1 The Quotient Field

Theorem 1.1

If D is an integral domain (a ring with no zero divisors; it's a ring with multiplicative cancellation), then there exists a field $\mathbb{F}$ that contains D as a subring.

Here are some examples:

1. Consider $\mathbb{Z} \subseteq \mathbb{Q}$. $\mathbb{Z}$ is an integral domain while $\mathbb{Q}$ is a field. We know that the integers look like:

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

   We can define the rationals like so:

$$\mathbb{Q} := \left\{ \frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\} \right\}$$

Definition 1.1

If D is an integral domain, we can define:

$$S = \{(a, b) \mid a, b \in D, b \neq 0\}$$

We can define an equivalence relation on S by $(a, b) \sim (c, d)$ if and only if $ad = bc$. Then, we can write $F = \frac{S}{\sim}$ and:

$$\frac{a}{b} = [(a, b)] = \{(c, d) \in S \mid (a, b) \sim (c, d)\}$$

Here, we use the operation:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$
$$\frac{a}{b}\frac{c}{d} = \frac{ac}{bd}$$

We call F the field of fractions or the field of quotients of D.

Remarks:

- In $\mathbb{Q}$, we know that $\frac{2}{4} = \frac{1}{2}$. So, we can say that $(2, 4)$ is "equal" to $(1, 2)$.
- The idea is that $\frac{a}{b} = cd \iff ad = bc$.

1.1 Equivalence Relation

We say that $(a, b) \sim (c, d)$ if and only if $ad = bc$.

- <u>Reflexive:</u> $(a, b) \sim (a, b)$ because $ab = ba$ as D is commutative.
- <u>Symmetric:</u> $(a, b) \sim (c, d) \implies ad = bc \implies cb = da \implies (c, d) \sim (a, b)$.
- <u>Transitive:</u> $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f) \implies ad = bc$ and $cf = de$. Then, $adf = bcf \implies adf = bde \implies daf = dbe \implies af = be$ since D is an integral domain. This tells us that $(a, b) \sim (e, f)$ as expected.

Thus, this equivalence relation is well-defined as a set.

1.1.1 Addition Well-Defined

Note that $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$. Remember that, depending on the representation (a, b) of $\frac{a}{b}$, we might get the same values. For example, $\frac{1}{2} = \frac{2}{4}$. So, suppose $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$. Then:

$$\begin{aligned}
(ad+bc)(b'd') &= adb'd' + bcb'd' \\
&= (ab')dd' + (cd')bb' && \text{Ring is commutative} \\
&= (a'b)dd' + (c'd)(bb') && \text{By the equivalence relation} \\
&= (a'd' + c'b')(bd)
\end{aligned}$$

Thus, $\frac{ad+bc}{bd} = \frac{a'd'+c'b'}{b'd'}$. Finally, if $\frac{a}{b}, \frac{c}{d} \in F$, then $b, d \neq 0$. This implies that $bd \neq 0$ since D is an integral domain. This tells us that $\frac{ad+bc}{bd} \in F$.

1.1.2 Addition Commutative

Here, we have:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} = \frac{cb+da}{db} = \frac{c}{d} + \frac{a}{b}$$

1.1.3 Addition Associative

This is similar to above.

1.1.4 Additive Identity

The identity is $\frac{0}{1} = \frac{0}{a} \in F$ for all $a \neq 0$. This is because:

$$\frac{0}{1} + \frac{a}{b} = \frac{0 \cdot b + 1 \cdot a}{1 \cdot b} = \frac{a}{b}$$

1.1.5 Additive Inverse

For an element $\frac{a}{b}$, its inverse is $\frac{-a}{b}$. This is because:

$$\frac{a}{b} + \frac{-a}{b} = \frac{ab + b(-a)}{b^2} = \frac{ab - ab}{b^2} = \frac{0}{b^2} = \frac{0}{1}$$

1.1.6 Multiplication Well-Defined

Let $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$. Then:

$$acb'd' = (ab')(cd') = (ba')(dc') = (a'c')(bd) \implies \frac{ac}{bd} = \frac{a'c'}{b'd'}$$

Also, $\frac{a}{b}, \frac{c}{d} \in F$ so $b, d \neq 0$ and thus $bd \neq 0$ since D is an integral domain. Thus, $\frac{ac}{bd} \in F$.

1.1.7 Multiplication Associative

$$\left(\frac{a}{b}\frac{c}{d}\right) = \frac{a}{b}\left(\frac{c}{d}\frac{e}{f}\right)$$

1.1.8 Multiplication Commutative

$$\frac{a}{b}\frac{c}{d} = \frac{ac}{bd} = \frac{ca}{db} = \frac{c}{d}\frac{a}{b}$$

1.1.9 Multiplication Unity

The unity is $\frac{1}{1} \in F$. This is because:

$$\frac{1}{1}\frac{a}{b} = \frac{1a}{1b} = \frac{a}{b}$$

1.1.10 Multiplicative Inverses

If $\frac{a}{b} \neq \frac{0}{1}$, then $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$. Note that:

$$\frac{a}{b} \neq \frac{0}{1} \implies a1 \neq b0$$

In other words, $a \neq 0$ and thus $\frac{b}{a} \in F$. Thus:

$$\frac{a}{b}\frac{b}{a} = \frac{ab}{ab} = \frac{1}{1}$$

1.1.11 Multiplication Distributive

This is left as an exercise.

1.2 Subring

How is D a subring of F? In $\mathbb{Q}$, we write $\frac{2}{1}$ as 2. Well:

$$a \in D \mapsto \frac{a}{1} \in F$$

In other words, we have a homomorphism.

1.3 Examples of Fields of Fractions

Here are some examples.

1. $\mathbb{Z} \mapsto \mathbb{Q}$.
2. $\mathbb{R}[x] \mapsto \mathbb{R}(x) = \left\{\frac{f(x)}{g(x)} \mid f, g \in \mathbb{R}[x], g \neq 0\right\}$.
3. $\mathbb{F}_p[x] \mapsto \mathbb{F}_p(x) = \left\{\frac{f(x)}{g(x)} \mid f, g \in \mathbb{F}_p[x], g \neq 0\right\}$. Note that $\mathbb{F}_p(x)$ has infinite size and has characteristic p. Additionally, $x+1 \in \mathbb{F}_p[x]$ has no multiplicative inverse.