langle\hat{x}^2\rangle = \frac{a^2 b}{4c}, \quad \langle\hat{p}^2\rangle = \frac{a^2}{4}$$

where $a = \sqrt{4mT}$. For this Gaussian state, expectation values of higher powers factorize according to Wick's theorem, leading to $\langle\hat{x}^4\rangle = 3\langle\hat{x}^2\rangle^2$ and $\langle\hat{p}^4\rangle = 3\langle\hat{p}^2\rangle^2$. All expectation values involving an odd number of $\hat{x}$ or $\hat{p}$ operators are zero.

Task: By setting up and solving the system of linear equations for the coefficients $c_T^{(i)}$ in the steady-state limit, determine the coefficient $c_T^{(x^2)}$ corresponding to the $\hat{x}^2$ operator. Express your result in terms of a, b, c, and γ.

Original Solution

$$c_T^{(x^2)} = \frac{4c\left(a^4b^2 + 4(bc + \gamma^2)\right)}{a^8b^4 - 16b^2c^2}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires deriving the SLD operator coefficients by enforcing a consistency condition on the master equation and its temperature derivative, involving operator expansions, Gaussian state properties, and solving a non-trivial system of linear equations.

Refined Problem

Refined Problem Statement: Background: A heavy point particle of mass m is confined by a harmonic potential with frequency ω and coupled to a thermal environment at temperature T. The coupling is characterized by a relaxation rate γ. The dynamics of the particle's reduced density matrix, $\hat{\rho}(T)$, is governed by the Caldeira-Leggett master equation (with $\hbar = 1$):

$$\partial_t \hat{\rho}(T) = -i[\hat{H}, \hat{\rho}] - i\gamma[\hat{x}, \{\hat{p}, \hat{\rho}\}] - 2\gamma m T[\hat{x}, [\hat{x}, \hat{\rho}]]$$

where $\hat{x}$ and $\hat{p}$ are the position and momentum operators satisfying $[\hat{x}, \hat{p}] = i$, and the Hamiltonian is $\hat{H} = b\hat{p}^2 + c\hat{x}^2$ with $b = 1/(2m)$ and $c = m\omega^2/2$. The Symmetric Logarithmic Derivative (SLD), $\hat{\Lambda}_T$, is an observable that quantifies the ultimate sensitivity to the parameter T and is implicitly defined by:

$$\partial_T \hat{\rho}(T) = \frac{1}{2}\left(\hat{\Lambda}_T \hat{\rho}(T) + \hat{\rho}(T)\hat{\Lambda}_T\right) - \hat{\rho}(T)\langle\hat{\Lambda}_T\rangle$$

The SLD can be found by enforcing the consistency condition $\partial_T(\partial_t\hat{\rho}) = \partial_t(\partial_T\hat{\rho})$. Consider an operator basis expansion for the SLD of the form $\hat{\Lambda}_T = \sum_i c_T^{(i)}\hat{A}_i$, where the basis operators are $\hat{A}_i \in \{\hat{x}, \hat{p}, \hat{x}^2, \hat{p}^2, \{\hat{x}, \hat{p}\}\}$. At late times, the system reaches a steady Gaussian state characterized by the following non-zero expectation values:

$$\langle\hat{x}^2\rangle = \frac{a^2 b}{4c}, \quad \langle\hat{p}^2\rangle = \frac{a^2}{4}$$

where $a = \sqrt{4mT}$. For this Gaussian state, expectation values of higher powers factorize according to Wick's theorem, leading to $\langle\hat{x}^4\rangle = 3\langle\hat{x}^2\rangle^2$ and $\langle\hat{p}^4\rangle = 3\langle\hat{p}^2\rangle^2$. All expectation values involving an odd number of $\hat{x}$ or $\hat{p}$ operators are zero.

Task: By setting up and solving the system of linear equations for the coefficients $c_T^{(i)}$ in the steady-state limit, determine the coefficient $c_T^{(x^2)}$ corresponding to the $\hat{x}^2$ operator. Express your result in terms of a, b, c, and γ.

Refined Solution:

$$c_T^{(x^2)} = \frac{4c\left(a^4b^2 + 4(bc + \gamma^2)\right)}{a^8b^4 - 16b^2c^2}$$

Problem 32 (Paper: 2506.23600v1, Index: 1)

Original Problem Statement

Background: Consider a heavy point particle of mass m confined in a one-dimensional harmonic potential, described by the Hamiltonian $\hat{H} = b\hat{p}^2 + c\hat{x}^2$, where $b = 1/(2m)$ and $c = m\omega^2/2$. The particle is coupled to a thermal environment at temperature T with a relaxation rate γ. The evolution of the particle's reduced density matrix $\hat{\rho}$ is governed by the Caldeira-Leggett master equation:

$$\partial_t\hat{\rho} = -i[\hat{H},\hat{\rho}] - i\gamma[\hat{x},\{\hat{p},\hat{\rho}\}] - \frac{\gamma T}{b}[\hat{x},[\hat{x},\hat{\rho}]]$$

where $[\cdot,\cdot]$ and $\{\cdot,\cdot\}$ are the commutator and anticommutator, respectively, and we set $\hbar = 1$ such that $[\hat{x},\hat{p}] = i$. The Symmetric Logarithmic Derivative (SLD) for temperature, $\hat{\Lambda}_T$, is an operator that quantifies the sensitivity of the state to changes in T and is implicitly defined by the relation $\partial_T\hat{\rho} = \frac{1}{2}(\hat{\Lambda}_T\hat{\rho}+\hat{\rho}\hat{\Lambda}_T)-\hat{\rho}\langle\hat{\Lambda}_T\rangle$. We expand the SLD in a basis of Hermitian operators $\hat{A}_i$: $\hat{\Lambda}_T = \sum_i c_T^{(i)}\hat{A}_i$. The coefficients $c_T^{(i)}$ can be found by solving the linear system $\sum_i M_{ji}c_T^{(i)} = D_j$, where the matrix elements M_{ji} are given by:

$$\begin{aligned}M_{ji} = &-\tfrac{1}{2}\left\langle\left\{\hat{A}_i,\left[\hat{H},\hat{A}_j\right]\right\}\right\rangle - \tfrac{i\gamma}{2}\left\langle\left\{\hat{A}_i,\left\{\hat{p},\left[\hat{A}_j,\hat{x}\right]\right\}\right\}\right\rangle - \tfrac{\gamma T}{2b}\left\langle\left\{\hat{A}_i,\left[\hat{x},\left[\hat{x},\hat{A}_j\right]\right]\right\}\right\rangle\\ &+\tfrac{i\gamma}{2}\left\langle\left\{\hat{p},\left[\left\{\hat{A}_i,\hat{A}_j\right\},\hat{x}\right]\right\}\right\rangle + \tfrac{\gamma T}{2b}\left\langle\left[\hat{x},\left[\hat{x},\left\{\hat{A}_i,\hat{A}_j\right\}\right]\right]\right\rangle - i\left\langle\hat{A}_j\right\rangle\left\langle\left[\hat{A}_i,\hat{H}\right]\right\rangle\\ &-i\gamma\left\langle\hat{A}_j\right\rangle\left\langle\left\{\hat{p},\left[\hat{A}_i,\hat{x}\right]\right\}\right\rangle - \tfrac{\gamma T}{b}\left\langle\hat{A}_j\right\rangle\left\langle\left[\hat{x},\left[\hat{x},\hat{A}_i\right]\right]\right\rangle\end{aligned}$$

The system is in a non-equilibrium squeezed Gaussian state characterized by $\langle\hat{x}\rangle = \langle\hat{p}\rangle = 0$. For such a state, expectation values of products of operators can be simplified using Wick's theorem. The relevant relations for Weyl-ordered operators $W(\hat{x}^n\hat{p}^m) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\hat{x}^{n-k}\hat{p}^m\hat{x}^k$ are:

$$\begin{aligned}\langle W(x^3p)\rangle &= \frac{3}{2}\langle x^2\rangle\langle\{\hat{x},\hat{p}\}\rangle\\ \langle W(xp^3)\rangle &= \frac{3}{2}\langle p^2\rangle\langle\{\hat{x},\hat{p}\}\rangle\\ \langle W(x^2p^2)\rangle &= \langle x^2\rangle\langle p^2\rangle + \frac{1}{2}\langle\{\hat{x},\hat{p}\}\rangle^2\end{aligned}$$

Also, for any Gaussian state with zero mean, $\langle\hat{O}_1\hat{O}_2\dots\hat{O}_{2k+1}\rangle = 0$ for operators $\hat{O}_i \in \{\hat{x},\hat{p}\}$.

Task: Using the operator basis $\{\hat{x},\hat{p},\hat{x}^2,\hat{p}^2,\{\hat{x},\hat{p}\}\}$, derive the matrix element $M_{\{\hat{x},\hat{p}\},\{\hat{x},\hat{p}\}}$. Express your final answer in terms of the expectation values $\langle\hat{x}^2\rangle$, $\langle\hat{p}^2\rangle$, $\langle\{\hat{x},\hat{p}\}\rangle$ and the parameters b, c, γ, T.

Original Solution

$$8\gamma\langle x^2\rangle\langle p^2\rangle - \frac{8\gamma T}{b}\langle x^2\rangle - 2\gamma + 8c\langle x^2\rangle\langle\{x,p\}\rangle - 8b\langle p^2\rangle\langle\{x,p\}\rangle + 2\gamma\langle\{x,p\}\rangle^2$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires a detailed multi-step derivation involving operator algebra, commutators, anti-commutators, and expectation values in a non-equilibrium Gaussian state, synthesizing several complex concepts and equations from the paper.

Refined Problem

Refined Problem Statement: Background: Consider a heavy point particle of mass m confined in a one-dimensional harmonic potential, described by the Hamiltonian $\hat{H} = b\hat{p}^2 + c\hat{x}^2$, where $b = 1/(2m)$ and $c = m\omega^2/2$. The particle is coupled to a thermal environment at temperature T with a relaxation rate γ. The evolution of the particle's reduced density matrix $\hat{\rho}$ is governed by the Caldeira-Leggett master equation:

$$\partial_t\hat{\rho} = -i[\hat{H},\hat{\rho}] - i\gamma[\hat{x},\{\hat{p},\hat{\rho}\}] - \frac{\gamma T}{b}[\hat{x},[\hat{x},\hat{\rho}]]$$

where $[\cdot,\cdot]$ and $\{\cdot,\cdot\}$ are the commutator and anticommutator, respectively, and we set $\hbar = 1$ such that $[\hat{x},\hat{p}] = i$. The Symmetric Logarithmic Derivative (SLD) for temperature, $\hat{\Lambda}_T$, is an operator that quantifies the sensitivity of the state to changes in T and is implicitly defined by the relation $\partial_T\hat{\rho} = \frac{1}{2}(\hat{\Lambda}_T\hat{\rho}+\hat{\rho}\hat{\Lambda}_T)-\hat{\rho}\langle\hat{\Lambda}_T\rangle$. We expand the SLD in a basis of Hermitian operators $\hat{A}_i$: $\hat{\Lambda}_T = \sum_i c_T^{(i)}\hat{A}_i$. The coefficients $c_T^{(i)}$ can be found by solving the linear system $\sum_i M_{ji}c_T^{(i)} = D_j$, where the matrix elements M_{ji} are given by:

$$\begin{aligned}M_{ji} = &-\tfrac{1}{2}\left\langle\left\{\hat{A}_i,\left[\hat{H},\hat{A}_j\right]\right\}\right\rangle - \tfrac{i\gamma}{2}\left\langle\left\{\hat{A}_i,\left\{\hat{p},\left[\hat{A}_j,\hat{x}\right]\right\}\right\}\right\rangle - \tfrac{\gamma T}{2b}\left\langle\left\{\hat{A}_i,\left[\hat{x},\left[\hat{x},\hat{A}_j\right]\right]\right\}\right\rangle\\ &+\tfrac{i\gamma}{2}\left\langle\left\{\hat{p},\left[\left\{\hat{A}_i,\hat{A}_j\right\},\hat{x}\right]\right\}\right\rangle + \tfrac{\gamma T}{2b}\left\langle\left[\hat{x},\left[\hat{x},\left\{\hat{A}_i,\hat{A}_j\right\}\right]\right]\right\rangle - i\left\langle\hat{A}_j\right\rangle\left\langle\left[\hat{A}_i,\hat{H}\right]\right\rangle\\ &-i\gamma\left\langle\hat{A}_j\right\rangle\left\langle\left\{\hat{p},\left[\hat{A}_i,\hat{x}\right]\right\}\right\rangle - \tfrac{\gamma T}{b}\left\langle\hat{A}_j\right\rangle\left\langle\left[\hat{x},\left[\hat{x},\hat{A}_i\right]\right]\right\rangle\end{aligned}$$

The system is in a non-equilibrium squeezed Gaussian state characterized by $\langle\hat{x}\rangle = \langle\hat{p}\rangle = 0$. For such a state, expectation values of products of operators can be simplified using Wick's theorem. The relevant relations for Weyl-ordered operators $W(\hat{x}^n\hat{p}^m) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\hat{x}^{n-k}\hat{p}^m\hat{x}^k$ are:

$$\begin{aligned}\langle W(x^3p)\rangle &= \frac{3}{2}\langle x^2\rangle\langle\{\hat{x},\hat{p}\}\rangle\\ \langle W(xp^3)\rangle &= \frac{3}{2}\langle p^2\rangle\langle\{\hat{x},\hat{p}\}\rangle\\ \langle W(x^2p^2)\rangle &= \langle x^2\rangle\langle p^2\rangle + \frac{1}{2}\langle\{\hat{x},\hat{p}\}\rangle^2\end{aligned}$$

Also, for any Gaussian state with zero mean, $\langle\hat{O}_1\hat{O}_2\ldots\hat{O}_{2k+1}\rangle = 0$ for operators $\hat{O}_i \in \{\hat{x},\hat{p}\}$.

Task: Using the operator basis $\{\hat{x},\hat{p},\hat{x}^2,\hat{p}^2,\{\hat{x},\hat{p}\}\}$, derive the matrix element $M_{\{\hat{x},\hat{p}\},\{\hat{x},\hat{p}\}}$. Express your final answer in terms of the expectation values $\langle\hat{x}^2\rangle$, $\langle\hat{p}^2\rangle$, $\langle\{\hat{x},\hat{p}\}\rangle$ and the parameters b, c, γ, T.

Refined Solution:

$$8\gamma\langle x^2\rangle\langle p^2\rangle - \frac{8\gamma T}{b}\langle x^2\rangle - 2\gamma + 8c\langle x^2\rangle\langle\{x,p\}\rangle - 8b\langle p^2\rangle\langle\{x,p\}\rangle + 2\gamma\langle\{x,p\}\rangle^2$$

Problem 33 (Paper: 2506.23604v1, Index: 0)

Original Problem Statement

Background: Consider a composite, isolated system described by a total Hamiltonian $H_{\text{tot}}(x, y, \lambda)$, where x represents the state variables of a small system of interest, y represents the state variables of a much larger thermal bath, and λ is an external control parameter. A key step in formulating a consistent thermodynamic description is to decompose the total Hamiltonian as $H_{\text{tot}} = H_X(x; \lambda) + H_Y(y; x, \lambda)$. A particularly useful decomposition is one where the bath energy becomes an adiabatic invariant, meaning it does not change under slow variations of the system state x and the control parameter λ. This is equivalent to requiring that the bath's Boltzmann entropy, $S_Y(E_Y) = \log \Omega_Y(E_Y)$, is independent of x and λ, where $\Omega_Y(E_Y; x, \lambda) = \int dy\, \delta(E_Y - H_Y(y; x, \lambda))$ is the microcanonical partition function for the bath.

Assume you are given an arbitrary initial decomposition of the total Hamiltonian, $H_{\text{tot}}(x, y, \lambda) = H_X^0(x; \lambda) + H_Y^0(y; x, \lambda)$. The bath entropy corresponding to this initial decomposition, $S_Y^0(E_Y; x, \lambda) = \log \int dy\, \delta(E_Y - H_Y^0(y; x, \lambda))$, generally depends on both x and λ. The bath temperature is defined as $T = 1/\beta = \left(\partial S_Y^0 / \partial E_Y\right)^{-1}$. Because the bath is large, its temperature T is a bulk property insensitive to local changes in x and λ. Furthermore, the change in S_Y^0 due to variations in x and λ is sub-extensive compared to the total bath energy E_Y.

Task: Your goal is to find a new decomposition, $H_{\text{tot}} = H_X(x; \lambda) + H_Y(y; x, \lambda)$, by modifying the original Hamiltonians. The new bath Hamiltonian H_Y must be constructed such that its corresponding Boltzmann entropy, $S_Y(E_Y)$, is independent of x and λ to first order in the correction term. Determine the expression for this new, adiabatically invariant bath Hamiltonian, $H_Y(y; x, \lambda, E_Y)$. Express your result in terms of the original bath Hamiltonian $H_Y^0(y; x, \lambda)$, its associated entropy $S_Y^0(E_Y; x, \lambda)$, the bath temperature T, and an arbitrary but fixed reference state (x_0, λ_0).

Original Solution

$$H_Y(y; x, \lambda, E_Y) = H_Y^0(y; x, \lambda) + T\left(S_Y^0(E_Y; x, \lambda) - S_Y^0(E_Y; x_0, \lambda_0)\right)$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing concepts of microcanonical entropy, adiabatic invariance, and Hamiltonian decomposition, involving a non-trivial derivation to construct a modified bath Hamiltonian ensuring entropy independence from system variables.

Refined Problem

Refined Problem Statement: Background: Consider a composite, isolated system described by a total Hamiltonian $H_{\text{tot}}(x, y, \lambda)$, where x represents the state variables of a small system of interest, y represents the state variables of a much larger thermal bath, and λ is an external control parameter. A key step in formulating a consistent thermodynamic description is to decompose the total Hamiltonian as $H_{\text{tot}} = H_X(x; \lambda) + H_Y(y; x, \lambda)$. A particularly useful decomposition is one where the bath energy becomes an adiabatic invariant, meaning it does not change under slow variations of the system state x and the control parameter λ. This is equivalent to requiring that the bath's Boltzmann entropy, $S_Y(E_Y) = \log \Omega_Y(E_Y)$, is independent of x and λ, where $\Omega_Y(E_Y; x, \lambda) = \int dy\, \delta(E_Y - H_Y(y; x, \lambda))$ is the microcanonical partition function for the bath.

Assume you are given an arbitrary initial decomposition of the total Hamiltonian, $H_{\text{tot}}(x, y, \lambda) = H_X^0(x; \lambda) + H_Y^0(y; x, \lambda)$. The bath entropy corresponding to this initial decomposition, $S_Y^0(E_Y; x, \lambda) = \log \int dy\, \delta(E_Y - H_Y^0(y; x, \lambda))$, generally depends on both x and λ. The bath temperature is defined as $T = 1/\beta = \left(\partial S_Y^0 / \partial E_Y\right)^{-1}$. Because the bath is large, its temperature T is a bulk property insensitive to local changes in x and λ. Furthermore, the change in S_Y^0 due to variations in x and λ is sub-extensive compared to the total bath energy E_Y.

Task: Your goal is to find a new decomposition, $H_{\text{tot}} = H_X(x; \lambda) + H_Y(y; x, \lambda)$, by modifying the original Hamiltonians. The new bath Hamiltonian H_Y must be constructed such that its corresponding Boltzmann entropy, $S_Y(E_Y)$, is independent of x and λ to first order in the correction term. Determine

the expression for this new, adiabatically invariant bath Hamiltonian, $H_Y(y; x, \lambda, E_Y)$. Express your result in terms of the original bath Hamiltonian $H_Y^0(y; x, \lambda)$, its associated entropy $S_Y^0(E_Y; x, \lambda)$, the bath temperature T, and an arbitrary but fixed reference state (x_0, λ_0).

Refined Solution:

$$H_Y(y; x, \lambda, E_Y) = H_Y^0(y; x, \lambda) + T\left(S_Y^0(E_Y; x, \lambda) - S_Y^0(E_Y; x_0, \lambda_0)\right)$$

Problem 34 (Paper: 2506.23604v1, Index: 1)

Original Problem Statement

Background: Consider an isolated composite system comprised of a small system of interest, with phase space coordinates $x = (q_x, p_x)$, and a much larger thermal bath, with coordinates $y = (q_y, p_y)$. The total Hamiltonian depends on a time-varying external control parameter λ and is decomposed as $H_{\rm tot}(x, y, \lambda) = H_X(x; \lambda) + H_Y(y; x, \lambda)$. This decomposition is specifically chosen such that the bath Hamiltonian H_Y is *adiabatically invariant*: its microcanonical density of states, $\Omega_Y(E_Y) = \int dy\, \delta(E_Y - H_Y(y; x, \lambda))$, and its Boltzmann entropy, $S_Y(E_Y) = \log \Omega_Y(E_Y)$, are independent of the system state x and the control parameter λ. The inverse temperature of the bath is defined as $\beta = \partial S_Y/\partial E_Y$. The dynamics is time-reversal symmetric, meaning $H_X(x; \lambda) = H_X(x^*; \lambda^*)$ and $H_Y(y; x, \lambda) = H_Y(y^*; x^*, \lambda^*)$, where c^* denotes the time-reversal of a state or parameter c.

A forward process is defined by the evolution from time $s = 0$ to $s = t$ under a protocol λ_s. The corresponding backward process evolves under the protocol $\tilde{\lambda}_s = \lambda^*_{t-s}$. The coarse-grained transition probability density for the joint system to evolve from a state with system coordinate x and bath energy E_Y to one with x' and E'_Y satisfies a generalized detailed balance (GDB) relation: $\langle x', E'_Y|x, E_Y\rangle_F = \langle x^*, E_Y|x'^*, E'_Y\rangle_B$.

The composite system is initially prepared in a microcanonical ensemble with a fixed total energy E_{XY} and control parameter λ_0. The initial probability density is $p^0_{XY}(x_0, y_0) = \delta(H_{\rm tot}(x_0, y_0, \lambda_0) - E_{XY})/\Omega_{XY}(E_{XY}, \lambda_0)$, where $\Omega_{XY}(E, \lambda) = \iint dxdy\, \delta(H_{\rm tot}(x, y, \lambda) - E)$ is the total microcanonical partition function. During the forward process, the work done on the system is $W = H_{\rm tot}(x_t, y_t, \lambda_t) - E_{XY}$, and the heat dissipated to the bath is $Q = H_Y(y_0; x_0, \lambda_0) - H_Y(y_t; x_t, \lambda_t)$. The joint probability density of observing work W, heat Q, final state x_t, and initial state x_0 is $p_F(W, Q, x_t, x_0)$. The corresponding density for the backward process, $p_B(-W, -Q, x^*_0, x^*_t)$, is defined for a process starting from a microcanonical ensemble with total energy $E_{XY} + W$ and control parameter λ_t. Assume that the system energy H_X is sub-extensive compared to the total energy E_{XY}, justifying a first-order Taylor expansion for the bath entropy where needed. The system's free energy is $F(\lambda) = -\beta^{-1} \log \int dx\, e^{-\beta H_X(x;\lambda)}$, and the change over the process is $\Delta F = F(\lambda_t) - F(\lambda_0)$.

Task: Derive an expression for the ratio of the joint probability densities for the forward and backward processes, $\frac{p_F(W,Q,x_t,x_0)}{p_B(-W,-Q,x^*_0,x^*_t)}$.

Original Solution

$$e^{\beta(W-\Delta F)}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing multiple advanced concepts including microcanonical ensembles, adiabatic invariance, time-reversal symmetry, generalized detailed balance, and nontrivial expansions of bath entropy to derive a fluctuation relation beyond straightforward application of known formulas.

Refined Problem

Refined Problem Statement: Background: Consider an isolated composite system comprised of a small system of interest, with phase space coordinates $x = (q_x, p_x)$, and a much larger thermal bath, with coordinates $y = (q_y, p_y)$. The total Hamiltonian depends on a time-varying external control parameter λ and is decomposed as $H_{\rm tot}(x, y, \lambda) = H_X(x; \lambda) + H_Y(y; x, \lambda)$. This decomposition is specifically chosen such that the bath Hamiltonian H_Y is *adiabatically invariant*: its microcanonical density of states, $\Omega_Y(E_Y) = \int dy\, \delta(E_Y - H_Y(y; x, \lambda))$, and its Boltzmann entropy, $S_Y(E_Y) = \log \Omega_Y(E_Y)$, are independent of the system state x and the control parameter λ. The inverse temperature of the bath is defined as $\beta = \partial S_Y/\partial E_Y$. The dynamics is time-reversal symmetric, meaning $H_X(x; \lambda) = H_X(x^*; \lambda^*)$ and $H_Y(y; x, \lambda) = H_Y(y^*; x^*, \lambda^*)$, where c^* denotes the time-reversal of a state or parameter c.

A forward process is defined by the evolution from time $s = 0$ to $s = t$ under a protocol λ_s. The corresponding backward process evolves under the protocol $\tilde{\lambda}_s = \lambda^*_{t-s}$. The coarse-grained transition probability density for the joint system to evolve from a state with system coordinate x and bath energy

E_Y to one with x' and E'_Y satisfies a generalized detailed balance (GDB) relation: $\langle x', E'_Y | x, E_Y \rangle_F = \langle x^*, E_Y | x'^*, E'_Y \rangle_B$.

The composite system is initially prepared in a microcanonical ensemble with a fixed total energy E_{XY} and control parameter λ_0. The initial probability density is $p^0_{XY}(x_0, y_0) = \delta(H_{\rm tot}(x_0, y_0, \lambda_0) - E_{XY})/\Omega_{XY}(E_{XY}, \lambda_0)$, where $\Omega_{XY}(E, \lambda) = \iint dxdy\, \delta(H_{\rm tot}(x, y, \lambda) - E)$ is the total microcanonical partition function. During the forward process, the work done on the system is $W = H_{\rm tot}(x_t, y_t, \lambda_t) - E_{XY}$, and the heat dissipated to the bath is $Q = H_Y(y_0; x_0, \lambda_0) - H_Y(y_t; x_t, \lambda_t)$. The joint probability density of observing work W, heat Q, final state x_t, and initial state x_0 is $p_F(W, Q, x_t, x_0)$. The corresponding density for the backward process, $p_B(-W, -Q, x^*_0, x^*_t)$, is defined for a process starting from a microcanonical ensemble with total energy $E_{XY} + W$ and control parameter λ_t. Assume that the system energy H_X is sub-extensive compared to the total energy E_{XY}, justifying a first-order Taylor expansion for the bath entropy where needed. The system's free energy is $F(\lambda) = -\beta^{-1} \log \int dx\, e^{-\beta H_X(x;\lambda)}$, and the change over the process is $\Delta F = F(\lambda_t) - F(\lambda_0)$.

Task: Derive an expression for the ratio of the joint probability densities for the forward and backward processes, $\frac{p_F(W,Q,x_t,x_0)}{p_B(-W,-Q,x^*_0,x^*_t)}$.

Refined Solution:

$$e^{\beta(W-\Delta F)}$$

Problem 35 (Paper: 2506.23609v1, Index: 0)

Original Problem Statement

Background: Consider a Dirac spinor field ψ with mass m in a four-dimensional metric-affine spacetime. The dynamics are described by the Lagrangian 4-form:

$$L_{\mathbb{D}} = \frac{i}{2}\left[\overline{\psi} * \gamma \wedge \mathbb{D}\psi + \mathbb{D}\overline{\psi} \wedge *\gamma\psi\right] + im\overline{\psi}\psi * 1$$

Here, $\overline{\psi} = \psi^\dagger\gamma_0$ is the Dirac adjoint, $*$ is the Hodge star operator, $\gamma = \gamma_a e^a$ is the Clifford algebra-valued 1-form built from the orthonormal coframe e^a and gamma matrices γ_a, and $*1$ is the volume form. The gamma matrices satisfy $\{\gamma_a, \gamma_b\} = 2\eta_{ab}I$.

The generalized covariant exterior derivative of the spinor, $\mathbb{D}\psi$, and its adjoint, $\mathbb{D}\overline{\psi}$, are defined as:

$$\mathbb{D}\psi = d\psi + \Big[\frac{1}{2}\sigma_{ab}\omega^{ab} + (a_1 I + a_2\gamma_5)Q + (a_3 I + a_4\gamma_5)P + (b_1 I + b_2\gamma_5)T + (b_3 I - b_4\gamma_5)\gamma\Big]\psi$$

$$\mathbb{D}\overline{\psi} = d\overline{\psi} + \overline{\psi}\Big[-\frac{1}{2}\sigma_{ab}\omega^{ab} + (a_1^\star I + a_2^\star\gamma_5)Q + (a_3^\star I + a_4^\star\gamma_5)P + (b_1^\star I + b_2^\star\gamma_5)T - (b_3^\star I + b_4^\star\gamma_5)\gamma\Big]$$

where $\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$, γ_5 is the chirality operator, and a_i, b_i are arbitrary complex coupling constants (* denotes complex conjugation). The geometric quantities are the full connection 1-form ω^{ab}, the first trace of non-metricity $Q = \eta_{cd}Q^{cd}$, the second trace of non-metricity $P = (\iota_c Q^{cd})e_d$, and the trace of torsion $T = \iota_c T^c$.

You are given the following two identities which hold up to an exact form $d(\dots)$ that does not contribute to the equations of motion: 1. The second kinetic term in the Lagrangian can be related to the first via:

$$\mathbb{D}\overline{\psi} \wedge *\gamma\psi \approx \overline{\psi} \wedge *\gamma \wedge \mathbb{D}'\psi - \overline{\psi}(D * e^a - \omega^{(ab)} \wedge *e_b)\gamma_a\psi$$

where $\approx$ denotes equality up to a boundary term, D is the covariant exterior derivative acting on forms, and $\mathbb{D}'\psi$ is given by:

$$\mathbb{D}'\psi = \Big\{d + \frac{1}{2}\sigma_{ab}\omega^{ab} + (-a_1^\star I + a_2^\star\gamma_5)Q + (-a_3^\star I + a_4^\star\gamma_5)P + (-b_1^\star I + b_2^\star\gamma_5)T + (b_3^\star I - b_4^\star\gamma_5)\gamma\Big\}\psi$$

2. The geometric term involving the covariant derivative of the Hodge dual of the coframe is:

$$D * e^a - \omega^{(ab)} \wedge *e_b = *e_a \wedge (Q + T - P)$$

where $\omega^{(ab)}$ is the symmetric part of the connection 1-form.

Task: By applying the variational principle to the Lagrangian $L_{\mathbb{D}}$, derive the equation of motion for the spinor field ψ. Your final answer should be the full variational Dirac equation, expressed in terms of the arbitrary complex constants a_i, b_i.

Original Solution

$$i*\gamma\wedge\Big\{d+\frac{1}{2}\sigma_{ab}\omega^{ab}+\frac{1}{2}[(a_1-a_1^\star-1)I+(a_2+a_2^\star)\gamma_5]Q+\frac{1}{2}[(a_3-a_3^\star+1)I+(a_4+a_4^\star)\gamma_5]P+\frac{1}{2}[(b_1-b_1^\star-1)I+(b_2+b_2^\star)\gamma_5]T+\frac{1}{2}[(b_3+b_3^\star)$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing multiple advanced concepts including spinor covariant derivatives with complex couplings, non-metricity and torsion contributions, and careful handling of boundary terms to derive the full variational Dirac equation.

Refined Problem

Refined Problem Statement: Background: Consider a Dirac spinor field ψ with mass m in a four-dimensional metric-affine spacetime. The dynamics are described by the Lagrangian 4-form:

$$L_{\mathbb{D}} = \frac{i}{2}\left[\overline{\psi} * \gamma \wedge \mathbb{D}\psi + \mathbb{D}\overline{\psi} \wedge *\gamma\psi\right] + im\overline{\psi}\psi * 1$$

Here, $\overline{\psi} = \psi^\dagger\gamma_0$ is the Dirac adjoint, $*$ is the Hodge star operator, $\gamma = \gamma_a e^a$ is the Clifford algebra-valued 1-form built from the orthonormal coframe e^a and gamma matrices γ_a, and $*1$ is the volume form. The gamma matrices satisfy $\{\gamma_a, \gamma_b\} = 2\eta_{ab}I$.

