1 Triangular Systems (1.3)

Generally, it's common practice to reduce general systems down to a triangular form, generally through a process known as Gaussian elimination. They are also easy to solve, and can be solved inexpensively.

1.1 Lower and Upper Triangular Matrices

We say that $L \in \mathbb{R}^{n\times n}$ is a **lower triangular** matrix if $\ell_{ij} = 0$ whenever $i < j$. Thus, a lower triangular matrix has the form

$$A = \begin{bmatrix} \ell_{11} & 0 & 0 & 0 & 0 & \dots & 0 \\ \ell_{21} & \ell_{22} & 0 & 0 & 0 & \dots & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} & 0 & 0 & \dots & 0 \\ \ell_{41} & \ell_{42} & \ell_{43} & \ell_{44} & 0 & \dots & 0 \\ \ell_{51} & \ell_{52} & \ell_{53} & \ell_{54} & \ell_{55} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & \ell_{n4} & \ell_{n5} & \dots & \ell_{nn} \end{bmatrix}.$$

Similarly, we say that $U \in \mathbb{R}^{n\times n}$ is an **upper triangular** matrix if $u_{ij} = 0$ whenever $i > j$; they have the form

$$A = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} & u_{15} & \dots & u_{1n} \\ 0 & u_{22} & u_{23} & u_{24} & u_{25} & \dots & u_{2n} \\ 0 & 0 & u_{33} & u_{34} & u_{35} & \dots & u_{3n} \\ 0 & 0 & 0 & u_{44} & u_{45} & \dots & u_{4n} \\ 0 & 0 & 0 & 0 & u_{55} & \dots & u_{5n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & u_{nn} \end{bmatrix}.$$

A **triangular matrix** is one that is either upper or lower triangular.

(Example.) Consider the following matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & -1 & 0 \\ -2 & 0 & 3 \end{bmatrix}.$$

This is a **lower triangular matrix**.

Remarks:

- We can have 0's in the places where there are normally nonzero numbers. This does not violate the definition of a lower- or upper-triangular matrix.
- Because of this, we can say that a diagonal matrix is both a lower- and upper-triangular matrix.

1.2 Uniqueness of Solution

When is there a unique system to $L\mathbf{x} = \mathbf{b}$ or $U\mathbf{x} = \mathbf{b}$?

A triangular system (lower or upper) has a unique solution if and only if *all diagonal entries* are non-zero.

In fact, as long as the *determinant* of the triangular matrix A is not 0, there will be a unique solution. With triangular matrices, computing the determinant is easy: we just need to multiply all the elements on the *diagonal* together. More technically, as long as

$$\det(A) = a_{11} \cdot a_{22} \cdot a_{33} \cdot \dots \cdot a_{nn} \neq 0,$$

then $A\mathbf{x} = \mathbf{b}$ will be a unique solution.

1.3 Solving Lower Triangular Systems: Forward Substitution

Suppose we want to solve the following system

$$\begin{bmatrix} \ell_{11} & 0 & 0 & \dots & 0 \\ \ell_{21} & \ell_{22} & 0 & \dots & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & \dots & \ell_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}.$$

Notice that

$$\ell_{11}x_1 = b_1 \implies x_1 = \frac{b_1}{\ell_{11}}.$$

Also notice that

$$\ell_{21}x_1 + \ell_{22}x_2 = b_2 \implies \ell_{22}x_2 = b_2 - \ell_{21}x_1 \implies x_2 = \frac{b_2 - \ell_{21}x_1}{\ell_{22}}.$$

And then notice that

$$\ell_{31}x_1 + \ell_{32}x_2 + \ell_{33}x_3 = b_3 \implies \ell_{33}x_3 = b_3 - \ell_{31}x_1 - \ell_{32}x_2 \implies x_3 = \frac{b_3 - \ell_{31}x_1 - \ell_{32}x_2}{\ell_{33}}.$$

Notice how we started off with a simple linear equation, which gave us the answer for x_1, and then we can use x_1 to find x_2 easily in the next equation, and so on. This algorithm is known as **forward substitution**. For $i = 1, \dots, n$, it makes use of the formula

$$\boxed{x_i = \frac{b_i - \ell_{i1}x_1 - \ell_{i2}x_2 - \dots - \ell_{i,i-1}x_{i-1}}{\ell_{ii}} = \frac{b_i - \sum_{j=1}^{i-1} \ell_{ij}x_j}{\ell_{ii}}.}$$

Roughly speaking, the algorithm looks like the following:

for $i = 1, \dots, n$ **do**
  for $j = 1, \dots, i-1$ **do**
    $x_i \leftarrow x_i - \ell_{ij}x_j$
  end for
  if $g_{ii} = 0$ **then**
    Set Error Flag, Exit
  end if
  $x_i \leftarrow x_i / \ell_{ii}$
end for