1 Distributions and Densities

1.1 Types of Continuous Probability Distributions

The following are some of the most important continuous probability distributions, some of which we've seen before.

- Uniform
- Exponential
- Gamma
- Normal
- Cauchy

1.1.1 Uniform Distribution

Definition 1.1: Uniform Distribution

U has **Uniform**$[a, b]$ distribution, for $a < b$, if its PDF is

$$f(u) = \frac{1}{b-a}$$

for $a \leq u \leq b$.

Remark: Note that $b - a$ is the *length* of $[a, b]$.
Note: There are also higher-dimensional uniform distributions, but then we replace length with area of volume.

1.1.2 Exponential Distribution

Definition 1.2: Exponential Distribution

X is Exponential(λ) with rate $\lambda > 0$ if its PDF is

$$f(x) = \lambda e^{-\lambda x}$$

for $x > 0$.

Note that there is an important connection between the Exponential and the Poisson, which we will now describe.

(Example: Busy Server.) Suppose that a single server queue (e.g. call center, bank, etc.) is very busy, so that there is always someone in the queue. Suppose that service times are independent and Exponential(λ)[a]. As soon as someone has been served, the next person in the queue starts being served immediately. Let $X_1, X_2, \ldots$ be an IID sequence of Exponential(λ) random variables. Then, the time T_n at which the point the nth person has been served is distributed as

$$\sum_{i=1}^{n} X_i.$$

This sum of n IID Exponential(λ) random variables has a special distribution, called the **Gamma**(n, λ) distribution. This has PDF[b]

$$g(x) = \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!}$$

for $x > 0$.

[a]As you wait longer, the time that you wait "decays," or decreases.
[b]When $n = 1$, this reduces to $\lambda e^{-\lambda x}$, the PDF of a single Exponential(λ).

The Poisson distribution arises when we ask: What is the distribution for the number N_t of people served by time $t > 0$? At any given time $t > 0$, we will (with probability 1, since the service times are continuous) be in the middle of serving someone. This person does not count towards N_t.

Note that $(N_t : t > 0)$ is called the **Poisson process**[1]. This is a *collection*, indexed by time, of random variables[2] In particular, the random variable N_t has the Poisson(λt) distribution.

Proof. Note that $N_t = k$ if and only if the kth person is served at some time $T_k = s \le t$, and then the next service $X_{k+1} > t - s$. In other words, we need to have finished serving k people and be in the middle of serving the $(k+1)$th person. Since

$$T_k = \sum_{i=1}^{k} X_i$$

and X_{k+1} are independent, it follows that

$$\mathbb{P}(N_t = k) = \int_0^t f_{T_k}(s)[1 - F_{X_{k+1}}(t-s)]ds.$$

Since T_k is Gamma(k, λ) and X_{k+1} is Exponential(λ), it follows that

$$\begin{aligned}
\mathbb{P}(N_t = k) &= \int_0^t f_{T_k}(s)[1 - F_{X_{k+1}}(t-s)]ds \\
&= \int_0^t \lambda e^{-\lambda s} \frac{(\lambda s)^{k-1}}{(k-1)!} \cdot e^{-\lambda(t-s)} ds \\
&= e^{-\lambda t} \frac{\lambda^k}{(k-1)!} \int_0^t s^{k-1} ds \\
&= e^{-\lambda t} \frac{(\lambda t)^k}{k!}.
\end{aligned}$$

Hence, this is the PMF of a Poisson, as claimed. □

1.1.3 Normal/Gaussian Distribution

Recall that a Binomial converges to a Poisson if

$$p = \lambda/n \mapsto 0$$

as

$$n \mapsto \infty.$$

On the other hand, if p is *fixed* (not converging to 0), the Binomial approaches a different distribution as $n \mapsto \infty$ called the **Normal** or **Gaussian** distribution. Indeed, as n goes to infinity, we see a bell-shaped curve.

[1]This is a fascinating mathematical object with many properties and applications, which won't be covered here.
[2]Such an object is called a **stochastic process**.

Definition 1.3: Normal Distribution

X is **Normal**(μ, σ^2) if its PDF is

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

for $-\infty < x < \infty$.

Here, μ is the "center" of the density and σ is the measure of the "spread" of the density.
When $\mu = 0$ and $\sigma^2 = 1$, X is called a standard normal, and its PDF is given the special notation

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

If X is not standard, then you can "standardize" by taking $Z = (X - \mu)/\sigma$. Then, X is Normal(μ, σ^2) if and only if Z is Normal$(0, 1)$.

1.1.4 Cauchy Distribution

Now, suppose that X and Y are two independent standard Normal random variables. A very interesting distribution arises if we consider the ratio

$$Z = X/Y.$$

Since X and Y are independent,

$$f_Z(z) = \int_{S_z} f_{X,Y}(x, y) dx dy,$$

where $S_z = \{\{(x, y) \mid x/y = z\}$. We make a change of variables $x = uz$ and $y = u$. Then, as u varies over $\mathbb{R}$, we get the whole set S_z. The Jacobian of this transformation is $|u|$, so

$$f_Z(z) = \int_{S_z} f_{X,Y}(x, y) dx dy = \int_{-\infty}^{\infty} |u| f_{X,Y}(uz, u) du.$$

This is the same as

$$2 \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_0^\infty u e^{-(uz)^2/2 - u^2/2} du.$$

It can be shown that

$$\int_0^\infty x e^{-cx^2} dx = \frac{1}{2c}.$$

Hence,

$$f_Z(z) = \frac{1}{\pi} \int_0^\infty u e^{-\frac{u^2(1+z^2)}{2}} du = \frac{1}{\pi(1 + z^2)}.$$

A random variable with this PDF is called a (standard) **Cauchy** random variable.

Note that the Cauchy distribution has some interesting properties. In partivular, it has no expected/average value. So, if you take an IID sequence $X_1, X_2, \ldots$ of Cauchy random variables, the limit

$$\lim_{n \mapsto \infty} \frac{1}{n} \sum_{i=1}^{n} X_i$$

does not exist.