The generalized covariant exterior derivative of the spinor, $\mathbb{D}\psi$, and its adjoint, $\mathbb{D}\overline{\psi}$, are defined as:

$$\mathbb{D}\psi = d\psi + \Big[\frac{1}{2}\sigma_{ab}\omega^{ab} + (a_1 I + a_2\gamma_5)Q + (a_3 I + a_4\gamma_5)P + (b_1 I + b_2\gamma_5)T + (b_3 I - b_4\gamma_5)\gamma\Big]\psi$$

$$\mathbb{D}\overline{\psi} = d\overline{\psi} + \overline{\psi}\Big[-\frac{1}{2}\sigma_{ab}\omega^{ab} + (a_1^\star I + a_2^\star\gamma_5)Q + (a_3^\star I + a_4^\star\gamma_5)P + (b_1^\star I + b_2^\star\gamma_5)T - (b_3^\star I + b_4^\star\gamma_5)\gamma\Big]$$

where $\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$, γ_5 is the chirality operator, and a_i, b_i are arbitrary complex coupling constants (* denotes complex conjugation). The geometric quantities are the full connection 1-form ω^{ab}, the first trace of non-metricity $Q = \eta_{cd}Q^{cd}$, the second trace of non-metricity $P = (\iota_c Q^{cd})e_d$, and the trace of torsion $T = \iota_c T^c$.

You are given the following two identities which hold up to an exact form $d(\dots)$ that does not contribute to the equations of motion: 1. The second kinetic term in the Lagrangian can be related to the first via:

$$\mathbb{D}\overline{\psi} \wedge *\gamma\psi \approx \overline{\psi} \wedge *\gamma \wedge \mathbb{D}'\psi - \overline{\psi}(D * e^a - \omega^{(ab)} \wedge *e_b)\gamma_a\psi$$

where $\approx$ denotes equality up to a boundary term, D is the covariant exterior derivative acting on forms, and $\mathbb{D}'\psi$ is given by:

$$\mathbb{D}'\psi = \Big\{d + \frac{1}{2}\sigma_{ab}\omega^{ab} + (-a_1^\star I + a_2^\star\gamma_5)Q + (-a_3^\star I + a_4^\star\gamma_5)P + (-b_1^\star I + b_2^\star\gamma_5)T + (b_3^\star I - b_4^\star\gamma_5)\gamma\Big\}\psi$$

2. The geometric term involving the covariant derivative of the Hodge dual of the coframe is:

$$D * e^a - \omega^{(ab)} \wedge *e_b = *e_a \wedge (Q + T - P)$$

where $\omega^{(ab)}$ is the symmetric part of the connection 1-form.

Task: By applying the variational principle to the Lagrangian $L_{\mathbb{D}}$, derive the equation of motion for the spinor field ψ. Your final answer should be the full variational Dirac equation, expressed in terms of the arbitrary complex constants a_i, b_i.

Refined Solution:

$$i*\gamma\wedge\Big\{d+\frac{1}{2}\sigma_{ab}\omega^{ab}+\frac{1}{2}[(a_1-a_1^\star-1)I+(a_2+a_2^\star)\gamma_5]Q+\frac{1}{2}[(a_3-a_3^\star+1)I+(a_4+a_4^\star)\gamma_5]P+\frac{1}{2}[(b_1-b_1^\star-1)I+(b_2+b_2^\star)\gamma_5]T+\frac{1}{2}[(b_3+b_3^\star)$$

Problem 36 (Paper: 2506.23609v1, Index: 1)

Original Problem Statement

Background: Consider a Dirac spinor field ψ with mass m in a four-dimensional metric-affine spacetime. The dynamics are described by the Lagrangian 4-form $L_D = \mathcal{H}er\left[i\overline{\psi} * \gamma \wedge \mathbb{D}\psi + im\overline{\psi}\psi * 1\right]$, where $\mathcal{H}er[Z] = \frac{1}{2}(Z + Z^\dagger)$ denotes the Hermitian part, $\overline{\psi} = \psi^\dagger\gamma_0$ is the Dirac adjoint, $*$ is the Hodge star operator, and $*1$ is the volume 4-form. The Clifford algebra-valued 1-form γ is defined as $\gamma = \gamma_a e^a$, where e^a is the orthonormal coframe and γ_a are the Dirac matrices satisfying $\{\gamma_a, \gamma_b\} = 2\eta_{ab}I$ with $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$. The generalized covariant exterior derivative $\mathbb{D}$ acting on the spinor is given by:

$$\mathbb{D}\psi = d\psi + \left[\frac{1}{2}\sigma_{ab}\omega^{ab} + (a_1 I + a_2\gamma_5)Q + (a_3 I + a_4\gamma_5)P + (b_1 I + b_2\gamma_5)T + (b_3 I - b_4\gamma_5)\gamma\right]\psi$$

Here, $\sigma_{ab} = \frac{1}{4}(\gamma_a\gamma_b - \gamma_b\gamma_a)$, $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$, ω^{ab} is the full connection 1-form, and Q, P, T are 1-forms constructed from the non-metricity and torsion tensors. The coefficients a_i are complex, while b_3 and b_4 are real constants. This covariant derivative can be split into two parts:

$$\mathbb{D}\psi = \widehat{\mathbb{D}}\psi + (b_3 I - b_4\gamma_5)\gamma\psi$$

where $\widehat{\mathbb{D}}\psi$ contains all terms not proportional to the coframe-related 1-form γ:

$$\widehat{\mathbb{D}}\psi = d\psi + \left[\frac{1}{2}\sigma_{ab}\omega^{ab} + (a_1 I + a_2\gamma_5)Q + (a_3 I + a_4\gamma_5)P + (b_1 I + b_2\gamma_5)T\right]\psi$$

You are given the following identities of Clifford algebra and differential forms: 1. $[\gamma_a, \gamma_5] = 2\gamma_a\gamma_5$, which implies $\gamma_a\gamma_5 = -\gamma_5\gamma_a$. 2. $\gamma_a\gamma^a = 4I$. 3. $*e^a \wedge e^b = -\eta^{ab} * 1$.

Task: By substituting the decomposition of $\mathbb{D}\psi$ into the Lagrangian L_D, manipulate the expression to show how the geometry-spinor coupling terms proportional to the coframe γ modify the mass term of the Lagrangian. Derive the final expression for the Lagrangian L_D written in terms of $\widehat{\mathbb{D}}\psi$ and an effective, chirality-dependent mass.

Original Solution

$$L_D = \mathcal{H}er\left[i\overline{\psi} * \gamma \wedge \widehat{\mathbb{D}}\psi + i\overline{\psi}(m - 4b_3 - 4b_4\gamma_5)\psi * 1\right]$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires a multi-step derivation involving Clifford algebra identities, differential form manipulations, and the decomposition of the covariant derivative to reveal how geometry-spinor coupling modifies the mass term, demonstrating a sophisticated synthesis of concepts.

Refined Problem

Refined Problem Statement: Background: Consider a Dirac spinor field ψ with mass m in a four-dimensional metric-affine spacetime. The dynamics are described by the Lagrangian 4-form $L_D = \mathcal{H}er\left[i\overline{\psi} * \gamma \wedge \mathbb{D}\psi + im\overline{\psi}\psi * 1\right]$, where $\mathcal{H}er[Z] = \frac{1}{2}(Z + Z^\dagger)$ denotes the Hermitian part, $\overline{\psi} = \psi^\dagger\gamma_0$ is the Dirac adjoint, $*$ is the Hodge star operator, and $*1$ is the volume 4-form. The Clifford algebra-valued 1-form γ is defined as $\gamma = \gamma_a e^a$, where e^a is the orthonormal coframe and γ_a are the Dirac matrices satisfying $\{\gamma_a, \gamma_b\} = 2\eta_{ab}I$ with $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$. The generalized covariant exterior derivative $\mathbb{D}$ acting on the spinor is given by:

$$\mathbb{D}\psi = d\psi + \left[\frac{1}{2}\sigma_{ab}\omega^{ab} + (a_1 I + a_2\gamma_5)Q + (a_3 I + a_4\gamma_5)P + (b_1 I + b_2\gamma_5)T + (b_3 I - b_4\gamma_5)\gamma\right]\psi$$

Here, $\sigma_{ab} = \frac{1}{4}(\gamma_a\gamma_b - \gamma_b\gamma_a)$, $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$, ω^{ab} is the full connection 1-form, and Q, P, T are 1-forms constructed from the non-metricity and torsion tensors. The coefficients a_i are complex, while b_3 and b_4 are real constants. This covariant derivative can be split into two parts:

$$\mathbb{D}\psi = \widehat{\mathbb{D}}\psi + (b_3 I - b_4\gamma_5)\gamma\psi$$

where $\widehat{\mathbb{D}}\psi$ contains all terms not proportional to the coframe-related 1-form γ:

$$\widehat{\mathbb{D}}\psi = d\psi + \Big[\frac{1}{2}\sigma_{ab}\omega^{ab} + (a_1 I + a_2\gamma_5)Q + (a_3 I + a_4\gamma_5)P + (b_1 I + b_2\gamma_5)T\Big]\psi$$

You are given the following identities of Clifford algebra and differential forms: 1. $[\gamma_a, \gamma_5] = 2\gamma_a\gamma_5$, which implies $\gamma_a\gamma_5 = -\gamma_5\gamma_a$. 2. $\gamma_a\gamma^a = 4I$. 3. $*e^a \wedge e^b = -\eta^{ab} * 1$.

Task: By substituting the decomposition of $\mathbb{D}\psi$ into the Lagrangian L_D, manipulate the expression to show how the geometry-spinor coupling terms proportional to the coframe γ modify the mass term of the Lagrangian. Derive the final expression for the Lagrangian L_D written in terms of $\widehat{\mathbb{D}}\psi$ and an effective, chirality-dependent mass.

Refined Solution:

$$L_D = \mathcal{H}er\left[i\overline{\psi} * \gamma \wedge \widehat{\mathbb{D}}\psi + i\overline{\psi}(m - 4b_3 - 4b_4\gamma_5)\psi * 1\right]$$

Problem 37 (Paper: 2506.23687v1, Index: 0)

Original Problem Statement

Background: Consider a q-state active Potts model on a two-dimensional square lattice. Each site i of the lattice is in a state $s_i \in \{0, 1, \ldots, q-1\}$. The system's dynamics are governed by single-site Metropolis Monte Carlo updates. The probability of accepting a flip from state s to s' at a site i is given by $\min(1, \exp(-\Delta E_{s\to s'}))$, where the total energy change is $\Delta E_{s\to s'} = \Delta H_{\text{int}} - h_{s,s'}$. The interaction energy is $H_{\text{int}} = -\sum_{\langle i,j\rangle} J_{s_i,s_j}$, where the sum is over nearest-neighbor pairs, and $J_{s,s'}$ is the contact energy between states s and s'. The term $h_{s,s'}$ represents a non-equilibrium driving potential. For this problem, we consider a $q = 4$ system with cyclic symmetry. The driving is non-zero only for transitions between adjacent states in the cycle $0 \to 1 \to 2 \to 3 \to 0$, such that $h_{k,[k+1]} = h$ and $h_{k,[k-1]} = -h$ for $k \in \{0, 1, 2, 3\}$, where $[k]$ denotes $k \pmod 4$. All other $h_{s,s'}$ are zero. The contact energies are symmetric, $J_{s,s'} = J_{s',s}$, and depend only on the cyclic distance between states, i.e., $J_{s,s'} = J_{k,k'}$ if $|s - s'| \pmod 4 = |k - k'| \pmod 4$.

This active Potts model can be related to an active Ashkin-Teller model. In this second model, each site i is described by a pair of Ising spins (σ_i^A, σ_i^B), where $\sigma_i^\alpha \in \{-1, 1\}$. The four states of the Potts model are mapped to the four spin-pair states as follows: $s = 0 \leftrightarrow (1, 1)$, $s = 1 \leftrightarrow (-1, 1)$, $s = 2 \leftrightarrow (-1, -1)$, and $s = 3 \leftrightarrow (1, -1)$. The dynamics of the Ashkin-Teller model are also governed by single-spin-flip Metropolis updates. The energy change for flipping spin σ_i^A is $\Delta E^A = 2J\sigma_i^A \sum_{j\in \text{nn}(i)} \sigma_j^A - 2K\sigma_i^A\sigma_i^B$, and for flipping spin σ_i^B is $\Delta E^B = 2J\sigma_i^B \sum_{j\in \text{nn}(i)} \sigma_j^B + 2K\sigma_i^B\sigma_i^A$, where J is the ferromagnetic coupling constant and K is a non-reciprocal coupling constant. Note that a transition $k \to [k+1]$ in the Potts model corresponds to a single spin flip in the Ashkin-Teller model (e.g., $s = 0 \to s = 1$ corresponds to flipping σ^A while σ^B is fixed).

Task: By requiring that the energy change for any single-site transition $s_i \to s_i'$ is identical in both models for any configuration of neighboring sites, determine the relationship between the parameters of the two models. Derive the general expression for the contact energy $J_{k,s}$ for an arbitrary pair of states (k, s) in the Potts model as a function of the Ashkin-Teller coupling J and the state indices k and s.

Original Solution

$$J_{k,s} = 2J \cos\left(\frac{\pi(k-s)}{2}\right)$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires mapping between two distinct models, equating energy changes for all transitions, and deriving a general expression for contact energies, involving multi-step reasoning and synthesis of model dynamics and symmetries.

Refined Problem

Refined Problem Statement: Background: Consider a q-state active Potts model on a two-dimensional square lattice. Each site i of the lattice is in a state $s_i \in \{0, 1, \ldots, q-1\}$. The system's dynamics are governed by single-site Metropolis Monte Carlo updates. The probability of accepting a flip from state s to s' at a site i is given by $\min(1, \exp(-\Delta E_{s\to s'}))$, where the total energy change is $\Delta E_{s\to s'} = \Delta H_{\text{int}} - h_{s,s'}$. The interaction energy is $H_{\text{int}} = -\sum_{\langle i,j\rangle} J_{s_i,s_j}$, where the sum is over nearest-neighbor pairs, and $J_{s,s'}$ is the contact energy between states s and s'. The term $h_{s,s'}$ represents a non-equilibrium driving potential. For this problem, we consider a $q = 4$ system with cyclic symmetry. The driving is non-zero only for transitions between adjacent states in the cycle $0 \to 1 \to 2 \to 3 \to 0$, such that $h_{k,[k+1]} = h$ and $h_{k,[k-1]} = -h$ for $k \in \{0, 1, 2, 3\}$, where $[k]$ denotes $k \pmod 4$. All other $h_{s,s'}$ are zero. The contact energies are symmetric, $J_{s,s'} = J_{s',s}$, and depend only on the cyclic distance between states, i.e., $J_{s,s'} = J_{k,k'}$ if $|s - s'| \pmod 4 = |k - k'| \pmod 4$.

This active Potts model can be related to an active Ashkin-Teller model. In this second model, each site i is described by a pair of Ising spins (σ_i^A, σ_i^B), where $\sigma_i^\alpha \in \{-1, 1\}$. The four states of the Potts model are mapped to the four spin-pair states as follows: $s = 0 \leftrightarrow (1, 1)$, $s = 1 \leftrightarrow (-1, 1)$, $s = 2 \leftrightarrow (-1, -1)$,

and $s = 3 \leftrightarrow (1, -1)$. The dynamics of the Ashkin-Teller model are also governed by single-spin-flip Metropolis updates. The energy change for flipping spin σ_i^A is $\Delta E^A = 2J\sigma_i^A \sum_{j\in \mathrm{nn}(i)} \sigma_j^A - 2K\sigma_i^A\sigma_i^B$, and for flipping spin σ_i^B is $\Delta E^B = 2J\sigma_i^B \sum_{j\in \mathrm{nn}(i)} \sigma_j^B + 2K\sigma_i^B\sigma_i^A$, where J is the ferromagnetic coupling constant and K is a non-reciprocal coupling constant. Note that a transition $k \to [k+1]$ in the Potts model corresponds to a single spin flip in the Ashkin-Teller model (e.g., $s = 0 \to s = 1$ corresponds to flipping σ^A while σ^B is fixed).

Task: By requiring that the energy change for any single-site transition $s_i \to s_i'$ is identical in both models for any configuration of neighboring sites, determine the relationship between the parameters of the two models. Derive the general expression for the contact energy $J_{k,s}$ for an arbitrary pair of states (k, s) in the Potts model as a function of the Ashkin-Teller coupling J and the state indices k and s.

Refined Solution:

$$J_{k,s} = 2J \cos\left(\frac{\pi(k-s)}{2}\right)$$

Problem 38 (Paper: 2506.23687v1, Index: 1)

Original Problem Statement

Background: A 4-state active Potts model on a lattice has sites with states $s_i \in \{0, 1, 2, 3\}$ and an interaction Hamiltonian $H_{\text{int}} = -\sum_{\langle ij \rangle} J_{s_i s_j}$, where the sum is over nearest-neighbor pairs. This system can be mapped onto an active 4-state vector Potts model, also known as the Ashkin-Teller model, where each site has two Ising spins, $\sigma^A = \pm 1$ and $\sigma^B = \pm 1$. The mapping between the Potts states and the spin pairs is defined as: $s = 0 \leftrightarrow (1, 1)$, $s = 1 \leftrightarrow (-1, 1)$, $s = 2 \leftrightarrow (-1, -1)$, and $s = 3 \leftrightarrow (1, -1)$. In the vector model, the interaction energy between two nearest-neighbor sites i and j is given by $H_{ij} = -J(\sigma_i^A \sigma_j^A + \sigma_i^B \sigma_j^B)$. By equating the interaction Hamiltonians of the two models, the Potts contact energies $J_{s_i s_j}$ can be expressed in terms of the parameter J and the spin states corresponding to s_i and s_j. Due to the cyclic nature of the state mapping, the contact energy $J_{k,k'}$ only depends on the "distance" between the states, i.e., $J_{k,k'} = J_{k,[k+(k'-k)]}$, where $[k]$ denotes $k \pmod 4$. We can therefore write the contact energy as $J_{k,[k+j]}$ where $j = (k' - k) \pmod 4$.

Task: Using the provided state mapping, derive the general expression for the contact energy $J_{k,[k+j]}$ in the Potts model as a function of J and the state separation $j \in \{0, 1, 2, 3\}$.

Original Solution

$$J_{k,[k+j]} = 2J \cos\left(\frac{\pi j}{2}\right)$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing the mapping between Potts states and Ising spins, applying the interaction Hamiltonian, and using modular arithmetic to derive a general expression for contact energies, involving multiple conceptual and algebraic steps.

Refined Problem

Refined Problem Statement: Background: A 4-state active Potts model on a lattice has sites with states $s_i \in \{0, 1, 2, 3\}$ and an interaction Hamiltonian $H_{\text{int}} = -\sum_{\langle ij \rangle} J_{s_i s_j}$, where the sum is over nearest-neighbor pairs. This system can be mapped onto an active 4-state vector Potts model, also known as the Ashkin-Teller model, where each site has two Ising spins, $\sigma^A = \pm 1$ and $\sigma^B = \pm 1$. The mapping between the Potts states and the spin pairs is defined as: $s = 0 \leftrightarrow (1, 1)$, $s = 1 \leftrightarrow (-1, 1)$, $s = 2 \leftrightarrow (-1, -1)$, and $s = 3 \leftrightarrow (1, -1)$. In the vector model, the interaction energy between two nearest-neighbor sites i and j is given by $H_{ij} = -J(\sigma_i^A \sigma_j^A + \sigma_i^B \sigma_j^B)$. By equating the interaction Hamiltonians of the two models, the Potts contact energies $J_{s_i s_j}$ can be expressed in terms of the parameter J and the spin states corresponding to s_i and s_j. Due to the cyclic nature of the state mapping, the contact energy $J_{k,k'}$ only depends on the "distance" between the states, i.e., $J_{k,k'} = J_{k,[k+(k'-k)]}$, where $[k]$ denotes $k \pmod 4$. We can therefore write the contact energy as $J_{k,[k+j]}$ where $j = (k' - k) \pmod 4$.

Task: Using the provided state mapping, derive the general expression for the contact energy $J_{k,[k+j]}$ in the Potts model as a function of J and the state separation $j \in \{0, 1, 2, 3\}$.

Refined Solution:

$$J_{k,[k+j]} = 2J \cos\left(\frac{\pi j}{2}\right)$$

Problem 39 (Paper: 2506.23792v1, Index: 0)

Original Problem Statement

Background: Consider a two-dimensional system where a particle of mass $m = 1$ moves with constant total energy E in a periodic potential $V(\boldsymbol{r})$. The potential consists of a triangular lattice of attractive wells, which act as "traps". The particle's motion can be modeled as a random walk between these traps. The diffusion coefficient D for this process is given by the random walk formula $D = \frac{l^2}{4\tau}$, where l is the fixed distance between the centers of adjacent traps and τ is the mean residence time of the particle within a single trap.

According to the Machta-Zwanzig approximation, the mean residence time is given by the ratio of the trap's phase space volume Ω to the total phase space flux ω out of the trap, i.e., $\tau \approx \Omega/\omega$. The phase space volume for a single trap is given by $\Omega = 2\pi \int_{V(r)\leq E} v(r)r$, where $v(\boldsymbol{r}) = \sqrt{2(E - V(\boldsymbol{r}))}$ is the particle's speed and the integral is over the energetically accessible region within the trap. The trap is part of a triangular lattice and has six identical exits to its neighbors. The total flux out of the trap is given by $\omega = 6 \int_{\text{exit}} \int_{-\pi/2}^{\pi/2} v(y)^2 \cos\theta\theta y$, where the spatial integral is over a single exit line, y is the coordinate along this line, and θ is the angle of the velocity vector relative to the exit's normal.

To express the diffusion coefficient in terms of averaged quantities, we define the following: - A_{trap} is the area of the unit cell corresponding to a single trap. - $v(r)_{\text{trap}} = \frac{1}{A_{\text{trap}}} \int_{V(r)\leq E} v(r)r$ is the average speed over the trap area. - $l_{\text{exit}} = \int_{V(y)\leq E} y$ is the effective length of a single exit. - $v(r)^2{}_{\text{exit}} = \frac{1}{l_{\text{exit}}} \int_{V(y)\leq E} v(y)^2 y$ is the mean squared speed over a single exit.

Task: Using the provided relations and definitions, derive an expression for the diffusion coefficient D in terms of the trap separation l, the trap area A_{trap}, the average trap speed $v(r)_{\text{trap}}$, the exit length l_{exit}, and the average exit speed squared $v(r)^2{}_{\text{exit}}$.

Original Solution

$$D = \frac{3l^2 l_{\text{exit}} v(r)^2{}_{\text{exit}}}{2\pi A_{\text{trap}} v(r)_{\text{trap}}}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing multiple integral expressions and physical concepts—phase space volume, flux, and averaging over spatial domains—to derive a nontrivial formula for the diffusion coefficient, involving careful manipulation of integrals and geometric factors.

Refined Problem

Refined Problem Statement: Background: Consider a two-dimensional system where a particle of mass $m = 1$ moves with constant total energy E in a periodic potential $V(\boldsymbol{r})$. The potential consists of a triangular lattice of attractive wells, which act as "traps". The particle's motion can be modeled as a random walk between these traps. The diffusion coefficient D for this process is given by the random walk formula $D = \frac{l^2}{4\tau}$, where l is the fixed distance between the centers of adjacent traps and τ is the mean residence time of the particle within a single trap.

According to the Machta-Zwanzig approximation, the mean residence time is given by the ratio of the trap's phase space volume Ω to the total phase space flux ω out of the trap, i.e., $\tau \approx \Omega/\omega$. The phase space volume for a single trap is given by $\Omega = 2\pi \int_{V(r)\leq E} v(r)r$, where $v(\boldsymbol{r}) = \sqrt{2(E - V(\boldsymbol{r}))}$ is the particle's speed and the integral is over the energetically accessible region within the trap. The trap is part of a triangular lattice and has six identical exits to its neighbors. The total flux out of the trap is given by $\omega = 6 \int_{\text{exit}} \int_{-\pi/2}^{\pi/2} v(y)^2 \cos\theta\theta y$, where the spatial integral is over a single exit line, y is the coordinate along this line, and θ is the angle of the velocity vector relative to the exit's normal.

To express the diffusion coefficient in terms of averaged quantities, we define the following: - A_{trap} is the area of the unit cell corresponding to a single trap. - $v(r)_{\text{trap}} = \frac{1}{A_{\text{trap}}} \int_{V(r)\leq E} v(r)r$ is the average speed over the trap area. - $l_{\text{exit}} = \int_{V(y)\leq E} y$ is the effective length of a single exit. - $v(r)^2{}_{\text{exit}} = \frac{1}{l_{\text{exit}}} \int_{V(y)\leq E} v(y)^2 y$ is the mean squared speed over a single exit.

Task: Using the provided relations and definitions, derive an expression for the diffusion coefficient D in terms of the trap separation l, the trap area A_{trap}, the average trap speed $v(r)_{\text{trap}}$, the exit length l_{exit}, and the average exit speed squared $v(r)^2{}_{\text{exit}}$.

Refined Solution:

$$D = \frac{3l^2 l_{\text{exit}} v(r)^2{}_{\text{exit}}}{2\pi A_{\text{trap}} v(r)_{\text{trap}}}$$

Problem 40 (Paper: 2506.23792v1, Index: 1)

Original Problem Statement

Background: Consider a system of a single particle moving in a two-dimensional periodic potential. The long-time dynamics are characterized by the diffusion coefficient D, which is obtained from the ensemble-averaged mean squared displacement (MSD), $\langle(\boldsymbol{r}(t)-\boldsymbol{r}(0))^2\rangle$, via the relation $D = \lim_{t\to\infty} \frac{\langle(\boldsymbol{r}(t)-\boldsymbol{r}(0))^2\rangle}{4t}$. The ensemble average is performed over a large set of initial conditions.

Within this system, trajectories can be partitioned into two classes. A fraction of trajectories, denoted as non-confined, exhibit hopping behavior between potential wells and contribute to diffusion. The remaining fraction consists of Confined Orbits (COs), which are trajectories that remain localized within a single potential well for all time. Let the number of non-confined trajectories in an ensemble be N and the number of COs be n. The fraction of confined orbits is thus $\rho_{\text{CO}} = \frac{n}{N+n}$.

Assume that the contribution of the COs to the long-time MSD is zero. We can define a diffusion coefficient, $D_{\text{non-CO}}$, which is calculated using an MSD that averages *only* over the N non-confined trajectories. The overall diffusion coefficient of the system, D, is calculated from the MSD averaged over all $N+n$ trajectories.

Task: Derive an expression for the overall diffusion coefficient D in terms of $D_{\text{non-CO}}$ and the fraction of confined orbits ρ_{CO}.

Original Solution

$$D = (1-\rho_{\text{CO}})D_{\text{non-CO}}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Trivial**

The problem reduces to a straightforward weighted average of diffusion coefficients based on fractions of trajectory types, requiring no complex derivation or synthesis of multiple concepts.

Refined Problem

Refined Problem Statement: Background: Consider a system of a single particle moving in a two-dimensional periodic potential. The long-time dynamics are characterized by the diffusion coefficient D, which is obtained from the ensemble-averaged mean squared displacement (MSD), $\langle(\boldsymbol{r}(t)-\boldsymbol{r}(0))^2\rangle$, via the relation $D = \lim_{t\to\infty} \frac{\langle(\boldsymbol{r}(t)-\boldsymbol{r}(0))^2\rangle}{4t}$. The ensemble average is performed over a large set of initial conditions.

Within this system, trajectories can be partitioned into two classes. A fraction of trajectories, denoted as non-confined, exhibit hopping behavior between potential wells and contribute to diffusion. The remaining fraction consists of Confined Orbits (COs), which are trajectories that remain localized within a single potential well for all time. Let the number of non-confined trajectories in an ensemble be N and the number of COs be n. The fraction of confined orbits is thus $\rho_{\text{CO}} = \frac{n}{N+n}$.

Assume that the contribution of the COs to the long-time MSD is zero. We can define a diffusion coefficient, $D_{\text{non-CO}}$, which is calculated using an MSD that averages *only* over the N non-confined trajectories. The overall diffusion coefficient of the system, D, is calculated from the MSD averaged over all $N+n$ trajectories.

Task: Derive an expression for the overall diffusion coefficient D in terms of $D_{\text{non-CO}}$ and the fraction of confined orbits ρ_{CO}.

Refined Solution:

$$D = (1-\rho_{\text{CO}})D_{\text{non-CO}}$$

Problem 41 (Paper: 2506.23796v1, Index: 0)

Original Problem Statement

Background: An interferometric scheme can be used to measure the out-of-time-ordered correlator ($\mathcal{F}$-OTOC) for an open quantum system. Consider a system (S) with an initial density matrix $\rho_S(0)$ and an auxiliary control qubit (c) initialized in the state $+_c = \frac{1}{\sqrt{2}}(0_c + 1_c)$. The total initial state is $\rho_{init} = \rho_S(0) \otimes ++_c$. The system's open dynamics are described by two general completely positive trace-preserving (CPTP) maps: a forward evolution map $\xi_f(t)$ and a backward evolution map $\xi_b(t)$. Let A_S and B_S be two unitary operators acting on the system's Hilbert space. The final state of the composite system, ρ_f, is obtained by applying the following sequence of five operations to ρ_{init}:

$$\begin{aligned}
\mathcal{S}_1 &= \mathcal{C}(\mathbf{I}_S \otimes 00_c + B_S \otimes 11_c) \\
\mathcal{S}_2 &= \xi_f(t) \otimes \mathcal{I}_c \\
\mathcal{S}_3 &= \mathcal{C}(A_S \otimes \mathbf{I}_c) \\
\mathcal{S}_4 &= \xi_b(t) \otimes \mathcal{I}_c \\
\mathcal{S}_5 &= \mathcal{C}(B_S \otimes 00_c + \mathbf{I}_S \ \otimes 11_c)
\end{aligned}$$

The final state is thus $\rho_f = \mathcal{S}_5 \cdot \mathcal{S}_4 \cdot \mathcal{S}_3 \cdot \mathcal{S}_2 \cdot \mathcal{S}_1 \cdot \rho_{init}$. Here, $\mathcal{C}(U)$ is a superoperator defined by its action on a density matrix ρ as $\mathcal{C}(U) \cdot \rho = U\rho U^\dagger$, and $\mathcal{I}_c$ is the identity superoperator on the control qubit space. The $\mathcal{F}$-OTOC is given by the expectation value of the Pauli operator σ_c^x on the control qubit in the final state.

Task: Derive the expression for the $\mathcal{F}$-OTOC, $\mathcal{F}(t, A, B) = \mathrm{Tr}(\sigma_c^x \rho_f)$. Express your final answer in terms of the operators A_S, B_S, the initial state $\rho_S(0)$, and the maps $\xi_f(t)$ and $\xi_b(t)$.

Original Solution

$$\Re \,\mathrm{Tr}\left[B_S^\dagger \xi_b(t)\left[A_S\left(\xi_f(t)\left[B_S\,\rho_S(0)\right]\right)A_S^\dagger\right]\right]$$

Critiques

Self-Containment Critique: **Status: Not self-contained**

The final solution expression uses the notation $\Re and \mathrm{Tr} without explicitly defining that \Re denotes the real part and \mathrm{Tr} denot$ without a matching opening environment.

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing multiple concepts including CPTP maps, controlled operations, and tracing over composite systems to derive a nontrivial expression for the F-OTOC, involving a multi-step chain of reasoning beyond direct application of known formulas.

Refined Problem

Refined Problem Statement: Background: An interferometric scheme can be used to measure the out-of-time-ordered correlator ($\mathcal{F}$-OTOC) for an open quantum system. Consider a system (S) with an initial density matrix $\rho_S(0)$ and an auxiliary control qubit (c) initialized in the state $+_c = \frac{1}{\sqrt{2}}(0_c + 1_c)$. The total initial state is $\rho_{\text{init}} = \rho_S(0) \otimes ++_c$. The system's open dynamics are described by two general completely positive trace-preserving (CPTP) maps: a forward evolution map $\xi_f(t)$ and a backward evolution map $\xi_b(t)$. Let A_S and B_S be two unitary operators acting on the system's Hilbert space. The final state of the composite system, ρ_f, is obtained by applying the following sequence of five operations to ρ_{init}:

$$\begin{aligned}
\mathcal{S}_1 &= \mathcal{C}(\mathbf{I}_S \otimes 00_c + B_S \otimes 11_c) \\
\mathcal{S}_2 &= \xi_f(t) \otimes \mathcal{I}_c \\
\mathcal{S}_3 &= \mathcal{C}(A_S \otimes \mathbf{I}_c) \\
\mathcal{S}_4 &= \xi_b(t) \otimes \mathcal{I}_c \\
\mathcal{S}_5 &= \mathcal{C}(B_S \otimes 00_c + \mathbf{I}_S \ \otimes 11_c)
\end{aligned}$$

The final state is thus $\rho_f = \mathcal{S}_5 \cdot \mathcal{S}_4 \cdot \mathcal{S}_3 \cdot \mathcal{S}_2 \cdot \mathcal{S}_1 \cdot \rho_{\text{init}}$. Here, $\mathcal{C}(U)$ is a superoperator defined by its action on a density matrix ρ as $\mathcal{C}(U) \cdot \rho = U\rho U^\dagger$, and $\mathcal{I}_c$ is the identity superoperator on the control qubit space.

