1 Trigonometric Interpolation (Section 6.12)

Recall, for one-dimensional interpolation, there is a unique $P(x_j) = f(x_j)$ for $0 \leq j \leq m-1$. However, when f is periodic, trigonometric functions for interpolation are appropriate.

1.1 Connection to Complex Variables

If f has a period of 2π and a continuous first derivative, then the **Fourier series/analysis**,

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos(kx) + b_k \sin(kx)),$$

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)dt,$$
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)dt.$$

A connection to complex variables is **Euler's formula**, which is

$$e^{i\theta} = \cos(\theta) + i\sin(\theta),$$

where $i = \sqrt{-1}$. The complex conjugate of Euler's formula is

$$\overline{e^{i\theta}} = \cos(\theta) - i\sin(\theta).$$

1.1.1 Special Exponential Functions

We can also define the function,

$$E_k(x) = e^{ikx}, \qquad k = 0, \pm 1, \pm 2, \pm 3, \ldots.$$

These are special because they form a "basis" of linearly independent functions.

1.1.2 Some Notation

Some notation to keep in mind:

- The inner product of two functions is defined by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx.$$

- The "discrete inner product" is defined by

$$\langle f, g\rangle_N = \frac{1}{N}\sum_{j=0}^{N-1} f\left(\frac{2\pi j}{N}\right)\overline{g\left(\frac{2\pi j}{N}\right)},$$

  with N being the number of points being used.

Using the special exponential function, we note that, for $k \neq m$,

$$\begin{aligned}
\langle E_k, E_m \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} \overline{e^{imx}} dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} e^{-imx} dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ix(k-m)} dx \\
&= \frac{1}{2\pi(k-m)} e^{ix(k-m)} |_{-\pi}^{\pi} \\
&= \frac{e^{i\pi(k-m)} - e^{-i\pi(k-m)}}{2\pi i(k-m)} \\
&= \dots \\
&= 0.
\end{aligned}$$

However, if $k = m$, we have

$$\begin{aligned}
\langle E_k, E_m \rangle &= \langle E_k, E_k \rangle \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 dx \\
&= 1.
\end{aligned}$$

So, $E_k(x)$ is a basis for trigonometric functions.

1.1.3 Exponential Polynomial & Interpolation

We can define $P(x)$ to be

$$P(x) = \sum_{k=0}^{m} c_k E_k(x) = \sum_{k=0}^{m} c_k e^{ikx} = \sum_{k=0}^{m} c_k \left(e^{ix}\right)^k.$$

Suppose now we want to interpolate a Polynomial at equally spaced points, $x_j = \frac{2\pi j}{N}$. Then, we can use the discrete inner product,

$$P(x) = \sum_{k=0}^{N-1} c_k E_k, \quad c_k = \langle f, E_k \rangle_N = \frac{1}{N} \sum_{j=0}^{N-1} f\left(\frac{2\pi j}{N}\right) \overline{E_k\left(\frac{2\pi j}{N}\right)}.$$

(Example.) Suppose $N = 2$. We want to find an explicit formula of P given f. So, $x_0 = 0$ and $x_1 = \frac{2\pi}{2} = \pi$ and

$$c_0 = \frac{1}{2} \sum_{j=0}^{2-1} f\left(\frac{2\pi j}{2}\right) e^{-i \cdot 0 \cdot j\pi} = \frac{1}{2}(f(0) + f(\pi)).$$

$$c_1 = \frac{1}{2} \sum_{j=0}^{2-1} f\left(\frac{2\pi j}{2}\right) e^{-i \cdot 2\pi} = \frac{1}{2}\left(f(0) + f(\pi) e^{-i\pi}\right) = \frac{1}{2}(f(0) - f(\pi)),$$

where $e^{-i\pi} = \cos(\pi) - i\sin(\pi) = -1 - i(0) = -1$. Then, combining this yields

$$P(x) = c_0 E_0(x) + c_1 E_1(x) = \frac{1}{2}(f(0) + f(\pi)) + \frac{1}{2}(f(0) - f(\pi)) e^{ix}.$$

1.2 Complex Fourier Series

Let

$$f(x) \sim \sum_{k=-\infty}^{\infty} \hat{f}(x)e^{ikx},$$

where

$$\hat{f}(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)(\cos(kt) - i\sin(kt))dt = \frac{1}{2}(a_k - ib_k).$$

If $f(t)$ is real, then it corresponds to the "real" Fourier series. Let $a_k = a_{-k}$ (i.e., even proeprty), $b_k = -b_{-k}$, $c_k = \frac{1}{2}(a_k - ib_k)$, $b_0 = 0$. Then,

$$f \sim \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx} = \sum_{k=-\infty}^{\infty} \frac{1}{2}(a_k - ib_k)(\cos(kt) + i\sin(kt)).$$

Here, the imaginary part is 0 (represented by the summation). In this sense,

$$\sum_{k=-\infty}^{\infty} \frac{1}{2}(a_k\sin(kx) - b_k\cos(kx))$$

is the imaginary part.