1 Order (Section 1.2)

Let's begin by looking at some different sequences. We'll be using **orders** to compare those sequences.

Definition 1.1: Big-$\mathcal{O}$

Let x_m and α_m be two sequences. We say that $x_m = \mathcal{O}(\alpha_m)$ if, for $m \mapsto \infty$, we have $\frac{x_m}{\alpha_m} \leq C$ for some constant C.

(Example.) Suppose $x_m = \frac{1}{m} + \frac{1}{m^2}$ and $\alpha_m = \frac{1}{m}$. Then,

$$\frac{x_m}{\alpha_m} = 1 + \frac{1}{m} \leq C$$

for $m \mapsto \infty$ and $C \in \mathbb{R}$. Therefore, we say that

$$x_m = \mathcal{O}\left(\frac{1}{m}\right) = \mathcal{O}(\alpha_m).$$

Definition 1.2: Little-o

Let x_m and α_m be two sequences, both tending to 0 as $m \mapsto \infty$. We say that $x_m = o(\alpha_m)$ if, for $m \mapsto \infty$, we have $\frac{x_m}{\alpha_m} \mapsto 0$.

Remarks:

- If something is little-o, then it will also be Big-$\mathcal{O}$.
- If $x_n \mapsto 0$ and $\alpha_n \mapsto 0$ and $x_n = \mathcal{O}(\alpha_n)$, then x_n converges to 0 at least as rapidly as α_n does. If $x_n = o(\alpha_n)$, then x_n converges to 0 more rapidly than α_n.

(Example.) Let $x_m = \frac{1}{m}$ and $\alpha_m = \frac{1}{\ln(m)}$. Then, as $m \mapsto \infty$, we have

$$\frac{x_m}{\alpha_m} = \frac{\ln(m)}{m} \mapsto 0.$$

Then, we can say

$$x_m = o\left(\frac{1}{\ln(m)}\right) = o(\alpha_m).$$

(Example.)

- $\frac{m+1}{m^2} = \frac{1}{m} + \frac{1}{m^2} = \mathcal{O}\left(\frac{1}{m}\right)$.
- $\frac{m+1}{\sqrt{m}} = \mathcal{O}(\sqrt{m})$.
- $\frac{1}{m\ln(m)} = o\left(\frac{1}{m}\right)$.

1.1 Functions

There are analogous definitions of Big-$\mathcal{O}$ and little-o for functions.

Definition 1.3: Big-$\mathcal{O}$ & Little-o

For functions $f(x)$, $g(x)$,

- we say that $f(x) = \mathcal{O}(g(x))$ if $\frac{f(x)}{g(x)} \leq C$ as $x \mapsto x^*$ and for some constant C.
- we say that $f(x) = o(g(x))$ if $\frac{f(x)}{g(x)} \mapsto 0$ as $x \mapsto x^*$.

(Example.) We know that the Taylor Series

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \dots \\ &= 1 + x + \dots + \mathcal{O}(x^2) \end{aligned}$$

for $|x| < 1$. Likewise,

$$\frac{1}{1-x} = 1 + x + o(x).$$

Here,

- In the first equation, the $\mathcal{O}(x^2)$ means that the remaining terms have order x^2.
- Likewise, $o(x^2)$ means that the remaining terms tend to 0 faster than x^2.

1.2 Polynomials

We know that polynomials can approximate functions (like we've seen through Taylor Series). That is, the polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_mx^m$$

can be used to approximate functions. Note that

$$\begin{aligned} p(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_mx^m \\ &= a_0 + x(a_1 + a_2x + a_3x^2 + \dots + a_mx^{m-1}) \\ &= a_0 + x(a_1 + x(a_2 + a_3x + \dots + a_mx^{m-2})) \end{aligned}$$

Suppose we want to evaluate this polynomial at some x. Using this process, known as **Horner's Method**, we can compute a polynomial at some x.

```
p = a_m
for i = m - 1 : 0
    p = px + a_i
end
```

This algorithm uses one loop, and does not store anything else (i.e., does not use any intermediate values). In terms of flop count, this is $\mathcal{O}(m)$ flops.

1.3 Mean-Value Theorem

Theorem 1.1: Mean-Value Theorem

For $\xi \in [a, b]$ and $f \in C^1$,

$$f(b) - f(a) = f'(\xi)(b - a).$$

1.4 Function Representation

Functions can be explicit or implicit. Generally, we'll consider functions in the explicit form,

$$y = f(x).$$

An implicit form may look like[1]

$$G(x, y) = 0.$$

(Example.)

- $y = x^2$ is a simple explicit function.
- $y^2 + x^2 = 0$ is a simple implicit function.

Essentially, an explicit function has one variable whereas an implicit function has two variables.

Theorem 1.2: Implicit Function Theorem

Let G be a function of two real variables defined and continuously differentiable in a neighborhood of (x_0, y_0). If $G(x_0, y_0) = 0$ and $\frac{\delta G}{\delta y} \neq 0$ at (x_0, y_0), then there is a continuously differentiable function f defined such that $f(x_0) = y_0$ and $G(x, f(x)) = 0$.

(Example.) Suppose $x^2+y^2 = 2$. Then, $G = x^2+y^2-2$ and we also know that, for example, $G(1, 1) = 0$. Then, $x_0 = y_0 = 1$ and

$$\frac{\partial G}{\partial y}(x_0, y_0) = 2y_0 = 2 \neq 0.$$

Then, there is an implicit function y around $(1, 1)$. We know that $G = 0$ and $y^2 = 2 - x^2 \implies y_1, y_2 = \pm\sqrt{2 - x^2}$, which means $x_0 = 1$.

[1] If we have $G(x, y) = C$ for some constant C, then we can subtract C on both sides to get $G(x, y) - C = 0$.