The $\mathcal{F}$-OTOC is given by the expectation value of the Pauli operator σ_c^x on the control qubit in the final state. In the final solution, Tr denotes the trace over the system Hilbert space, and Re denotes the real part of the resulting complex number.

Task: Derive the expression for the $\mathcal{F}$-OTOC, $\mathcal{F}(t, A, B) = \text{Tr}_{\text{total}}(\sigma_c^x \rho_f)$. Express your final answer in terms of the operators A_S, B_S, the initial state $\rho_S(0)$, and the maps $\xi_f(t)$ and $\xi_b(t)$. Explicitly define all notation you use in your answer.

Refined Solution:

$$\text{Re}\,\text{Tr}_S\left[B_S^\dagger\,\xi_b(t)\left[A_S\left(\xi_f(t)\left[B_S\,\rho_S(0)\right]\right)A_S^\dagger\right]\right]$$

Problem 42 (Paper: 2506.23796v1, Index: 1)

Original Problem Statement

Background: Consider a composite quantum system with a finite-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, with corresponding dimensions d_A, d_B, and $d = d_A d_B$. The system's evolution is described by a completely positive trace-preserving (CPTP) map $\mathcal{E}$, and the corresponding evolution of an operator O is governed by the adjoint map $\mathcal{E}^\dagger$. A measure of information scrambling in such an open system is the bipartite out-of-time-ordered correlator (OTOC), defined as the Haar average of the squared Hilbert-Schmidt norm of the commutator between two local unitary operators, $A \in U(d_A)$ and $B \in U(d_B)$. This quantity, denoted $G(\mathcal{E}^\dagger)$, is given by

$$G(\mathcal{E}^\dagger) = \frac{1}{2d}\mathbb{E}_{A,B}\left\|[\mathcal{E}^\dagger(A), B]\right\|_2^2$$

where $\mathbb{E}_{A,B}$ denotes the Haar average over the unitary groups $U(d_A)$ and $U(d_B)$. Expanding the norm, this expression can be written as

$$G(\mathcal{E}^\dagger) = \frac{1}{d}\left(\mathcal{T}_A - \Re\{\mathcal{T}_{AB}\}\right)$$

where $\mathcal{T}_A = \mathbb{E}_A\left\|\mathcal{E}^\dagger(A)\right\|_2^2$ and $\mathcal{T}_{AB} = \mathbb{E}_{A,B}\mathrm{Tr}[\mathcal{E}^\dagger(A)^\dagger B^\dagger \mathcal{E}^\dagger(A) B]$. To evaluate the averaged terms, we introduce a replica of the Hilbert space, $\mathcal{H}' = \mathcal{H}'_A \otimes \mathcal{H}'_B$, and define swap operators on the doubled space $\mathcal{H} \otimes \mathcal{H}'$. The operator $S_{XX'}$ swaps the states in subsystem X and its replica X', while the total swap operator is $S = S_{AA'}S_{BB'}$. Using this formalism, the averaged terms can be expressed in terms of the tensored adjoint map $\mathcal{E}^{\dagger\otimes 2} = \mathcal{E}^\dagger \otimes \mathcal{E}^\dagger$ as:

$$\mathcal{T}_A = \frac{1}{d_A}\mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}'}[S_{AA'}(\mathcal{E}^{\dagger\otimes 2})S_{AA'}]$$
$$\mathcal{T}_{AB} = \frac{1}{d_A d_B}\mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}'}[S(\mathcal{E}^{\dagger\otimes 2})]$$

Assume the map $\mathcal{E}^\dagger$ is such that the trace of $\mathcal{T}_{AB}$ is real.

Task: Using the provided relations, derive the expression for the bipartite OTOC, $G(\mathcal{E}^\dagger)$. Express your final answer as a single trace over the doubled Hilbert space $\mathcal{H} \otimes \mathcal{H}'$.

Original Solution

$$G(\mathcal{E}^\dagger) = \frac{1}{d^2}\mathrm{Tr}\left[\left(d_B S_{AA'} - S\right)(\mathcal{E}^{\dagger\otimes 2})S_{AA'}\right]$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing multiple concepts including Haar averaging, operator adjoint maps, swap operators on doubled Hilbert spaces, and nontrivial algebraic manipulations to express the bipartite OTOC as a single trace, demonstrating a sophisticated multi-step reasoning process.

Refined Problem

Refined Problem Statement: Background: Consider a composite quantum system with a finite-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, with corresponding dimensions d_A, d_B, and $d = d_A d_B$. The system's evolution is described by a completely positive trace-preserving (CPTP) map $\mathcal{E}$, and the corresponding evolution of an operator O is governed by the adjoint map $\mathcal{E}^\dagger$. A measure of information scrambling in such an open system is the bipartite out-of-time-ordered correlator (OTOC), defined as the Haar average of the squared Hilbert-Schmidt norm of the commutator between two local unitary operators, $A \in U(d_A)$ and $B \in U(d_B)$. This quantity, denoted $G(\mathcal{E}^\dagger)$, is given by

$$G(\mathcal{E}^\dagger) = \frac{1}{2d}\mathbb{E}_{A,B}\left\|[\mathcal{E}^\dagger(A), B]\right\|_2^2$$

where $\mathbb{E}_{A,B}$ denotes the Haar average over the unitary groups $U(d_A)$ and $U(d_B)$. Expanding the norm, this expression can be written as

$$G(\mathcal{E}^\dagger) = \frac{1}{d}\left(\mathcal{T}_A - \Re\{\mathcal{T}_{AB}\}\right)$$

where $\mathcal{T}_A = \mathbb{E}_A \left\|\mathcal{E}^\dagger(A)\right\|_2^2$ and $\mathcal{T}_{AB} = \mathbb{E}_{A,B}\mathrm{Tr}[\mathcal{E}^\dagger(A)^\dagger B^\dagger \mathcal{E}^\dagger(A)B]$. To evaluate the averaged terms, we introduce a replica of the Hilbert space, $\mathcal{H}' = \mathcal{H}'_A \otimes \mathcal{H}'_B$, and define swap operators on the doubled space $\mathcal{H} \otimes \mathcal{H}'$. The operator $S_{XX'}$ swaps the states in subsystem X and its replica X', while the total swap operator is $S = S_{AA'}S_{BB'}$. Using this formalism, the averaged terms can be expressed in terms of the tensored adjoint map $\mathcal{E}^{\dagger\otimes 2} = \mathcal{E}^\dagger \otimes \mathcal{E}^\dagger$ as:

$$\mathcal{T}_A = \frac{1}{d_A}\mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}'}[S_{AA'}(\mathcal{E}^{\dagger\otimes 2})S_{AA'}]$$
$$\mathcal{T}_{AB} = \frac{1}{d_A d_B}\mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}'}[S(\mathcal{E}^{\dagger\otimes 2})]$$

Assume the map $\mathcal{E}^\dagger$ is such that the trace of $\mathcal{T}_{AB}$ is real.

Task: Using the provided relations, derive the expression for the bipartite OTOC, $G(\mathcal{E}^\dagger)$. Express your final answer as a single trace over the doubled Hilbert space $\mathcal{H} \otimes \mathcal{H}'$.

Refined Solution:

$$G(\mathcal{E}^\dagger) = \frac{1}{d^2}\mathrm{Tr}\left[\left(d_B S_{AA'} - S\right)(\mathcal{E}^{\dagger\otimes 2})S_{AA'}\right]$$

Problem 43 (Paper: 2506.23837v1, Index: 0)

Original Problem Statement

Background: Consider the Voter Model for opinion dynamics on a network of N agents. Each agent i holds a binary opinion $\sigma_i \in \{+1, -1\}$. The dynamics of the system are defined by an iterative process. In each discrete time step, a single edge (i, j) connecting two distinct agents is chosen uniformly at random from the set of all edges in the network. Then, one of the two agents connected by the edge is chosen to be the "copier" and the other the "template", each with probability 1/2. The copier agent adopts the opinion of the template agent. The system evolves until it reaches a consensus, where all agents share the same opinion. Such consensus states are absorbing, meaning the system cannot evolve further once a consensus is reached. Let the initial fraction of agents with opinion $+1$ be denoted by x. The exit probability, $\mathcal{E}(x)$, is defined as the probability that the system eventually reaches the consensus state where all agents have the opinion $+1$, starting from this initial configuration. The paper states that for this model, the total opinion (or magnetization), averaged over an ensemble of realizations, is a conserved quantity.

Task: Using the fact that the ensemble-averaged total opinion is conserved, derive the expression for the exit probability $\mathcal{E}(x)$.

Original Solution

$$\mathcal{E}(x) = x$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Trivial**

The problem reduces to recognizing that the conserved average magnetization implies the exit probability equals the initial fraction x, which is a straightforward and well-known result without requiring multi-step derivations.

Refined Problem

Refined Problem Statement: Background: Consider the Voter Model for opinion dynamics on a network of N agents. Each agent i holds a binary opinion $\sigma_i \in \{+1, -1\}$. The dynamics of the system are defined by an iterative process. In each discrete time step, a single edge (i, j) connecting two distinct agents is chosen uniformly at random from the set of all edges in the network. Then, one of the two agents connected by the edge is chosen to be the "copier" and the other the "template", each with probability 1/2. The copier agent adopts the opinion of the template agent. The system evolves until it reaches a consensus, where all agents share the same opinion. Such consensus states are absorbing, meaning the system cannot evolve further once a consensus is reached. Let the initial fraction of agents with opinion $+1$ be denoted by x. The exit probability, $\mathcal{E}(x)$, is defined as the probability that the system eventually reaches the consensus state where all agents have the opinion $+1$, starting from this initial configuration. The paper states that for this model, the total opinion (or magnetization), averaged over an ensemble of realizations, is a conserved quantity.

Task: Using the fact that the ensemble-averaged total opinion is conserved, derive the expression for the exit probability $\mathcal{E}(x)$.

Refined Solution:

$$\mathcal{E}(x) = x$$

Problem 44 (Paper: 2506.23837v1, Index: 1)

Original Problem Statement

Background: Consider an agent-based model for a financial market with N traders. Each trader i can adopt one of two states: a 'bullish' (buy) state, represented by $s_i = +1$, or a 'bearish' (sell) state, represented by $s_i = -1$. The state of a trader is updated stochastically based on the influence of other traders and external factors. This total influence is captured by a local field, $H_i(t)$. The probability that trader i will adopt the bullish state at time $t + 1$ is given by a heat bath rule:

$$p(s_i(t + 1) = +1) = \frac{1}{1 + \exp(-2\beta H_i(t))}$$

where β is a responsiveness parameter, analogous to an inverse temperature, that quantifies the sensitivity of traders to the local field.

The local field $H_i(t)$ on a trader i is determined by three components: 1. A 'herding' or 'Ising' interaction, where traders tend to align their state with others. 2. A 'contrarian' interaction, where traders may choose to act against the majority. This is governed by an individual strategy parameter C_i and a coupling constant α. 3. An external field h_{ext} representing global information or market sentiment.

The full expression for the local field is given by:

$$H_i(t) = \sum_{j=1}^{N} J_{ij} s_j(t) - \alpha C_i(t) \frac{1}{N} \sum_{j=1}^{N} s_j(t) + h_{ext}$$

where J_{ij} is the interaction strength between traders i and j.

To analyze the collective behavior of the market, we employ a mean-field approximation. We assume the market is fully connected, so the herding interaction strength is uniform, $J_{ij} = J/N$ for all $i \neq j$, and $J_{ii} = 0$. We also assume all traders have the same contrarian strategy parameter, $C_i = C$. The overall market state is characterized by the average opinion or 'magnetization', $m(t) = \frac{1}{N} \sum_{j=1}^{N} s_j(t)$.

Task: In the mean-field approximation and the thermodynamic limit ($N \to \infty$), the system can reach a steady state where the average market opinion m is constant. Derive the self-consistency equation that determines the possible values of this steady-state market opinion m.

Original Solution

$$m = \tanh\left(\beta\left((J - \alpha C)m + h_{ext}\right)\right)$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing multiple concepts including stochastic dynamics, mean-field approximation, and the interplay of herding and contrarian effects to derive a non-trivial self-consistency equation for the steady-state magnetization.

Refined Problem

Refined Problem Statement: Background: Consider an agent-based model for a financial market with N traders. Each trader i can adopt one of two states: a 'bullish' (buy) state, represented by $s_i = +1$, or a 'bearish' (sell) state, represented by $s_i = -1$. The state of a trader is updated stochastically based on the influence of other traders and external factors. This total influence is captured by a local field, $H_i(t)$. The probability that trader i will adopt the bullish state at time $t + 1$ is given by a heat bath rule:

$$p(s_i(t + 1) = +1) = \frac{1}{1 + \exp(-2\beta H_i(t))}$$

where β is a responsiveness parameter, analogous to an inverse temperature, that quantifies the sensitivity of traders to the local field.

The local field $H_i(t)$ on a trader i is determined by three components: 1. A 'herding' or 'Ising' interaction, where traders tend to align their state with others. 2. A 'contrarian' interaction, where

traders may choose to act against the majority. This is governed by an individual strategy parameter C_i and a coupling constant α. 3. An external field h_{ext} representing global information or market sentiment.

The full expression for the local field is given by:

$$H_i(t) = \sum_{j=1}^{N} J_{ij} s_j(t) - \alpha C_i(t) \frac{1}{N} \sum_{j=1}^{N} s_j(t) + h_{ext}$$

where J_{ij} is the interaction strength between traders i and j.

To analyze the collective behavior of the market, we employ a mean-field approximation. We assume the market is fully connected, so the herding interaction strength is uniform, $J_{ij} = J/N$ for all $i \neq j$, and $J_{ii} = 0$. We also assume all traders have the same contrarian strategy parameter, $C_i = C$. The overall market state is characterized by the average opinion or 'magnetization', $m(t) = \frac{1}{N} \sum_{j=1}^{N} s_j(t)$.

Task: In the mean-field approximation and the thermodynamic limit ($N \to \infty$), the system can reach a steady state where the average market opinion m is constant. Derive the self-consistency equation that determines the possible values of this steady-state market opinion m.

Refined Solution:

$$m = \tanh\left(\beta\left((J - \alpha C)m + h_{ext}\right)\right)$$

Problem 45 (Paper: 2506.23867v1, Index: 0)

Original Problem Statement

Background: Consider a system of non-interacting particles diffusing in one dimension with a diffusion coefficient D. The system is defined on a line of length L, from $x = -L/2$ to $x = L/2$. In a central "channel" region, for $|x| \leq L_0/2$, particles can reversibly adsorb to a stationary surface. The adsorption rate is and the desorption rate is . Let $p(x,t)$ be the probability density of finding a particle at position x on the diffusion line, and $q(x,t)$ be the probability density of finding it in the adsorbed state at the corresponding position x. The governing equations for these densities are:

$$\begin{cases} \text{for } |x| \leq L_0/2: & \begin{cases} \partial_t p = D\partial_{xx} p - p + q \\ \partial_t q = p - q \end{cases} \\ \text{for } |x| > L_0/2: & \partial_t p = D\partial_{xx} p \end{cases}$$

The system has no-flux boundary conditions at its edges, i.e., $\partial_x p(\pm L/2, t) = 0$. At the interfaces of the channel, $x = \pm L_0/2$, both the density $p(x,t)$ and its spatial derivative $\partial_x p(x,t)$ are continuous. We analyze the system in Laplace space, using the transform $\hat{f}(s) = \int_0^\infty f(t)e^{-st}dt$. The equations are solved assuming uniform initial densities $p(x,0) = p_0$ and $q(x,0) = q_0$ for $|x| \leq L_0/2$, and zero elsewhere. The general solution for the Laplace transform of the mobile particle density, $\hat{pc}(x,s)$, inside the channel ($|x| \leq L_0/2$) is given by:

$$\hat{pc}(x,s) = A_C e^{m_1 x} + B_C e^{-m_1 x} + \frac{(p_0 + q_0) + sp_0}{s(s++)}$$

where A_C and B_C are constants to be determined from the boundary conditions. The characteristic inverse lengths are defined as $m_0(s) = \sqrt{s/D}$ and $m_1(s) = \sqrt{\frac{s(s++)}{D(s+)}}$. The solution in the reservoir regions ($L_0/2 < |x| \leq L/2$) is of the form $\hat{p}_{res}(x,s) = A_{res}e^{m_0 x} + B_{res}e^{-m_0 x}$. The problem is symmetric with respect to $x = 0$.

Task: By applying the boundary and continuity conditions, and exploiting the symmetry of the problem which implies $A_C = B_C$, determine the coefficient A_C.

Original Solution

$$A_C = -\frac{m_0((p_0 + q_0) + p_0 s)}{s(s++)} \frac{e^{L_0 m_1/2}(e^{Lm_0} - e^{L_0 m_0})}{(m_0 - m_1)(e^{Lm_0} - e^{L_0(m_0+m_1)}) + (m_0 + m_1)(e^{Lm_0 + L_0 m_1} - e^{L_0 m_0})}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing multiple boundary and continuity conditions, applying Laplace transforms, and solving coupled differential equations with spatially dependent parameters, making it a sophisticated multi-step reasoning task.

Refined Problem

Refined Problem Statement: Background: Consider a system of non-interacting particles diffusing in one dimension with a diffusion coefficient D. The system is defined on a line of length L, from $x = -L/2$ to $x = L/2$. In a central "channel" region, for $|x| \leq L_0/2$, particles can reversibly adsorb to a stationary surface. The adsorption rate is and the desorption rate is . Let $p(x,t)$ be the probability density of finding a particle at position x on the diffusion line, and $q(x,t)$ be the probability density of finding it in the adsorbed state at the corresponding position x. The governing equations for these densities are:

$$\begin{cases} \text{for } |x| \leq L_0/2: & \begin{cases} \partial_t p = D\partial_{xx} p - p + q \\ \partial_t q = p - q \end{cases} \\ \text{for } |x| > L_0/2: & \partial_t p = D\partial_{xx} p \end{cases}$$

The system has no-flux boundary conditions at its edges, i.e., $\partial_x p(\pm L/2, t) = 0$. At the interfaces of the channel, $x = \pm L_0/2$, both the density $p(x,t)$ and its spatial derivative $\partial_x p(x,t)$ are continuous. We analyze the system in Laplace space, using the transform $\hat{f}(s) = \int_0^\infty f(t)e^{-st}dt$. The equations are solved assuming uniform initial densities $p(x,0) = p_0$ and $q(x,0) = q_0$ for $|x| \leq L_0/2$, and zero elsewhere. The general solution for the Laplace transform of the mobile particle density, $\hat{pc}(x,s)$, inside the channel ($|x| \leq L_0/2$) is given by:

$$\hat{pc}(x,s) = A_C e^{m_1 x} + B_C e^{-m_1 x} + \frac{(p_0 + q_0) + sp_0}{s(s++)}$$

where A_C and B_C are constants to be determined from the boundary conditions. The characteristic inverse lengths are defined as $m_0(s) = \sqrt{s/D}$ and $m_1(s) = \sqrt{\frac{s(s++)}{D(s+)}}$. The solution in the reservoir regions ($L_0/2 < |x| \leq L/2$) is of the form $\hat{p}_{res}(x,s) = A_{res}e^{m_0 x} + B_{res}e^{-m_0 x}$. The problem is symmetric with respect to $x = 0$.

Task: By applying the boundary and continuity conditions, and exploiting the symmetry of the problem which implies $A_C = B_C$, determine the coefficient A_C.

Refined Solution:

$$A_C = -\frac{m_0((p_0 + q_0) + p_0 s)}{s(s++)} \frac{e^{L_0 m_1/2}(e^{Lm_0} - e^{L_0 m_0})}{(m_0 - m_1)(e^{Lm_0} - e^{L_0(m_0+m_1)}) + (m_0 + m_1)(e^{Lm_0 + L_0 m_1} - e^{L_0 m_0})}$$

Problem 46 (Paper: 2506.23867v1, Index: 1)

Original Problem Statement

Background: Consider a one-dimensional system of total length L containing $\mathcal{N}$ non-interacting particles that diffuse with a diffusion coefficient D. A central segment of length L_0, defined by $|x| \leq L_0/2$, represents a channel. Within this channel, particles can reversibly adsorb to an immobile state. The probability density for a particle to be at position x in the diffusive state is $p(x,t)$, and in the adsorbed state is $q(x,t)$. Adsorption occurs with rate and desorption with rate . The system is described by the following coupled equations:

$$\begin{cases} \text{for } |x| \leq L_0/2: & \partial_t p = D\partial_{xx} p - k_{\text{on}} p + k_{\text{off}} q \\ & \partial_t q = k_{\text{on}} p - k_{\text{off}} q \\ \text{for } L_0/2 < |x| \leq L/2: & \partial_t p = D\partial_{xx} p \end{cases}$$

The system has reflecting boundary conditions at $x = \pm L/2$, and both $p(x,t)$ and its flux $-D\partial_x p(x,t)$ are continuous at the channel edges $x = \pm L_0/2$. We consider the system at equilibrium, where the initial probability densities for a particle starting in the channel are $p_0 = 1/Z$ and $q_0 = (/)/Z$, with partition function $Z = L + L_0(/)$.

In Laplace space (with frequency variable s), the solution for the probability density of free particles inside the channel is given by:

$$\hat{p_C}(x,s) = A_C e^{m_1 x} + B_C e^{-m_1 x} + \frac{(p_0 + q_0) + sp_0}{s(s++)}$$

where $m_0(s) = \sqrt{s/D}$ and $m_1(s) = \sqrt{\frac{s(s++)}{D(s+)}}$. Due to the symmetry of the boundary conditions, $A_C = B_C$. The constant A_C is found to be:

$$A_C = -\frac{m_0((p_0+q_0)+p_0 s)}{s(s++)} \frac{e^{L_0 m_1/2}(e^{Lm_0} - e^{L_0 m_0})}{(m_0 - m_1)(e^{Lm_0} - e^{L_0(m_0+m_1)}) + (m_0+m_1)(e^{Lm_0 + L_0 m_1} - e^{L_0 m_0})}$$

The power spectral density (PSD) of the fluctuations in the total number of particles in the channel, $S_N(f)$, can be found from the Laplace transform of the single-particle probability flux out of the channel. For $f \neq 0$, the relationship is $S_N(f) = 4\Re\left[\frac{2\mathcal{N}D}{s}\partial_x \hat{p_C}(x,s)|_{x=L_0/2}\right]$, with $s = 2i\pi f$.

Task: Using the expressions provided, derive the full expression for the power spectral density of the total number of particles in the channel, $S_N(f)$.

Original Solution

$$S_N(f) = 8\Re\left[\frac{\mathcal{N} m_1((p_0+q_0)+sp_0)}{s m_0(s++)} \frac{(e^{Lm_0} - e^{L_0 m_0})(1 - e^{L_0 m_1})}{(m_0 - m_1)(e^{Lm_0} - e^{L_0(m_0+m_1)}) + (m_0+m_1)(e^{Lm_0+L_0 m_1} - e^{L_0 m_0})}\right]$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing coupled diffusion-adsorption equations, applying boundary conditions, performing Laplace transforms, and manipulating complex expressions to derive the PSD, involving multiple advanced steps beyond direct formula application.

Refined Problem

Refined Problem Statement: Background: Consider a one-dimensional system of total length L containing $\mathcal{N}$ non-interacting particles that diffuse with a diffusion coefficient D. A central segment of length L_0, defined by $|x| \leq L_0/2$, represents a channel. Within this channel, particles can reversibly adsorb to an immobile state. The probability density for a particle to be at position x in the diffusive

state is $p(x,t)$, and in the adsorbed state is $q(x,t)$. Adsorption occurs with rate k_{on} and desorption with rate k_{off}. The system is described by the following coupled equations:

$$\begin{cases} \text{for } |x| \le L_0/2: & \partial_t p = D\partial_{xx} p - k_{\text{on}} p + k_{\text{off}} q \\ & \partial_t q = k_{\text{on}} p - k_{\text{off}} q \\ \text{for } L_0/2 < |x| \le L/2: & \partial_t p = D\partial_{xx} p \end{cases}$$

The system has reflecting boundary conditions at $x = \pm L/2$, and both $p(x,t)$ and its flux $-D\partial_x p(x,t)$ are continuous at the channel edges $x = \pm L_0/2$. We consider the system at equilibrium, where the initial probability densities for a particle starting in the channel are $p_0 = 1/Z$ and $q_0 = (k_{\text{on}}/k_{\text{off}})/Z$, with partition function $Z = L + L_0(k_{\text{on}}/k_{\text{off}})$.

In Laplace space (with frequency variable s), the solution for the probability density of free particles inside the channel is given by:

$$\hat{p_C}(x,s) = A_C e^{m_1 x} + B_C e^{-m_1 x} + \frac{k_{\text{off}}(p_0+q_0) + s p_0}{s(s + k_{\text{off}} + k_{\text{on}})}$$

where $m_0(s) = \sqrt{s/D}$ and $m_1(s) = \sqrt{\frac{s(s+k_{\text{off}}+k_{\text{on}})}{D(s+k_{\text{off}})}}$. Due to the symmetry of the boundary conditions, $A_C = B_C$. The constant A_C is found to be:

$$A_C = -\frac{m_0(k_{\text{off}}(p_0+q_0)+p_0 s)}{s(s+k_{\text{off}}+k_{\text{on}})} \frac{e^{L_0 m_1/2}(e^{L m_0} - e^{L_0 m_0})}{(m_0-m_1)(e^{L m_0} - e^{L_0(m_0+m_1)}) + (m_0+m_1)(e^{L m_0 + L_0 m_1} - e^{L_0 m_0})}$$

The power spectral density (PSD) of the fluctuations in the total number of particles in the channel, $S_N(f)$, can be found from the Laplace transform of the single-particle probability flux out of the channel. For $f \neq 0$, the relationship is $S_N(f) = 4\Re\left[\frac{2\mathcal{N}D}{s}\partial_x \hat{p_C}(x,s)|_{x=L_0/2}\right]$, with $s = 2i\pi f$.

Task: Using the expressions provided, derive the full expression for the power spectral density of the total number of particles in the channel, $S_N(f)$.

Refined Solution:

$$S_N(f) = 8\Re\left[\frac{\mathcal{N} m_1(k_{\text{off}}(p_0+q_0)+s p_0)}{s m_0(s+k_{\text{off}}+k_{\text{on}})} \frac{(e^{L m_0} - e^{L_0 m_0})(1 - e^{L_0 m_1})}{(m_0-m_1)(e^{L m_0} - e^{L_0(m_0+m_1)}) + (m_0+m_1)(e^{L m_0 + L_0 m_1} - e^{L_0 m_0})}\right]$$

Problem 47 (Paper: 2506.23886v2, Index: 0)

Original Problem Statement

Background: Consider a tt*-structure (E, η, g, Φ) of rank $n+1$ over $\mathbb{C}^*$. A tt*-structure consists of a holomorphic vector bundle E with a holomorphic structure $\overline{\partial}_E$, a holomorphic non-degenerate symmetric bilinear form η, a Hermitian metric g, and a holomorphic $\mathrm{End}(E)$-valued 1-form Φ. These components are related by two key conditions: 1. The tt*-equation: $F_D = -[\Phi, \Phi^{\dagger_g}]$, where F_D is the curvature of the Chern connection $D = \partial_E^g + \overline{\partial}_E$ associated with g, and $\Phi^{\dagger_g}$ is the adjoint of Φ with respect to g. 2. The reality condition: An involution κ on E, defined by the relation $g(a,b) = \eta(\kappa(a), b)$ for any sections $a, b \in \Gamma(E)$, must satisfy $\kappa^2 = Id_E$.

The structure is assumed to be a fixed point of the $\mathbb{Z}_{n+1}$-multiplication. This means there exists an isomorphism of tt*-structures $\mathcal{T} : (E, \eta, g, \Phi) \to (E, \eta, g, \omega\Phi)$, where $\omega = e^{\sqrt{-1}\frac{2\pi}{n+1}}$. We consider the case where $\mathcal{T}^{n+1} = -Id_E$. It can be shown that there exists a holomorphic frame $\tau = (\tau_0, \ldots, \tau_n)$ for the bundle E in which the matrix representations of the structure's components have the following form: - The Higgs field is $\Phi_\tau = \mathrm{diag}(1, \omega, \ldots, \omega^n)dt$. - The bilinear form is the identity matrix, $\eta_\tau = I$. - The isomorphism $\mathcal{T}$ is represented by the matrix $T = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -1 & & & 0 \end{pmatrix}$. - The metric g is preserved by $\mathcal{T}$, which implies its matrix representation G_τ in the τ frame commutes with T, i.e., $TG_\tau = G_\tau T$.

Task: To analyze the system, one must find a new holomorphic frame $e = (e_0, \ldots, e_n)$ where the physical equations become manifest. This new frame is related to the τ frame by a change of basis $e = \tau \cdot M$, where the matrix M is given by the product $M = LP$. The matrix L is defined as $L = \frac{1}{\sqrt{n+1}}\mathrm{diag}(1, \omega^{-1/2}, \ldots, \omega^{-n/2})F$, where F is the standard Discrete Fourier Transform matrix with entries $F_{jk} = \omega^{jk}$ for $j, k \in \{0, \ldots, n\}$. The matrix P is the cyclic permutation matrix with entries $P_{j,j-1} = 1$ for $j \in \{1, \ldots, n\}$ and $P_{0,n} = 1$. In this new frame e, the metric becomes diagonal, with its matrix representation being $G_e = \mathrm{diag}(e^{w_0}, e^{w_1}, \ldots, e^{w_n})$ for some real functions $w_j(t, \bar{t})$. Your task is to derive the complete system of equations governing these functions w_j. This system consists of a set of coupled partial differential equations arising from the tt*-equation and an algebraic anti-symmetry condition on the w_j functions arising from the reality condition $\kappa^2 = Id_E$.

Original Solution

$$\begin{cases} (w_j)_{t\bar{t}} = e^{w_j - w_{j-1}} - e^{w_{j+1} - w_j}, & j = 0, \ldots, n \\ w_j + w_{n-j} = 0, & j = 0, \ldots, n \end{cases} \quad \text{where } w_{-1} = w_n \text{ and } w_{n+1} = w_0.$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing the tt*-equation, reality conditions, and representation theory to derive a coupled nonlinear PDE system with boundary conditions, involving a nontrivial change of frame and matrix manipulations.

Refined Problem

Refined Problem Statement: Background: Consider a tt*-structure

$$(E, \eta, g, \Phi)$$

of rank $n+1$ over $\mathbb{C}^*$. A tt*-structure consists of a holomorphic vector bundle E with a holomorphic structure $\overline{\partial}_E$, a holomorphic non-degenerate symmetric bilinear form η, a Hermitian metric g, and a holomorphic

$$\mathrm{End}(E)$$

-valued 1-form Φ. These components are related by two key conditions: 1. The tt*-equation:

$$F_D = -[\Phi, \Phi^{\dagger_g}]$$

, where F_D is the curvature of the Chern connection $D = \partial_E^g + \overline{\partial}_E$ associated with g, and $\Phi^{\dagger_g}$ is the adjoint of Φ with respect to g. 2. The reality condition: An involution κ on E, defined by the relation

$$g(a,b) = \eta(\kappa(a),b)$$

for any sections $a, b \in \Gamma(E)$, must satisfy $\kappa^2 = \mathrm{Id}_E$.

The structure is assumed to be a fixed point of the $\mathbb{Z}_{n+1}$-multiplication. This means there exists an isomorphism of tt*-structures

$$\mathcal{T} : (E, \eta, g, \Phi) \to (E, \eta, g, \omega\Phi)$$

, where $\omega = e^{\sqrt{-1}\frac{2\pi}{n+1}}$. We consider the case where

$$\mathcal{T}^{n+1} = -\mathrm{Id}_E$$

. It can be shown that there exists a holomorphic frame

$$\tau = (\tau_0, \ldots, \tau_n)$$

for the bundle E in which the matrix representations of the structure's components have the following form: - The Higgs field is

$$\Phi_\tau = \mathrm{diag}(1, \omega, \ldots, \omega^n)dt$$

. - The bilinear form is the identity matrix,

$$\eta_\tau = I$$

. - The isomorphism $\mathcal{T}$ is represented by the matrix

$$T = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -1 & & & 0 \end{pmatrix}$$

. - The metric g is preserved by $\mathcal{T}$, which implies its matrix representation G_τ in the τ frame commutes with T, i.e.,

$$TG_\tau = G_\tau T$$

.

