CSE 100 Notes

Advanced Data Structures

Fall 2021
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1   A Brief Introduction

In this course, we will primarily be building off of our prior knowledge of data structures (CSE 12). In particular, we will:

- Analyze data structures for both time and space complexity.

- Describe the strengths and weaknesses of a data structure.

- Implement complex data structures correctly and efficiently.

1.1   Data Structures vs. Abstract Data Types

When talking about data, we often hear about data structures and abstract data types.

Data Structures (DS)	**Abstract Data Type** (ADT)
Data structures are collections that contain: • Data values. • Relationships among the data. • Operations applied to the data. It also describes how the data are organized and how tasks are performed. So, a data structure defines every single detail about anything relating to the data.	Abstract data types are defined primarily by its <u>behavior</u> from the view of the <u>user</u>. So, not necessarily how the operations are done, but rather what operations it must have from a completely abstract point of view. Specifically, it describes only what needs to be done, not how it's done.

Consider the `ArrayList` (DS) vs. the `List` (ADT).

- A `List` will most likely have the following operations:

  - `add`: Adds an element to the list.
  - `find`: Does an element exist in the list?
  - `remove`: Remove an element from the list.
  - `size`: How many elements are in this list?
  - `ordered`: Each element should be ordered in the way we added it. For example, if we added `5`, and *then* added `3`, and *then* added `10`, our list should look like: `[5, 3, 10]`.

  Of course, as an abstract data type, `List` isn't going to define how these operations work. It just lists all operations that any implementing data structure must have. In other words, we can think of `List`, or any abstract data type, as a *blueprint* for future data structures.

- An `ArrayList` is simply an array that is expandable. It is internally backed by an <u>array</u>. So, we can perform the following operations:

  - We can `add` an element to the `ArrayList`. In this case, we add the element to the next available slot in the array, expanding the array if necessary.
  - We can `find` an element in the `ArrayList`. In this case, we can search through each slot of the array until we find the array or we reach the end of the array.
  - We can `remove` an element from the `ArrayList`. In this case, we can simply move every element after the specified element back one slot.
  - We can get the `size` of the `ArrayList`. In this case, this is as simple as seeing how many elements are in this `ArrayList`.
  - And, we know that the `ArrayList` is `ordered`. In this case, this is already done via the `add` and `remove` methods.

Notice how `ArrayList` specifies how each operation defined by `List` works. In this sense, we say that `ArrayList` essentially implements `List` because we need to define *how* the tasks defined by `List` are performed.

So, the key takeaways are:

- An abstract data type (in our case, `List`) specifies what needs to be done without specifying how it's done.

- A data structure (in our case, `ArrayList`) actually defines **how** the data is organized, how the different operations are performed, and how exactly everything is represented.

2   Introduction to C++

Here, we will talk about C++, the programming language that we will use in this course.

2.1   Data Types

First, we'll compare the data types in Java and C++.

Data Type	Java	C++
byte	1 byte	1 byte
short	2 bytes	2 bytes
int	4 bytes	4 bytes
long	8 bytes	8 bytes
long long		16 bytes
float	4 bytes	4 bytes
double	8 bytes	8 bytes
boolean	Usually 1 byte	
bool		Usually 1 byte
char	2 bytes	1 byte

It should be mentioned that:

- In Java, you can only have signed data types.

- In C++, you can have both signed and unsigned data types.

- `boolean` (Java) and `bool` (C++) are effectively the same thing: they represent either `true` or `false`.

2.2   Strings

There are some major differences between strings in Java and C++, which we will discuss below.

2.2.1   Representation

In Java, strings are represented by the `String` class. In C++, strings are represented by the `string` type.

2.2.2   Mutability

Strings in Java are <u>immutable</u>. The moment you create a string, you won't be able to modify them. The only way to change a string variable is by creating a new string and reassigning them.

In C++, strings are actually <u>mutable</u>. You can modify strings in-place.

2.2.3   Concatenation

In Java, you can concatenate any type to a string. For example, the following is valid:

```
String a = "this is a string" + 123;
```

In C++, you can only concatenate strings with other strings. So, if you wanted to convert an integer (or any other type) to a string, you would have to *first* convert that integer to a string (or use a string stream).

2.2.4 Substring Method

In Java, we can take the substring of a string using the `substring` method. The method signature is:

```
String#substring(beginIndex, endIndex);
```

In C++, we can take the substring using the `substr` method. The method signature is:

```
string#substr(beginIndex, length);
```

An important distinction to make here is that Java's `substring` method takes in an **end index** for the second parameter, whereas C++'s `substr` method takes in a **length** for the second parameter.

2.3 Comparing Non-Primitive Objects

Suppose `a` and `b` are two non-primitive objects.

In Java, if we want to compare these two objects, we have to make use of the methods:

```
a.equals(b)
a.compareTo(b)
```

If we tried using the relational operators like `==` or `!=`, Java would compare the memory addresses of the two objects, which is often something that we aren't looking for.

In C++, even if `a` and `b` are objects, we can make use of the relational operators:

```
a == b        a != b
a < b         a <= b
a > b         a >= b
```

This is done through something called **operator overloading**, where we write a custom class and define how these operators should function.

2.4 Variables

Now, we will briefly discuss how variables function in both C++ and Java.

2.4.1 Initialization

In Java, variable initialization is **checked**. Consider the following code:

```
int fast;
int furious;
int fastFurious = fast + furious;
```

Because `fast` and `furious` aren't initialized, the Java compiler will throw a compilation error.

In C++, variable initialization is **not checked**. Consider the same code, which will compile:

```
int fast;
int furious;
int fastFurious = fast + furious;
```

Here, this would result in **undefined** behavior.

