1 Multivariable Interpolation (Section 6.10)

In this section, we'll focus on interpolation in multiple variables, in particular two variables. Our data is represented by

- the points (i.e., nodes),
$$\mathcal{N} = \{(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)\}.$$
Note that each node is unique.
- the values associated with each node,
$$f(x_1, y_1), f(x_2, y_2), \ldots, f(x_m, y_m).$$

To interpolate each point in $\mathcal{N}$, we seek an interpolation formula $F : \mathbb{R}^2 \mapsto \mathbb{R}$ so that

$$F(x_i, y_i) = f(x_i, y_i), \qquad 1 \le i \le m.$$

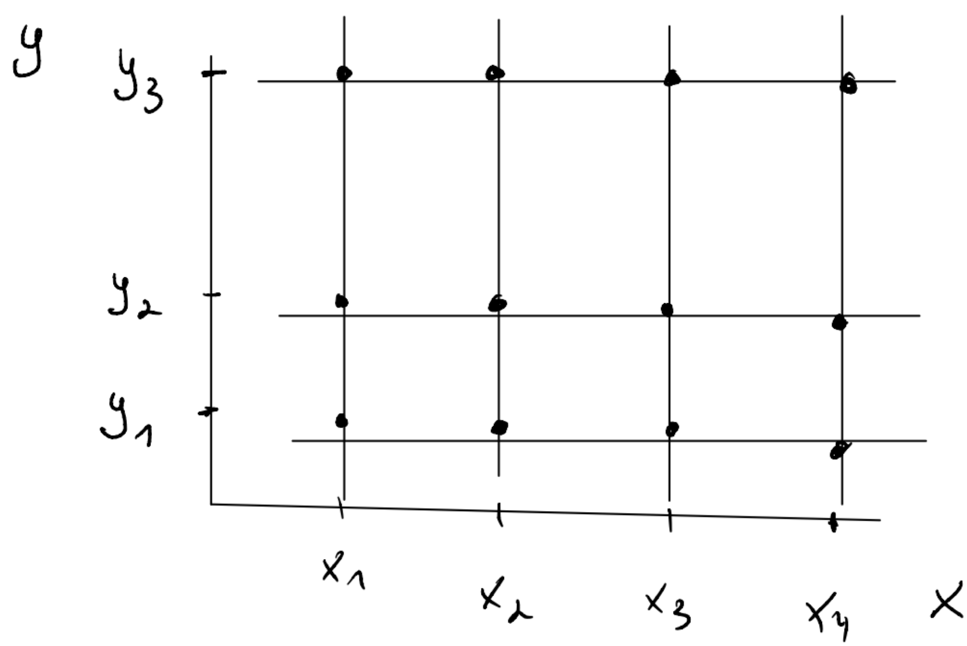
For arbitrary sets of nodes, this process is not always well-defined. So, we'll consider a special case.

1.1 Interpolating in the Special Case

Our special case deals with $\mathcal{N}$ being a cartesian product, i.e.,

$$\mathcal{N} = \{(x_i, y_j) : 1 \le i \le p, 1 \le j \le q\}.$$

(Example.) If $p = 4$ and $q = 3$, we might have the points such that



We are interpolating along the vertical and horizontal lines here.

Remark: Points do not need to be equally spaced. In other words, it's not necessarily true that $y_2 - y_1 = y_1 - y_0$.

1.1.1 Cardinal Functions

Additionally, we only want to consider "cardinal" functions, i.e., functions $u_i(x)$ and $v_j(y)$ so that

$$u_i(x_j) = \delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \qquad v_j(x_i) = \delta_{j,i} = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

for $1 \leq i, j \leq p$ and $1 \leq i, j \leq q$, respectively. We can construct such a cardinal function by

$$u_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{p} \frac{x - x_j}{x_i - x_j} \tag{1}$$

$$v_j(y) = \prod_{\substack{i=1 \\ i \neq j}}^{q} \frac{y - y_i}{y_j - y_i} \tag{2}$$

We can define the function

$$(\overline{P}f)(x, y) = \sum_{i=1}^{p} f(x, y) u_i(x) = P_i(x, y), \qquad P_i(x_i, y) = f(x_i, y),$$

which interpolates f on the *vertical* lines, $L_i = \{(x_i, y) : -\infty < y < \infty\}$. Likewise, we have

$$(\overline{Q}f)(x, y) = \sum_{j=1}^{q} f(x, y_j) v_j(y),$$

which interpolates f on all the *horizontal* lines, $L^i = \{(x, y_i) : -\infty < x < \infty\}$.

1.1.2 Constructing Interpolating Functions

There are two options for evaluating this:

- Option 1: Consider the tensor product. Then,

$$\begin{aligned} \overline{P} \otimes \overline{Q} &= (\overline{P}(\overline{Q}f))(x, y) \\ &= \sum_{i=1}^{p} \left(\sum_{j=1}^{q} f(x_i, y_j) v_j(y) \right) u_i(x) \\ &= \sum_{i=1}^{p} \sum_{j=1}^{q} f(x_i, y_i) v_j(y) u_i(x) \\ &= F(x, y). \end{aligned}$$

  where the u and v functions can be derived from (1) and (2), respectively. This interpolates $f(x_i, y_j)$.

- Option 2: Consider the boolean sum. Then,

$$\begin{aligned} \overline{P} \oplus \overline{Q} = \overline{P} + \overline{Q} - \overline{PQ} \\ &= \sum_{i=1}^{p} f(x_i, y) u_i(x) + \sum_{j=1}^{q} f(x, y_j) v_j(y) - \sum_{i=1}^{p} \sum_{j=1}^{q} f(x_i, y_j) v_j(y) u_i(x) \\ &= F(x, y). \end{aligned}$$

  where the u and v functions can be derived from (1) and (2), respectively. This is also an interpolation.