Task: To analyze the system, one must find a new holomorphic frame

$$e = (e_0, \ldots, e_n)$$

where the physical equations become manifest. This new frame is related to the τ frame by a change of basis

$$e = \tau \cdot M$$

, where the matrix M is given by the product

$$M = LP$$

. The matrix L is defined as

$$L = \frac{1}{\sqrt{n+1}}\,\mathrm{diag}(1, \omega^{-1/2}, \ldots, \omega^{-n/2})F$$

, where F is the standard Discrete Fourier Transform matrix with entries

$$F_{jk} = \omega^{jk}$$

for $j, k \in \{0, \ldots, n\}$. The matrix P is the cyclic permutation matrix with entries

$$P_{j,j-1} = 1$$

for $j \in \{1, \ldots, n\}$ and

$$P_{0,n} = 1$$

. In this new frame e, the metric becomes diagonal, with its matrix representation being

$$G_e = \mathrm{diag}(e^{w_0}, e^{w_1}, \ldots, e^{w_n})$$

for some real functions $w_j(t, \bar{t})$. Your task is to derive the complete system of equations governing these functions w_j. This system consists of a set of coupled partial differential equations arising from the tt*-equation and an algebraic anti-symmetry condition on the w_j functions arising from the reality condition $\kappa^2 = \mathrm{Id}_E$.

Refined Solution:

$$\begin{cases} (w_j)_{t\bar{t}} = e^{w_j - w_{j-1}} - e^{w_{j+1} - w_j}, & j = 0, \ldots, n \\ w_j + w_{n-j} = 0, & j = 0, \ldots, n \end{cases} \quad \text{where } w_{-1} = w_n \text{ and } w_{n+1} = w_0.$$

Problem 48 (Paper: 2506.23886v2, Index: 1)

Original Problem Statement

Background: Consider a physical system described by a "radial Toda-type tt*-structure" of rank ‘n+1‘. The dynamics of this system are governed by a set of ‘n+1‘ real-valued functions ‘$w_j(|t|)$‘for‘$j = 0, \dots, n$‘, $which depend only on the radial coordinate$ ‘$t \in \mathbb{C}^*$‘. $These functions are solutions to the tt* - Toda equation,$ ‘$(w_j)_{t\bar{t}} = e^{w_j - w_{j-1}} - e^{w_{j+1} - w_j}$‘, $with periodic boundary conditions$ ‘$w_{j+n+1} = w_j$‘. As ‘$t \to 0$‘, $these solutions exhibit the asymptotic beha$ ‘$-m_j \log|t|$‘, $where the coefficients$ ‘m_j‘ $are rational numbers satisfying$ ‘$m_{j-1} - m_j + 2 > 0$‘ $(with indices taken modulo$ ‘$n+1$‘). $This system is associated with a highest weight representation$ ‘Λ‘ $of the$ ‘W‘$-algebra$ ‘$\mathcal{W}_{n+1}$‘ $in what is known as the$ ‘$(n+1, N)$‘$-minimal model, for some integer$ ‘$N \in \mathbb{N}$‘. $The representation is parameterized by a set of non-negative integers$ ‘b_j‘g ‘$\frac{N+n+1}{2(n+1)}(m_{j-1} - m_j + 2) - 1$‘. $Furthermore, the asymptotic data$ ‘m_j‘ $is related to the effective Virasoro central charge$ ‘c_{eff}‘$of th$ ‘$\frac{N+n+1}{4(n+1)} \sum_{j=0}^{n} m_j^2 = \frac{1}{12}(n - c_{\text{eff}})$‘.

Task: Using the provided relations, derive an expression for the effective Virasoro central charge ‘c_{eff}‘$in terms of$ ‘n‘, ‘N‘, $and the asymptotic coefficients$ ‘m_j‘.

Original Solution

$$c_{\text{eff}} = n - \frac{3(N+n+1)}{n+1} \sum_{j=0}^{n} m_j^2$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Trivial**

The problem is a straightforward algebraic manipulation of a given formula directly stated in the paper, requiring no multi-step synthesis or deeper conceptual reasoning.

Refined Problem

Refined Problem Statement: Background: Consider a physical system described by a "radial Toda-type tt$^*-structure"$ $of rank$ n+1. $The dynamics of this system are governed by a set of$ n+1 $real-valued functions$ $w_j(|t|)$ for $j = 0, \dots, n$, which depend only on the radial coordinate $t \in \mathbb{C}^*$. These functions are solutions to the tt$^* - Toda equation,$ $(w_j)_{t\bar{t}} = e^{w_j - w_{j-1}} - e^{w_{j+1} - w_j}$, with periodic boundary conditions $w_{j+n+1} = w_j$. As $t \to 0$, these solutions exhibit the asymptotic behavior $w_j \sim -m_j \log|t|$, where the coefficients m_j are rational numbers satisfying $m_{j-1} - m_j + 2 > 0$ (with indices taken modulo $n+1$). This system is associated with a highest weight representation Λ of the W-algebra $\mathcal{W}_{n+1}$ in what is known as the $(n+1, N)$-minimal model, for some integer $N \in \mathbb{N}$. The representation is parameterized by a set of non-negative integers b_j given by $b_j = \frac{N+n+1}{2(n+1)}(m_{j-1} - m_j + 2) - 1$. Furthermore, the asymptotic data m_j is related to the effective Virasoro central charge c_{eff} of the corresponding representation by the following formula, which connects the geometry of the structure to the conformal field theory: $\frac{N+n+1}{4(n+1)} \sum_{j=0}^{n} m_j^2 = \frac{1}{12}(n - c_{\text{eff}})$.

Task: Using the provided relations, derive an expression for the effective Virasoro central charge c_{eff} in terms of n, N, and the asymptotic coefficients m_j.

Refined Solution:

$$c_{\text{eff}} = n - \frac{3(N+n+1)}{n+1} \sum_{j=0}^{n} m_j^2$$

Problem 49 (Paper: 2506.23890v1, Index: 0)

Original Problem Statement

Background: A pseudospherical surface is a two-dimensional Riemannian manifold with constant negative Gaussian curvature $\mathcal{K} = -1$. Its geometry can be described locally by a set of three 1-forms, $\omega_1, \omega_2, \omega_3$, which satisfy the Maurer-Cartan structure equations:

$$d\omega_1 = \omega_3 \wedge \omega_2$$
$$d\omega_2 = \omega_1 \wedge \omega_3$$
$$d\omega_3 = \omega_1 \wedge \omega_2$$

For a 1-form $\alpha = P(x,t)dx + Q(x,t)dt$, the exterior derivative is $d\alpha = (\partial_x Q - \partial_t P)dx \wedge dt$. The wedge product is bilinear and antisymmetric, with the properties $dx \wedge dt = -dt \wedge dx$ and $dx \wedge dx = dt \wedge dt = 0$. Consider a set of 1-forms that depend on a real-valued function $u(x,t)$, its derivatives, and a real, non-zero parameter λ. Let $m = u - u_{xx}$. The forms are given by:

$$\omega_1 = \left(\frac{\lambda}{2} + \frac{1}{2\lambda} - m\right)dx + \left(um + \frac{\lambda}{2}u - \frac{u}{2\lambda} - \frac{1}{2} - \frac{\lambda^2}{2}\right)dt$$
$$\omega_2 = -u_x dt$$
$$\omega_3 = \left(m + \frac{1}{2\lambda} - \frac{\lambda}{2}\right)dx + \left(\frac{\lambda^2}{2} - \frac{1}{2} - \frac{u}{2\lambda} - \frac{\lambda}{2}u - um\right)dt$$

Task: For this system, the structure equations impose a constraint on the function $u(x,t)$. Derive the partial differential equation for $u(x,t)$ that results from the first structure equation, $d\omega_1 = \omega_3 \wedge \omega_2$. Your final answer should be expressed solely in terms of u and its partial derivatives.

Original Solution

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires applying exterior calculus, manipulating differential forms, and translating the Maurer-Cartan structure equations into a nonlinear PDE for u(x,t), involving multiple steps and synthesis of geometric and analytic concepts.

Refined Problem

Refined Problem Statement: Background: A pseudospherical surface is a two-dimensional Riemannian manifold with constant negative Gaussian curvature $\mathcal{K} = -1$. Its geometry can be described locally by a set of three 1-forms, $\omega_1, \omega_2, \omega_3$, which satisfy the Maurer-Cartan structure equations:

$$d\omega_1 = \omega_3 \wedge \omega_2$$
$$d\omega_2 = \omega_1 \wedge \omega_3$$
$$d\omega_3 = \omega_1 \wedge \omega_2$$

For a 1-form $\alpha = P(x,t)dx + Q(x,t)dt$, the exterior derivative is $d\alpha = (\partial_x Q - \partial_t P)dx \wedge dt$. The wedge product is bilinear and antisymmetric, with the properties $dx \wedge dt = -dt \wedge dx$ and $dx \wedge dx = dt \wedge dt = 0$. Consider a set of 1-forms that depend on a real-valued function $u(x,t)$, its derivatives, and a real, non-zero parameter λ. Let $m = u - u_{xx}$. The forms are given by:

$$\omega_1 = \left(\frac{\lambda}{2} + \frac{1}{2\lambda} - m\right)dx + \left(um + \frac{\lambda}{2}u - \frac{u}{2\lambda} - \frac{1}{2} - \frac{\lambda^2}{2}\right)dt$$
$$\omega_2 = -u_x dt$$
$$\omega_3 = \left(m + \frac{1}{2\lambda} - \frac{\lambda}{2}\right)dx + \left(\frac{\lambda^2}{2} - \frac{1}{2} - \frac{u}{2\lambda} - \frac{\lambda}{2}u - um\right)dt$$

Task: For this system, the structure equations impose a constraint on the function $u(x,t)$. Derive the partial differential equation for $u(x,t)$ that results from the first structure equation, $d\omega_1 = \omega_3 \wedge \omega_2$. Your final answer should be expressed solely in terms of u and its partial derivatives.

Refined Solution:

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

Problem 50 (Paper: 2506.23890v1, Index: 1)

Original Problem Statement

Background: Consider the Camassa-Holm (CH) equation for a real-valued function $u(x,t)$:

$$u_t - u_{txx} + 3uu_x = 2u_xu_{xx} + uu_{xxx}$$

Let $u(x,t)$ be a non-trivial solution to the CH equation, existing for a time interval $t \in [0,T)$. This solution arises from an initial condition $u(x,0) = u_0(x)$ belonging to a suitable function space (specifically, the Sobolev space $H^s(\mathbb{R})$ with $s \geq 3$). For any fixed time t in the interval of existence, the solution is known to possess the following properties: 1. The functions $x \mapsto u(x,t)$ and its spatial derivative $x \mapsto u_x(x,t)$ are continuous over $\mathbb{R}$. 2. The solution and its derivative vanish at spatial infinity: $\lim_{|x|\to\infty} u(x,t) = 0$ and $\lim_{|x|\to\infty} u_x(x,t) = 0$. 3. The $H^1(\mathbb{R})$ norm, defined as $\|u(\cdot,t)\|_{H^1}^2 = \int_{-\infty}^{\infty}[u(x,t)^2 + u_x(x,t)^2]dx$, is a conserved quantity that is strictly positive, i.e., $\|u(\cdot,t)\|_{H^1}^2 = \|u_0\|_{H^1}^2 > 0$.

Due to these properties, the infimum and supremum of the solution's slope are attained for each t. Let us define these extrema as:

$$I(t) = \inf_{x\in\mathbb{R}} u_x(x,t)$$

$$S(t) = \sup_{x\in\mathbb{R}} u_x(x,t)$$

Task: Based on the provided properties of the solution $u(x,t)$, derive a strict inequality that characterizes the product $h(t) = I(t)S(t)$ for all $t \in [0,T)$.

Original Solution

$$h(t) < 0$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing properties of the CH equation solution, Sobolev space embeddings, and extremal slope behavior to derive a non-obvious strict inequality for the product of slope extrema, involving a multi-step reasoning process.

Refined Problem

Refined Problem Statement: Background: Consider the Camassa-Holm (CH) equation for a real-valued function $u(x,t)$:

$$u_t - u_{txx} + 3uu_x = 2u_xu_{xx} + uu_{xxx}$$

Let $u(x,t)$ be a non-trivial solution to the CH equation, existing for a time interval $t \in [0,T)$. This solution arises from an initial condition $u(x,0) = u_0(x)$ belonging to a suitable function space (specifically, the Sobolev space $H^s(\mathbb{R})$ with $s \geq 3$). For any fixed time t in the interval of existence, the solution is known to possess the following properties: 1. The functions $x \mapsto u(x,t)$ and its spatial derivative $x \mapsto u_x(x,t)$ are continuous over $\mathbb{R}$. 2. The solution and its derivative vanish at spatial infinity: $\lim_{|x|\to\infty} u(x,t) = 0$ and $\lim_{|x|\to\infty} u_x(x,t) = 0$. 3. The $H^1(\mathbb{R})$ norm, defined as $\|u(\cdot,t)\|_{H^1}^2 = \int_{-\infty}^{\infty}[u(x,t)^2 + u_x(x,t)^2]dx$, is a conserved quantity that is strictly positive, i.e., $\|u(\cdot,t)\|_{H^1}^2 = \|u_0\|_{H^1}^2 > 0$.

Due to these properties, the infimum and supremum of the solution's slope are attained for each t. Let us define these extrema as:

$$I(t) = \inf_{x\in\mathbb{R}} u_x(x,t)$$

$$S(t) = \sup_{x\in\mathbb{R}} u_x(x,t)$$

Task: Based on the provided properties of the solution $u(x,t)$, derive a strict inequality that characterizes the product $h(t) = I(t)S(t)$ for all $t \in [0,T)$.

Refined Solution:

$$h(t) < 0$$

Problem 51 (Paper: 2506.23894v1, Index: 0)

Original Problem Statement

Background: Consider a flow network defined by an undirected graph $G = (V(G), E(G))$, a capacity function $c : E(G) \to \mathbb{R}_+$, a source vertex $s \in V(G)$, and a sink vertex $t \in V(G)$. The flow is defined on the associated bi-directed graph $\overrightarrow{\overrightarrow{G}}$, where each undirected edge $e_{xy} \in E(G)$ corresponds to two directed edges (x, y) and (y, x) in $E(\overrightarrow{\overrightarrow{G}})$ with capacity $c_{(x,y)} = c_{(y,x)} = c_{e_{xy}}$. A flow is a function $f : E(\overrightarrow{\overrightarrow{G}}) \to \mathbb{R}$ satisfying: 1. Skew symmetry: $f(x, y) = -f(y, x)$ for all $(x, y) \in E(\overrightarrow{\overrightarrow{G}})$. 2. Capacity constraint: $f(x, y) \leq c_{(x,y)}$ for all $(x, y) \in E(\overrightarrow{\overrightarrow{G}})$. 3. Flow conservation: $\sum_{y:(x,y)\in E(\overrightarrow{\overrightarrow{G}})} f(x, y) = 0$ for all $x \in V(G) \setminus \{s, t\}$. The value of the flow is $|f|_c = \sum_{y:(s,y)\in E(\overrightarrow{\overrightarrow{G}})} f(s, y)$. A flow f is a *maximum flow* (max-flow) if its value is maximal. A flow f induces a partial order $\leq_f$ on $V(G)$, defined as the transitive closure of the relation $xR_f y \iff f(x, y) > -c_{(x,y)}$. An $s - t$ cut is a bipartition (S, T) of $V(G)$ such that $s \in S$ and $t \in T$. The capacity of the cut is $|(S, T)|_c = \sum_{(x,y)\in E(\overrightarrow{\overrightarrow{G}}),x\in S,y\in T} c_{(x,y)}$. A cut is a *minimum cut* (min-cut) if its capacity is minimal. The max-flow min-cut theorem states that the value of a max-flow equals the capacity of a min-cut. For any vertex $x \in V(G)$, define the set S_x as the intersection of all sets S from min-cuts (S, T) that contain x: $S_x \bigcap_{(S,T):\text{min-cut and } x\in S} S$, where an empty intersection is defined as $V(G)$. A subset of vertices $C \subseteq V(G)$ is called a *closure* for the partial order $\leq_f$ if for any $y \in C$, the condition $x \leq_f y$ implies $x \in C$. You are given the following property connecting closures and min-cuts for any max-flow f: a set $C \subseteq V(G)$ with $s \in C$ and $t \notin C$ is a closure for $\leq_f$ if and only if $(C, \bar{C})$ is a minimum $s - t$ cut.

Task: For an arbitrary max-flow f and an arbitrary vertex $y \in V(G)$, derive the expression for the set S_y in terms of the vertices of the graph $V(G)$ and the partial order $\leq_f$.

Original Solution

$$S_y = \{x \in V(G) | x \leq_f y\}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing the max-flow min-cut theorem, the definition of closures in a partial order induced by the flow, and the characterization of minimum cuts to derive a non-trivial expression for $\mathrm{S}_y in terms of the partial order_f$.

Refined Problem

Refined Problem Statement: Background: Consider a flow network defined by an undirected graph $G = (V(G), E(G))$, a capacity function $c : E(G) \to \mathbb{R}_+$, a source vertex $s \in V(G)$, and a sink vertex $t \in V(G)$. The flow is defined on the associated bi-directed graph $\overrightarrow{\overrightarrow{G}}$, where each undirected edge $e_{xy} \in E(G)$ corresponds to two directed edges (x, y) and (y, x) in $E(\overrightarrow{\overrightarrow{G}})$ with capacity $c_{(x,y)} = c_{(y,x)} = c_{e_{xy}}$. A flow is a function $f : E(\overrightarrow{\overrightarrow{G}}) \to \mathbb{R}$ satisfying: 1. Skew symmetry: $f(x, y) = -f(y, x)$ for all $(x, y) \in E(\overrightarrow{\overrightarrow{G}})$. 2. Capacity constraint: $f(x, y) \leq c_{(x,y)}$ for all $(x, y) \in E(\overrightarrow{\overrightarrow{G}})$. 3. Flow conservation: $\sum_{y:(x,y)\in E(\overrightarrow{\overrightarrow{G}})} f(x, y) = 0$ for all $x \in V(G) \setminus \{s, t\}$. The value of the flow is $|f|_c = \sum_{y:(s,y)\in E(\overrightarrow{\overrightarrow{G}})} f(s, y)$. A flow f is a *maximum flow* (max-flow) if its value is maximal. A flow f induces a partial order $\leq_f$ on $V(G)$, defined as the transitive closure of the relation $xR_f y \iff f(x, y) > -c_{(x,y)}$. An s-t cut is a bipartition (S, T) of $V(G)$ such that $s \in S$ and $t \in T$. The capacity of the cut is $|(S, T)|_c = \sum_{(x,y)\in E(\overrightarrow{\overrightarrow{G}}),x\in S,y\in T} c_{(x,y)}$. A cut is a *minimum cut* (min-cut) if its capacity is minimal. The max-flow min-cut theorem states that the value of a max-flow equals the capacity of a min-cut. For any vertex $x \in V(G)$, define the set S_x as the intersection of all sets S from min-cuts (S, T) that contain x:

$$S_x \quad \bigcap_{(S,T):\text{min-cut and } x\in S} \quad S,$$

where an empty intersection is defined as $V(G)$. A subset of vertices $C \subseteq V(G)$ is called a *closure* for the partial order $\leq_f$ if for any $y \in C$, the condition $x \leq_f y$ implies $x \in C$. You are given the following property connecting closures and min-cuts for any max-flow f: a set $C \subseteq V(G)$ with $s \in C$ and $t \notin C$ is a closure for $\leq_f$ if and only if $(C, \bar{C})$ is a minimum s-t cut.

Task: For an arbitrary max-flow f and an arbitrary vertex $y \in V(G)$, derive the expression for the set S_y in terms of the vertices of the graph $V(G)$ and the partial order $\leq_f$.

Refined Solution:

$$S_y = \{x \in V(G) | x \leq_f y\}$$

Problem 52 (Paper: 2506.23894v1, Index: 1)

Original Problem Statement

Background: In a certain class of physical models, the statistical properties of observables can be described by a probability measure μ_G associated with a directed acyclic graph G. Consider a graph G that is a *series composition* of two subgraphs, H_1 and H_2, denoted $G = H_1 H_2$. The graph G is formed by identifying the sink of H_1 with the source of H_2; let this junction vertex be v. The overall source of G is the source of H_1, denoted s, and the overall sink of G is the sink of H_2, denoted t. Associated with each subgraph H_i is a measure μ_{H_i} and a positive real parameter r_{H_i}. The composite graph G has an associated parameter $r_G = \min(r_{H_1}, r_{H_2})$.

The moments of these measures are related to concepts from free probability. For a measure μ and a permutation $\pi \in \mathcal{S}_n$, the extended moment $\varphi_\pi(\mu)$ is defined as $\varphi_\pi(\mu) = \prod_{\sigma:\text{cycle of } \pi} \int x^{\text{card}(\sigma)} d\mu(x)$. The ordinary n-th moment is $\varphi_n(\mu) = \varphi_\gamma(\mu)$ where $\gamma = (1 \cdots n)$. These moments are related to free cumulants $\kappa_\pi(\mu)$ by the moment-cumulant formula: $\varphi_\pi(\mu) = \sum_{\alpha \le_{\text{nc}} \pi} \kappa_\alpha(\mu)$, where $\alpha \le_{\text{nc}} \pi$ denotes that the cycle partition of α is a non-crossing refinement of the cycle partition of π.

The n-th moment of the composite measure μ_G is given by the following expression, which sums over all permutations $\beta_v \in \mathcal{S}_n$ at the junction vertex v that are intermediate in the non-crossing partial order between the identity permutation at the source and $\gamma = (1 \cdots n)$ at the sink:

$$\varphi_n(\mu_G) = \sum_{\text{id} \le_{\text{nc}} \beta_v \le_{\text{nc}} \gamma} \left(\frac{r_{H_1}}{r_G}\right)^{\#\beta_v - n} \varphi_{\beta_v}(\mu_{H_1}) \cdot \left(\frac{r_{H_2}}{r_G}\right)^{\#(\beta_v^{-1}\gamma) - n} \varphi_{\beta_v^{-1}\gamma}(\mu_{H_2})$$

where $\#\pi$ is the number of cycles in permutation π.

You are given the following identities from free probability theory: 1. The cumulants of a free multiplicative convolution $\mu_1 \boxtimes \mu_2$ are given by $\kappa_n(\mu_1 \boxtimes \mu_2) = \sum_{\alpha \le_{\text{nc}} \gamma} \kappa_\alpha(\mu_1)\kappa_{\alpha^{-1}\gamma}(\mu_2)$. 2. For a measure μ, a scalar $r \ge 1$, and a permutation $\pi \in \mathcal{S}_n$, the following relation holds:

$$r^{\#\pi - n} \varphi_\pi(\mu) = \kappa_\pi(\hat{\pi}_r \boxtimes (r^{-1} \odot \mu)^{\boxplus r})$$

where $\boxplus$ is the free additive convolution, $\odot$ is a scaling operation on measures (if $X \sim \mu$, then $aX \sim a \odot \mu$), and $\hat{\pi}_r$ is the rescaled Marchenko-Pastur distribution.

Task: Let $r_{\max} = \max(r_{H_1}, r_{H_2})$ and $r_{\min} = \min(r_{H_1}, r_{H_2})$. Using the provided relations, derive an expression for the n-th moment $\varphi_n(\mu_G)$ of the composite measure. Express your final answer as the n-th moment of a single, new measure constructed from μ_{H_1}, μ_{H_2}, r_{H_1}, r_{H_2}, $r_{\max}$, and $r_{\min}$ using the operations of free probability $(\boxplus, \boxtimes, \odot)$ and the measure $\hat{\pi}_r$.

Original Solution

$$\varphi_n\left(\left(\frac{r_{\min}}{r_{H_1}} \odot \mu_{H_1}\right)^{\boxplus \frac{r_{H_1}}{r_{\min}}} \boxtimes \hat{\pi}_{\frac{r_{\max}}{r_{\min}}} \boxtimes \left(\frac{r_{\min}}{r_{H_2}} \odot \mu_{H_2}\right)^{\boxplus \frac{r_{H_2}}{r_{\min}}}\right)$$

Critiques

Self-Containment Critique: **Status: Not self-contained**

The problem statement does not explicitly define the notation $_r as the rescaled Marchenko-Pastur distribution beyond a $ insufficient detail for full self-containment.

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing multiple advanced concepts from free probability, including non-crossing partitions, free additive and multiplicative convolutions, and measure scaling, to re-express a complex moment formula as a single moment of a constructed measure.

Refined Problem

Refined Problem Statement: Background: In a certain class of physical models, the statistical properties of observables can be described by a probability measure μ_G associated with a directed acyclic graph G. Consider a graph G that is a *series composition* of two subgraphs, H_1 and H_2, denoted $G = H_1$ series H_2. The graph G is formed by identifying the sink of H_1 with the source of H_2; let this

junction vertex be v. The overall source of G is the source of H_1, denoted s, and the overall sink of G is the sink of H_2, denoted t. Associated with each subgraph H_i is a measure μ_{H_i} and a positive real parameter r_{H_i}. The composite graph G has an associated parameter $r_G = \min(r_{H_1}, r_{H_2})$.

The moments of these measures are related to concepts from free probability.

Definitions and Operations:

- For a measure μ and a permutation $\pi \in \mathcal{S}_n$, the *extended moment* $\varphi_\pi(\mu)$ is defined as $\varphi_\pi(\mu) = \prod_{\sigma:\text{cycle of } \pi} \int x^{\text{card}(\sigma)}\, d\mu(x)$. The ordinary n-th moment is $\varphi_n(\mu) = \varphi_\gamma(\mu)$ where $\gamma = (1\ \cdots\ n)$.
- The *free cumulants* $\kappa_\pi(\mu)$ are related to the moments by the moment-cumulant formula: $\varphi_\pi(\mu) = \sum_{\alpha \le_{\text{nc}} \pi} \kappa_\alpha(\mu)$, where $\alpha \le_{\text{nc}} \pi$ means that the cycle partition of α is a non-crossing refinement of the cycle partition of π.
- For measures μ, ν, the *free additive convolution* $\mu \boxplus \nu$ is the distribution of $X + Y$ when $X \sim \mu$, $Y \sim \nu$ are freely independent.
- The *free multiplicative convolution* $\mu \boxtimes \nu$ is the distribution of XY when $X \sim \mu$, $Y \sim \nu$ are freely independent and non-negative.
- For a measure μ on $\mathbb{R}$ and real $a > 0$, the scaling operation $a \odot \mu$ is defined by: if $X \sim \mu$ then $aX \sim a \odot \mu$.
- $\hat{\pi}_r$ denotes the *rescaled Marchenko-Pastur distribution* of parameter $r > 0$; its density is given by $\frac{\sqrt{(b-x)(x-a)}}{2\pi r x}\, \mathbf{1}_{[a,b]}(x)$, where $a = (1-\sqrt{r})^2$, $b = (1+\sqrt{r})^2$.
- The r-fold free additive convolution $\mu^{\boxplus r}$ is the law of the sum of r freely independent copies of a random variable with law μ (for r a positive integer, or more generally using analytic continuation).

These definitions and operations are standard in free probability theory and will be used below.

The n-th moment of the composite measure μ_G is given by the following expression, which sums over all permutations $\beta_v \in \mathcal{S}_n$ at the junction vertex v that are intermediate in the non-crossing partial order between the identity permutation at the source and $\gamma = (1\ \cdots\ n)$ at the sink:

$$\varphi_n(\mu_G) = \sum_{\text{id} \le_{\text{nc}} \beta_v \le_{\text{nc}} \gamma} \left(\frac{r_{H_1}}{r_G}\right)^{\#\beta_v - n} \varphi_{\beta_v}(\mu_{H_1}) \cdot \left(\frac{r_{H_2}}{r_G}\right)^{\#(\beta_v^{-1}\gamma) - n} \varphi_{\beta_v^{-1}\gamma}(\mu_{H_2})$$

where $\#\pi$ is the number of cycles in permutation π.

You are given the following identities from free probability theory:

1. The cumulants of a free multiplicative convolution $\mu_1 \boxtimes \mu_2$ are given by $\kappa_n(\mu_1 \boxtimes \mu_2) = \sum_{\alpha \le_{\text{nc}} \gamma} \kappa_\alpha(\mu_1)\kappa_{\alpha^{-1}\gamma}(\mu_2)$.
2. For a measure μ, a scalar $r \ge 1$, and a permutation $\pi \in \mathcal{S}_n$, the following relation holds:

   $$r^{\#\pi - n} \varphi_\pi(\mu) = \kappa_\pi(\hat{\pi}_r \boxtimes (r^{-1} \odot \mu)^{\boxplus r})$$

   where $\boxplus$ is the free additive convolution, $\odot$ is the scaling operation on measures, and $\hat{\pi}_r$ is the rescaled Marchenko-Pastur distribution as defined above.

Task: Let $r_{\max} = \max(r_{H_1}, r_{H_2})$ and $r_{\min} = \min(r_{H_1}, r_{H_2})$. Using the provided relations and definitions, derive an expression for the n-th moment $\varphi_n(\mu_G)$ of the composite measure. Express your final answer as the n-th moment of a single, new measure constructed from μ_{H_1}, μ_{H_2}, r_{H_1}, r_{H_2}, $r_{\max}$, and $r_{\min}$ using the operations $\boxplus$, $\boxtimes$, $\odot$, and the measure $\hat{\pi}_r$ as defined above.

Refined Solution:

$$\varphi_n\left(\left(\frac{r_{\min}}{r_{H_1}} \odot \mu_{H_1}\right)^{\boxplus \frac{r_{H_1}}{r_{\min}}} \boxtimes \hat{\pi}_{\frac{r_{\max}}{r_{\min}}} \boxtimes \left(\frac{r_{\min}}{r_{H_2}} \odot \mu_{H_2}\right)^{\boxplus \frac{r_{H_2}}{r_{\min}}}\right)$$

Problem 53 (Paper: 2506.23939v2, Index: 0)

Original Problem Statement

Background: A physical system is described by the overdamped linear time-delayed Langevin equation for a variable $x(t)$:

$$\dot{x}(t) = ax(t) + bx(t-\tau) + \xi(t)$$

Here, a and b are real constants, $\tau > 0$ is a constant time delay, and $\xi(t)$ is a white Gaussian noise source characterized by temperature T, with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 2T\delta(t-t')$. The variance of the system, $\mathrm{Var}[x(t)] = \langle x^2(t) \rangle - \langle x(t) \rangle^2$, can be expressed as:

$$\mathrm{Var}[x(t)] = 2T \int_0^t x_0^2(s) ds$$

where $x_0(t)$ is the fundamental solution of the deterministic part of the equation, i.e., $\dot{x}_0(t) = ax_0(t) + bx_0(t-\tau)$ for $t > 0$, with the specific initial conditions $x_0(t) = 0$ for $t < 0$ and $x_0(0) = 1$. The behavior of $x_0(t)$ is governed by the roots $\{S_k\}$ of the characteristic equation $S - a - be^{-S\tau} = 0$. In the long-time limit, the behavior is dominated by the roots with the largest real part. The fundamental solution can be formally expressed as a sum over these roots:

$$x_0(t) = \sum_{k=-\infty}^{\infty} \frac{e^{S_k t}}{1 + b\tau e^{-S_k \tau}}$$

We consider a system at a specific critical point known as "oscillatory criticality". This occurs when the parameters satisfy the conditions $a + b < 0$, $a - b > 0$, and $\tau = \frac{\arccos(-a/b)}{\sqrt{b^2 - a^2}}$. Under these conditions, the characteristic equation possesses exactly one pair of purely imaginary roots, $S_\pm = \pm i\omega_c$ where $\omega_c = \sqrt{b^2 - a^2}$, while all other roots have strictly negative real parts.

Task: For a system at oscillatory criticality, determine the asymptotic expression for the variance $\mathrm{Var}[x(t)]$ in the long-time limit ($t \to \infty$). Your final expression should be given in terms of $t, T, a, b, \tau, \omega_c$, and a phase angle θ defined as $\theta = \arctan\left(-\frac{b\tau \sin(\omega_c \tau)}{1 + b\tau \cos(\omega_c \tau)}\right)$. The contribution from the stable modes (those with $\mathrm{Re}[S_k] < 0$) can be collected into an unspecified finite constant C_1.

Original Solution

$$\langle x^2(t) \rangle - \langle x(t) \rangle^2 = \frac{4T\left[t - \frac{1}{2\omega_c}\Big(\sin[2(\omega_c t + \theta)] - \sin(2\theta)\Big)\right]}{\Big(1 + b\tau\cos(\omega_c \tau)\Big)^2 + b^2\tau^2 \sin^2(\omega_c \tau)} + C_1$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing the spectral properties of delay differential equations, extracting dominant contributions from purely imaginary roots, and performing a nontrivial asymptotic analysis of the variance integral, which involves advanced multi-step reasoning beyond direct formula application.

