1 Reduced QR (3.4)

Let's begin with an example from a few sections ago. Suppose we have the following **full QR decomposition**

$$\underbrace{\begin{bmatrix} -1 & -1 & 1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{bmatrix}}_{A\in\mathbb{R}^{4\times 4}} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 2 & 8 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{2} \underbrace{\begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{Q\in\mathbb{R}^{4\times 4}} \underbrace{\begin{bmatrix} 2 & 4 & 2 \\ 0 & 2 & 8 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}}_{R\in\mathbb{R}^{4\times 3}}.$$

Here, Q is orthogonal and R is a tall matrix. Let's look at R. Notice how the last row of R are just 0's. In particular, the last column of the matrix Q and the last row of R yields 0's everywhere; it's not helpful. So, what if we throw away the last row of R and corresponding columns of Q? This brings us to the topic of **reduced QR**. In particular,

$$\underbrace{\begin{bmatrix} -1 & -1 & 1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{bmatrix}}_{A\in\mathbb{R}^{4\times 4}} = \frac{1}{2} \underbrace{\begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}_{\hat{Q}\in\mathbb{R}^{4\times 3}} \underbrace{\begin{bmatrix} 2 & 4 & 2 \\ 0 & 2 & 8 \\ 0 & 0 & 4 \end{bmatrix}}_{\hat{R}\in\mathbb{R}^{3\times 3}}.$$

Remark: $\hat{Q}$ is not a square matrix anymore; it's a tall matrix. The concept of orthogonal matrices does not make sense here anymore. Instead, note that $\hat{Q}$ is an **isometry**;

$$\underbrace{\hat{Q}^T}_{3\times 4} \underbrace{\hat{Q}}_{4\times 3} = \underbrace{I}_{3\times 3}.$$

(Compare this to orthogonal, where we have $Q^TQ = QQ^T = I$.)

Theorem 1.1: Reduced QR

Suppose $A \in \mathbb{R}^{n\times m}$ such that $n \geq m$. Then, there exists a $\hat{Q} \in \mathbb{R}^{n\times m}$ isometry and $\hat{R} \in \mathbb{R}^{m\times m}$ upper-triangular such that

$$A = \hat{Q}\hat{R}.$$

Remark: The reduced QR decomposition is unique if $\text{rank}(A) = m$ and we choose $r_{ii} > 0$ (entry on diagonal of $\hat{R}$).

1.1 Orthonormal Set

Before we talk about how to obtain the reduced QR decomposition, we first introduce orthonormal sets.

Definition 1.1: Orthonormal Set

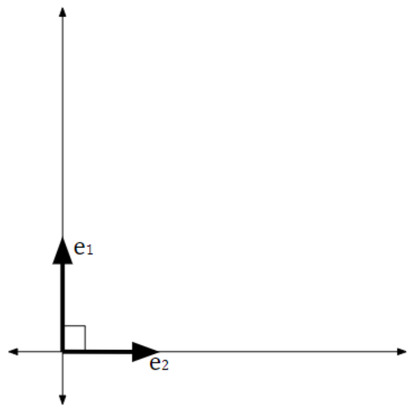
We say that[a]

$$\{q_1, q_2, \ldots, q_m\}$$

is called **orthonormal** if $\langle q_i, q_j\rangle = 0$ whenever $i \neq j$ and $\langle q_i, q_i\rangle = 1$.

[a] Note that q_i is a *vector*.

Remark: If Q is orthogonal (isometry), then the columns are orthonormal. For example, if the set $\{e_1, e_2\}$ is orthonormal, then this might visually look like



1.2 Gram-Schmidt

With the idea of orthonormal sets in mind, the idea is to use the **Gram-Schmidt** algorithm to make the columns of A into an orthonormal set $\{q_1, q_2, \ldots, q_m\}$. This represents $\hat{Q}$.

Notationally, assuming A has full rank (i.e., linearly independent), we can say that

$$\{a_1, a_2, \ldots, a_m\}$$

represents the columns of A.

