1 The Power Method (5.3)

Let $A \in \mathbb{C}^{n\times n}$, and assume that A is semisimple. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues associated with $v_1, \ldots, v_n$, respectively. Assume that the vectors are ordered so that $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$. If $|\lambda_1| > |\lambda_2|$, then λ_1 is called the **dominant eigenvalue**[1] and v_1 is called the **dominant eigenvector** of A.

1.1 The Iterative Power Method

Assuming we have $|\lambda_1| > |\lambda_2|$ as described above (otherwise, this method may not work), the general idea behind the iterative power method is that we can pick $q \in \mathbb{R}^n$ randomly. Then, we can form the sequence of vectors

$$q, Aq, A^2q, A^3q, \ldots.$$

To calculate this sequence, we don't necessarily need to form the powers of A explicitly. Each vector in the sequence can be obtained by multiplying the previous vector by A, e.g., $A^{j+1}q = A(A^jq)$. It's easy to show that the sequence converge, in a sense, to a dominant eigenvector, for almost all choices of q. Since $v_1, \ldots, v_n$ form a basis for $\mathbb{C}^n$, there exists constants $c_1, \ldots, c_n$ such that

$$q = c_1v_1 + c_2v_2 + \ldots + c_nv_n.$$

We don't know what $v_1, \ldots, v_n$ are, so we don't know what $c_1, \ldots, c_n$ are, either. However, it's clear that, for any choice of q, c_1 will be nonzero. The argument that follows is valid for every q for which $c_1 \neq 0$; multiplying by A, we have

$$\begin{aligned} Aq &= c_1Av_1 + c_2Ac_2 + \ldots + c_nAv_n \\ &= c_1\lambda_1v_1 + c_2\lambda_2v_2 + \ldots + c_n\lambda_nv_n. \end{aligned}$$

Similarly,

$$\begin{aligned} A^2q &= A(c_1\lambda_1v_1 + c_2\lambda_2v_2 + \ldots + c_n\lambda_nv_n) \\ &= c_1\lambda_1(Av_1) + c_2\lambda_2(Av_2) + \ldots + c_n\lambda_n(Av_n) \\ &= c_1\lambda_1(\lambda_1v_1) + c_2\lambda_2(\lambda_2v_2) + \ldots + c_n\lambda_n(\lambda_nv_n) \\ &= c_1\lambda_1^2v_1 + c_2\lambda_2^2v_2 + \ldots + c_n\lambda_n^2v_n. \end{aligned}$$

In general, we have

$$\begin{aligned} A^jq &= c_1\lambda_1^jv_1 + c_2\lambda_2^jv_2 + \ldots + c_n\lambda_n^jv_n \\ &= \lambda_1^j\left(c_1v_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^jv_2 + \ldots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^jv_n\right). \end{aligned}$$

In particular, we have

$$\frac{1}{\lambda_1^j}A^iq = c_1v_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^jv_2 + \ldots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^jv_n.$$

Notice that $\lim_{j\mapsto\infty}\left(\frac{\lambda_i}{\lambda_1}\right)^j = 0$, so

$$\lim_{j\mapsto\infty}\frac{1}{\lambda_1^j}A^jq = c_1v_1,$$

the dominant eigenvector.

Remark: This only works if λ_1 is known.

[1] Basically, the largest absolute eigenvalue.