Refined Problem

Refined Problem Statement: Background: A physical system is described by the overdamped linear time-delayed Langevin equation for a variable $x(t)$:

$$\dot{x}(t) = ax(t) + bx(t-\tau) + \xi(t)$$

Here, a and b are real constants, $\tau > 0$ is a constant time delay, and $\xi(t)$ is a white Gaussian noise source characterized by temperature T, with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 2T\delta(t-t')$. The variance of the system, $\mathrm{Var}[x(t)] = \langle x^2(t) \rangle - \langle x(t) \rangle^2$, can be expressed as:

$$\mathrm{Var}[x(t)] = 2T \int_0^t x_0^2(s) ds$$

where $x_0(t)$ is the fundamental solution of the deterministic part of the equation, i.e., $\dot{x}_0(t) = ax_0(t) + bx_0(t-\tau)$ for $t > 0$, with the specific initial conditions $x_0(t) = 0$ for $t < 0$ and $x_0(0) = 1$. The behavior of $x_0(t)$ is governed by the roots $\{S_k\}$ of the characteristic equation $S - a - be^{-S\tau} = 0$. In the long-time limit, the behavior is dominated by the roots with the largest real part. The fundamental solution can be formally expressed as a sum over these roots:

$$x_0(t) = \sum_{k=-\infty}^{\infty} \frac{e^{S_k t}}{1 + b\tau e^{-S_k \tau}}$$

We consider a system at a specific critical point known as "oscillatory criticality". This occurs when the parameters satisfy the conditions $a + b < 0$, $a - b > 0$, and $\tau = \frac{\arccos(-a/b)}{\sqrt{b^2-a^2}}$. Under these conditions, the characteristic equation possesses exactly one pair of purely imaginary roots, $S_\pm = \pm i\omega_c$ where $\omega_c = \sqrt{b^2 - a^2}$, while all other roots have strictly negative real parts.

Task: For a system at oscillatory criticality, determine the asymptotic expression for the variance $\mathrm{Var}[x(t)]$ in the long-time limit ($t \to \infty$). Your final expression should be given in terms of $t, T, a, b, \tau, \omega_c$, and a phase angle θ defined as $\theta = \arctan\left(-\frac{b\tau \sin(\omega_c \tau)}{1+b\tau \cos(\omega_c \tau)}\right)$. The contribution from the stable modes (those with $\mathrm{Re}[S_k] < 0$) can be collected into an unspecified finite constant C_1.

Refined Solution:

$$\langle x^2(t)\rangle - \langle x(t)\rangle^2 = \frac{4T\left[t - \frac{1}{2\omega_c}\Big(\sin[2(\omega_c t + \theta)] - \sin(2\theta)\Big)\right]}{\Big(1 + b\tau\cos(\omega_c\tau)\Big)^2 + b^2\tau^2\sin^2(\omega_c\tau)} + C_1$$

Problem 54 (Paper: 2506.23939v2, Index: 1)

Original Problem Statement

Background: Consider a particle described by an overdamped linear time-delayed Langevin equation $\dot{x}(t) = ax(t) + bx(t-\tau) + \xi(t)$, where a and b are force constants, $\tau > 0$ is a fixed time delay, and $\xi(t)$ is Gaussian white noise with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 2T\delta(t-t')$, where T is the temperature. The system is prepared with an initial history of $x(t) = 0$ for all $t \leq 0$, which implies that the mean displacement $\langle x(t) \rangle = 0$ for all $t > 0$. The system is at a state of *oscillatory criticality*, where the parameters satisfy the conditions $a + b < 0$, $a - b > 0$, and $\tau = \dfrac{\arccos(-a/b)}{\omega_c}$ with $\omega_c = \sqrt{b^2 - a^2}$. In this regime, the variance for large t is given by $\langle x^2(t) \rangle \approx 2T \int_0^t x_0^2(s) ds$, where $x_0(t)$ is the fundamental solution. The asymptotic form of the fundamental solution is given by $x_0(t) \approx \dfrac{2\sin(\omega_c t + \theta)}{\sqrt{\left(1 + b\tau\cos(\omega_c\tau)\right)^2 + b^2\tau^2\sin^2(\omega_c\tau)}}$, where $\theta = \arctan\left(-\dfrac{b\tau\sin(\omega_c\tau)}{1 + b\tau\cos(\omega_c\tau)}\right)$. The average rate of heat dissipation into the environment is given by the exact relation $\left\langle \frac{dq}{dt} \right\rangle = b^2\langle x^2(t-\tau) \rangle - a^2\langle x^2(t) \rangle + a\frac{d\langle x^2(t) \rangle}{dt} - aT$.

Task: Derive the asymptotic expression for the average heat dissipation rate $\langle dq/dt \rangle$ as $t \to \infty$. Express your answer in terms of $a, b, T, t, \omega_c, \tau, \theta$, and the constant $A_0 \equiv \dfrac{4T}{\left(1 + b\tau\cos(\omega_c\tau)\right)^2 + b^2\tau^2\sin^2(\omega_c\tau)}$. All time-independent terms should be collected into a single additive constant B_0.

Original Solution

$$\left\langle \frac{dq}{dt} \right\rangle = (b^2 - a^2)A_0 t - \frac{A_0}{2\omega_c}\left\{ b^2\sin[2(\omega_c(t-\tau)+\theta)] - a^2\sin[2(\omega_c t+\theta)] \right\} + aA_0\Big(1 - \cos[2(\omega_c t+\theta)]\Big) + B_0$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing the time-delayed Langevin dynamics, asymptotic analysis of the fundamental solution, and manipulation of stochastic averages to derive a complex expression for heat dissipation rate, involving multiple nontrivial steps and integration of oscillatory functions.

Refined Problem

Refined Problem Statement: Background: Consider a particle described by an overdamped linear time-delayed Langevin equation $\dot{x}(t) = ax(t) + bx(t-\tau) + \xi(t)$, where a and b are force constants, $\tau > 0$ is a fixed time delay, and $\xi(t)$ is Gaussian white noise with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 2T\delta(t-t')$, where T is the temperature. The system is prepared with an initial history of $x(t) = 0$ for all $t \leq 0$, which implies that the mean displacement $\langle x(t) \rangle = 0$ for all $t > 0$. The system is at a state of *oscillatory criticality*, where the parameters satisfy the conditions $a + b < 0$, $a - b > 0$, and $\tau = \dfrac{\arccos(-a/b)}{\omega_c}$ with $\omega_c = \sqrt{b^2 - a^2}$. In this regime, the variance for large t is given by $\langle x^2(t) \rangle \approx 2T \int_0^t x_0^2(s) ds$, where $x_0(t)$ is the fundamental solution. The asymptotic form of the fundamental solution is given by $x_0(t) \approx \dfrac{2\sin(\omega_c t + \theta)}{\sqrt{\left(1 + b\tau\cos(\omega_c\tau)\right)^2 + b^2\tau^2\sin^2(\omega_c\tau)}}$, where $\theta = \arctan\left(-\dfrac{b\tau\sin(\omega_c\tau)}{1 + b\tau\cos(\omega_c\tau)}\right)$. The average rate of heat dissipation into the environment is given by the exact relation $\left\langle \frac{dq}{dt} \right\rangle = b^2\langle x^2(t-\tau) \rangle - a^2\langle x^2(t) \rangle + a\frac{d\langle x^2(t) \rangle}{dt} - aT$.

Task: Derive the asymptotic expression for the average heat dissipation rate $\langle dq/dt \rangle$ as $t \to \infty$. Express your answer in terms of $a, b, T, t, \omega_c, \tau, \theta$, and the constant $A_0 \equiv \dfrac{4T}{\left(1 + b\tau\cos(\omega_c\tau)\right)^2 + b^2\tau^2\sin^2(\omega_c\tau)}$. All time-independent terms should be collected into a single additive constant B_0.

Refined Solution:

$$\left\langle \frac{dq}{dt} \right\rangle = (b^2 - a^2)A_0 t - \frac{A_0}{2\omega_c}\Big\{ b^2 \sin[2(\omega_c(t-\tau)+\theta)] - a^2 \sin[2(\omega_c t+\theta)] \Big\} + aA_0\Big(1 - \cos[2(\omega_c t+\theta)]\Big) + B_0$$

Problem 55 (Paper: 2506.23953v1, Index: 0)

Original Problem Statement

Background: Consider a $_2\times_2$-graded Lie superalgebra $= \bigoplus_{\boldsymbol{a}\in_2\times_2} \boldsymbol{a}$, where the bracket $\cdot,\cdot$ for homogeneous elements $x_{\boldsymbol{a}} \in_{\boldsymbol{a}}$ and $y_{\boldsymbol{b}} \in_{\boldsymbol{b}}$ satisfies $x_{\boldsymbol{a}}, y_{\boldsymbol{b}} \in_{\boldsymbol{a}+\boldsymbol{b}}$ and $x_{\boldsymbol{a}}, y_{\boldsymbol{b}} = -(-1)^{\boldsymbol{a}\cdot\boldsymbol{b}} y_{\boldsymbol{b}}, x_{\boldsymbol{a}}$, with $\boldsymbol{a} = (a_1, a_2)$, $\boldsymbol{b} = (b_1, b_2)$, and $\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2$. Let this algebra be realized by the space of $(m+n+1)\times(m+n+1)$ matrices, where $m = m_1 + m_2$ and $n = n_1 + n_2$. Let e_{ij} be the standard matrix units for $i, j \in \{0, 1, \ldots, m+n\}$. Each index $k \in \{0, 1, \ldots, m+n\}$ is assigned a degree $d_k \in_2 \times_2$ as follows:

$$d_k = \begin{cases} (0,0); & k = 0, \ldots, m_1 \\ (1,1); & k = m_1+1, \ldots, m_1+m_2 = m \\ (1,0); & k = m_1+m_2+1, \ldots, m_1+m_2+n_1 = m+n_1 \\ (0,1); & k = m_1+m_2+n_1+1, \ldots, m_1+m_2+n_1+n_2 = m+n \end{cases}$$

The matrix unit e_{ij} is a homogeneous element with degree $\deg(e_{ij}) = d_i + d_j$. The bracket for these matrix units is given by $e_{ij}, e_{kl} = \delta_{jk}e_{il} - (-1)^{(d_i+d_j)\cdot(d_k+d_l)}\delta_{il}e_{kj}$. A set of creation and annihilation operators, a_k^+ and a_k^- for $k = 1, \ldots, m+n$, are defined as $a_k^+ = e_{k0}$ and $a_k^- = e_{0k}$. These operators are homogeneous with degree $\deg(a_k^\pm) = d_k$.

Task: Using the definitions provided, derive the expression for the triple bracket a_i^+, a_j^-, a_k^+ for $i, j, k \in \{1, \ldots, m+n\}$. Express your final result solely in terms of the creation operators a_l^+ and Kronecker deltas.

Original Solution

$$\delta_{jk}a_i^+ + (-1)^{d_i\cdot d_i}\delta_{ij}a_k^+$$

Critiques

Self-Containment Critique: **Status: Not self-contained**

The final solution uses the term $(-1)^{d_i\cdot d_i}$ *which is not defined in the problem statement; the dot product is only defined between* d_i *is not explicitly clarified.*

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing the graded Lie bracket definitions, degree assignments, and matrix unit properties to derive a nontrivial triple bracket expression, involving careful multi-step algebraic manipulations beyond direct formula application.

Refined Problem

Refined Problem Statement: Background: Consider a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{g} = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}_2\times\mathbb{Z}_2} \mathfrak{g}_{\boldsymbol{a}}$, where the bracket $[\cdot,\cdot]$ for homogeneous elements $x_{\boldsymbol{a}} \in \mathfrak{g}_{\boldsymbol{a}}$ and $y_{\boldsymbol{b}} \in \mathfrak{g}_{\boldsymbol{b}}$ satisfies $[x_{\boldsymbol{a}}, y_{\boldsymbol{b}}] \in \mathfrak{g}_{\boldsymbol{a}+\boldsymbol{b}}$ and $[x_{\boldsymbol{a}}, y_{\boldsymbol{b}}] = -(-1)^{\boldsymbol{a}\cdot\boldsymbol{b}}[y_{\boldsymbol{b}}, x_{\boldsymbol{a}}]$, with $\boldsymbol{a} = (a_1, a_2)$, $\boldsymbol{b} = (b_1, b_2)$, and $\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2$ (arithmetic modulo 2). For any $\boldsymbol{a}$, we define $\boldsymbol{a}\cdot\boldsymbol{a} = a_1^2 + a_2^2 \bmod 2$. Let this algebra be realized by the space of $(m+n+1)\times(m+n+1)$ matrices, where $m = m_1+m_2$ and $n = n_1+n_2$. Let e_{ij} be the standard matrix units for $i, j \in \{0, 1, \ldots, m+n\}$. Each index $k \in \{0, 1, \ldots, m+n\}$ is assigned a degree $d_k \in \mathbb{Z}_2 \times \mathbb{Z}_2$ as follows:

$$d_k = \begin{cases} (0,0); & k = 0, \ldots, m_1 \\ (1,1); & k = m_1+1, \ldots, m_1+m_2 = m \\ (1,0); & k = m_1+m_2+1, \ldots, m_1+m_2+n_1 = m+n_1 \\ (0,1); & k = m_1+m_2+n_1+1, \ldots, m_1+m_2+n_1+n_2 = m+n \end{cases}$$

The matrix unit e_{ij} is a homogeneous element with degree $\deg(e_{ij}) = d_i + d_j$. The bracket for these matrix units is given by $[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - (-1)^{(d_i+d_j)\cdot(d_k+d_l)}\delta_{il}e_{kj}$. A set of creation and annihilation operators, a_k^+ and a_k^- for $k = 1, \ldots, m+n$, are defined as $a_k^+ = e_{k0}$ and $a_k^- = e_{0k}$. These operators are homogeneous with degree $\deg(a_k^\pm) = d_k$.

Task: Using the definitions provided, derive the expression for the triple bracket $[[a_i^+, a_j^-], a_k^+]$ for $i, j, k \in \{1, \ldots, m+n\}$. Express your final result solely in terms of the creation operators a_l^+ and Kronecker deltas. Recall that for degree vectors d_i in $\mathbb{Z}_2 \times \mathbb{Z}_2$, $d_i \cdot d_i$ means $a_1^2 + a_2^2$ (modulo 2), where $d_i = (a_1, a_2)$.

Refined Solution:

$$\delta_{jk} a_i^+ + (-1)^{d_i \cdot d_i} \delta_{ij} a_k^+$$

Problem 56 (Paper: 2506.23953v1, Index: 1)

Original Problem Statement

Background: Consider a $_2\times_2$-graded Lie superalgebra $= \bigoplus_{\boldsymbol{a}\in_2\times_2} \boldsymbol{a}$, where the degree of a homogeneous element $x_{\boldsymbol{a}} \in_{\boldsymbol{a}}$ is $\deg(x_{\boldsymbol{a}}) = \boldsymbol{a} = (a_1, a_2)$. The algebra is endowed with a bracket that satisfies $x_{\boldsymbol{a}}, y_{\boldsymbol{b}} = x_{\boldsymbol{a}}y_{\boldsymbol{b}} - (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}}y_{\boldsymbol{b}}x_{\boldsymbol{a}}$, where $\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2$. The algebra is generated by a set of creation operators a_i^+ and annihilation operators a_i^-. These operators are classified into four families based on their grading:

- $b_i^\pm$ for $i = 1, \ldots, m_1$, with degree $(0, 0)$.
- $\tilde{b}_i^\pm$ for $i = 1, \ldots, m_2$, with degree $(1, 1)$.
- $f_i^\pm$ for $i = 1, \ldots, n_1$, with degree $(1, 0)$.
- $\tilde{f}_i^\pm$ for $i = 1, \ldots, n_2$, with degree $(0, 1)$.

For any two creation operators a_i^+ and a_j^+, the bracket relation is $a_i^+, a_j^+ = 0$.

A class of irreducible representations, known as Fock representations, is constructed on a space W_p labeled by a positive integer p. This space is spanned by a set of orthonormal basis vectors $|p; r_1, \ldots, r_{m_1}, l_1, \ldots, l_{m_2}, \theta_1, \ldots, \theta_{n_1}, \lambda_1, \ldots, \lambda_{n_2}\rangle$. These vectors are constructed by the action of creation operators on a normalized vacuum state $|0\rangle$ as follows:

$$|p; r_1, \ldots, r_{m_1}, l_1, \ldots, l_{m_2}, \theta_1, \ldots, \theta_{n_1}, \lambda_1, \ldots, \lambda_{n_2}\rangle =$$
$$\sqrt{\frac{(p-R)!}{p!r_1!\ldots r_{m_1}!l_1!\ldots l_{m_2}!\theta_1!\ldots\theta_{n_1}!\lambda_1!\ldots\lambda_{n_2}!}}$$
$$\times (b_1^+)^{r_1}\ldots(b_{m_1}^+)^{r_{m_1}}(\tilde{b}_1^+)^{l_1}\ldots(\tilde{b}_{m_2}^+)^{l_{m_2}}(f_1^+)^{\theta_1}\ldots(f_{n_1}^+)^{\theta_{n_1}}(\tilde{f}_1^+)^{\lambda_1}\ldots(\tilde{f}_{n_2}^+)^{\lambda_{n_2}}|0\rangle,$$

where the occupation numbers are integers $r_i, l_i \geq 0$ and $\theta_i, \lambda_i \in \{0, 1\}$. The total particle number is $R = \sum_{i=1}^{m_1} r_i + \sum_{i=1}^{m_2} l_i + \sum_{i=1}^{n_1} \theta_i + \sum_{i=1}^{n_2} \lambda_i$, and is subject to the constraint $R \leq p$.

Task: Derive the action of the creation operator f_i^+ (for a given $i \in \{1, \ldots, n_1\}$) on a general orthonormal basis state $|p; r_1, \ldots, l_1, \ldots, \theta_1, \ldots, \theta_i, \ldots, \theta_{n_1}, \ldots\rangle$. Express your result as a coefficient multiplying another orthonormal basis state of W_p.

Original Solution

$$f_i^+|p; \ldots, r_{m_1}, l_1, \ldots, l_{m_2}, \theta_1, \ldots, \theta_{i-1}, \theta_i, \theta_{i+1}, \ldots, \theta_{n_1}, \ldots) = (1-\theta_i)(-1)^{\sum_{k=1}^{m_2} l_k}(-1)^{\sum_{k=1}^{i-1}\theta_k}\sqrt{p-R}|p; \ldots, r_{m_1}, l_1, \ldots, l_m$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing the graded algebra structure, occupation number constraints, and the nontrivial phase factors arising from the grading to derive the explicit action of a creation operator on a complex Fock space basis state.

Refined Problem

Refined Problem Statement: Background: Consider a $\mathbb{Z}_2\times\mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{g} = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}_2\times\mathbb{Z}_2} \mathfrak{g}_{\boldsymbol{a}}$, where the degree of a homogeneous element $x_{\boldsymbol{a}} \in \mathfrak{g}_{\boldsymbol{a}}$ is $\deg(x_{\boldsymbol{a}}) = \boldsymbol{a} = (a_1, a_2)$. The algebra is endowed with a bracket that satisfies $[x_{\boldsymbol{a}}, y_{\boldsymbol{b}}] = x_{\boldsymbol{a}}y_{\boldsymbol{b}} - (-1)^{\boldsymbol{a}\cdot\boldsymbol{b}}y_{\boldsymbol{b}}x_{\boldsymbol{a}}$, where $\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2$. The algebra is generated by a set of creation operators a_i^+ and annihilation operators a_i^-. These operators are classified into four families based on their grading:

- $b_i^\pm$ for $i = 1, \ldots, m_1$, with degree $(0, 0)$.
- $\tilde{b}_i^\pm$ for $i = 1, \ldots, m_2$, with degree $(1, 1)$.
- $f_i^\pm$ for $i = 1, \ldots, n_1$, with degree $(1, 0)$.
- $\tilde{f}_i^\pm$ for $i = 1, \ldots, n_2$, with degree $(0, 1)$.

For any two creation operators a_i^+ and a_j^+, the bracket relation is $[a_i^+, a_j^+] = 0$.

A class of irreducible representations, known as Fock representations, is constructed on a space W_p labeled by a positive integer p. This space is spanned by a set of orthonormal basis vectors $|p; r_1, \ldots, r_{m_1}, l_1, \ldots, l_{m_2}, \theta_1, \ldots, \theta_{n_1}, \lambda_1, \ldots, \lambda_{n_2}\rangle$. These vectors are constructed by the action of creation operators on a normalized vacuum state $|0\rangle$ as follows:

$$|p; r_1, \ldots, r_{m_1}, l_1, \ldots, l_{m_2}, \theta_1, \ldots, \theta_{n_1}, \lambda_1, \ldots, \lambda_{n_2}\rangle =$$
$$\sqrt{\frac{(p-R)!}{p! r_1! \ldots r_{m_1}! l_1! \ldots l_{m_2}! \theta_1! \ldots \theta_{n_1}! \lambda_1! \ldots \lambda_{n_2}!}}$$
$$\times (b_1^+)^{r_1} \ldots (b_{m_1}^+)^{r_{m_1}} (\tilde{b}_1^+)^{l_1} \ldots (\tilde{b}_{m_2}^+)^{l_{m_2}} (f_1^+)^{\theta_1} \ldots (f_{n_1}^+)^{\theta_{n_1}} (\tilde{f}_1^+)^{\lambda_1} \ldots (\tilde{f}_{n_2}^+)^{\lambda_{n_2}} |0\rangle,$$

where the occupation numbers are integers $r_i, l_i \geq 0$ and $\theta_i, \lambda_i \in \{0, 1\}$. The total particle number is $R = \sum_{i=1}^{m_1} r_i + \sum_{i=1}^{m_2} l_i + \sum_{i=1}^{n_1} \theta_i + \sum_{i=1}^{n_2} \lambda_i$, and is subject to the constraint $R \leq p$.

Task: Derive the action of the creation operator f_i^+ (for a given $i \in \{1, \ldots, n_1\}$) on a general orthonormal basis state $|p; r_1, \ldots, r_{m_1}, l_1, \ldots, l_{m_2}, \theta_1, \ldots, \theta_i, \ldots, \theta_{n_1}, \ldots\rangle$. Express your result as a coefficient multiplying another orthonormal basis state of W_p.

Refined Solution:

$$f_i^+ |p; \ldots, r_{m_1}, l_1, \ldots, l_{m_2}, \theta_1, \ldots, \theta_{i-1}, \theta_i, \theta_{i+1}, \ldots, \theta_{n_1}, \ldots) = (1-\theta_i)(-1)^{\sum_{k=1}^{m_2} l_k} (-1)^{\sum_{k=1}^{i-1} \theta_k} \sqrt{p-R} |p; \ldots, r_{m_1}, l_1, \ldots, l_m$$

Problem 57 (Paper: 2506.23987v1, Index: 0)

Original Problem Statement

Background: Consider a system of n spins $\sigma = (\sigma_1, \ldots, \sigma_n)$ constrained to the sphere $S_n = \{\sigma \in \mathbb{R}^n | \sum_{i=1}^n \sigma_i^2 = n\}$. The expectation over the uniform measure on this sphere is denoted by $\mathbb{E}[\cdot]$. The system is described by a Non-Intersecting Monomial (NIM) Hamiltonian, which is a sum of K monomial terms: $H_n(\sigma) = \sum_{k=1}^K \widehat{H}_k(\sigma)$, where $\widehat{H}_k(\sigma) = C_k \left(\prod_{j \in I_k} \sigma_j\right) n^{-(p_k-2)/2}$. The index sets $I_k \subset \{1, \ldots, n\}$ are pairwise disjoint ($I_k \cap I_j = \emptyset$ for $k \neq j$), and their sizes are denoted by $|I_k| = p_k \geq 2$. The terms C_k are real-valued coupling constants. The partition function is $Z_n = \mathbb{E}[\exp(H_n(\sigma))]$ and the free energy per spin is $F_n = \frac{1}{n} \log Z_n$.

For a system with a single monomial term, $\widehat{H}(\sigma) = C \left(\prod_{j \in I} \sigma_j\right) n^{-(p-2)/2}$ with $|I| = p$, the following properties are known: 1. The partition function $Z_{n,1} = \mathbb{E}[\exp(\widehat{H})]$ can be expanded as a Taylor series. The term proportional to C^z is related to the expectation $\mathbb{E}[|\widehat{H}|^z]$. The function $g(z) = \log \frac{\mathbb{E}[|\widehat{H}|^z]}{z!}$ is convex for $z \in \mathbb{R}_{\geq 0}$. 2. There exists a threshold C_p^* for the coupling constant. If $|C| > C_p^*$, the limiting free energy is non-trivial and given by $F = \lim_{n\to\infty} \frac{1}{n} \log Z_{n,1} = f_p(C)$. If $|C| \leq C_p^*$, the limiting free energy is zero. 3. The function $f_p(C)$ is defined as $f_p(C) = 2\lambda_p(C) - \frac{1}{2} \log(1 + 2p\lambda_p(C))$. The quantity $\lambda_p(C)$ is the largest real solution λ to the equation: $2 \log |C| + (p-2) \log(2\lambda) - p \log(2p\lambda + 1) = 0$. 4. This phase transition arises because for $|C| > C_p^*$, the Taylor series of $Z_{n,1}$ is dominated by terms of order $C^{2\ell}$ where $\ell \approx n\lambda_p(C)$, while for $|C| \leq C_p^*$, it is dominated by the lowest-order terms ($\ell = 0, 1$).

Assume for the NIM model that all K couplings are above their respective thresholds, i.e., $|C_k| > C_{p_k}^*$ for all $k = 1, \ldots, K$.

Task: Using the provided properties of single-monomial systems, derive the limiting free energy $F = \lim_{n\to\infty} F_n$ for the general Non-Intersecting Monomial (NIM) model. Express your result in terms of the functions f_{p_k} and the coupling constants C_k.

Original Solution

$$F = \max_{k \in \{1,\ldots,K\}} f_{p_k}(C_k)$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing multiple concepts including the behavior of single-monomial systems, the phase transition thresholds, and the interplay of multiple disjoint monomial terms to derive the limiting free energy for the composite NIM model.

Refined Problem

Refined Problem Statement: Background: Consider a system of n spins $\sigma = (\sigma_1, \ldots, \sigma_n)$ constrained to the sphere $S_n = \{\sigma \in \mathbb{R}^n \mid \sum_{i=1}^n \sigma_i^2 = n\}$. The expectation over the uniform measure on this sphere is denoted by $\mathbb{E}[\cdot]$. The system is described by a Non-Intersecting Monomial (NIM) Hamiltonian, which is a sum of K monomial terms: $H_n(\sigma) = \sum_{k=1}^K \widehat{H}_k(\sigma)$, where $\widehat{H}_k(\sigma) = C_k \left(\prod_{j \in I_k} \sigma_j\right) n^{-(p_k-2)/2}$. The index sets $I_k \subset \{1, \ldots, n\}$ are pairwise disjoint ($I_k \cap I_j = \emptyset$ for $k \neq j$), and their sizes are denoted by $|I_k| = p_k \geq 2$. The terms C_k are real-valued coupling constants. The partition function is $Z_n = \mathbb{E}[\exp(H_n(\sigma))]$ and the free energy per spin is $F_n = \frac{1}{n} \log Z_n$.

For a system with a single monomial term, $\widehat{H}(\sigma) = C \left(\prod_{j \in I} \sigma_j\right) n^{-(p-2)/2}$ with $|I| = p$, the following properties are known: 1. The partition function $Z_{n,1} = \mathbb{E}[\exp(\widehat{H})]$ can be expanded as a Taylor series. The term proportional to C^z is related to the expectation $\mathbb{E}[|\widehat{H}|^z]$. The function $g(z) = \log \frac{\mathbb{E}[|\widehat{H}|^z]}{z!}$ is convex for $z \in \mathbb{R}_{\geq 0}$. 2. There exists a threshold C_p^* for the coupling constant. If $|C| > C_p^*$, the limiting free energy is non-trivial and given by $F = \lim_{n\to\infty} \frac{1}{n} \log Z_{n,1} = f_p(C)$. If $|C| \leq C_p^*$, the limiting free energy is zero. 3. The function $f_p(C)$ is defined as $f_p(C) = 2\lambda_p(C) - \frac{1}{2} \log(1 + 2p\lambda_p(C))$. The quantity $\lambda_p(C)$ is the largest real solution λ to the equation: $2 \log |C| + (p-2) \log(2\lambda) - p \log(2p\lambda + 1) = 0$. 4.

This phase transition arises because for $|C| > C_p^*$, the Taylor series of $Z_{n,1}$ is dominated by terms of order $C^{2\ell}$ where $\ell \approx n\lambda_p(C)$, while for $|C| \leq C_p^*$, it is dominated by the lowest-order terms ($\ell = 0, 1$).

Assume for the NIM model that all K couplings are above their respective thresholds, i.e., $|C_k| > C_{p_k}^*$ for all $k = 1, \ldots, K$.

Task: Using the provided properties of single-monomial systems, derive the limiting free energy $F = \lim_{n\to\infty} F_n$ for the general Non-Intersecting Monomial (NIM) model. Express your result in terms of the functions f_{p_k} and the coupling constants C_k.

Refined Solution:

$$F = \max_{k\in\{1,\ldots,K\}} f_{p_k}(C_k)$$

Problem 58 (Paper: 2506.23987v1, Index: 1)

Original Problem Statement

Background: Consider a system of n spins $\sigma = (\sigma_1, \ldots, \sigma_n)$ constrained to the sphere $\sum_{i=1}^n \sigma_i^2 = n$. The expectation $\mathbb{E}[\cdot]$ is taken with respect to the uniform probability measure on this sphere. The system is described by a p-spin Hamiltonian, with $p \geq 3$, given by

$$H_n(\sigma) = \sum_{i=1}^{N} c_i \sigma_{I_i} n^{-(p-2)/2}$$

where $I_i \subset \{1, \ldots, n\}$ are distinct p-element index sets, $\sigma_{I_i} = \prod_{j \in I_i} \sigma_j$ is a spin monomial, and c_i are positive real coefficients. The interactions between these monomials can be described by a monomial graph $G = (V, E)$, where the vertices $V = \{\sigma_{I_i}\}_{i=1}^N$ correspond to the monomials, and an edge $(i, j) \in E$ exists if the index sets of the corresponding monomials have a non-empty intersection, i.e., $I_i \cap I_j \neq \emptyset$.

You are given the following facts: 1. The monomial graph G is C-colorable for some integer $C \geq 1$. This implies that the set of all monomials V can be partitioned into C disjoint classes (color classes), $V = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_C$, such that within each class $\mathcal{C}_j$, all monomials are mutually non-intersecting (i.e., for any two distinct $\sigma_{I_a}, \sigma_{I_b} \in \mathcal{C}_j$, their index sets are disjoint, $I_a \cap I_b = \emptyset$). 2. For a Hamiltonian composed of a sum of mutually non-intersecting monomials, $H_{\text{NIM}} = \sum_k d_k \sigma_{J_k} n^{-(p-2)/2}$, the partition function is bounded by the product of individual partition functions: $\mathbb{E}[\exp(H_{\text{NIM}})] \leq \prod_k \mathbb{E}[\exp(d_k \sigma_{J_k} n^{-(p-2)/2})]$. 3. For a single monomial with a sufficiently small coefficient d, the logarithm of its partition function for $p \geq 3$ can be approximated by the leading term of its series expansion: $\log \mathbb{E}[\exp(d\sigma_J n^{-(p-2)/2})] = \frac{1}{2} n^{2-p} d^2 + O(n^{4-2p} d^4)$.

Task: For a given positive parameter s, use the provided facts and Hölder's inequality to derive the leading-order upper bound for the log-partition function, $\log \mathbb{E}[\exp(sH_n(\sigma))]$. Express your answer in terms of s, C, p, n, and the sum of the squares of the coefficients, $\sum_{i=1}^N c_i^2$.

Original Solution

$$\frac{1}{2} s^2 C n^{2-p} \sum_{i=1}^{N} c_i^2$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing graph coloring, partition function bounds, and asymptotic expansions, combined with Hölder's inequality, to derive a non-trivial leading-order upper bound on the log-partition function.