1.2.1 Classical Algorithm

Given A, we want to find $\hat{Q}$ and $\hat{R}$ such that $A = \hat{Q}\hat{R}$. As mentioned above, we can write A as a set of linearly independent columns,

$$\begin{bmatrix} a_1, a_2, \ldots, a_m \end{bmatrix}.$$

We can also write $\hat{Q}$ in the same way:

$$\begin{bmatrix} q_1, q_2, \ldots, q_m \end{bmatrix}.$$

We can write $\hat{R}$ like so:

$$\begin{bmatrix} r_{11} & r_{12} & \ldots & r_{1m} \\ 0 & r_{22} & \ldots & r_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & r_{mm} \end{bmatrix}.$$

Combining this, we end up with

$$\begin{bmatrix} a_1, a_2, \ldots, a_m \end{bmatrix} = \begin{bmatrix} q_1, q_2, \ldots, q_m \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \ldots & r_{1m} \\ 0 & r_{22} & \ldots & r_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & r_{mm} \end{bmatrix}.$$

Then, we note that

$$a_1 = q_1 r_{11} \implies q_1 = \frac{a_1}{r_{11}} \implies r_{11} = ||a_1||_2.$$

$$a_2 = q_1 r_{12} + q_2 r_{22}.$$

$$a_3 = q_1 r_{13} + q_2 r_{23} + q_3 r_{33}.$$

Eventually, we'll end up with

$$a_m = q_1 r_{1m} + q_2 r_{2m} + \ldots + q_m r_{mm}.$$

So, this is the basic idea: processing each column one at a time. Writing this out as steps, we have:

1. $r_{11} = ||a_1||_2$, $q_1 = \frac{a_1}{r_{11}} = \frac{a_1}{||a_1||_2}$. It follows that
$$||q_1||_2 = 1.$$
2. $a_2 = r_{12}q_1 + r_{22}q_2$. Then, we can multiply q_1 on both sides:
$$\begin{aligned}\langle a_2, q_1\rangle &= \langle r_{12}q_1 + r_{22}q_2, q_1\rangle \\ &= r_{12}\underbrace{\langle q_1, q_1\rangle}_{1} + r_{22}\underbrace{\langle q_2, q_1\rangle}_{0} \\ &= r_{12}.\end{aligned}$$
Note that we got the 0 and 1 from the properties of orthonormal sets. In any case, it follows that
$$q_2 = \frac{a_2 - r_{12}q_1}{r_{22}}.$$
Setting $r_{22} = ||a_2 - r_{12}q_1||_2$, it follows that $||q_2||_2 = 1$.
3. We start from a_3 and determine q_3 and r_{13}, r_{23}, and r_{33}.

Notice that we essentially keep going like this. Let's try to generalize this. The formula for $\hat{R}$ is given by

$$\hat{R} = (r_{ji})$$

for $j < i$. Then,

$$\hat{Q} = \begin{bmatrix} q_1 & q_2 & \ldots & q_m \end{bmatrix}$$

and we can get $A = \hat{Q}\hat{R}$. Remember that

$$r_{12} = \langle a_2, q_1\rangle.$$

Note that the 1 in r index corresponds to the 1 in q_1 and the 2 in the r index corresponds to the 2 in a_2. We also know that

$$r_{22} = ||a_2 - r_{12}q_1||_2$$

and

$$q_2 = \frac{a_2 - r_{12}q_1}{r_{22}}.$$

Analogously, notice that

$$r_{13} = \langle a_3, q_1\rangle$$

and

$$r_{23} = \langle a_3, q_2\rangle.$$

We also know that

$$a_{33} = ||a_3 - r_{13}q_1 - r_{23}q_2||_2$$

and

$$q_3 = \frac{a_3 - \sum_{j=1}^{2} r_{j3}q_j}{r_{33}}.$$

<u>So, to conclude, we can generalize the formula:</u>

$$r_{ii} = \left|\left| a_i - \sum_{j=1}^{i-1} r_{ji}q_j \right|\right|_2.$$

$$r_{ji} = \langle a_i, q_j\rangle \qquad j < i.$$

$$q_i = \frac{a_i - \sum_{j=1}^{i-1} r_{ji}q_j}{r_{ii}}.$$