1 Sums of Random Variables

We will now work towards the Law of Large Numbers and the Central Limit Theorem. Before we do this, we need to first talk about sums of random variables.

1.1 Discrete Case

Theorem 1.1

Suppose that X and Y are independent discrete random variables with PMFs p_X and p_Y. Then, the PMF of their sum $X+Y$ is the **convolution** of p_X and p_Y. That is,

$$p_{X+Y}(z) = \sum_x p_X(x) p_Y(z-x).$$

Remark: We want to find the probability that $X+Y=z$. To do this, we can take the sum of all possible values X can take. Then, X will take on some value and Y will take the rest of the value $z-x$.

More generally, if $X_1, \dots, X_n$ are independent, then the PMF for their sum

$$S_n = \sum_{i=1}^n X_i$$

is the n**-fold convolution**

$$p_{S_n}(z) = \sum_{x_1+\cdots+x_n=z} \left(\prod_{i=1}^n p_i(x_i) \right).$$

Alternatively, we note that (this is often useful for an induction proof)

$$p_{S_n}(z) = \sum_x p_{S_{n-1}} p_n(z-x)$$

is the convolution of $p_{S_{n-1}}$ and p_n.

(Example.) Let $X_1, X_2, \dots$ be the result of independent dice rolls. Let S_2 be the sum of the first *two* rolls. To find $\{S_2 = 5\}$, we note that

Roll 1 (x)	Roll 2 ($5-x$)
1	4
2	3
3	2
4	1

Then,

$$\mathbb{P}(S_2 = 5) = \sum_x p_1(x) p_2(5-x) = \sum_{x=1}^4 \frac{1}{6}\frac{1}{6} = \frac{4}{36} = \frac{1}{9}.$$

(Example.) Let's suppose that we now want to find $\{S_3 = 4\}$. There are two ways to do this.

- Approach 1: We note that

$$\mathbb{P}(S_3 = 4) = \sum_{x_1+x_2+x_3=4} \frac{1}{6^3} = \frac{3}{6^3},$$

since the only possibilities are $\{112, 121, 211\}$.

- Approach 2: We also note that

$$\begin{aligned}
\mathbb{P}(S_3 = 4) &= \sum_x \mathbb{P}(S_2 = x)\mathbb{P}(X_3 = 4 - x) \\
&= \sum_{x=2}^{3} \mathbb{P}(S_2 = x)\mathbb{P}(X_3 = 4 - x) \\
&= \frac{1}{6^2}\frac{1}{6} + \frac{2}{6^2}\frac{1}{6} \\
&= \frac{3}{6^3}.
\end{aligned}$$

To see how we got this, note that S_2 represents the sum of the first two rolls. The minimum value S_2 can take is 2 (since the minimum value each die has is 1). The maximum value S_2 can take is 3 (since we need to account for the third roll as well). So, we have:

Roll 1 & 2 ($S_2 = x$)	Roll 3 ($4 - x$)
2	2
3	1

(Example.) Recall the convolution of k independent Geometric RVs is a Negative Binomial RV (the number of trials until the kth "success.") What is the convolution of two independent Binomial RVs with the same probability parameter p?

Recall that a Binomial random variable with parameters n and p is the distribution of the number of successes in a sequence of n independent experiments, where each experiment is a Bernoulli trial.

If X is a Binomial random variable with parameters n and p, then we can represent it like

$$X = B_1 + B_2 + \cdots + B_n.$$

Likewise, if Y is a Binomial random variable with parameters m and p, then

$$Y = B_1' + B_2' + \cdots + B_m'.$$

Thus, the convolution is given by

$$X + Y = B_1 + B_2 + \cdots + B_n + B_1' + B_2' + \cdots + B_m'.$$

Notice that this is also a Binomial random variable with parameters $n + m$ and p.

1.2 Continuous Case

The continuous case is very similar to the discrete case, except we make use of integration.

Theorem 1.2

Suppose that X and Y are **independent** continuous RVs with PDFs f_X and f_Y. Then, the PDF of their sum $X+Y$ is the **convolution** of f_X and f_Y. That is,

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx.$$

We can generalize this to sums

$$S_n = \sum_{i=1}^{n} X_i$$

of independent RVs, as before.

(Example.) Recall the example on the sum $S = M + N$ of two independent Uniform[0, 1] RVs. We found that

$$f(s) = \begin{cases} s & s \in [0,1] \\ 2-s & s \in (1,2] \\ 0 & \text{Otherwise} \end{cases}.$$

It is somewhat easier, although essentially equivalent, to do this with convolutions. To do this, note that

$$f(s) = \int_{-\infty}^{\infty} f_M(u) f_N(s-u) du.$$

Note that $f_M(u)f_N(s-u) = 1$ if and only if $0 \leq u, s-u \leq 1$ if and only if $u \in [0,1] \cap [s-1, s]$. Therefore,

$$f(s) = \min\{1, s\} - \max\{0, s-1\} = \begin{cases} s & s \in [0,1] \\ 2-s & s \in (1,2] \\ 0 & \text{Otherwise} \end{cases},$$

as expected.

1.3 Normal Random Variables

The sum of independent Normal RVs is still Normal. Moreover, we add the means and add the variances.

Theorem 1.3

Suppose that $X_1, \dots, X_n$ are independent Normal RVs with means μ_i and variances σ_i^2. Then, their sum

$$S_n = \sum_{i=1}^{n} X_i$$

is normal with mean

$$\mu = \sum_{i=1}^{n} \mu_i$$

and

$$\sigma^2 = \sum_{i=1}^{n} \sigma_i^2.$$

Remark: Note that S_n having this sum and variance comes from LoE and the fact that they are independent (so we can add the variances).