Refined Problem

Refined Problem Statement: Background: Consider a system of n spins $\sigma = (\sigma_1, \ldots, \sigma_n)$ constrained to the sphere $\sum_{i=1}^n \sigma_i^2 = n$. The expectation $\mathbb{E}[\cdot]$ is taken with respect to the uniform probability measure on this sphere. The system is described by a p-spin Hamiltonian, with $p \geq 3$, given by

$$H_n(\sigma) = \sum_{i=1}^{N} c_i \sigma_{I_i} n^{-(p-2)/2}$$

where $I_i \subset \{1, \ldots, n\}$ are distinct p-element index sets, $\sigma_{I_i} = \prod_{j \in I_i} \sigma_j$ is a spin monomial, and c_i are positive real coefficients. The interactions between these monomials can be described by a monomial graph $G = (V, E)$, where the vertices $V = \{\sigma_{I_i}\}_{i=1}^N$ correspond to the monomials, and an edge $(i, j) \in E$ exists if the index sets of the corresponding monomials have a non-empty intersection, i.e., $I_i \cap I_j \neq \emptyset$.

You are given the following facts: 1. The monomial graph G is C-colorable for some integer $C \geq 1$. This implies that the set of all monomials V can be partitioned into C disjoint classes (color classes), $V = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_C$, such that within each class $\mathcal{C}_j$, all monomials are mutually non-intersecting (i.e., for any two distinct $\sigma_{I_a}, \sigma_{I_b} \in \mathcal{C}_j$, their index sets are disjoint, $I_a \cap I_b = \emptyset$). 2. For a Hamiltonian composed

of a sum of mutually non-intersecting monomials, $H_{\mathrm{NIM}} = \sum_k d_k \sigma_{J_k} n^{-(p-2)/2}$, the partition function is bounded by the product of individual partition functions: $\mathbb{E}[\exp(H_{\mathrm{NIM}})] \leq \prod_k \mathbb{E}[\exp(d_k \sigma_{J_k} n^{-(p-2)/2})]$. 3. For a single monomial with a sufficiently small coefficient d, the logarithm of its partition function for $p \geq 3$ can be approximated by the leading term of its series expansion: $\log \mathbb{E}[\exp(d\sigma_J n^{-(p-2)/2})] = \frac{1}{2} n^{2-p} d^2 + O(n^{4-2p} d^4)$.

Task: For a given positive parameter s, use the provided facts and Hölder's inequality to derive the leading-order upper bound for the log-partition function, $\log \mathbb{E}[\exp(sH_n(\sigma))]$. Express your answer in terms of s, C, p, n, and the sum of the squares of the coefficients, $\sum_{i=1}^N c_i^2$.

Refined Solution:

$$\frac{1}{2} s^2 C n^{2-p} \sum_{i=1}^{N} c_i^2$$

Problem 59 (Paper: 2506.23991v1, Index: 0)

Original Problem Statement

Background: Consider a non-dissipative two-fluid plasma model described by the variables $(\boldsymbol{E}, \boldsymbol{B}, n_i, n_e, \boldsymbol{u}_e, \boldsymbol{u}_i)$, where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields, and $n_\sigma, \boldsymbol{u}_\sigma$ are the number density and fluid velocity for species $\sigma \in \{i, e\}$ (ions, electrons). The system possesses a Hamiltonian structure defined by a Poisson bracket. This structure can be described by an anchor map $\pi^\#$, which maps the functional derivatives of any functional F, denoted $F_{\chi_i} = F\chi_i$, to the corresponding Hamiltonian vector field $X_F = (\dot{\boldsymbol{B}}, \dot{\boldsymbol{E}}, \dot{\boldsymbol{u}}_i, \dot{\boldsymbol{u}}_e, \dot{n}_i, \dot{n}_e)$. The components of this vector field are given by:

$$
\begin{aligned}
\dot{\boldsymbol{B}} &= -\frac{1}{\epsilon\epsilon_o}\nabla \times F_{\boldsymbol{E}} \\
\dot{\boldsymbol{E}} &= \frac{1}{\epsilon\epsilon_o}\nabla \times F_{\boldsymbol{B}} - \frac{q_e}{\epsilon^2\epsilon_o m_e}F_{\boldsymbol{u}_e} + \frac{Z_i q_e}{\epsilon^2\epsilon_o m_i}F_{\boldsymbol{u}_i} \\
\dot{\boldsymbol{u}}_i &= -\frac{Z_i q_e}{\epsilon m_i^2 n_i}(F_{\boldsymbol{u}_i} \times \boldsymbol{B}) - \frac{Z_i q_e}{\epsilon^2\epsilon_o m_i}F_{\boldsymbol{E}} - \frac{1}{m_i}\nabla F_{n_i} + \frac{1}{m_i n_i}F_{\boldsymbol{u}_i} \times (\nabla \times \boldsymbol{u}_i) \\
\dot{\boldsymbol{u}}_e &= \frac{q_e}{\epsilon m_e^2 n_e}(F_{\boldsymbol{u}_e} \times \boldsymbol{B}) + \frac{q_e}{\epsilon^2\epsilon_o m_e}F_{\boldsymbol{E}} - \frac{1}{m_e}\nabla F_{n_e} + \frac{1}{m_e n_e}F_{\boldsymbol{u}_e} \times (\nabla \times \boldsymbol{u}_e) \\
\dot{n}_i &= -m_i^{-1}\nabla \cdot F_{\boldsymbol{u}_i} \\
\dot{n}_e &= -m_e^{-1}\nabla \cdot F_{\boldsymbol{u}_e}
\end{aligned}
$$

Here, m_σ is the species mass, q_e is the electron charge, Z_i is the ion atomic number, and ϵ, ϵ_o are physical constants. We wish to reduce this system by imposing Gauss's Law as a constraint. This defines a submanifold N where the electron number density n_e is no longer an independent variable, but is determined by the other fields:

$$n_e = Z_i n_i + \epsilon^2\epsilon_o q_e^{-1}\nabla \cdot \boldsymbol{E}$$

The state on this submanifold N is described by the variables $(\boldsymbol{E}, \boldsymbol{B}, n_i, \boldsymbol{u}_e, \boldsymbol{u}_i)$. It can be shown that N is a Poisson-Dirac submanifold. For this specific case, the submanifold is a Poisson submanifold, which implies that the 'π-orthogonal complement' of its tangent space, $(T_pN)^{\pi\perp}$, is trivial. This property leads to a significant simplification in the procedure for finding the reduced Poisson bracket on N: the reduced anchor map $\pi_N^\#$ can be obtained by taking the full anchor map $\pi^\#$ and applying it to functionals F that have no explicit dependence on the eliminated variable n_e. This is equivalent to setting the functional derivative F_{n_e} to zero in the expressions for the components of the vector field X_F.

Task: Using the simplification described above, determine the components of the reduced Hamiltonian vector field X_F on the submanifold N. The vector field on N has components $(\dot{\boldsymbol{B}}, \dot{\boldsymbol{E}}, \dot{\boldsymbol{u}}_i, \dot{\boldsymbol{u}}_e, \dot{n}_i)$. In your final expression, any remaining instances of n_e should be understood as being determined by the constraint equation.

Original Solution

$$
X_F = \begin{pmatrix}
-\frac{1}{\epsilon\epsilon_o}\nabla \times F_{\boldsymbol{E}} \\
\frac{1}{\epsilon\epsilon_o}\nabla \times F_{\boldsymbol{B}} - \frac{q_e}{\epsilon^2\epsilon_o m_e}F_{\boldsymbol{u}_e} + \frac{Z_i q_e}{\epsilon^2\epsilon_o m_i}F_{\boldsymbol{u}_i} \\
-\frac{Z_i q_e}{\epsilon m_i^2 n_i}(F_{\boldsymbol{u}_i} \times \boldsymbol{B}) - \frac{Z_i q_e}{\epsilon^2\epsilon_o m_i}F_{\boldsymbol{E}} - m_i^{-1}\nabla F_{n_i} + \frac{1}{m_i n_i}F_{\boldsymbol{u}_i} \times (\nabla \times \boldsymbol{u}_i) \\
\frac{q_e}{\epsilon m_e^2 n_e}(F_{\boldsymbol{u}_e} \times \boldsymbol{B}) + \frac{q_e}{\epsilon^2\epsilon_o m_e}F_{\boldsymbol{E}} + \frac{1}{m_e n_e}F_{\boldsymbol{u}_e} \times (\nabla \times \boldsymbol{u}_e) \\
-m_i^{-1}\nabla \cdot F_{\boldsymbol{u}_i}
\end{pmatrix}
$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing the full Hamiltonian structure, applying a nontrivial constraint to reduce the phase space, and carefully modifying the functional derivatives and vector field components accordingly, which involves advanced understanding of Poisson geometry and constrained Hamiltonian systems.

Refined Problem

Refined Problem Statement: Background: Consider a non-dissipative two-fluid plasma model described by the variables

$$(\boldsymbol{E}, \boldsymbol{B}, n_i, n_e, \boldsymbol{u}_e, \boldsymbol{u}_i)$$

, where

$$\boldsymbol{E}$$

and

$$\boldsymbol{B}$$

are the electric and magnetic fields, and

$$n_\sigma, \boldsymbol{u}_\sigma$$

are the number density and fluid velocity for species

$$\sigma \in \{i, e\}$$

(ions, electrons). The system possesses a Hamiltonian structure defined by a Poisson bracket. This structure can be described by an anchor map

$$\pi^\#$$

, which maps the functional derivatives of any functional

$$F$$

, denoted

$$F_{\chi_i} = F\chi_i$$

, to the corresponding Hamiltonian vector field

$$X_F = (\dot{\boldsymbol{B}}, \dot{\boldsymbol{E}}, \dot{\boldsymbol{u}}_i, \dot{\boldsymbol{u}}_e, \dot{n}_i, \dot{n}_e)$$

. The components of this vector field are given by:

$$\begin{aligned}
\dot{\boldsymbol{B}} &= -\frac{1}{\epsilon\epsilon_o}\nabla \times F_{\boldsymbol{E}} \\
\dot{\boldsymbol{E}} &= \frac{1}{\epsilon\epsilon_o}\nabla \times F_{\boldsymbol{B}} - \frac{q_e}{\epsilon^2\epsilon_o m_e}F_{\boldsymbol{u}_e} + \frac{Z_i q_e}{\epsilon^2\epsilon_o m_i}F_{\boldsymbol{u}_i} \\
\dot{\boldsymbol{u}}_i &= -\frac{Z_i q_e}{\epsilon m_i^2 n_i}(F_{\boldsymbol{u}_i} \times \boldsymbol{B}) - \frac{Z_i q_e}{\epsilon^2\epsilon_o m_i}F_{\boldsymbol{E}} - \frac{1}{m_i}\nabla F_{n_i} + \frac{1}{m_i n_i}F_{\boldsymbol{u}_i} \times (\nabla \times \boldsymbol{u}_i) \\
\dot{\boldsymbol{u}}_e &= \frac{q_e}{\epsilon m_e^2 n_e}(F_{\boldsymbol{u}_e} \times \boldsymbol{B}) + \frac{q_e}{\epsilon^2\epsilon_o m_e}F_{\boldsymbol{E}} - \frac{1}{m_e}\nabla F_{n_e} + \frac{1}{m_e n_e}F_{\boldsymbol{u}_e} \times (\nabla \times \boldsymbol{u}_e) \\
\dot{n}_i &= -m_i^{-1}\nabla \cdot F_{\boldsymbol{u}_i} \\
\dot{n}_e &= -m_e^{-1}\nabla \cdot F_{\boldsymbol{u}_e}
\end{aligned}$$

Here,

$$m_\sigma$$

is the species mass,

$$q_e$$

is the electron charge,

$$Z_i$$

is the ion atomic number, and

$$\epsilon, \epsilon_o$$

are physical constants. We wish to reduce this system by imposing Gauss's Law as a constraint. This defines a submanifold

$$N$$

where the electron number density

$$n_e$$

is no longer an independent variable, but is determined by the other fields:

$$n_e = Z_i n_i + \epsilon^2 \epsilon_o q_e^{-1} \nabla \cdot \boldsymbol{E}$$

The state on this submanifold

$$N$$

is described by the variables

$$(\boldsymbol{E}, \boldsymbol{B}, n_i, \boldsymbol{u}_e, \boldsymbol{u}_i)$$

. It can be shown that

$$N$$

is a Poisson-Dirac submanifold. For this specific case, the submanifold is a Poisson submanifold, which implies that the '

$$\pi$$

-orthogonal complement' of its tangent space,

$$(T_p N)^{\pi \perp}$$

, is trivial. This property leads to a significant simplification in the procedure for finding the reduced Poisson bracket on

$$N$$

: the reduced anchor map

$$\pi_N^{\#}$$

can be obtained by taking the full anchor map

$$\pi^{\#}$$

and applying it to functionals

$$F$$

that have no explicit dependence on the eliminated variable

$$n_e$$

. This is equivalent to setting the functional derivative

$$F_{n_e}$$

to zero in the expressions for the components of the vector field

$$X_F$$

.

Task: Using the simplification described above, determine the components of the reduced Hamiltonian vector field

$$X_F$$

on the submanifold

$$N$$

. The vector field on

$$N$$

has components

$$(\dot{\boldsymbol{B}}, \dot{\boldsymbol{E}}, \dot{\boldsymbol{u}_i}, \dot{\boldsymbol{u}_e}, \dot{n_i})$$

. In your final expression, any remaining instances of

$$n_e$$

should be understood as being determined by the constraint equation.

Solution:

Refined Solution:

$$X_F = \begin{pmatrix} -\frac{1}{\epsilon\epsilon_o}\nabla \times F_{\boldsymbol{E}} \\ \frac{1}{\epsilon\epsilon_o}\nabla \times F_{\boldsymbol{B}} - \frac{q_e}{\epsilon^2\epsilon_o m_e}F_{\boldsymbol{u}_e} + \frac{Z_i q_e}{\epsilon^2\epsilon_o m_i}F_{\boldsymbol{u}_i} \\ -\frac{Z_i q_e}{\epsilon m_i^2 n_i}(F_{\boldsymbol{u}_i} \times \boldsymbol{B}) - \frac{Z_i q_e}{\epsilon^2\epsilon_o m_i}F_{\boldsymbol{E}} - m_i^{-1}\nabla F_{n_i} + \frac{1}{m_i n_i}F_{\boldsymbol{u}_i} \times (\nabla \times \boldsymbol{u}_i) \\ \frac{q_e}{\epsilon m_e^2 n_e}(F_{\boldsymbol{u}_e} \times \boldsymbol{B}) + \frac{q_e}{\epsilon^2\epsilon_o m_e}F_{\boldsymbol{E}} + \frac{1}{m_e n_e}F_{\boldsymbol{u}_e} \times (\nabla \times \boldsymbol{u}_e) \\ -m_i^{-1}\nabla \cdot F_{\boldsymbol{u}_i} \end{pmatrix}$$

Problem 60 (Paper: 2506.23991v1, Index: 1)

Original Problem Statement

Background: Consider the ideal two-fluid plasma model, whose state is described by the fields $(\boldsymbol{E}, \boldsymbol{B}, n_i, n_e, \boldsymbol{u}_e, \boldsymbol{u}_i)$ representing the electric field, magnetic field, ion and electron number densities, and ion and electron fluid velocities, respectively. The dynamics are Hamiltonian, governed by a Poisson bracket $\{F, G\}$ for any two functionals F and G of the state variables. The associated Poisson anchor map $\pi^{\#}$, which maps a covector (functional derivative) δF to a vector (time derivative) $X_F = \dot{z}$, can be expanded in a small parameter ϵ as $\epsilon^2 \pi_\epsilon^{\#} = \pi_0 + \epsilon \pi_1 + \epsilon^2 \pi_2 + \mathcal{O}(\epsilon^3)$. The electron number density n_e is determined by the other fields via Gauss's Law, $\epsilon_0 \nabla \cdot \boldsymbol{E} = q_e n_e + q_i n_i$, where $q_i = -Z_i q_e$ is the ion charge. The phase space can be described by the independent variables $z = (n_i, \boldsymbol{u}_i, \boldsymbol{B}, \boldsymbol{u}_e, \boldsymbol{E})$. The components of the vector $X_F = (\delta\boldsymbol{B}, \delta\boldsymbol{E}, \delta\boldsymbol{u}_i, \delta\boldsymbol{u}_e, \delta n_i)^T$ for each order in ϵ are given by:

$$\pi_0(\delta F) = \begin{pmatrix} 0 \\ -\frac{q_e}{\epsilon_o m_e} F_{\boldsymbol{u}_e} + \frac{Z_i q_e}{\epsilon_o m_i} F_{\boldsymbol{u}_i} \\ -\frac{Z_i q_e}{\epsilon_o m_i} F_{\boldsymbol{E}} \\ \frac{q_e}{\epsilon_o m_e} F_{\boldsymbol{E}} \\ 0 \end{pmatrix},$$

$$\pi_1(\delta F) = \begin{pmatrix} -\frac{1}{\epsilon_o} \nabla \times F_{\boldsymbol{E}} \\ \frac{1}{\epsilon_o} \nabla \times F_{\boldsymbol{B}} \\ \frac{-Z_i q_e}{m_i^2 n_i} (F_{\boldsymbol{u}_i} \times \boldsymbol{B}) \\ \frac{q_e}{m_e^2 n_e} (F_{\boldsymbol{u}_e} \times \boldsymbol{B}) \\ 0 \end{pmatrix},$$

$$\pi_2(\delta F) = \begin{pmatrix} 0 \\ 0 \\ -m_i^{-1} \nabla F_{n_i} + \frac{1}{m_i n_i} F_{\boldsymbol{u}_i} \times (\nabla \times \boldsymbol{u}_i) \\ \frac{1}{m_e n_e} F_{\boldsymbol{u}_e} \times (\nabla \times \boldsymbol{u}_e) \\ -m_i^{-1} \nabla \cdot F_{\boldsymbol{u}_i} \end{pmatrix}.$$

We are interested in the dynamics on the "slow manifold" N, which is a submanifold of the full phase space defined by the constraints $\boldsymbol{u}_e = \boldsymbol{u}_i$ and $\boldsymbol{E} = -\boldsymbol{u}_i \times \boldsymbol{B}$. The variables on N are $(n_i, \boldsymbol{u}_i, \boldsymbol{B})$. The induced Poisson bracket on N is given by $\{F, G\}_N = \{\tilde{F}, \tilde{G}\}|_N$, where $\tilde{F}$ is an extension of the functional F from N to the full phase space. An extension $\tilde{F}$ must satisfy $\tilde{F}|_N = F$ and its associated Hamiltonian vector field $X_{\tilde{F}} = \pi^{\#}(\delta\tilde{F})$ must be tangent to N. The one-form $\delta\tilde{F}$ for an extension can be written as

$$\delta\tilde{F} = \int (F_{n_i} \delta n_i + (F_{\boldsymbol{u}_i} - \Theta_{\boldsymbol{E}} \times \boldsymbol{B} - \Theta_{\boldsymbol{u}_e}) \cdot \delta\boldsymbol{u}_i + (\Theta_{\boldsymbol{E}} \times \boldsymbol{u}_i + F_{\boldsymbol{B}}) \cdot \delta\boldsymbol{B} + \Theta_{\boldsymbol{u}_e} \cdot \delta\boldsymbol{u}_e + \Theta_{\boldsymbol{E}} \cdot \delta\boldsymbol{E})\, d^3\boldsymbol{x},$$

where $F_{n_i}, F_{\boldsymbol{u}_i}, F_{\boldsymbol{B}}$ are the functional derivatives of F on N, and the functionals $\Theta_{\boldsymbol{u}_e}$ and $\Theta_{\boldsymbol{E}}$ are determined by enforcing the tangency condition on $X_{\tilde{F}}$ perturbatively in ϵ. In your calculations, you should expand quantities involving n_e as a power series in ϵ, using $n_e = Z_i n_i - (\epsilon^2 \epsilon_0 / q_e) \nabla \cdot \boldsymbol{E}$ and the on-manifold relation for $\boldsymbol{E}$. Define the mass ratio $\nu = m_e / m_i$ and the total mass $\mathsf{m} = (1 + \nu Z_i) m_i$.

Task: By solving for the extension functionals $\Theta_{\boldsymbol{u}_e}$ and $\Theta_{\boldsymbol{E}}$ perturbatively and computing the induced bracket, determine the $\mathcal{O}(1)$ Poisson bracket $\{F, G\}_N$ on the slow manifold N. This corresponds to the $\mathcal{O}(\epsilon^2)$ part of the full bracket $\{\tilde{F}, \tilde{G}\}$. Your final answer should be a single integral expression for $\{F, G\}_N$ in terms of the functional derivatives $F_{n_i}, F_{\boldsymbol{u}_i}, F_{\boldsymbol{B}}$ and their counterparts for G.

Original Solution

$$\{F, G\}_N = \int \Bigg(-\mathsf{m}^{-1} (G_{\boldsymbol{u}_i} \cdot \nabla F_{n_i} + G_{n_i} \nabla \cdot F_{\boldsymbol{u}_i}) + \frac{(\nabla \times \boldsymbol{u}_i)}{\mathsf{m} n_i} \cdot (G_{\boldsymbol{u}_i} \times F_{\boldsymbol{u}_i}) + \frac{1}{\mathsf{m} n_i} \boldsymbol{B} \cdot (G_{\boldsymbol{u}_i} \times (\nabla \times F_{\boldsymbol{B}}) - F_{\boldsymbol{u}_i} \times (\nabla \times G_{\boldsymbol{B}}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires a multi-step perturbative expansion, enforcing tangency conditions, and synthesizing the Poisson bracket structure on a constrained slow manifold, which involves advanced functional derivatives and vector calculus beyond direct application of known formulas.

Refined Problem

Refined Problem Statement: Background: Consider the ideal two-fluid plasma model, whose state is described by the fields $(\boldsymbol{E}, \boldsymbol{B}, n_i, n_e, \boldsymbol{u}_e, \boldsymbol{u}_i)$ representing the electric field, magnetic field, ion and electron number densities, and ion and electron fluid velocities, respectively. The dynamics are Hamiltonian, governed by a Poisson bracket $\{F, G\}$ for any two functionals F and G of the state variables. The associated Poisson anchor map $\pi^{\#}$, which maps a covector (functional derivative) δF to a vector (time derivative) $X_F = \dot{z}$, can be expanded in a small parameter ϵ as $\epsilon^2 \pi_\epsilon^{\#} = \pi_0 + \epsilon\pi_1 + \epsilon^2\pi_2 + \mathcal{O}(\epsilon^3)$. The electron number density n_e is determined by the other fields via Gauss's Law, $\epsilon_0 \nabla \cdot \boldsymbol{E} = q_e n_e + q_i n_i$, where $q_i = -Z_i q_e$ is the ion charge. The phase space can be described by the independent variables $z = (n_i, \boldsymbol{u}_i, \boldsymbol{B}, \boldsymbol{u}_e, \boldsymbol{E})$. The components of the vector $X_F = (\delta\boldsymbol{B}, \delta\boldsymbol{E}, \delta\boldsymbol{u}_i, \delta\boldsymbol{u}_e, \delta n_i)^T$ for each order in ϵ are given by:

$$\pi_0(\delta F) = \begin{pmatrix} 0 \\ -\frac{q_e}{\epsilon_o m_e} F_{\boldsymbol{u}_e} + \frac{Z_i q_e}{\epsilon_o m_i} F_{\boldsymbol{u}_i} \\ -\frac{Z_i q_e}{\epsilon_o m_i} F_{\boldsymbol{E}} \\ \frac{q_e}{\epsilon_o m_e} F_{\boldsymbol{E}} \\ 0 \end{pmatrix},$$

$$\pi_1(\delta F) = \begin{pmatrix} -\frac{1}{\epsilon_o} \nabla \times F_{\boldsymbol{E}} \\ \frac{1}{\epsilon_o} \nabla \times F_{\boldsymbol{B}} \\ \frac{-Z_i q_e}{m_i^2 n_i} (F_{\boldsymbol{u}_i} \times \boldsymbol{B}) \\ \frac{q_e}{m_e^2 n_e} (F_{\boldsymbol{u}_e} \times \boldsymbol{B}) \\ 0 \end{pmatrix},$$

$$\pi_2(\delta F) = \begin{pmatrix} 0 \\ 0 \\ -m_i^{-1} \nabla F_{n_i} + \frac{1}{m_i n_i} F_{\boldsymbol{u}_i} \times (\nabla \times \boldsymbol{u}_i) \\ \frac{1}{m_e n_e} F_{\boldsymbol{u}_e} \times (\nabla \times \boldsymbol{u}_e) \\ -m_i^{-1} \nabla \cdot F_{\boldsymbol{u}_i} \end{pmatrix}.$$

We are interested in the dynamics on the "slow manifold" N, which is a submanifold of the full phase space defined by the constraints $\boldsymbol{u}_e = \boldsymbol{u}_i$ and $\boldsymbol{E} = -\boldsymbol{u}_i \times \boldsymbol{B}$. The variables on N are $(n_i, \boldsymbol{u}_i, \boldsymbol{B})$. The induced Poisson bracket on N is given by $\{F, G\}_N = \{\tilde{F}, \tilde{G}\}|_N$, where $\tilde{F}$ is an extension of the functional F from N to the full phase space. An extension $\tilde{F}$ must satisfy $\tilde{F}|_N = F$ and its associated Hamiltonian vector field $X_{\tilde{F}} = \pi^{\#}(\delta\tilde{F})$ must be tangent to N. The one-form $\delta\tilde{F}$ for an extension can be written as

$$\delta\tilde{F} = \int \left(F_{n_i}\delta n_i + (F_{\boldsymbol{u}_i} - \Theta_{\boldsymbol{E}} \times \boldsymbol{B} - \Theta_{\boldsymbol{u}_e}) \cdot \delta\boldsymbol{u}_i + (\Theta_{\boldsymbol{E}} \times \boldsymbol{u}_i + F_{\boldsymbol{B}}) \cdot \delta\boldsymbol{B} + \Theta_{\boldsymbol{u}_e} \cdot \delta\boldsymbol{u}_e + \Theta_{\boldsymbol{E}} \cdot \delta\boldsymbol{E}\right) d^3\boldsymbol{x},$$

where $F_{n_i}, F_{\boldsymbol{u}_i}, F_{\boldsymbol{B}}$ are the functional derivatives of F on N, and the functionals $\Theta_{\boldsymbol{u}_e}$ and $\Theta_{\boldsymbol{E}}$ are determined by enforcing the tangency condition on $X_{\tilde{F}}$ perturbatively in ϵ. In your calculations, you should expand quantities involving n_e as a power series in ϵ, using $n_e = Z_i n_i - (\epsilon^2\epsilon_0/q_e)\nabla \cdot \boldsymbol{E}$ and the on-manifold relation for $\boldsymbol{E}$. Define the mass ratio $\nu = m_e/m_i$ and the total mass $\mathsf{m} = (1 + \nu Z_i) m_i$.

Task: By solving for the extension functionals $\Theta_{\boldsymbol{u}_e}$ and $\Theta_{\boldsymbol{E}}$ perturbatively and computing the induced bracket, determine the $\mathcal{O}(1)$ Poisson bracket $\{F, G\}_N$ on the slow manifold N. This corresponds to the $\mathcal{O}(\epsilon^2)$ part of the full bracket $\{\tilde{F}, \tilde{G}\}$. Your final answer should be a single integral expression for $\{F, G\}_N$ in terms of the functional derivatives $F_{n_i}, F_{\boldsymbol{u}_i}, F_{\boldsymbol{B}}$ and their counterparts for G.

Refined Solution:

$$\{F, G\}_N = \int \Bigg(-\mathsf{m}^{-1}(G_{\boldsymbol{u}_i} \cdot \nabla F_{n_i} + G_{n_i} \nabla \cdot F_{\boldsymbol{u}_i}) + \frac{(\nabla \times \boldsymbol{u}_i)}{\mathsf{m} n_i} \cdot (G_{\boldsymbol{u}_i} \times F_{\boldsymbol{u}_i}) + \frac{1}{\mathsf{m} n_i} \boldsymbol{B} \cdot (G_{\boldsymbol{u}_i} \times (\nabla \times F_{\boldsymbol{B}}) - F_{\boldsymbol{u}_i} \times (\nabla \times G_{\boldsymbol{B}}$$

Problem 61 (Paper: 2506.24021v1, Index: 0)

Original Problem Statement

Background: Consider a physical system whose state is described by a particle's position $\in \mathbb{R}^d$. The system is in contact with a thermal bath at temperature T. A process of duration τ transforms the probability distribution of the particle's position from an initial state $\rho_0()$ to a final state $\rho_\tau()$. The minimum work required for this transformation is bounded by the thermodynamic speed limit: $W \geq T(\rho_\tau||\rho_0) + \frac{1}{\tau}{}^2(\rho_\tau, \rho_0)$, where $(\nu||\mu) = \int \nu() \log(\frac{\nu()}{\mu()})$ is the Kullback-Leibler (KL) divergence, and ${}^2(\nu, \mu) = \inf_{\pi \in \Pi(\nu,\mu)} \int ||-||^2 \pi(,)$ is the squared 2-Wasserstein distance over all joint distributions π with marginals ν and μ.

We define two logical operations based on this principle: 1. **One-bit erasure (1BE)**: The system is one-dimensional ($d = 1$). A bit is encoded by the sign of the particle's position x. The initial distribution $\rho_0^x(x)$ is symmetric, i.e., $\rho_0^x(x) = \rho_0^x(-x)$. The erasure operation constrains the final distribution $\rho_\tau^x(x)$ to be supported only on the positive real axis, i.e., its support is a subset of $\mathbb{R}^+$. The minimal work for this operation is given by the infimum over all such valid final distributions: $W^{\text{1BE}}(\rho_0^x) = \inf_{\rho_\tau^x : (\rho_\tau^x) \subseteq \mathbb{R}^+} \left[T(\rho_\tau^x||\rho_0^x) + \frac{1}{\tau}{}^2(\rho_\tau^x, \rho_0^x) \right]$.

2. **Two-bit erasure (2BE)**: The system is two-dimensional ($d = 2$). Two bits are encoded by the signs of the coordinates (x, y). The initial distribution is a product of two independent, symmetric 1D distributions, $\rho_0(x, y) = \rho_0^x(x)\rho_0^y(y)$, where $\rho_0^x(x) = \rho_0^x(-x)$ and $\rho_0^y(y) = \rho_0^y(-y)$. The erasure operation constrains the final distribution $\rho_\tau(x, y)$ to be supported only in the quadrant where $x > 0$ and $y < 0$. The minimal work is $W^{\text{2BE}}(\rho_0) = \inf_{\rho_\tau : (\rho_\tau) \subseteq \mathbb{R}^+ \times \mathbb{R}^-} \left[T(\rho_\tau||\rho_0) + \frac{1}{\tau}{}^2(\rho_\tau, \rho_0) \right]$.

You may use the following properties without proof: - For any distribution $p(x, y)$ and a product distribution $q(x, y) = q_x(x)q_y(y)$, the KL divergence can be decomposed as: $(p||q) = (p||p_x p_y) + (p_x||q_x) + (p_y||q_y)$, where p_x, p_y are the marginals of p. The term $(p||p_x p_y)$, representing mutual information, is non-negative. - For any two distributions π and μ on $\mathbb{R}^2$, ${}^2(\pi, \mu) \geq {}^2(\pi_x, \mu_x) + {}^2(\pi_y, \mu_y)$, where subscripts denote marginal distributions. - The squared 2-Wasserstein distance between two product measures $\mu = \mu_x \otimes \mu_y$ and $\nu = \nu_x \otimes \nu_y$ is additive: ${}^2(\mu, \nu) = {}^2(\mu_x, \nu_x) + {}^2(\mu_y, \nu_y)$.

Task: For a system with an initial distribution $\rho_0(x, y) = \rho_0^x(x)\rho_0^y(y)$, derive the exact relationship between the minimal work for two-bit erasure, $W^{\text{2BE}}(\rho_0)$, and the minimal works for one-bit erasure of the corresponding marginals, $W^{\text{1BE}}(\rho_0^x)$ and $W^{\text{1BE}}(\rho_0^y)$.

Original Solution

$$W^{\text{2BE}}(\rho_0^x \rho_0^y) = W^{\text{1BE}}(\rho_0^x) + W^{\text{1BE}}(\rho_0^y)$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing properties of KL divergence, Wasserstein distances, and constraints on support to derive a non-trivial decomposition of minimal work for two-bit erasure into one-bit erasures, involving multi-step reasoning and careful use of inequalities and equalities.

