1 Divided Differences (Section 6.2)

We'll begin by briefly reviewing Newton's Form from the previous section. Recall the coefficients in the Newton form of interpolating polynomial, $P(x)$, where $f(x)$ is interpoled by $P(x)$, is c_i. Then,

$$P(x_i) = \underbrace{f(x_i)}_{\text{Data}} \quad 0 \leq i \leq m.$$

Newton's Form looks like

$$P(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + \ldots + c_m(x - x_0)(x - x_1) \ldots (x - x_{m-1}).$$

Then,

$$\begin{aligned} P(x_0) &= f(x_0) = c_0 \\ P(x_1) &= f(x_1) = c_0 + c_1(x_1 - x_0) \\ P(x_2) &= f(x_2) = c_0 + c_1(x_2 - x_0) + c_2(x_2 - x_0)(x_2 - x_1) \end{aligned} \tag{1}$$

We can represent (1) as a system of equations, modeled using matrices, like so:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & (x_1 - x_0) & 0 \\ 1 & (x_2 - x_0) & (x_2 - x_0)(x_2 - x_1) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{bmatrix}.$$

This is a triangular system. We can represent the c_i's in terms of the $f(x_i)$'s. Thus, the **divided difference** is defined by

$$f[x_0, x_1, x_2, \ldots, x_m] \equiv c_m. \tag{2}$$

We say that $f[x_0, x_1, x_2, \ldots, x_m]$ is the coefficient of x^m in the polynomial of degree at most n that interpolates f at $x_0, x_1, \ldots, x_n$. With this in mind, some explicit formulas include

- if $m = 0$, then $c_0 = f[x_0] = f(x_0)$.
- if $m = 1$, then $c_1 = f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$.

Thus, the Newton interpolating form can be rewritten as

$$P(x) = \sum_{k=0}^{m} f[x_0, x_1, x_2, \ldots, x_k] \prod_{j=0}^{k-1} (x - x_j).$$

1.1 Higher Order Differences

Theorem 1.1: Divided Difference

Divided differences satisfy the equation,

$$f[x_0, x_1, \ldots, x_m] = \frac{f[x_1, x_2, \ldots, x_m] - f[x_0, x_1, \ldots, x_{m-1}]}{x_m - x_0}.$$

Proof. Let $P_k(x)$ be a polynomial of degree at most k interpolating $x_0, x_1, \ldots, x_k$. Further, let $Q_k(x)$ be a polynomial of degree at most $m-1$ interpolating $x_1, x_2, \ldots, x_m$. Then, we have $P_k(x_i) - f(x_i)$ for $0 \leq i \leq k$ and $q(x_i) = f(x_i)$ for $1 \leq i \leq m$. So,

$$P_m(x) = q(x) + \frac{(x - x_m)}{x_m - x_0}(q(x) - P_{m-1}(x)).$$

We can verify that any data point we substitute in will satisfy the above equation. In any case, the left- and right-hand sides match at $x_0, x_1, \ldots, x_m$. Then, the coefficients on the left- and right- for x^m are

$$f[x_0, x_1, \ldots, x_m] = \frac{f[x_1, x_2, \ldots, x_m] - f[x_0, x_1, \ldots, x_{m-1}]}{x_m - x_0},$$

