

1 Ring Homomorphism

Ring homomorphism is very similar in nature to group homomorphisms. Here, a ring homomorphism preserves the ring operations.

Definition 1.1: Ring Homomorphism

A **ring homomorphism** φ from a ring R to a ring S is a mapping from R to S that preserves the ring operation. That is, for all $a, b \in R$:

$$\varphi(a + b) = \varphi(a) + \varphi(b) \quad \varphi(ab) = \varphi(a)\varphi(b)$$

Remark: As is the case for groups, the operations on the left of the equal signs are those of R , while the operations on the right side are those of S .

Along with ring homomorphisms, there is also ring isomorphisms.

Definition 1.2: Ring Isomorphism

A **ring isomorphism** is a ring homomorphism that is both one-to-one and onto (i.e. bijective).

1.1 Properties of Ring Homomorphisms

Theorem 1.1

Let φ be a ring homomorphism from a ring R to a ring S , and let A be a subring of R and let B be an ideal of S .

1. For any $r \in R$ and any positive integer n , $\varphi(nr) = n\varphi(r)$ and $\varphi(r^n) = (\varphi(r))^n$.
2. $\varphi(A) = \{\varphi(a) \mid a \in A\}$ is a subring of S .
3. If A is an ideal and φ is onto S , then $\varphi(A)$ is an ideal.
4. $\varphi^{-1}(B) = \{r \in R \mid \varphi(r) \in B\}$ is an ideal of R .
5. If R is commutative, then $\varphi(R)$ is commutative.
6. If R has a unity 1 , $S \neq \{0\}$, and φ is onto, then $\varphi(1)$ is the unity of S .
7. φ is an isomorphism if and only if φ is onto and $\ker(\varphi) = \{r \in R \mid \varphi(r) = 0\} = \{0\}$.
8. If φ is an isomorphism from R onto S , then φ^{-1} is an isomorphism from S onto R .

1.2 Examples of Ring Homomorphism

Here are some examples of ring homomorphisms.

1.2.1 Example 1: Integers and Modulo

Consider the mapping:

$$k \mapsto k \pmod{n}$$

This is a ring homomorphism from \mathbb{Z} onto \mathbb{Z}_n , and is called the natural homomorphism from \mathbb{Z} to \mathbb{Z}_n .

1.2.2 Example 2: Complex Numbers

Consider the mapping:

$$a + bi \mapsto a - bi$$

This is a ring homomorphism from the complex numbers onto the complex numbers.

1.2.3 Example 3: Functions

Consider the ring of all polynomials with real coefficients $\mathbb{R}[x]$. Consider the mapping:

$$f(x) \mapsto f(1)$$

This is a ring homomorphism from $\mathbb{R}[x]$ onto \mathbb{R} .