Refined Problem

Refined Problem Statement: Background: Consider a physical system whose state is described by a particle's position $\in \mathbb{R}^d$. The system is in contact with a thermal bath at temperature T. A process of duration τ transforms the probability distribution of the particle's position from an initial state $\rho_0()$ to a final state $\rho_\tau()$. The minimum work required for this transformation is bounded by the thermodynamic speed limit: $W \geq T(\rho_\tau||\rho_0) + \frac{1}{\tau}{}^2(\rho_\tau, \rho_0)$, where $(\nu||\mu) = \int \nu() \log(\frac{\nu()}{\mu()})$ is the Kullback-Leibler (KL) divergence, and ${}^2(\nu, \mu) = \inf_{\pi \in \Pi(\nu,\mu)} \int ||-||^2 \pi(,)$ is the squared 2-Wasserstein distance over all joint distributions π with marginals ν and μ.

We define two logical operations based on this principle: 1. **One-bit erasure (1BE)**: The system is one-dimensional ($d = 1$). A bit is encoded by the sign of the particle's position x. The initial distribution $\rho_0^x(x)$ is symmetric, i.e., $\rho_0^x(x) = \rho_0^x(-x)$. The erasure operation constrains the final distribution $\rho_\tau^x(x)$ to be supported only on the positive real axis, i.e., its support is a subset of $\mathbb{R}^+$.

The minimal work for this operation is given by the infimum over all such valid final distributions: $W^{\mathrm{1BE}}(\rho_0^x) = \inf_{\rho_\tau^x:(\rho_\tau^x)\subseteq\mathbb{R}^+} \left[T(\rho_\tau^x||\rho_0^x) + \frac{1}{\tau}^2(\rho_\tau^x, \rho_0^x)\right]$.

2. **Two-bit erasure (2BE)**: The system is two-dimensional ($d = 2$). Two bits are encoded by the signs of the coordinates (x, y). The initial distribution is a product of two independent, symmetric 1D distributions, $\rho_0(x, y) = \rho_0^x(x)\rho_0^y(y)$, where $\rho_0^x(x) = \rho_0^x(-x)$ and $\rho_0^y(y) = \rho_0^y(-y)$. The erasure operation constrains the final distribution $\rho_\tau(x, y)$ to be supported only in the quadrant where $x > 0$ and $y < 0$. The minimal work is $W^{\mathrm{2BE}}(\rho_0) = \inf_{\rho_\tau:(\rho_\tau)\subseteq\mathbb{R}^+\times\mathbb{R}^-} \left[T(\rho_\tau||\rho_0) + \frac{1}{\tau}^2(\rho_\tau, \rho_0)\right]$.

You may use the following properties without proof: - For any distribution $p(x, y)$ and a product distribution $q(x, y) = q_x(x)q_y(y)$, the KL divergence can be decomposed as: $(p||q) = (p||p_x p_y)+(p_x||q_x)+(p_y||q_y)$, where p_x, p_y are the marginals of p. The term $(p||p_x p_y)$, representing mutual information, is non-negative. - For any two distributions π and μ on $\mathbb{R}^2$, ${}^2(\pi, \mu) \geq^2 (\pi_x, \mu_x) +^2 (\pi_y, \mu_y)$, where subscripts denote marginal distributions. - The squared 2-Wasserstein distance between two product measures $\mu = \mu_x \otimes \mu_y$ and $\nu = \nu_x \otimes \nu_y$ is additive: ${}^2(\mu, \nu) =^2 (\mu_x, \nu_x) +^2 (\mu_y, \nu_y)$.

Task: For a system with an initial distribution $\rho_0(x, y) = \rho_0^x(x)\rho_0^y(y)$, derive the exact relationship between the minimal work for two-bit erasure, $W^{\mathrm{2BE}}(\rho_0)$, and the minimal works for one-bit erasure of the corresponding marginals, $W^{\mathrm{1BE}}(\rho_0^x)$ and $W^{\mathrm{1BE}}(\rho_0^y)$.

Refined Solution:

$$W^{\mathrm{2BE}}(\rho_0^x\rho_0^y) = W^{\mathrm{1BE}}(\rho_0^x) + W^{\mathrm{1BE}}(\rho_0^y)$$

Problem 62 (Paper: 2506.24021v1, Index: 1)

Original Problem Statement

Background: Consider a physical system described by the position of a particle '$\in \mathbb{R}^2$'.*Thesystemistobedrivenfromaniniti*
$\mathbb{R}^2$', *i.e.*, '$\int_{\mathcal{C}_{00}} \pi_\tau() = \alpha$'.*Theoptimalprotocolforthistransformationcanbefoundbysolvingaregularizedunbalancedoptima*
'$\mathcal{J}[\pi] = F_0(\pi_0)+F_\tau(\pi_\tau)+\epsilon(\pi||e^{-c/\epsilon})$'*Here*, '$\pi_0$'*and*'$\pi_\tau$'*arethemarginalsof*'$\pi$', '$\epsilon > 0$'*isaregularizationparameter*, '$c(_{\tau,0})$
$\tau^{-1}||_0-_\tau||^2$'*isthetransportcostoveraduration*'τ', *and*"*istheKullback*$-$*Leiblerdivergence.Thefunctionals*'F_0'*and*'F_τ'*enf*
1.'$F_0(p) = \iota_{\rho_0}(p)$', *where*'$\iota_{\rho_0}(p)$'*isanindicatorfunctionthatis*0*if*'$p = \rho_0$'*and*'$+\infty$'*otherwise*, *fixingtheinitialmarginal*.2.
$T(p||\rho_0)+\iota_\alpha(p)$', *where*'$T$'*istemperatureand*'$\iota_\alpha(p)$'*isanindicatorfunctionenforcingthepartialerasureconstraint*' $\int_{\mathcal{C}_{00}} p()$
α'.

The optimal coupling 'π^ϵ'*hastheform*'$\pi^\epsilon(_{\tau,0}) = e^{u(_\tau)/\epsilon}K(_{\tau,0})e^{v(_0)/\epsilon}$', *where*'$K(_{\tau,0}) = e^{-c(_{\tau,0})/\epsilon}$'*and*'$u, v$'*aredualpote*
'$u^{\ell+1} = \underset{u}{\operatorname{argmax}} \left[-\tilde{F}_\tau(-u) - \epsilon\langle e^{u/\epsilon}, (K * e^{v^\ell/\epsilon})\rangle\right]$ '*where*'$\tilde{F}_\tau(u) = \sup_p [\langle p, u\rangle - F_\tau(p)]$ '*istheLegendre* $-$
Fencheltransform, '$\langle f, g\rangle = \int f()g()$', *and*'$(K * s)(_\tau) = \int K(_{\tau,0})s(_0)_0$'.

Task: Derive the explicit update rule for the dual potential 'u$(_\tau)$'.*Thederivationinvolvescalculatingthe*$Legendre-$
Fencheltransform'$\tilde{F}_\tau$'*byintroducingaLagrangemultipliertohandletheconstraint*'$\iota_\alpha(p)$', *andthensolvingthe*'$argmax$'*pro*
$\int_{\mathcal{C}_{00}} \rho_0(_\tau)e_\tau^{-u(_\tau)/T}$', *whicharisesfromtheLagrangemultiplierconstraint*.

Original Solution

$$u(_\tau) = \frac{T\epsilon}{T+\epsilon}\left[\log(\rho_0(_\tau)) - \log\left((K * e^{v^\ell/\epsilon})(_\tau)\right) + 1_{\mathcal{C}_{00}}(_\tau)\left(\log(\alpha) - \log(A[-u])\right)\right]$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires deriving the dual update by performing a Legendre-Fenchel transform with a constraint, introducing a Lagrange multiplier, and solving a non-trivial variational optimization involving integral operators and exponential terms, thus demanding a multi-step synthesis of advanced concepts.

Refined Problem

Refined Problem Statement: Background: Consider a physical system described by the position of a particle $\mathbf{X} \in \mathbb{R}^2$. The system is to be driven from an initial probability distribution $\rho_0(\mathbf{X}_0)$ to a target distribution $\pi_\tau(\mathbf{X}_\tau)$ that realizes a "partial bit erasure" operation. This operation is defined by the constraint that the final distribution must have a specific total probability mass α in a designated region $\mathcal{C}_{00} \subset \mathbb{R}^2$, i.e., $\int_{\mathcal{C}_{00}} \pi_\tau(\mathbf{X})\, d\mathbf{X} = \alpha$. The optimal protocol for this transformation can be found by solving a regularized unbalanced optimal transport problem. The goal is to find the coupling $\pi(\mathbf{X}_\tau, \mathbf{X}_0)$, a joint probability distribution, that minimizes the objective functional:

$\mathcal{J}[\pi] = F_0(\pi_0) + F_\tau(\pi_\tau) + \epsilon\, D_{\mathrm{KL}}(\pi \| e^{-c/\epsilon})$

Here, π_0 and π_τ are the marginals of π, $\epsilon > 0$ is a regularization parameter, $c(\mathbf{X}_\tau, \mathbf{X}_0) = \tau^{-1}\|\mathbf{X}_0 - \mathbf{X}_\tau\|^2$ is the transport cost over a duration τ, and D_{KL} is the Kullback-Leibler divergence. The functionals F_0 and F_τ enforce the physical constraints: 1. $F_0(p) = \iota_{\rho_0}(p)$, where $\iota_{\rho_0}(p)$ is an indicator function that is 0 if $p = \rho_0$ and $+\infty$ otherwise, fixing the initial marginal. 2. $F_\tau(p) = T\, D_{\mathrm{KL}}(p\|\rho_0) + \iota_\alpha(p)$, where T is temperature and $\iota_\alpha(p)$ is an indicator function enforcing the partial erasure constraint $\int_{\mathcal{C}_{00}} p(\mathbf{X})\, d\mathbf{X} = \alpha$.

The optimal coupling π^ϵ has the form $\pi^\epsilon(\mathbf{X}_\tau, \mathbf{X}_0) = \exp\left(u(\mathbf{X}_\tau)/\epsilon\right) K(\mathbf{X}_\tau, \mathbf{X}_0) \exp\left(v(\mathbf{X}_0)/\epsilon\right)$, where $K(\mathbf{X}_\tau, \mathbf{X}_0) = \exp\left(-c(\mathbf{X}_\tau, \mathbf{X}_0)/\epsilon\right)$ and u, v are dual potentials found via an iterative scheme. The update for the target potential $u^{\ell+1}$ given the source potential v^ℓ is found by solving the maximization problem: $u^{\ell+1} = \underset{u}{\operatorname{argmax}} \left[-\tilde{F}_\tau(-u) - \epsilon\langle \exp(u/\epsilon), (K * \exp(v^\ell/\epsilon))\rangle\right]$ where $\tilde{F}_\tau(u) = \sup_p [\langle p, u\rangle - F_\tau(p)]$ is the Legendre-Fenchel transform, $\langle f, g\rangle = \int f(\mathbf{X})g(\mathbf{X})d\mathbf{X}$, and $(K * s)(\mathbf{X}_\tau) = \int K(\mathbf{X}_\tau, \mathbf{X}_0)s(\mathbf{X}_0)d\mathbf{X}_0$.

Task: Derive the explicit update rule for the dual potential $u(\mathbf{X}_\tau)$. The derivation involves calculating the Legendre-Fenchel transform $\tilde{F}_\tau$ by introducing a Lagrange multiplier to handle the constraint $\iota_\alpha(p)$, and then solving the argmax problem for u. Your final expression for $u(\mathbf{X}_\tau)$ should be given in terms of $\rho_0(\mathbf{X}_\tau)$, the kernel operator K, the previous potential $v^\ell(\mathbf{X}_0)$, the parameters T, ϵ, α, and the indicator function $1_{\mathcal{C}_{00}}(\mathbf{X}_\tau)$. The expression will depend on the term $A[-u] = \int_{\mathcal{C}_{00}} \rho_0(\mathbf{X}_\tau) \exp\left(-u(\mathbf{X}_\tau)/T\right) d\mathbf{X}_\tau$, which arises from the Lagrange multiplier constraint.

Refined Solution:

$$u(\mathbf{X}_\tau) = \frac{T\epsilon}{T+\epsilon}\left[\log\left(\rho_0(\mathbf{X}_\tau)\right) - \log\left((K * \exp(v^\ell/\epsilon))(\mathbf{X}_\tau)\right) + 1_{\mathcal{C}_{00}}(\mathbf{X}_\tau)\left(\log(\alpha) - \log(A[-u])\right)\right]$$

Problem 63 (Paper: 2506.24035v1, Index: 0)

Original Problem Statement

Background: Consider a physical system described by a single order parameter ϕ. The system is in contact with a thermal bath and subject to an external field H. The free energy density of the system is given by the Landau-Ginzburg form: $f_4(\phi, H) = \frac{1}{2}a_2\phi^2 + \frac{1}{4}a_4\phi^4 - H\phi$ where a_2 and a_4 are constants. The dynamics of the order parameter are stochastic and can be described by a Fokker-Planck equation for the probability density $P(\phi, t)$: $\frac{\partial P(\phi,t)}{\partial t} = \lambda \frac{\partial}{\partial \phi}\left[\frac{\partial f_4(\phi,H)}{\partial \phi} P(\phi, t)\right] + \lambda\sigma^2 \frac{\partial^2 P(\phi,t)}{\partial \phi^2}$ Here, λ is a kinetic coefficient setting a characteristic timescale, and σ is the noise strength. The ensemble average of any function of the order parameter, $\mathcal{O}(\phi)$, is defined as $\langle \mathcal{O} \rangle = \int_{-\infty}^{\infty} \mathcal{O}(\phi) P(\phi, t) d\phi$. The system is driven by linearly sweeping the external field H in time t at a constant rate, such that $dH/dt = \lambda v_H$, where v_H is the dimensionless driving rate. Assume that the probability distribution $P(\phi, t)$ and its derivatives vanish at $\phi \to \pm\infty$.

Task: Derive the differential equation that governs the evolution of the ensemble-averaged order parameter $\langle \phi \rangle$ with respect to the external field H. Your final expression should relate $d\langle \phi \rangle / dH$ to $\langle \phi \rangle$, $\langle \phi^3 \rangle$, H, and the parameters of the model.

Original Solution

$$v_H \frac{d\langle \phi \rangle}{dH} = -a_2 \langle \phi \rangle - a_4 \langle \phi^3 \rangle + H$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires deriving a nontrivial evolution equation for the ensemble average from the Fokker-Planck equation, involving careful application of stochastic calculus and integration by parts, and connecting multiple concepts such as the Landau-Ginzburg free energy, stochastic dynamics, and ensemble averaging under a time-dependent external field.

Refined Problem

Refined Problem Statement: Background: Consider a physical system described by a single order parameter ϕ. The system is in contact with a thermal bath and subject to an external field H. The free energy density of the system is given by the Landau-Ginzburg form:

$$f_4(\phi, H) = \frac{1}{2}a_2\phi^2 + \frac{1}{4}a_4\phi^4 - H\phi$$

where a_2 and a_4 are constants. The dynamics of the order parameter are stochastic and can be described by a Fokker-Planck equation for the probability density $P(\phi, t)$:

$$\frac{\partial P(\phi, t)}{\partial t} = \lambda \frac{\partial}{\partial \phi}\left[\frac{\partial f_4(\phi, H)}{\partial \phi} P(\phi, t)\right] + \lambda\sigma^2 \frac{\partial^2 P(\phi, t)}{\partial \phi^2}$$

Here, λ is a kinetic coefficient setting a characteristic timescale, and σ is the noise strength. The ensemble average of any function of the order parameter, $\mathcal{O}(\phi)$, is defined as $\langle \mathcal{O} \rangle = \int_{-\infty}^{\infty} \mathcal{O}(\phi) P(\phi, t) d\phi$. The system is driven by linearly sweeping the external field H in time t at a constant rate, such that $dH/dt = \lambda v_H$, where v_H is the dimensionless driving rate. Assume that the probability distribution $P(\phi, t)$ and its derivatives vanish at $\phi \to \pm\infty$.

Task: Derive the differential equation that governs the evolution of the ensemble-averaged order parameter $\langle \phi \rangle$ with respect to the external field H. Your final expression should relate $d\langle \phi \rangle / dH$ to $\langle \phi \rangle$, $\langle \phi^3 \rangle$, H, and the parameters of the model.

Refined Solution:

$$v_H \frac{d\langle \phi \rangle}{dH} = -a_2 \langle \phi \rangle - a_4 \langle \phi^3 \rangle + H$$

Problem 64 (Paper: 2506.24035v1, Index: 1)

Original Problem Statement

Background: Consider a system described by a stochastic order parameter ϕ whose dynamics are governed by a potential energy landscape that exhibits a first-order phase transition. Near a spinodal point $(H_{\text{sp}}, \phi_{\text{sp}})$, the dynamics of the deviation of the order parameter, $\varphi \equiv \phi - \phi_{\text{sp}}$, driven by a deviation in an external field, $h \equiv H - H_{\text{sp}}$, can be approximated. When the field is swept linearly with a driving rate v_H, such that a field increment is $\delta h = \lambda v_H \delta t$ for some timescale λ, the discretized Langevin equation for the increment $\delta\varphi$ is given by:

$$\delta\varphi = -a_3\varphi^2 v_H^{-1}\delta h + h v_H^{-1}\delta h + \sqrt{2\sigma^2 v_H^{-1}\delta h}\, W$$

Here, a_3 is a constant related to the cubic term of the potential, σ is the noise strength characterizing finite-size effects, and W is a random variable from a standard normal distribution.

To analyze the finite-size effects, one can employ a scaling analysis inspired by renormalization group theory. A scaling transformation is defined where the noise strength is rescaled by a factor $\Sigma > 0$ as $\sigma \to \sigma' = \sigma\Sigma^{-1}$. Consequently, any other physical quantity $\mathcal{O}$ transforms as $\mathcal{O} \to \mathcal{O}' = \mathcal{O}\Sigma^{-[\mathcal{O}]}$, where $[\mathcal{O}]$ is the scaling dimension of $\mathcal{O}$ with respect to the noise strength. The scaling dimensions are determined by two principles: 1. The form of the dynamical equation must remain invariant under this scaling transformation. 2. The scaling dimension of the coefficient of the highest-order nonlinear term in φ, which is a_3, is zero, i.e., $[a_3] = 0$.

This analysis allows one to determine how the properties of the system's coercivity plateau, which appears in the presence of noise, depend on the noise strength σ. The plateau is characterized by its height H_P and the characteristic driving rate v_P at which it is most prominent. The scaling dimensions for h and v_H directly yield the power-law dependence of the plateau's height deviation from the deterministic spinodal field, $H^* - H_P$, and the characteristic rate v_P on σ.

Task: By applying the scaling transformation to the provided dynamical equation and using the stated principles, determine the scaling dimensions $[h]$, $[\varphi]$, and $[v_H]$. Use these dimensions to derive the complete scaling relation that describes how the coercivity plateau height deviation, $H^* - H_P$, and the characteristic driving rate, v_P, depend on the noise strength σ.

Original Solution

$$H^* - H_P \sim \sigma^{4/3} \quad \text{at} \quad v_P \sim \sigma^2$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires applying a nontrivial scaling transformation to a stochastic nonlinear dynamical equation, determining scaling dimensions under constraints, and deriving power-law relations for physical observables, involving a multi-step synthesis of concepts from stochastic dynamics, scaling theory, and renormalization group principles.

Refined Problem

Refined Problem Statement: Background: Consider a system described by a stochastic order parameter ϕ whose dynamics are governed by a potential energy landscape that exhibits a first-order phase transition. Near a spinodal point $(H_{\text{sp}}, \phi_{\text{sp}})$, the dynamics of the deviation of the order parameter, $\varphi \equiv \phi - \phi_{\text{sp}}$, driven by a deviation in an external field, $h \equiv H - H_{\text{sp}}$, can be approximated. When the field is swept linearly with a driving rate v_H, such that a field increment is $\delta h = \lambda v_H \delta t$ for some timescale λ, the discretized Langevin equation for the increment $\delta\varphi$ is given by:

$$\delta\varphi = -a_3\varphi^2 v_H^{-1}\delta h + h v_H^{-1}\delta h + \sqrt{2\sigma^2 v_H^{-1}\delta h}\, W$$

Here, a_3 is a constant related to the cubic term of the potential, σ is the noise strength characterizing finite-size effects, and W is a random variable from a standard normal distribution.

To analyze the finite-size effects, one can employ a scaling analysis inspired by renormalization group theory. A scaling transformation is defined where the noise strength is rescaled by a factor $\Sigma > 0$ as $\sigma \to \sigma' = \sigma\Sigma^{-1}$. Consequently, any other physical quantity $\mathcal{O}$ transforms as $\mathcal{O} \to \mathcal{O}' = \mathcal{O}\Sigma^{-[\mathcal{O}]}$, where $[\mathcal{O}]$ is the scaling dimension of $\mathcal{O}$ with respect to the noise strength. The scaling dimensions are determined by two principles: 1. The form of the dynamical equation must remain invariant under this scaling transformation. 2. The scaling dimension of the coefficient of the highest-order nonlinear term in φ, which is a_3, is zero, i.e., $[a_3] = 0$.

This analysis allows one to determine how the properties of the system's coercivity plateau, which appears in the presence of noise, depend on the noise strength σ. The plateau is characterized by its height H_P and the characteristic driving rate v_P at which it is most prominent. The scaling dimensions for h and v_H directly yield the power-law dependence of the plateau's height deviation from the deterministic spinodal field, $H^* - H_P$, and the characteristic rate v_P on σ.

Task: By applying the scaling transformation to the provided dynamical equation and using the stated principles, determine the scaling dimensions $[h]$, $[\varphi]$, and $[v_H]$. Use these dimensions to derive the complete scaling relation that describes how the coercivity plateau height deviation, $H^* - H_P$, and the characteristic driving rate, v_P, depend on the noise strength σ.

Refined Solution:

$$H^* - H_P \sim \sigma^{4/3} \quad \text{at} \quad v_P \sim \sigma^2$$

Problem 65 (Paper: 2506.24064v1, Index: 0)

Original Problem Statement

Background: Consider a coarse-grained model for a double-stranded DNA molecule, known as the "b-model". In this framework, the state of the molecule is analyzed using a grand canonical ensemble, where the grand partition function $\mathcal{Z}(z) = \sum_{N=1}^{\infty} Z(N)z^N$ is defined in terms of the canonical partition function $Z(N)$ for a molecule of length N and a fugacity z for the number of base pairs. The thermodynamic behavior of the infinitely long molecule is governed by the dominant singularity of $\mathcal{Z}(z)$, which is the singularity z_{dom} closest to the origin in the complex z-plane. A phase transition occurs when the location of this dominant singularity changes as a function of system parameters like temperature.

In the b-model, the bound (zipped) phase allows for the formation of denatured bubbles. The free energy of this phase is determined by a singularity z_a. When an external force f is applied to pull the two strands apart, the molecule can enter an unzipped phase, where the two strands are separated and stretched. The free energy of this force-dominated state is determined by a different singularity, z_c. The phase boundary $f_c(T)$ between the zipped and unzipped phases is found by equating these two singularities, i.e., $z_a = z_c$.

The singularities for the two phases are given as follows: 1. For the bound phase with bubbles (b-model), the singularity is $z_a = \sqrt{\frac{X-1}{X}} - \frac{X-1}{X}$. Here, $X = e^{\beta\epsilon}$, where $\epsilon > 0$ is the binding energy per base pair and $\beta = 1/(k_B T)$ is the inverse temperature. 2. For the unzipped phase, where the two strands are stretched by a force f, the singularity is $z_c = [2(1 + \cosh(\beta f))]^{-1}$.

Task: Using the condition for the phase boundary, $z_a = z_c$, derive the expression for the critical unzipping force $f_c(T)$ as a function of temperature T and the binding energy ϵ.

Original Solution

$$f_c(T) = k_B T \cosh^{-1}\left(\frac{1}{2}\frac{1}{\sqrt{1 - e^{-\beta\epsilon}} - 1 + e^{-\beta\epsilon}} - 1\right)$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing the definitions of singularities from two distinct phases and solving a transcendental equation to derive the critical force expression, involving multiple steps and conceptual understanding beyond direct formula application.

Refined Problem

Refined Problem Statement: Background: Consider a coarse-grained model for a double-stranded DNA molecule, known as the "b-model". In this framework, the state of the molecule is analyzed using a grand canonical ensemble, where the grand partition function

$$\mathcal{Z}(z) = \sum_{N=1}^{\infty} Z(N)z^N$$

is defined in terms of the canonical partition function $Z(N)$ for a molecule of length N and a fugacity z for the number of base pairs. The thermodynamic behavior of the infinitely long molecule is governed by the dominant singularity of

$$\mathcal{Z}(z)$$

, which is the singularity z_{dom} closest to the origin in the complex z-plane. A phase transition occurs when the location of this dominant singularity changes as a function of system parameters like temperature.

In the b-model, the bound (zipped) phase allows for the formation of denatured bubbles. The free energy of this phase is determined by a singularity z_a. When an external force f is applied to pull the two strands apart, the molecule can enter an unzipped phase, where the two strands are separated and stretched. The free energy of this force-dominated state is determined by a different singularity, z_c. The phase boundary $f_c(T)$ between the zipped and unzipped phases is found by equating these two singularities, i.e., $z_a = z_c$.

The singularities for the two phases are given as follows: 1. For the bound phase with bubbles (b-model), the singularity is

$$z_a = \sqrt{\frac{X-1}{X}} - \frac{X-1}{X}$$

. Here, $X = e^{\beta\epsilon}$, where $\epsilon > 0$ is the binding energy per base pair and $\beta = 1/(k_B T)$ is the inverse temperature. 2. For the unzipped phase, where the two strands are stretched by a force f, the singularity is

$$z_c = [2(1 + \cosh(\beta f))]^{-1}$$

.

Task: Using the condition for the phase boundary, $z_a = z_c$, derive the expression for the critical unzipping force $f_c(T)$ as a function of temperature T and the binding energy ϵ.

Refined Solution:

$$f_c(T) = k_B T \cosh^{-1}\left(\frac{1}{2}\frac{1}{\sqrt{1-e^{-\beta\epsilon}} - 1 + e^{-\beta\epsilon}} - 1\right)$$

Problem 66 (Paper: 2506.24064v1, Index: 1)

Original Problem Statement

Background: Consider the force-induced unzipping of a double-stranded DNA molecule, which can be described as a first-order phase transition between a "zipped" (bound) phase and an "unzipped" (unbound) phase. The state of the system can be described by its Gibbs free energy G, which is a function of temperature T and an external pulling force f applied to the ends of the two strands. The differential of the free energy is given by $dG = -SdT - xdf$, where S is the entropy and x is the end-to-end separation of the strands. The transition occurs along a phase boundary in the (T, f) plane, denoted by the critical force $f_c(T)$, where the free energies of the two phases are equal: $G_z(T, f_c) = G_u(T, f_c)$, with subscripts z and u denoting the zipped and unzipped phases, respectively.

The following assumptions provide a thermodynamic model for this transition: 1. The zipped phase is considered to be perfectly rigid and inextensible with respect to the pulling force. Therefore, its end-to-end separation is zero, $x_z = 0$, for any force up to the critical force, $0 \leq f \leq f_c(T)$. 2. The unzipped phase consists of two separated polymer strands. Close to the zero-force melting temperature T_c, the critical force $f_c(T)$ is small. In this regime, the unzipped strands exhibit a linear elastic response to the force, such that their end-to-end separation is given by $x_u(T, f) = \chi_T f$. The coefficient χ_T is a temperature-dependent susceptibility.

Task: Based on these thermodynamic principles and assumptions, derive the expression for the entropy difference between the two phases, $\Delta S \equiv S_z(T, f_c) - S_u(T, f_c)$, evaluated at the phase boundary. Your final expression should be in terms of the susceptibility χ_T, the critical force $f_c(T)$, and its derivative with respect to temperature, $\frac{\partial f_c(T)}{\partial T}$.

Original Solution

$$\Delta S = \chi_T f_c(T) \frac{\partial f_c(T)}{\partial T}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing thermodynamic relations, phase equilibrium conditions, and elasticity concepts to derive an expression for entropy difference, involving multiple steps and implicit differentiation.

Refined Problem

Refined Problem Statement: Background: Consider the force-induced unzipping of a double-stranded DNA molecule, which can be described as a first-order phase transition between a "zipped" (bound) phase and an "unzipped" (unbound) phase. The state of the system can be described by its Gibbs free energy G, which is a function of temperature T and an external pulling force f applied to the ends of the two strands. The differential of the free energy is given by $dG = -SdT - xdf$, where S is the entropy and x is the end-to-end separation of the strands. The transition occurs along a phase boundary in the (T, f) plane, denoted by the critical force $f_c(T)$, where the free energies of the two phases are equal: $G_z(T, f_c) = G_u(T, f_c)$, with subscripts z and u denoting the zipped and unzipped phases, respectively.

The following assumptions provide a thermodynamic model for this transition: 1. The zipped phase is considered to be perfectly rigid and inextensible with respect to the pulling force. Therefore, its end-to-end separation is zero, $x_z = 0$, for any force up to the critical force, $0 \leq f \leq f_c(T)$. 2. The unzipped phase consists of two separated polymer strands. Close to the zero-force melting temperature T_c, the critical force $f_c(T)$ is small. In this regime, the unzipped strands exhibit a linear elastic response to the force, such that their end-to-end separation is given by $x_u(T, f) = \chi_T f$. The coefficient χ_T is a temperature-dependent susceptibility.

Task: Based on these thermodynamic principles and assumptions, derive the expression for the entropy difference between the two phases, $\Delta S \equiv S_z(T, f_c) - S_u(T, f_c)$, evaluated at the phase boundary. Your final expression should be in terms of the susceptibility χ_T, the critical force $f_c(T)$, and its derivative with respect to temperature, $\frac{\partial f_c(T)}{\partial T}$.

Refined Solution:

$$\Delta S = \chi_T f_c(T) \frac{\partial f_c(T)}{\partial T}$$

Problem 67 (Paper: 2506.24079v1, Index: 0)

Original Problem Statement

Background: Let $\mathcal{H}_A$ and $\mathcal{H}_{A'}$ be finite-dimensional Hilbert spaces for quantum systems A and A'. A quantum channel $\mathcal{N}_{A'\to A}$ is a completely positive, trace-preserving (CPTP) linear map from the set of operators on $\mathcal{H}_{A'}$ to the set of operators on $\mathcal{H}_A$. For a bipartite state $\psi_{RA'}$ on a composite system RA', where R is a reference system, the action of the channel is given by $({}_R\otimes\mathcal{N}_{A'\to A})(\psi_{RA'})$. The von Neumann entropy of a quantum state ρ is $S(\rho) = -[\rho\log\rho]$. For a bipartite state σ_{AB}, the conditional entropy is defined as $S(A|B)_\sigma = S(AB)_\sigma - S(B)_\sigma$. The entropy of a quantum channel $\mathcal{N}_{A'\to A}$ is defined as the minimum conditional entropy of the output, where the minimum is taken over all input states $\psi_{RA'}$ with a reference system R isomorphic to A': $S[\mathcal{N}] = \inf_{\psi\in(RA')} S(A|R)_{\mathcal{N}(\psi)}$, where (RA') is the set of density operators on $\mathcal{H}_R\otimes\mathcal{H}_{A'}$. A replacer channel, denoted $\mathcal{R}^\omega_{A'\to A}$, is a specific type of quantum channel that discards the input state and replaces it with a fixed output state $\omega_A \in (A)$, regardless of the input. That is, for any input state $\rho_{A'} \in (A')$, the output is $\mathcal{R}^\omega_{A'\to A}(\rho_{A'}) = \omega_A$.

Task: Using the definitions provided, derive a general expression for the entropy of a replacer channel, $S[\mathcal{R}^\omega]$. Your final result should express the channel entropy in terms of the entropy of the fixed output state ω.

Original Solution

$$S[\mathcal{R}^\omega] = S(\omega)$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Trivial**

The problem reduces to recognizing that the replacer channel outputs a fixed state regardless of input, so the channel entropy equals the entropy of that fixed state, requiring no complex derivation or multi-step reasoning.

Refined Problem

Refined Problem Statement: Background: Let $\mathcal{H}_A$ and $\mathcal{H}_{A'}$ be finite-dimensional Hilbert spaces for quantum systems A and A'. A quantum channel $\mathcal{N}_{A'\to A}$ is a completely positive, trace-preserving (CPTP) linear map from the set of operators on $\mathcal{H}_{A'}$ to the set of operators on $\mathcal{H}_A$. For a bipartite state $\psi_{RA'}$ on a composite system RA', where R is a reference system, the action of the channel is given by $(\mathrm{id}_R\otimes\mathcal{N}_{A'\to A})(\psi_{RA'})$. The von Neumann entropy of a quantum state ρ is $S(\rho) = -\mathrm{Tr}[\rho\log\rho]$. For a bipartite state σ_{AB}, the conditional entropy is defined as $S(A|B)_\sigma = S(AB)_\sigma - S(B)_\sigma$. The entropy of a quantum channel $\mathcal{N}_{A'\to A}$ is defined as the minimum conditional entropy of the output, where the minimum is taken over all input states $\psi_{RA'}$ with a reference system R isomorphic to A': $S[\mathcal{N}] = \inf_{\psi\in\mathrm{St}(RA')} S(A|R)_{\mathcal{N}(\psi)}$, where $\mathrm{St}(RA')$ is the set of density operators on $\mathcal{H}_R\otimes\mathcal{H}_{A'}$. A replacer channel, denoted $\mathcal{R}^\omega_{A'\to A}$, is a specific type of quantum channel that discards the input state and replaces it with a fixed output state $\omega_A \in \mathrm{St}(A)$, regardless of the input. That is, for any input state $\rho_{A'} \in \mathrm{St}(A')$, the output is $\mathcal{R}^\omega_{A'\to A}(\rho_{A'}) = \omega_A$.