as desired. $\square$

So, with this formula, if we were to consider $m = 0$, then

$$f[x_0] = f(x_0).$$

Likewise, if we consider $m = 1$, we have

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}.$$

Finally, for $m = 3$, we have

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}.$$

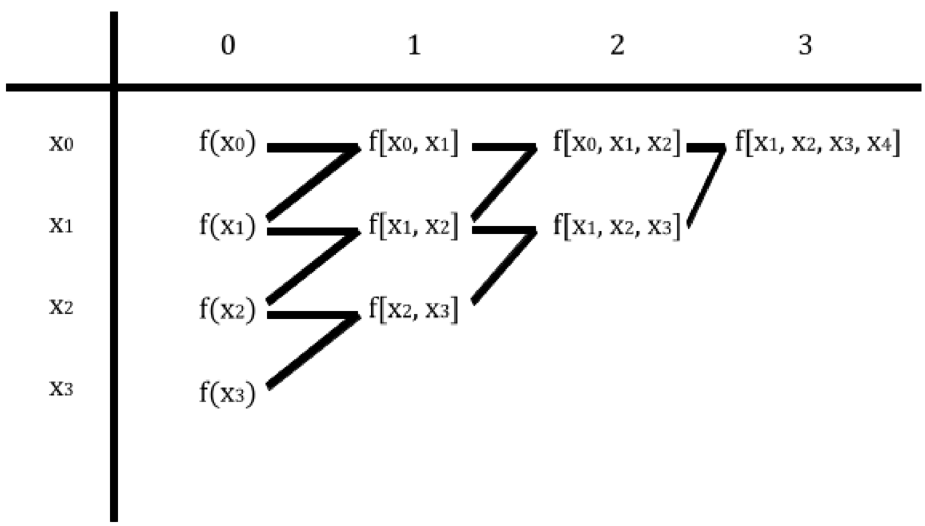
The recursive relation is given by

$$f[x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+j}] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+j}] - f[x_i, x_{i+1}, \ldots, x_{i+j-1}]}{x_{i+j} - x_i}. \tag{3}$$

We can describe this recursive relation in terms of a table. For a table of function values $(x_i, f(x_i))$,

	0	1	2	3
x_0	$f(x_0)$	$f[x_0, x_1]$	$f[x_0, x_1, x_2]$	$f[x_0, x_1, x_2, x_3]$
x_1	$f(x_1)$	$f[x_1, x_2]$	$f[x_1, x_2, x_3]$	
x_2	$f(x_2)$	$f[x_2, x_3]$		
x_3	$f(x_3)$			

Note that $f(x_i) = f[x_i]$. So, the top row of the table corresponds to the coefficients in Newton's form. Each columns represents the difference of orders. The recursive relationship is given by



For example, the calculation of $f[x_1, x_2, x_3]$ depends on the result of $f[x_1, x_2]$ and $f[x_2, x_3]$; to see why, from (3), note that

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}.$$

(Exercise.) Given the set of points,

x	3	1	5	6
$f(x)$	1	-3	2	4

Compute a divided difference table for these function values.

We have the table

x_i	$f(x_i)$			
3	1	$f[x_0, x_1]$	$f[x_0, x_1, x_2]$	$f[x_0, x_1, x_2, x_3]$
1	-3	$f[x_1, x_2]$	$f[x_1, x_2, x_3]$	
5	2	$f[x_2, x_3]$		
6	4			

Using the table above, we find that,

- for the first column, i.e., $f[x_i, x_j]$,

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{-3-1}{1-3} = \frac{-4}{-2} = 2.$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{2-(-3)}{5-1} = \frac{5}{4}.$$

$$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2} = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{4-2}{6-5} = \frac{2}{1} = 2.$$

- for the second column, i.e, $f[x_i, x_j, x_k]$,

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{5}{4} - 2}{5-3} = -\frac{3}{8}.$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{2 - \frac{5}{4}}{6-1} = \frac{3}{20}.$$

- for the third column, i.e., $f[x_i, x_j, x_k, x_\ell]$,

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{\frac{3}{20} - \left(-\frac{3}{8}\right)}{6-3} = \frac{7}{40}.$$

Therefore, the divided difference table looks like

x_i	$f(x_i)$			
3	1	2	$-\frac{3}{8}$	$\frac{7}{40}$
1	-3	$\frac{5}{4}$	$\frac{3}{20}$	
5	2	2		
6	4			

Recall that the top row of the table corresponds to the coefficients in Newton's form, so we find that the interpolating polynomial $P(x)$ is

$$\begin{aligned} P(x) &= c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + c_3(x - x_0)(x - x_1)(x - x_2) \\ &= 1 + 2(x-3) - \frac{3}{8}(x-3)(x-1) + \frac{7}{40}(x-3)(x-1)(x-5) \end{aligned}.$$