Task: Using the definitions provided, derive a general expression for the entropy of a replacer channel, $S[\mathcal{R}^\omega]$. Your final result should express the channel entropy in terms of the entropy of the fixed output state ω.

Refined Solution:

$$S[\mathcal{R}^\omega] = S(\omega)$$

Problem 68 (Paper: 2506.24079v1, Index: 1)

Original Problem Statement

Background: A quantum process is described by a quantum channel, which is a completely positive, trace-preserving (CPTP) linear map $\mathcal{N}_{A'\to A}$ acting on density operators. Consider a quantum system A with a corresponding bounded Hamiltonian $\widehat{H}_A$. The mean energy of a channel $\mathcal{N}_{A'\to A}$ is defined as the maximum possible mean energy of its output states:

$$\langle \widehat{H} \rangle_{\mathcal{N}} := \sup_{\rho \in (A')} [\widehat{H}_A \mathcal{N}(\rho_{A'})]$$

where (A') is the set of all density operators on the input Hilbert space $\mathcal{H}_{A'}$.

The entropy of a quantum channel, $S[\mathcal{N}]$, is a measure of the uncertainty it generates. It is related to the von Neumann entropy $S(\sigma) = -[\sigma \log \sigma]$ of its possible output states by the following properties: 1. For any channel $\mathcal{N}_{A'\to A}$, its entropy is bounded by the minimum entropy of any of its output states:

$$S[\mathcal{N}] \leq \inf_{\rho \in (A')} S(\mathcal{N}(\rho_{A'}))$$

2. A special class of channels are the "replacer channels" $\mathcal{R}^{\omega}_{A'\to A}$, which produce a fixed output state $\omega_A \in (A)$ regardless of the input state, i.e., $\mathcal{N}(\rho_{A'}) = \omega_A$ for all $\rho \in (A')$. For these channels, the inequality above is saturated, and the channel entropy is simply the entropy of the fixed output state, $S[\mathcal{R}^{\omega}] = S(\omega_A)$. The mean energy of a replacer channel is $\langle \widehat{H} \rangle_{\mathcal{R}^{\omega}} = [\widehat{H}_A \omega_A]$.

Furthermore, it is a known principle that for any given mean energy E, the state $\sigma_A \in (A)$ that maximizes the entropy $S(\sigma_A)$ subject to the constraint $[\widehat{H}_A \sigma_A] = E$ is the unique thermal state γ^{β}, defined as:

$$\gamma^{\beta} = \frac{\exp(-\beta \widehat{H}_A)}{[\exp(-\beta \widehat{H}_A)]}$$

where the parameter β (inverse temperature) is chosen such that $\langle \widehat{H} \rangle_{\gamma^{\beta}} = [\widehat{H}_A \gamma^{\beta}] = E$.

Task: Consider the set of all quantum channels $\mathcal{N}_{A'\to A}$ that have a fixed mean energy $\langle \widehat{H} \rangle_{\mathcal{N}} = E$. Using the principles and definitions provided, determine the maximum possible value of the channel entropy $S[\mathcal{N}]$ for this set of channels. Express your answer in terms of the von Neumann entropy of the thermal state γ^{β} corresponding to the mean energy E.

Original Solution

$$S(\gamma^{\beta})$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing concepts of quantum channel entropy, energy constraints, and the characterization of thermal states to derive a maximal entropy condition, involving a multi-step reasoning beyond direct formula application.

Refined Problem

Refined Problem Statement: Background: A quantum process is described by a quantum channel, which is a completely positive, trace-preserving (CPTP) linear map $\mathcal{N}_{A'\to A}$ acting on density operators. Consider a quantum system A with a corresponding bounded Hamiltonian $\widehat{H}_A$. The mean energy of a channel $\mathcal{N}_{A'\to A}$ is defined as the maximum possible mean energy of its output states:

$$\langle \widehat{H} \rangle_{\mathcal{N}} := \sup_{\rho \in (A')} [\widehat{H}_A \mathcal{N}(\rho_{A'})]$$

where (A') is the set of all density operators on the input Hilbert space $\mathcal{H}_{A'}$.

The entropy of a quantum channel, $S[\mathcal{N}]$, is a measure of the uncertainty it generates. It is related to the von Neumann entropy $S(\sigma) = -[\sigma \log \sigma]$ of its possible output states by the following properties:
1. For any channel $\mathcal{N}_{A' \to A}$, its entropy is bounded by the minimum entropy of any of its output states:

$$S[\mathcal{N}] \leq \inf_{\rho \in (A')} S(\mathcal{N}(\rho_{A'}))$$

2. A special class of channels are the "replacer channels" $\mathcal{R}^{\omega}_{A' \to A}$, which produce a fixed output state $\omega_A \in (A)$ regardless of the input state, i.e., $\mathcal{N}(\rho_{A'}) = \omega_A$ for all $\rho \in (A')$. For these channels, the inequality above is saturated, and the channel entropy is simply the entropy of the fixed output state, $S[\mathcal{R}^{\omega}] = S(\omega_A)$. The mean energy of a replacer channel is $\langle \widehat{H} \rangle_{\mathcal{R}^{\omega}} = [\widehat{H}_A \omega_A]$.

Furthermore, it is a known principle that for any given mean energy E, the state $\sigma_A \in (A)$ that maximizes the entropy $S(\sigma_A)$ subject to the constraint $[\widehat{H}_A \sigma_A] = E$ is the unique thermal state γ^{β}, defined as:

$$\gamma^{\beta} = \frac{\exp(-\beta \widehat{H}_A)}{[\exp(-\beta \widehat{H}_A)]}$$

where the parameter β (inverse temperature) is chosen such that $\langle \widehat{H} \rangle_{\gamma^{\beta}} = [\widehat{H}_A \gamma^{\beta}] = E$.

Task: Consider the set of all quantum channels $\mathcal{N}_{A' \to A}$ that have a fixed mean energy $\langle \widehat{H} \rangle_{\mathcal{N}} = E$. Using the principles and definitions provided, determine the maximum possible value of the channel entropy $S[\mathcal{N}]$ for this set of channels. Express your answer in terms of the von Neumann entropy of the thermal state γ^{β} corresponding to the mean energy E.

Refined Solution:

$$S(\gamma^{\beta})$$

Problem 69 (Paper: 2506.24097v1, Index: 0)

Original Problem Statement

Background: Consider a one-dimensional quantum circuit on an infinite lattice of qubits. The dynamics over one time step are described by a unitary propagator U, and the corresponding Heisenberg super-operator is $\mathcal{U}(A)U^\dagger AU$. The system is translationally invariant by s sites. Let $\mathcal{S}$ be the super-operator for a 1-site translation to the right. The Heisenberg propagator commutes with translations, i.e., $\mathcal{U}\mathcal{S}^j = \mathcal{S}^j\mathcal{U}$. We define a basis of "extensive observables" with a well-defined quasi-momentum $k \in [0, 2\pi)$. These are constructed from "local densities" $\{b\}$, which are operators with finite support. The basis elements for extensive observables are labeled by an intra-cell position $m \in \{0, \ldots, s-1\}$ and a local density b, and are defined as $B_k^{(m,b)} \sum_{j=-\infty}^{\infty} -kj\mathcal{S}^{sj+m}(b)$. We define two Hilbert-Schmidt inner products. The "local" inner product for local operators a, b' is $ab' \lim_{N\to\infty} \frac{1}{2^N}(a^\dagger b')$. This inner product is translationally invariant: $\mathcal{S}^j(a)\mathcal{S}^j(b') = ab'$. The "extensive" inner product for extensive observables A, B is defined as $AB \lim_{N\to\infty} \frac{s}{N} AB$. The basis $\{B_k^{(m,b)}\}$ is orthonormal with respect to this extensive inner product, i.e., $B_k^{(m,b)} B_k^{(m',b')} = \delta_{m,m'}\delta_{b,b'}$.

Task: The dynamics of extensive observables are described by the propagator of extensive observables, $\mathcal{U}_k$, which is the Heisenberg propagator $\mathcal{U}$ represented in the basis of extensive observables $\{B_k^{(m,b)}\}$. Find the general expression for the matrix elements $[\mathcal{U}_k]_{(m,b),(m',b')}\, B_k^{(m,b)}\mathcal{U}\left(B_k^{(m',b')}\right)$ in terms of the local inner product and local densities.

Original Solution

$$[\mathcal{U}_k]_{(m,b),(m',b')} = \sum_j kj\mathcal{S}^{sj+m}(b)\mathcal{U}\left(\mathcal{S}^{m'}(b')\right)$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing the definitions of extensive observables, translational invariance, and two types of inner products to derive a non-trivial expression for the matrix elements of the propagator in momentum space.

Refined Problem

Refined Problem Statement: Background: Consider a one-dimensional quantum circuit on an infinite lattice of qubits. The dynamics over one time step are described by a unitary propagator U, and the corresponding Heisenberg super-operator is $\mathcal{U}(A)U^\dagger AU$. The system is translationally invariant by s sites. Let $\mathcal{S}$ be the super-operator for a 1-site translation to the right. The Heisenberg propagator commutes with translations, i.e., $\mathcal{U}\mathcal{S}^j = \mathcal{S}^j\mathcal{U}$. We define a basis of "extensive observables" with a well-defined quasi-momentum $k \in [0, 2\pi)$. These are constructed from "local densities" $\{b\}$, which are operators with finite support. The basis elements for extensive observables are labeled by an intra-cell position $m \in \{0, \ldots, s-1\}$ and a local density b, and are defined as $B_k^{(m,b)} \sum_{j=-\infty}^{\infty} e^{-ikj}\mathcal{S}^{sj+m}(b)$. We define two Hilbert-Schmidt inner products. The "local" inner product for local operators a, b' is $\langle a, b'\rangle \lim_{N\to\infty} \frac{1}{2^N}\mathrm{tr}(a^\dagger b')$. This inner product is translationally invariant: $\langle \mathcal{S}^j(a), \mathcal{S}^j(b')\rangle = \langle a, b'\rangle$. The "extensive" inner product for extensive observables A, B is defined as $\langle\langle A, B\rangle\rangle \lim_{N\to\infty} \frac{s}{N}\langle A, B\rangle$. The basis $\{B_k^{(m,b)}\}$ is orthonormal with respect to this extensive inner product, i.e., $\langle\langle B_k^{(m,b)}, B_k^{(m',b')}\rangle\rangle = \delta_{m,m'}\delta_{b,b'}$.

Task: The dynamics of extensive observables are described by the propagator of extensive observables, $\mathcal{U}_k$, which is the Heisenberg propagator $\mathcal{U}$ represented in the basis of extensive observables $\{B_k^{(m,b)}\}$. Find the general expression for the matrix elements $[\mathcal{U}_k]_{(m,b),(m',b')}\, \langle\langle B_k^{(m,b)}, \mathcal{U}\left(B_k^{(m',b')}\right)\rangle\rangle$ in terms of the local inner product and local densities.

Refined Solution:

$$[\mathcal{U}_k]_{(m,b),(m',b')} = \sum_j e^{ikj} \langle \mathcal{S}^{sj+m}(b), \mathcal{U}\left(\mathcal{S}^{m'}(b')\right) \rangle$$

Problem 70 (Paper: 2506.24097v1, Index: 1)

Original Problem Statement

Background: Consider a one-dimensional many-body system in a statistical equilibrium state, characterized by an expectation value ... . The system possesses a single conserved quantity, whose local density is given by the operator $\sigma^z(x,t)$ at position x and time t. The dynamics of this density is governed by a continuity equation,

$$\partial_t \sigma^z(x,t) = -\partial_x j(x,t), \tag{1}$$

where $j(x,t)$ is the corresponding local current density. On large spatio-temporal scales, the transport of the conserved quantity is diffusive. This is reflected in the equilibrium auto-correlation function of the local density, which is given by the Green's function of the diffusion equation:

$$\sigma^z(x,t)\sigma^z(0,0) = \frac{1}{\sqrt{4\pi D t}} - \frac{x^2}{4Dt}, \tag{2}$$

where D is the diffusion constant. Assume that the equilibrium state is time-translationally and space-translationally invariant.

Task: Using the provided relations for the diffusive dynamics of the conserved density, derive an expression for the equilibrium auto-correlation function of the local current, $j(x,t)j(0,0)$.

Original Solution

$$\frac{x^2 - 2Dt}{8\sqrt{\pi D}t^{5/2}} - \frac{x^2}{4Dt}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires deriving the current auto-correlation function from the density correlation using the continuity equation and properties of diffusion, involving multiple steps of differentiation and understanding of stochastic hydrodynamics.

Refined Problem

Refined Problem Statement: Background: Consider a one-dimensional many-body system in a statistical equilibrium state, characterized by an expectation value ... . The system possesses a single conserved quantity, whose local density is given by the operator $\sigma^z(x,t)$ at position x and time t. The dynamics of this density is governed by a continuity equation,

$$\partial_t \sigma^z(x,t) = -\partial_x j(x,t), \tag{3}$$

where $j(x,t)$ is the corresponding local current density. On large spatio-temporal scales, the transport of the conserved quantity is diffusive. This is reflected in the equilibrium auto-correlation function of the local density, which is given by the Green's function of the diffusion equation:

$$\sigma^z(x,t)\sigma^z(0,0) = \frac{1}{\sqrt{4\pi D t}} - \frac{x^2}{4Dt}, \tag{4}$$

where D is the diffusion constant. Assume that the equilibrium state is time-translationally and space-translationally invariant.

Task: Using the provided relations for the diffusive dynamics of the conserved density, derive an expression for the equilibrium auto-correlation function of the local current, $j(x,t)j(0,0)$.

Solution:

Refined Solution:

$$\frac{x^2 - 2Dt}{8\sqrt{\pi D}t^{5/2}} - \frac{x^2}{4Dt}$$

Problem 71 (Paper: 2506.24115v1, Index: 0)

Original Problem Statement

Background: Consider a 2D lattice model realizing the $D(S_3)$ topological phase, which is enriched by a global $\mathbb{Z}_2$ symmetry. The symmetry group of the underlying gauge theory is $S_3 = \{1, r, r^2, s, sr, rs\}$, with relations $r^3 = s^2 = (sr)^2 = 1$. The system possesses two distinct symmetry sectors, labeled + and −, which are interchanged by the action of the non-trivial $\mathbb{Z}_2$ symmetry generator, denoted $\mathcal{G}_{em}$. The excitations of this model are anyons. We are interested in the non-Abelian anyon of type D. While the anyon type D as a whole is invariant under the symmetry, its internal degrees of freedom transform non-trivially. The internal Hilbert space of a D-type anyon is 3-dimensional, spanned by an orthonormal basis $\{D_p\}$, where the label p is a "flux type" from the set $\{s, rs, sr\}$. A complete basis state for a D-anyon in a given sector is denoted D_p, σ, where $p \in \{s, rs, sr\}$ and $\sigma \in \{+, -\}$.

The action of the symmetry operator $\mathcal{G}_{em}$ on the basis states of a D-anyon in the + sector is given by a 3×3 unitary matrix $\rho^D(\mathcal{G}_{em})$:

$$\mathcal{G}_{em} D_p, + = \sum_{q \in \{s, rs, sr\}} [\rho^D(\mathcal{G}_{em})]_{qp} D_q, -$$

Physically, this transformation is realized when an anyon crosses a domain wall separating the + and − sectors. There are two types of domain walls, α and β. The transformation $\mathcal{G}_{em}$ is defined by the process of an anyon crossing an α-type domain wall. The amplitudes for these processes are given by the half-braiding tensors, or z-tensors, of the form $z_{a,b,c}^{(D,D);k}$. Here, a and b are the initial and final internal states of the anyon (e.g., s_{++} denotes state $D_s, +$), and c and k are the initial and final domain wall types (e.g., α_{+-} denotes an α-wall separating the + sector on the left from the − sector on the right). The matrix element $[\rho^D(\mathcal{G}_{em})]_{qp}$ is given by the z-tensor for the process where an anyon in state p_{++} crosses an α_{+-} domain wall and emerges in state q_{--}. Crucially, for the D-type anyon, this specific transformation process also changes the domain wall from type α to type β.

The non-zero values of the relevant z-tensors for the (D, D) anyon interacting with domain walls are given below. Note that p_{++} indicates a flux p in the + sector, q_{--} indicates a flux q in the − sector, and so on.

$$z_{s_{++},s_{--},\alpha_{+-}}^{(D,D);\beta_{+-}} = \frac{1}{\sqrt{3}} \qquad z_{s_{++},s_{--},\beta_{+-}}^{(D,D);\alpha_{+-}} = \frac{1}{\sqrt{3}}$$

$$z_{s_{++},rs_{--},\alpha_{+-}}^{(D,D);\beta_{+-}} = \frac{1}{\sqrt{3}} \qquad z_{s_{++},rs_{--},\beta_{+-}}^{(D,D);\alpha_{+-}} = \frac{1}{\sqrt{3}}$$

$$z_{s_{++},sr_{--},\alpha_{+-}}^{(D,D);\beta_{+-}} = \frac{1}{\sqrt{3}} \qquad z_{s_{++},sr_{--},\beta_{+-}}^{(D,D);\alpha_{+-}} = \frac{1}{\sqrt{3}}$$

$$z_{rs_{++},s_{--},\alpha_{+-}}^{(D,D);\beta_{+-}} = \frac{1}{\sqrt{3}} \qquad z_{rs_{++},s_{--},\beta_{+-}}^{(D,D);\alpha_{+-}} = \frac{1}{\sqrt{3}}$$

$$z_{rs_{++},rs_{--},\alpha_{+-}}^{(D,D);\beta_{+-}} = \frac{e^{i\frac{2\pi}{3}}}{\sqrt{3}} \qquad z_{rs_{++},rs_{--},\beta_{+-}}^{(D,D);\alpha_{+-}} = \frac{e^{-i\frac{2\pi}{3}}}{\sqrt{3}}$$

$$z_{rs_{++},sr_{--},\alpha_{+-}}^{(D,D);\beta_{+-}} = \frac{e^{-i\frac{2\pi}{3}}}{\sqrt{3}} \qquad z_{rs_{++},sr_{--},\beta_{+-}}^{(D,D);\alpha_{+-}} = \frac{e^{i\frac{2\pi}{3}}}{\sqrt{3}}$$

$$z_{sr_{++},s_{--},\alpha_{+-}}^{(D,D);\beta_{+-}} = \frac{1}{\sqrt{3}} \qquad z_{sr_{++},s_{--},\beta_{+-}}^{(D,D);\alpha_{+-}} = \frac{1}{\sqrt{3}}$$

$$z_{sr_{++},rs_{--},\alpha_{+-}}^{(D,D);\beta_{+-}} = \frac{e^{-i\frac{2\pi}{3}}}{\sqrt{3}} \qquad z_{sr_{++},rs_{--},\beta_{+-}}^{(D,D);\alpha_{+-}} = \frac{e^{i\frac{2\pi}{3}}}{\sqrt{3}}$$

$$z_{sr_{++},sr_{--},\alpha_{+-}}^{(D,D);\beta_{+-}} = \frac{e^{i\frac{2\pi}{3}}}{\sqrt{3}} \qquad z_{sr_{++},sr_{--},\beta_{+-}}^{(D,D);\alpha_{+-}} = \frac{e^{-i\frac{2\pi}{3}}}{\sqrt{3}}$$

Additional non-zero tensors exist for the reverse process (crossing from − to +), such as $z_{s_{--},s_{++},\alpha_{-+}}^{(D,D);\beta_{-+}}$, but are not listed here.

Task: Using the basis ordering (s, rs, sr) for the rows (final states q) and columns (initial states p), determine the transformation matrix $\rho^D(\mathcal{G}_{em})$.

Original Solution

$$\rho^D(\mathcal{G}_{em}) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{2\pi}{3}} & e^{-i\frac{2\pi}{3}} \\ 1 & e^{-i\frac{2\pi}{3}} & e^{i\frac{2\pi}{3}} \end{pmatrix}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing the abstract algebraic structure of the symmetry group, the anyon fusion and braiding data, and the half-braiding tensors to explicitly construct a non-trivial unitary transformation matrix, involving careful indexing and phase factors, thus demanding a multi-step, advanced reasoning process.

Refined Problem

Refined Problem Statement: Background: Consider a 2D lattice model realizing the $D(S_3)$ topological phase, which is enriched by a global $\mathbb{Z}_2$ symmetry. The symmetry group of the underlying gauge theory is $S_3 = \{1, r, r^2, s, sr, rs\}$, with relations $r^3 = s^2 = (sr)^2 = 1$. The system possesses two distinct symmetry sectors, labeled $+$ and $-$, which are interchanged by the action of the non-trivial $\mathbb{Z}_2$ symmetry generator, denoted $\mathcal{G}_{em}$. The excitations of this model are anyons. We are interested in the non-Abelian anyon of type D. While the anyon type D as a whole is invariant under the symmetry, its internal degrees of freedom transform non-trivially. The internal Hilbert space of a D-type anyon is 3-dimensional, spanned by an orthonormal basis $\{D_p\}$, where the label p is a "flux type" from the set $\{s, rs, sr\}$. A complete basis state for a D-anyon in a given sector is denoted D_p, σ, where $p \in \{s, rs, sr\}$ and $\sigma \in \{+, -\}$.

The action of the symmetry operator $\mathcal{G}_{em}$ on the basis states of a D-anyon in the $+$ sector is given by a 3×3 unitary matrix $\rho^D(\mathcal{G}_{em})$:

$$\mathcal{G}_{em} D_p, + = \sum_{q \in \{s,rs,sr\}} [\rho^D(\mathcal{G}_{em})]_{qp} D_q, -$$

Physically, this transformation is realized when an anyon crosses a domain wall separating the $+$ and $-$ sectors. There are two types of domain walls, α and β. The transformation $\mathcal{G}_{em}$ is defined by the process of an anyon crossing an α-type domain wall. The amplitudes for these processes are given by the half-braiding tensors, or z-tensors, of the form $z_{a,b,c}^{(D,D);k}$. Here, a and b are the initial and final internal states of the anyon (e.g., s_{++} denotes state $D_s, +$), and c and k are the initial and final domain wall types (e.g., α_{+-} denotes an α-wall separating the $+$ sector on the left from the $-$ sector on the right). The matrix element $[\rho^D(\mathcal{G}_{em})]_{qp}$ is given by the z-tensor for the process where an anyon in state p_{++} crosses an α_{+-} domain wall and emerges in state q_{--}. Crucially, for the D-type anyon, this specific transformation process also changes the domain wall from type α to type β.

The non-zero values of the relevant z-tensors for the (D, D) anyon interacting with domain walls are given below. Note that p_{++} indicates a flux p in the $+$ sector, q_{--} indicates a flux q in the $-$ sector,

and so on.

$$z^{(D,D);\beta_{+-}}_{s_{++},s_{--},\alpha_{+-}} = \frac{1}{\sqrt{3}} \qquad z^{(D,D);\alpha_{+-}}_{s_{++},s_{--},\beta_{+-}} = \frac{1}{\sqrt{3}}$$

$$z^{(D,D);\beta_{+-}}_{s_{++},rs_{--},\alpha_{+-}} = \frac{1}{\sqrt{3}} \qquad z^{(D,D);\alpha_{+-}}_{s_{++},rs_{--},\beta_{+-}} = \frac{1}{\sqrt{3}}$$

$$z^{(D,D);\beta_{+-}}_{s_{++},sr_{--},\alpha_{+-}} = \frac{1}{\sqrt{3}} \qquad z^{(D,D);\alpha_{+-}}_{s_{++},sr_{--},\beta_{+-}} = \frac{1}{\sqrt{3}}$$

$$z^{(D,D);\beta_{+-}}_{rs_{++},s_{--},\alpha_{+-}} = \frac{1}{\sqrt{3}} \qquad z^{(D,D);\alpha_{+-}}_{rs_{++},s_{--},\beta_{+-}} = \frac{1}{\sqrt{3}}$$

$$z^{(D,D);\beta_{+-}}_{rs_{++},rs_{--},\alpha_{+-}} = \frac{e^{i\frac{2\pi}{3}}}{\sqrt{3}} \qquad z^{(D,D);\alpha_{+-}}_{rs_{++},rs_{--},\beta_{+-}} = \frac{e^{-i\frac{2\pi}{3}}}{\sqrt{3}}$$

$$z^{(D,D);\beta_{+-}}_{rs_{++},sr_{--},\alpha_{+-}} = \frac{e^{-i\frac{2\pi}{3}}}{\sqrt{3}} \qquad z^{(D,D);\alpha_{+-}}_{rs_{++},sr_{--},\beta_{+-}} = \frac{e^{i\frac{2\pi}{3}}}{\sqrt{3}}$$

$$z^{(D,D);\beta_{+-}}_{sr_{++},s_{--},\alpha_{+-}} = \frac{1}{\sqrt{3}} \qquad z^{(D,D);\alpha_{+-}}_{sr_{++},s_{--},\beta_{+-}} = \frac{1}{\sqrt{3}}$$

$$z^{(D,D);\beta_{+-}}_{sr_{++},rs_{--},\alpha_{+-}} = \frac{e^{-i\frac{2\pi}{3}}}{\sqrt{3}} \qquad z^{(D,D);\alpha_{+-}}_{sr_{++},rs_{--},\beta_{+-}} = \frac{e^{i\frac{2\pi}{3}}}{\sqrt{3}}$$

$$z^{(D,D);\beta_{+-}}_{sr_{++},sr_{--},\alpha_{+-}} = \frac{e^{i\frac{2\pi}{3}}}{\sqrt{3}} \qquad z^{(D,D);\alpha_{+-}}_{sr_{++},sr_{--},\beta_{+-}} = \frac{e^{-i\frac{2\pi}{3}}}{\sqrt{3}}$$

Additional non-zero tensors exist for the reverse process (crossing from $-$ to $+$), such as $z^{(D,D);\beta_{-+}}_{s_{--},s_{++},\alpha_{-+}}$, but are not listed here.

Task: Using the basis ordering (s, rs, sr) for the rows (final states q) and columns (initial states p), determine the transformation matrix $\rho^D(\mathcal{G}_{em})$.

Refined Solution:

$$\rho^D(\mathcal{G}_{em}) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{i\frac{2\pi}{3}} & e^{-i\frac{2\pi}{3}} \\ 1 & e^{-i\frac{2\pi}{3}} & e^{i\frac{2\pi}{3}} \end{pmatrix}$$

Problem 72 (Paper: 2506.24115v1, Index: 1)

Original Problem Statement

Background: Consider a symmetry-enriched topological (SET) phase featuring non-Abelian anyons. One such anyon, of type H, possesses a two-dimensional internal Hilbert space. A basis for this space is given by the states $\{H_r, H_{r^2}\}$, where the labels r and r^2 are elements of the group S_3 satisfying $r^3 = 1$. The system is endowed with a global $\mathbb{Z}_2$ symmetry, whose non-trivial generator is denoted by $\mathcal{G}_{em}$. The action of this symmetry on the internal basis states of the H anyon is given by a matrix representation $\rho^H(\mathcal{G}_{em})$. In the basis (H_r, H_{r^2}), the transformation is:

$$\mathcal{G}_{em} H_r = e^{-i\frac{2\pi}{3}} H_{r^2}$$
$$\mathcal{G}_{em} H_{r^2} = e^{-i\frac{2\pi}{3}} H_r$$

Due to the categorical nature of the symmetry in this topological phase, the composition of two successive symmetry transformations, $\mathcal{G}_{em} \circ \mathcal{G}_{em}$, does not correspond to a simple matrix multiplication of their representations. Instead, the matrix representation of the composite transformation, which we denote $\rho^H(\mathcal{G}_{em} \circ \mathcal{G}_{em})$, is defined by a nonlinear composition rule. Its matrix elements are given by:

$$(\rho^H(\mathcal{G}_{em} \circ \mathcal{G}_{em}))_{ac} = \sum_{b \in \{r, r^2\}} \omega_{ab} (\rho^H(\mathcal{G}_{em}))_{ab} (\rho^H(\mathcal{G}_{em}))_{bc}$$

where $a, b, c \in \{r, r^2\}$ are the basis-state labels. The rule involves a rank-2 tensor ω, which encodes the effects of the underlying lattice geometry (Pachner moves). The relevant non-zero components of this tensor for the internal space of the H anyon are:

$$\omega_{r,r} = e^{i\frac{2\pi}{3}}$$
$$\omega_{r^2,r^2} = e^{i\frac{2\pi}{3}}$$
$$\omega_{r,r^2} = \omega_{r^2,r} = e^{-i\frac{2\pi}{3}}$$

The composition of two $\mathcal{G}_{em}$ transformations is equivalent to the identity transformation, $\mathcal{G}_{em} \circ \mathcal{G}_{em} = 1$, whose representation is the identity matrix $\rho^H(1) =$.

Task: Using the provided nonlinear composition rule and the representation of a single symmetry transformation, derive the matrix representation for the composite transformation $\mathcal{G}_{em} \circ \mathcal{G}_{em}$ acting on the internal space of the H-type anyon.

Original Solution

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Critiques

Self-Containment Critique: **Status: Self-contained**

Difficulty Critique: **Status: Non-trivial**

The problem requires synthesizing a nonlinear composition rule involving a tensor and matrix representations, demanding multi-step algebraic manipulation beyond direct application of standard group representation theory.

Refined Problem

Refined Problem Statement: Background: Consider a symmetry-enriched topological (SET) phase featuring non-Abelian anyons. One such anyon, of type H, possesses a two-dimensional internal Hilbert space. A basis for this space is given by the states $\{H_r, H_{r^2}\}$, where the labels r and r^2 are elements of the group S_3 satisfying $r^3 = 1$. The system is endowed with a global $\mathbb{Z}_2$ symmetry, whose non-trivial generator is denoted by $\mathcal{G}_{em}$. The action of this symmetry on the internal basis states of the H anyon is given by a matrix representation $\rho^H(\mathcal{G}_{em})$. In the basis (H_r, H_{r^2}), the transformation is:

$$\mathcal{G}_{em} H_r = e^{-i\frac{2\pi}{3}} H_{r^2}$$
$$\mathcal{G}_{em} H_{r^2} = e^{-i\frac{2\pi}{3}} H_r$$

Due to the categorical nature of the symmetry in this topological phase, the composition of two successive symmetry transformations, $\mathcal{G}_{em} \circ \mathcal{G}_{em}$, does not correspond to a simple matrix multiplication of their representations. Instead, the matrix representation of the composite transformation, which we denote $\rho^H(\mathcal{G}_{em} \circ \mathcal{G}_{em})$, is defined by a nonlinear composition rule. Its matrix elements are given by:

$$(\rho^H(\mathcal{G}_{em} \circ \mathcal{G}_{em}))_{ac} = \sum_{b \in \{r, r^2\}} \omega_{ab} (\rho^H(\mathcal{G}_{em}))_{ab} (\rho^H(\mathcal{G}_{em}))_{bc}$$

where $a, b, c \in \{r, r^2\}$ are the basis-state labels. The rule involves a rank-2 tensor ω, which encodes the effects of the underlying lattice geometry (Pachner moves). The relevant non-zero components of this tensor for the internal space of the H anyon are:

$$\begin{aligned} \omega_{r,r} &= e^{i\frac{2\pi}{3}} \\ \omega_{r^2,r^2} &= e^{i\frac{2\pi}{3}} \\ \omega_{r,r^2} &= \omega_{r^2,r} = e^{-i\frac{2\pi}{3}} \end{aligned}$$

The composition of two $\mathcal{G}_{em}$ transformations is equivalent to the identity transformation, $\mathcal{G}_{em} \circ \mathcal{G}_{em} = 1$, whose representation is the identity matrix $\rho^H(1) =$.

Task: Using the provided nonlinear composition rule and the representation of a single symmetry transformation, derive the matrix representation for the composite transformation $\mathcal{G}_{em} \circ \mathcal{G}_{em}$ acting on the internal space of the H-type anyon.

Refined Solution:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$