2.4.2   Narrowing

In Java, if we have a higher variable type and then try to cast this type to a smaller type, we would get a compilation error. Consider the following code:

```
int x = 40_000;
short y = x;
```

This code would result in a compilation error. If we didn't want a compilation error, we would have to explicitly *cast* the bigger variable type to the smaller type. The following Java code would compile just fine:

```
int x = 40_000;
short y = (short) x;
```

In C++, no compilation error would occur; that is, the following code would compile:

```
int x = 40_000;
short y = x;
```

What would actually happen is that `x` would get **truncated** when it is assigned to `y`, resulting in integer overflow.

2.4.3   Variable Declaration

In Java, variables **cannot** be declared outside of a class. The following Java code would result in a compile error:

```
// MyClass.java

int meaningOfLife = 42;
class MyClass {
    // some code
}
```

In order for this to compile, you have to put variable declarations inside the class space (as an instance variable) or in a method inside a class (as a local variable).

In C++, variables **can** be declared outside of a class. The following C++ code would compile completely fine:

```
// MyClass.cpp

int meaningOfLife = 42;
class MyClass {
    // some code
}
```

Here, `meaningOfLife` is a **global variable**. Anything in this file can access this variable. In general, it is considered poor practice to use global variables except in cases of constants.

2.5   Classes, Source Code, and Headers

Another thing that is important is the concept of classes (which leads to the topic of object-oriented programming). That being said, Java and C++ has some differences with regards to how classes function.

2.5.1   Class Declaration

There are some key differences in how methods and instance variables are laid out in Java and C++.
In Java, a typical class would look like:

```
class Student {
    public static int numStudents = 0;
    private String name;

    public Student(String n) { /* Code */ }

    public void setName(String n) { /* Code */ }
    public String getName() { /* Code */ }
}
```

And in C++, a typical class would look like:

```
class Student {
    public:
        static int numStudents;

        Student(string n);

        void setName(string n);
        string getName() const;

    private:
        string name;
}

int Student::numStudents = 0;
Student::Student(string n) { /* Code */ }
void Student::setName(string n) { /* Code */ }
string Student::getName() const { /* Code */ }
```

There are several notable differences:

- **Modifiers:** In Java, if you want your method or instance variable to have an access modifier, you explicitly state the access modifier. In C++, you have a region for your access modifier. That is, there is a `public` region, `private` region, etc. Any methods or instance variables listed under these regions will take on that access modifier. For instance, `setName` is in the `public` region, so `setName` is public.

- **Implementation:** In Java, directly after declaring a method or constructor in a class, we need to provide the implementation code. In C++, we can "declare" the methods and the constructor, and then outside of the class we can implement the methods.

Now, consider the following C++ code:

```
class Point {
    private:
        int x;
        int y;

    public:
        Point(int i, int j);
}

Point::Point(int i, int j) {
```

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```
        x = i;
        y = j;
    }
```

Here, we're initializing the `x` and `y` instance variables directly from the constructor implementation. However, we can initialize these instance variables directly like so:

```
class Point {
    private:
        int x;
        int y;

    public:
        Point(int i, int j);
}

    Point::Point(int i, int j) : x(i), y(j) {}
```

This is called the **member initializer list**.

2.5.2   Source vs. Header Files

Consider the following class:

```
class Student {
    public:
        static int numStudents;
        Student(string n);

    private:
        string name;
}

int Student::numStudents = 0;
Student::Student(string n) : name(n) {
    numStudents++;
}
```

We can choose to break this up into two separate files; a **source** (usually `.cpp`) file and a **header** (usually `.h`) file. The header file contains the class and the method *declaration*; the source file contains the implementations for those methods. So, the above code can be written like so:

```
// The header file
// Student.h
class Student {
    public:
        static int numStudents;
        Student(string n);

    private:
        string name;
}

// The source file
// Student.cpp
int Student::numStudents = 0;
Student::Student(string n) : name(n) {
```

```
        numStudents++;
    }
```

2.6 Memory Diagrams

Consider the following Java code:

```
    Student s1 = new Student("Niema");
    Student s2 = s1;
```

Here, `s1` is a *reference* to a `Student` object. This `Student` object contains a *reference* to a `string` object with the content `Niema`. That is:

```
Student object
|----------|
| |----|   |    (Reference)  |-------|
| |name|------------------> | Niema |
| |----|   |                 |-------|
|----------|   <------         String object
   /|\                \
    |                  \
    |                   \ (Reference)
    |                   |
    | (Reference)       |
  s1 -/                 s2
```
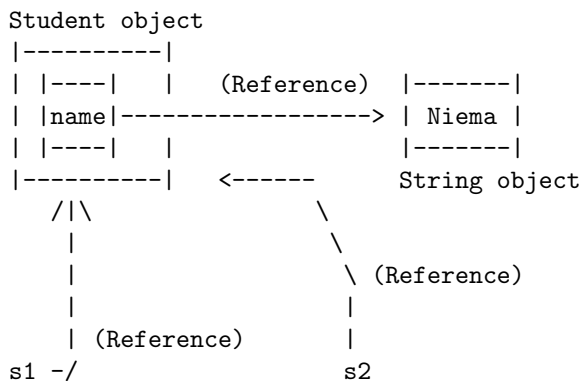
It also follows that `s2` is a reference to the same object that `s1` is referring to.

Now, consider the following C++ code:

```
    Student s1("Niema");
    Student s2 = s1;
```

Here, `s1` is a `Student` *object*. The `Student` object contains a `string` object with the content `Niema`. That is:

```
|-----------------|      |-----------------|
| |-----------|   |      | |-----------|   |
| |name: Niema|   |      | |name: Niema|   |
| |-----------|   |      | |-----------|   |
|   string object  |      |   string object  |
|-----------------|      |-----------------|
Student object           Student object
    s1                        s2
```
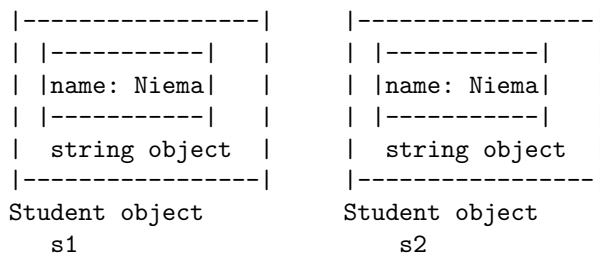
Additionally, when we assign `s1` to `s2`, we actually make a copy of said object. So, `s2` is its own object; it does not share a reference with `s1`.

In other words, in Java, `s1` and `s2` are both references to the same object; in C++, `s1` *is* the object and `s2` is *another* object.

2.6.1 References

Consider the following C++ code:

```
    Student s1 = Student("Niema");
    Student & s2 = s1;
    Student s3 = s2;
```

The memory diagram looks like this:

```
|----------------|        |----------------|
| |----------|   |        | |----------|   |
| |name: Niema|  |        | |name: Niema|  |
| |----------|   |        | |----------|   |
|   string object |       |   string object |
|----------------|        |----------------|
   Student object            Student object
     s1        s2                 s3
```

Here, `s2` can be seen as *another* way to call `s1` (think of `s2` as another name for `s1`). `s3` would be a copy of `s1`.

2.6.2   Pointers

Pointers are similar to Java references. Consider the following C++ code:

```
Student s = Student("Niema");
// * in this case means pointer
// & means memory address
// So, ptr stores a memory address to some object. In other words,
// it points to the object s.
Student* ptr = &s;
Student** ptrPtr = &ptr;
```
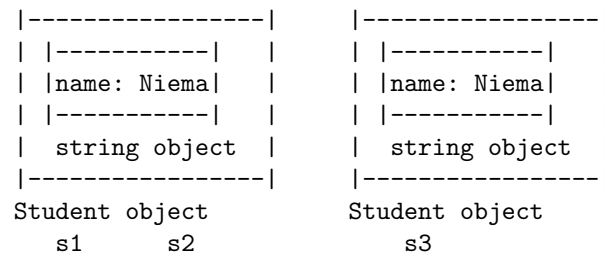
The memory diagram would look like:

```
Mem. Address: 9500
|----------------|
| |----------|   |        |----------|         |----------|
| |name: Niema|  |  <--- |   9500   |  <--- |          |
| |----------|   |        |----------|         |----------|
|   string object |        Student* (pointer)   Student** (pointer to pointer)
|----------------|             ptr                      ptrPtr
   Student object
        s
```
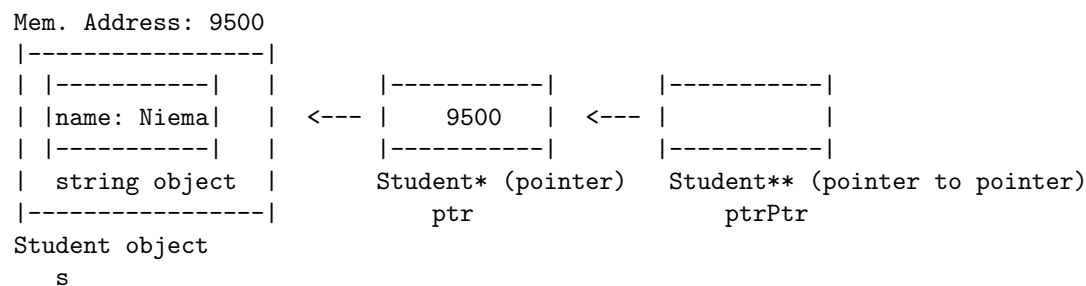
If we wanted to access an object through a pointer, we can do this in several ways.

1. Deferencing a pointer.

    ```
    // * in this case dereferences the pointer
    // Think of the * as following the arrow
    (*ptr).name;
    ```

2. Arrow dereferencing.

    ```
    // ptr->x is the same thing as (*ptr).x
    ptr->name;
    ```

2.6.3   Memory Management

Consider the following C++ code:

```
Student s1 = Student("Niema");
Student* s2 = new Student("Ryan");
```

The corresponding memory diagram is:
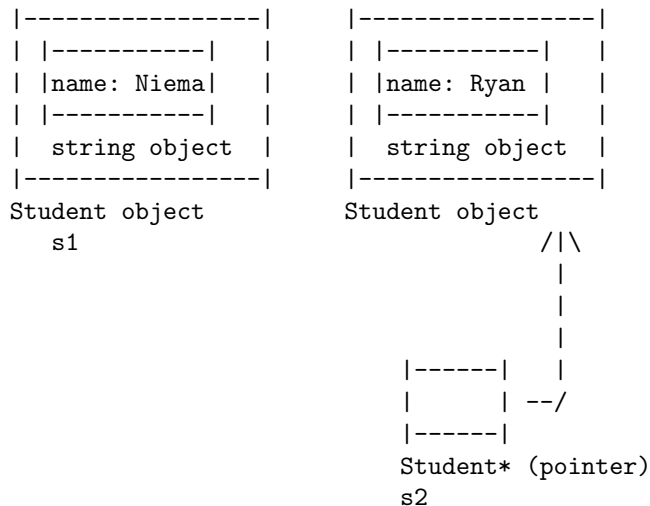
```
|-----------------|       |-----------------|
| |-----------|   |       | |-----------|   |
| |name: Niema|   |       | |name: Ryan |   |
| |-----------|   |       | |-----------|   |
|  string object  |       |  string object  |
|-----------------|       |-----------------|
Student object            Student object
   s1                                    /|\
                                          |
                                          |
                                          |
                            |------|      |
                            |      | --/
                            |------|
                            Student* (pointer)
                            s2
```

Here, `s1` is allocated on the *stack*; once the method returns, `s1` is automatically destroyed.

`s2` is allocated through the `new` keyword. This is known as dynamic memory allocation. So, `s2` is a pointer to the newly allocated memory. Because this object was created using the `new` keyword, we need to deallocate it ourselves. To do so, we need to explicitly call `delete` on this object:

```
delete s2;
```

`delete` takes in a memory address (i.e. pointer). This is very similar to `free` (in C). If we don't free this, we run into what is called a **memory leak**.

2.7   Constant Keyword

In C++, the `const` keyword means that the variable can never be reassigned. Consider the following:

```
const int a = 42;
int const b = 42;
```

If we tried reassigning `a` (e.g. `a = 41;`), we would get a compiler error.

The second line (`int const`) is identical to the first line.

2.7.1   `const` and Pointers

Consider the following C++ code:

```
int a = 42;                   // a
const int* ptr1 = &a;         // b
int const* ptr2 = &a;         // c
int* const ptr3 = &a;         // d
const int* const ptr4 = &a;   // e
```

- For lines (b) and (c), the pointer cannot modify the object that it is pointing to. But, we can reassign the pointer to point to a different object.

- For line (d), we cannot reassign the pointer to point to a different object. However, we can modify the object that the pointer is pointing to.

- For (e), we cannot reassign the pointer to point to a different object *or* modify the object that the pointer is pointing to.

In general:

```
const type* const varName = ...;
----------- --------------
    (a)           (b)
```

- Segment (A): The `const` next to `type*` means that we cannot modify the object or value behind the pointer.

- Segment (B): The `const` next to `varName` (the variable name) means that we cannot reassign the pointer to point to a different object or value.

2.7.2   `const` **and References**

Suppose we have the following C++ code:

```
int a = 42;
const int & ref1 = a;        // a
int const & ref2 = a;        // b
```

- In (a), the `const` means that we cannot modify the variable through the constant reference. So:

```
        a = 21;          // Allowed.
        ref1 = 20;       // Compile error!
```

- (b) is the same exact thing is (a).

2.7.3   `const` **Functions**

Recall the `Student` class from earlier:

```
class Student {
    public:
        Student(string n);
        string getName() const;

    private:
        string name;
}

Student::Student(string n) : name(n) {}
string Student::getName() const {
    return name;
}
```

What does the `const` in `getName()` do? Well, the `const` keyword after the function declaration means that the function cannot modify *this* object. So:

- You cannot do any assignments to instance variables.

- You can only call other `const` functions.

So, effectively, `const` after a function name means that we are guaranteeing that we aren't changing the object's state in any way.

2.8 Functions

In C++, we can have global functions (functions that are defined outside of classes). For instance, the main method (shown below) is a global function (and is required to be):

```cpp
int main() {
    /* Do stuff */
}

class MyClass {
    /* Some code */
}
```

2.8.1 Passing by Value vs. Reference

In C++, you can pass parameters either by value or reference.

When passing by value, the function makes a **copy** of the values that you passed in. Some example code is shown below:

```cpp
void swap(int a, int b) {
    int tmp = a;
    a = b;
    b = tmp;
}
```

These copies are destroyed once the function returns (the stack frame is destroyed).

When passing by reference, the function takes in *references* to the variables. Some example code is shown below:

```cpp
void swap(int & a, int & b) {
    int tmp = a;
    a = b;
    b = tmp;
}
```

Effectively, whatever you change with the references will be reflected with the actual variables. So, in the above `swap` method, `a` and `b` will be updated after the function is done.

2.9 Vectors

A C++ `vector` is very similar in nature to Java's `ArrayList` class and arrays. Consider the following code, which demonstrates some common operations:

```cpp
// Creates a new vector.
vector<int> a;
// Adds 42 to end of vector. Looks like: [42]
a.push_back(42);
// Adds 21 to end of vector. Looks like: [42, 21]
a.push_back(21);
// Removes 21 from vector. Looks like [42]
a.pop_back(); // returns 21
// We can access the first element (0th index).
a[0];
```

Like Java arrays or `ArrayList`, elements in a C++ vector are stored contiguously; that is, they are stored after the previous element.

We know that if we assign an object to another variable, the other variable will get a full copy of that object. The same applies to vectors; we can also create a copy of a vector simply be reassigning it:

```
vector<int> a;
a.push_back(42);
vector<int> b = a;
// a: [42]
// b: [42]
```

2.10   Input and Output

Consider the following code:

```
int n;
cout << "Enter a number: ";
cin >> n;

string message;
cout << "Enter a message: ";
getline(cin, message);

if (cin.fail()) {
    cerr << "Bad input!" << endl;
}
```

Here:

- `cin` represents standard input (`stdin`).

- `cout` represents standard output (`stdout`).

- `cerr` represents standard error (`stderr`).

In C++, we can use `istream` to handle input stream and `ostream` to handle output stream. `cin` is an example of an `istream`; `cout` is an example of an `ostream`.

We can make use of the overloaded `<<` and `>>` operators to write to standard output and read from standard input, respectively. So:

- `cout << "Enter a number"` effectively means to write this message to standard output.

- `cin >> n` effectively means to read from the standard input and store that input into `n`. We aren't necessarily restricted to `int`; we could use `long`, `double`, `string`, etc.

- We can also use `getline` to read from standard input and then store the result into a variable. In our example above, we called `getline(cin, message)`. `cin` is where we are reading the input from and `message` is the variable where we store the result of reading from `cin`.

- `endl` means `end line` and, in our use case here, writes a new line to standard error. In reality, we can use `endl` to write a newline to standard output or error.

2.11   Templates

Templates introduce the notion of *generic programming*. Consider the following code in Java:

```
class Node<Data> {
    public final Data data;
    public Node(Data d) {
        data = d;
    }
}

Node<String> a = new Node<String>(s);
Node<Integer> a = new Node<Integer>(s);
```

The generic type is `Data` (though you can rename it to whatever you want). We can use this type either as a parameter type or a return type. When creating a new object with a generic type, we simply put the type between the `<>` (like with the `Node` examples).

Consider the equivalent C++ example:

```
template<typename Data>
class Node {
    public:
        Data const data;
        Node(const Data & d) : data(d) {}
}

Node<string> a(s);
Node<int> b(n);
```

Here, we can use templates to achieve similar results (compared to the Java example). Functionality-wise, this is similar to Java.

2.12   Iterators

Consider the following C++ code:

```
for (string name : names) {
    cout << name << endl;
}
```

What is `names`?

- Is it a `vector`?

- Is it a `set`?

- Is it an `unordered_set`?

- Is it another collection that C++ has?

Well, it doesn't matter! Regardless of what collection we are using, how we use it doesn't matter when it comes to iterating over it. This functionality is made possible by something called **iterators**.

2.12.1   Iterating Over Arrays

Consider the following code:

```
void printInorder(int* p, int size) {
    for (int i = 0; i < size; ++i) {
        cout << *p << endl;
        ++p;
    }
}
```

The `*p` deferences the pointer, giving the value at the location that the pointer is pointing to.

The `++p` is an example of pointer arithmetic; this will add whatever the size of the type is to the pointer. In this case, this will make the pointer point to the memory address of the next element in the array.

Here, we know that `p` is (initially) a pointer to the first element in the array:

```
0   4   8   12  16  20  24  28          Memory Address
-------------------------------         (sizeof(int) = 4)
[10, 20, 25, 30, 46, 50, 55, 60]        Array
 ^

 p                                      Pointer
```

Dereferencing `p` (`*p`) gives us 10.

When we do `++p`, we made the pointer point to the next memory address:

```
0   4   8   12  16  20  24  28          Memory Address
-------------------------------
[10, 20, 25, 30, 46, 50, 55, 60]        Array
     ^

     p                                  Pointer
```

Dereferencing `p` (`*p`) gives us 20.

2.12.2   Using Iterators

Consider the following C++ code:

```
vector<string> names;
// populate with data

vector<string>::iterator itr = names.begin();
vector<string>::iterator end = names.end();

while (itr != end) {
    cout << *itr << endl;
    ++itr;
}
```

Here, we note a few things.

- `iterator` is simply a class that handles, well, iteration. So, `itr` and `end` are instances of the `iterator` class that is iterating over `names`.

- The `!=` operator (in `itr != end`) has been overloaded. This checks the `curr` property in the `iterator` class to see if it is equal (or, more specifically, not equal) to the `curr` property of the other index. In this case, `itr != end` is effectively comparing `itr.curr` with `itr.end`.

- The * dereferencing operator (in *itr) has also been overloaded. This operator has been overloaded to return whatever the value is at the curr index. So, in our case,*itr would return whatever value is at the specified curr index in the array that we are iterating through.

- The ++ operator (in ++itr) is also overloaded. This will increment the curr property in the iterator instance.

Suppose names has the following:

```
    0       1       2           // Index
["Niema", "Ryan", "Felix"]      // names array
```

Essentially, vector<string>::iterator will look something like:

```
|---------------------------| |---------------------------|
|          |----|           | |            |----|         |
|   curr: | 0  |            | |   curr: | 3  |           |
|          |----|           | |            |----|         |
|            int            | |              int          |
|---------------------------| |---------------------------|
             itr                           end
```

Calling *itr will basically give us names[curr] (or, more specifically, names[0]). Comparing itr != end is basically the same as checking 0 != 3.

When we call ++itr, we now have:

```
|---------------------------| |---------------------------|
|          |----|           | |            |----|         |
|   curr: | 1  |            | |   curr: | 3  |           |
|          |----|           | |            |----|         |
|            int            | |              int          |
|---------------------------| |---------------------------|
             itr                           end
```

Calling *itr now will basically give us names[curr] (or, more specifically, names[1]). Comparing itr != end is basically the same as checking 1 != 3.

2.12.3   Linked List Iterator

Consider the following code, which is essentially the same code as the previous one:
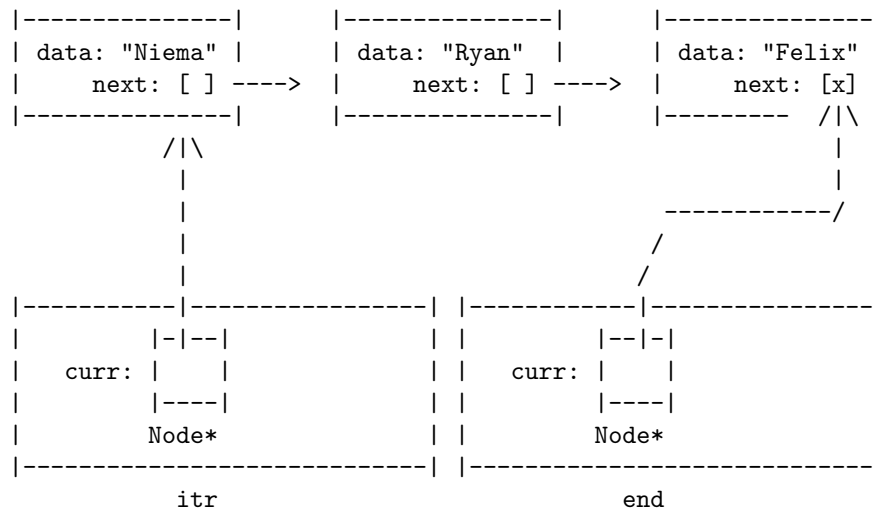
```
LinkedList<string> names;
// populate with data

LinkedList<string>::iterator itr = names.begin();
LinkedList<string>::iterator end = names.end();

while (itr != end) {
    cout << *itr << endl;
    ++itr;
}
```

The only difference is that we're now using a LinkedList instead of vector. However, the way the data is structured is very different. Suppose names has the following:

```
|---------------|     |---------------|     |---------------|
| data: "Niema" |     | data: "Ryan"  |     | data: "Felix" |
|    next: [ ] ---->  |    next: [ ] ---->  |    next: [x]  |
|---------------|     |---------------|     |---------------|
```

Here, `[x]` (in the `next` property of the last node) is a `nullptr`.

How does using nodes change our iterator? Well, `LinkedList<string>::iterator` will look something like:

```
|---------------|       |---------------|        |---------------|
| data: "Niema" |       | data: "Ryan"  |        | data: "Felix" |
|     next: [ ] ---->   |     next: [ ] ---->    |     next: [x] |
|---------------|       |---------------|        |---------  /|\ |
        /|\                                                   |
         |                                                    |
         |                                      -----------/
         |                                     /
         |                                    /
|-----------|---------------|  |-----------|---------------|
|           |-|--|          |  | |           |--|-|          |
|   curr:   |    |          |  | |   curr:   |    |          |
|           |----|          |  | |           |----|          |
|           Node*           |  | |           Node*           |
|---------------------------|  |---------------------------|
          itr                           end
```

Going back to the code:

```
while (itr != end) {
    cout << *itr << endl;
    ++itr;
}
```

It should be noted that:

- `!=` is once again overloaded to compare the values of the node's `data`.

- `*itr` is once again overloaded to return `data`. It would look like:

        ```
        return curr->data;
        ```

- `++itr` is once again overloaded to make the iterator move to the next node. This would look like:

        ```
        curr = curr->next;
        ```

2.12.4   Creating an Iterator Class

When creating data structures, we'll often need to create our own Iterator classes.

First, we'll talk about the operators associated with the iterator class:

- `==`: `true` if the iterators are pointing to the same item and `false` otherwise.

- `!=`: `true` if the iterators are pointing to the different item and `false` otherwise.

- `*` (dereference): Return a reference to the current data value.

- `++` (pre- and post-increment): Move the iterator to the next item.

And, we also need to talk about what functions are in the <u>data structure class</u> so we can make use of the iterator:

- `begin()`: Returns an iterator to the first element.

- `end()`: Returns an iterator to the element just after the last element (not the last element, but *after* the last element).

So, in the Linked List example:

```
[] -> [] -> []
^             ^
begin()      end()
```

And in any array-based structures:

```
[a, b, c, d, e]
 ^             ^
 begin()      end()
```

3   Time and Space Complexity

One of the key things computer scientists try to do is automate competitive tasks, and of course, that requires *performance*. So, that begs the question: how can we measure the performance of our program?

- How many hours does it take to run?

- Minutes?

- Nanoseconds?

These are all metrics of *human time*. However, a program has two aspects:

- The implementation.

- The algorithm behind that program.

While these different metrics of human time are good at measuring the actual implementation of a program, they don't do a good job describing how fast the *idea*, the algorithm itself, is. For instance, running the algorithm on two different devices, both which have wildly different hardware, will result in a significant difference in how fast your algorithm runs.

That being said, we want to know how fast an algorithm is. The best way to do so is by figuring out the performance in terms of number of operations with respect to the input size n (instead of the amount of time).

3.1   Notation of Complexity

Consider the following notations:

- Big-O: Upper bound.

- Big-Ω: Lower bound.

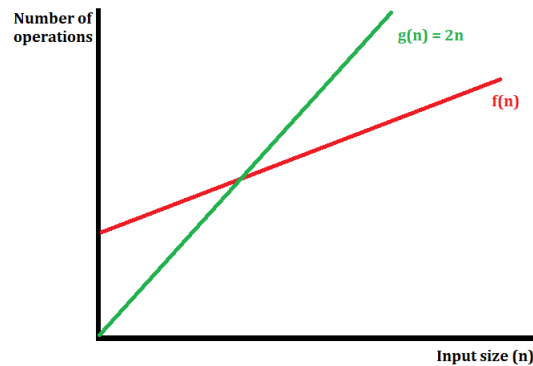- Big-θ: Both upper and lower bound.

Consider the following graph:
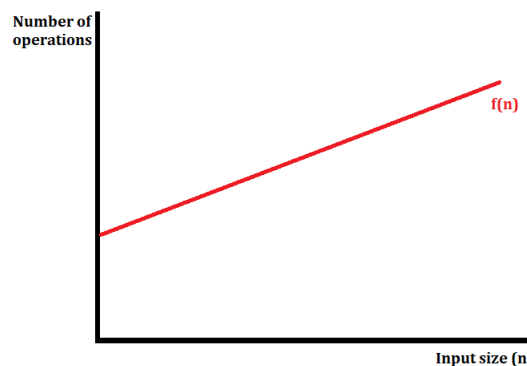
Where $f(n)$ describes the number of operations of your algorithm for some n.

- We say that $f(n)$ is $O(g(n))$ if, for some constant a, we have $a * g(n) \geq f(n)$ as $n \rightarrow \infty$.

  Consider the following graph:

Here, we see that the intersection of the red and the green line occurs at some point, and that after that point the green line will always be greater than the red line. In other words, at that point, we can say that $f(n)$ will never be bigger than $g(n)$ beyond that point. Therefore, we say that $f(n)$ is $O(2n)$, or simply $O(n)$.

- Big-Ω works similarly. We say that $f(n)$ is $\Omega(g(n))$ if, for some constant b, $b * g(n) \leq f(n)$ as $n \to \infty$.

Consider the following graph:
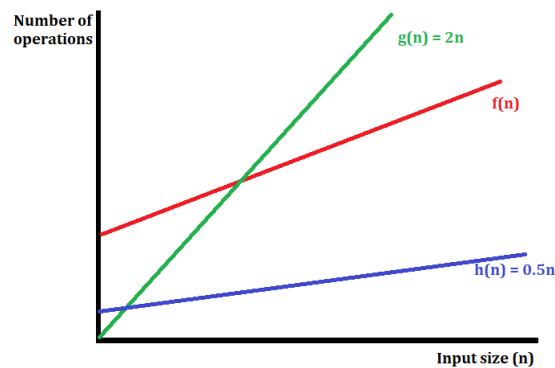
Here, we see that the blue line h is strictly lower than the red line. In other words, $f(n)$ will never be smaller than $h(n)$. Therefore, we say that $f(n)$ is $\Omega(0.5n)$, or simply $\Omega(n)$.

- We say that $f(n)$ is $\theta(g(n))$ if $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$. Mathematically:

$$b * g(n) \leq f(n) \leq a * g(n)$$

In the graphs above, we already found the b and a constants. So:

$$0.5n \leq f(n) \leq 2n$$

Therefore, we can say $f(n)$ is $\theta(n)$.

Remarks:

- Your bigger or smaller functions do not need to be strictly (i.e. always) bigger or smaller than your $f(n)$. They just need to be strictly bigger or smaller beyond some n.

- We will almost always use Big-O.

3.2   Finding Big-O Time Complexity

Given some algorithm, how do we find the Big-O time complexity of it?

1) Determine $f(n)$, or the number of operations our algorithm performs to solve an input of size n.

2) Drop all lower terms of n. In other words, we're only interested in the highest term of n.

3) Drop the constant coefficient.

3.2.1   Example: Grades

Consider the following easy example.

- Input: List of n students, like so:

```
|---------|---------|---------|
| Niema   | Ryan    | Felix   |
| A+      | A       | A       |
|---------|---------|---------|
```

- Algorithm:

```
Print length of the list (n)
For each student x in the list:
    Print x's name
    Print x's grade
```

- Output:

```
3
Niema   A+
Ryan    A
Felix   A
```

Here, we note a few things:

- Regardless of what the length of the list is, printing the length of the list n is a constant time operation. So, this is exactly 1 operation.

- For one student, we print the student's name and grade. Both of these are 1 operation each, for a total of 2 operations. So, for each student, we need to do 2 operations. Therefore, we need to do $2n$ operations for the loop.

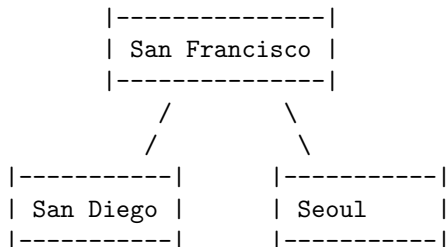- So, the number of operations that we need to do is:

$$f(n) = 2n + 1$$

- Of course, we need to drop the lowest terms and the coefficient of the remaining term:

$$\boxed{f(n) = n}$$

3.2.2   Example: Flight Network

Consider the following medium example.

- Input: Flight network of n cities, like so:

```
              |---------------|
              | San Francisco |
              |---------------|
                 /         \
                /           \
     |-----------|        |-----------|
     | San Diego |        | Seoul     |
     |-----------|        |-----------|
```

  Here, we have a direct flight from San Diego to San Francisco and a direct flight from Seoul to San Francisco. We do not have a direct flight from San Diego to Seoul.

- Algorithm:

```
// Header line = sets up some file
Print header line.
For each unique pair of cities:
    Print the city names
    Print whether or not a direct flight exists
```

- Output:

```
city1, city2, direct
San Diego, San Francisco, yes
San Francisco, Seoul, yes
San Diego, Seoul, no
```

Here, we note a few things again:

- When printing the header line, we need to print three things: `city 1`, `city 2`, and `direct`. Thus, there are 3 operations that are involved.

- For one pair of cities, we need to print three things: the first city name in the pair; the second city name in the pair; and whether or not there is a direct flight between the two cities.

  So, for each pair of cities, there are three operations. We also know that there are $\binom{n}{2}$ ways to get every possible pair of cities. Therefore, we need to do $3\binom{n}{2} = 3\frac{n(n-1)}{2} = 3\left(\frac{n^2}{2} - \frac{n}{2}\right)$ operations for the loop.

- So, the number of operations that we need to do is:

$$f(n) = \frac{3n^2}{2} - \frac{3n}{2} + 3$$

- And, finally, we can drop the lower terms and the coefficient of the remaining term.

$$f(n) = n^2$$

3.2.3   Example: Loops

Consider the following hard example.

- Algorithm:

```
1:  void foo(unsigned int n) {
2:      unsigned int count = 0;
3:      while (n > 0) {
4:          for (int i = 0; i < n; ++i) {
5:              cout << ++count << endl;
6:          }
7:          n /= 2;
8:      }
9:  }
```

And, as usual, the notes:

- This is considered a hard example because the inner loop depends on the outer loop. In the previous examples, we only had to consider one loop and whatever was in this loop.

- As usual, in the `count` declaration (line 2), we only have one operation.

- Before we discuss the two loops, let's first consider the following notes:

  - Let's assume that the print statement is one operation.
  - Whatever `n` is in the while loop, we will iterate from 0 to `n` in the inner for-loop.
  - The division operator (line 7) is considered to be one operation.

  We now consider the actual loops:

  - Whatever n currently is in the while loop, we're going to iterate n times (0 to $n - 1$) in the for loop.
  - After the for loop, we do one additional operation for division.

  We just need to figure out how many times the inner loop is iterating overall across all iterations of the while loop, that'll be our answer. So, whatever `n` currently is, the inner loop will be iterating `n` times. In this sense, we only really need to consider all cases of `n` that the inner loop will encounter.

  In our first iteration, the <u>inner loop</u> will iterate n times. The next iteration will iterate $\frac{n}{2}$ times. The next iteration will iterate $\frac{n}{4}$ times, and so on. Essentially, for the inner loop:

$$\overbrace{n}^{\text{1st iteration}} + \underbrace{\frac{n}{2}}_{\text{2nd iteration}} + \overbrace{\frac{n}{4}}^{\text{3rd iteration}} + \underbrace{\frac{n}{8}}_{\text{4th iteration}} + \cdots + 4 + 2 + 1$$
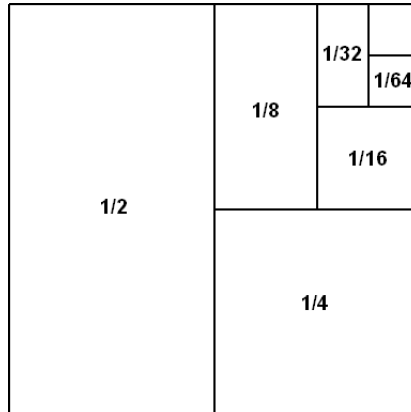
  If we pull out n, we have:

$$n\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right)$$

  Now, we need to figure out what the sum of everything in the parenthesis. Effectively, we note that we're working with:

$$n\left(1 + \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n\right) = 2$$

  Where the summation was evaluated due to the infinite geometric sum formula $S_\infty = \frac{a_1}{1-r}$.

For a better visualization of the summation, we note that the summation can be represented by[1]:

- So, the number of operations is:
$$f(n) = 2n$$

- Taking out the constant, we have:
$$f(n) = n$$

So, this algorithm runs in $O(n)$ time.

Remark: It's important to not jump straight to conclusions. Most people (when they saw this algorithm) would have assumed an $O(n^2)$ or $O(n \log(n))$ algorithm.

---

[1]Taken from Wikipedia

3.3   Common Big-O Time Complexity

It's good to know of some Big-O time complexities.

Big-O	Common Name	Notes
$O(1)$	Constant	The time complexity does not depend on the input size n.
$O(\log n)$	Logarithmic	If the input size is doubled, the number of operations is increased by a constant. One common example is binary search: if we have 8 elements, it would take 3 operations. Doubling the number of elements would result in 4 operations. Also, it does not matter what base the logarithmic function is.
$O(n)$	Linear	Your algorithm scales with the number of elements linearly. For example, twice as many elements roughly means twice as slow.
$O(n \log n)$		
$O(n^2)$	Quadratic	If the input size is doubled, we'll have quadruple the amount of elements.
$O(n^3)$	Cubic	Similarly to quadratic or linear, if the input size is doubled, the number of operations are multiplied by 8.
$O(n^a)$	Polynomial	For some constant a, this is known as polynomial time. All of the Big-O time complexities above are considered polynomial. Anything that is upper-bounded by $O(n^a)$ is called polynomial.
$O(k^n)$	Exponential	For some constant k.
$O(n!)$	Factorial	

Remark: Anything algorithm that runs in polynomial time is considered "good." Exponential and factorial time complexities are considered "bad."

3.4   Space Complexity

We can also describes algorithms by space complexity – how much space does an algorithm need for some input size n? Just like time complexity, we often use Big-O notation.

Consider a singly linked list:

```
|------|      |------|      |------|      |------|
| Head | --> |      | --> |      |      | <-- | Tail |
|------|      |------|      |------|      |------|
```
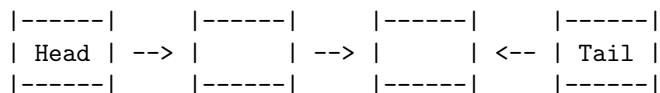
Suppose it takes k bytes to store a node. If we have n nodes, it would take roughly $k \cdot n$ bytes to represent all of these nodes. We will always have one head and one tail nodes (regardless of how many inputs we have), so these remain constant. Thus, the space complexity for a singly linked list is:
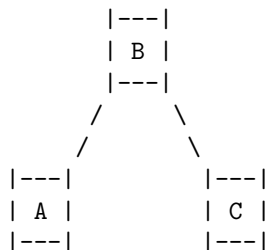
$$f(n) = kn + 2c$$

Where:

- c is a constant that represents the number of bytes needed to store the head and tail pointer.

- k is a constant that represents the number of bytes needed to store a node.

4   Trees

Before we can talk about trees, we need to talk about graphs.

4.1   Graphs

A graph is a collection of nodes and edges. For instance, here is a simple graph:

```
        |---|
        | B |
        |---|
       /     \
      /       \
 |---|         |---|
 | A |         | C |
 |---|         |---|
```

We have nodes A, B, and C, that have some connection to each other. We also have edges, or links, that connect their nodes.

There are two types of edges.

- A directed edge, where we can go from one node to another node, but not the other way around. In other words, we can think of a directed edge as an *one-way street*.

```
 |---|             |---|
 | A | ------> | B |
 |---|             |---|
```
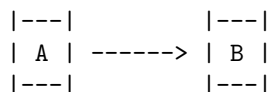
- An undirected edge, where we can go from one node to another node and vice versa. In other words, we can think of an undirected edge as a *two-way street*.

```
 |---|             |---|
 | A | ------- | B |
 |---|             |---|
```

With that said, we should observe that the simple graph that we drew above (containing nodes A, B, and C) could represent a linked list. In particular:

- If the edges were undirected, we would have a doubly linked list.

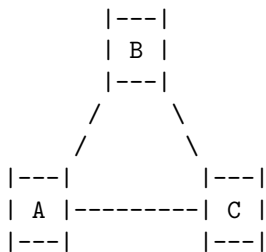- If the edges were directed (directional), we would have a singly linked list.

So, a linked list is essentially a chain of nodes in sequence. We have n nodes and $n - 1$ edges.

4.2   What are Trees?

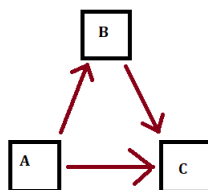A tree is a graph with two properties:

- It has no undirected cycles.

  In graphs, we could theoretically have something like:

```
        |---|
        | B |
        |---|
        /     \
       /       \
   |---|        |---|
   | A |--------| C |
   |---|        |---|
```

This is known as a **cycle** (specifically, an undirected cycle). Essentially, we can go from A to B, B to C, and then back to A from C.

We can also have something like this:

This is not a cycle because we cannot start at A and end up back at A (and the same applies with B). This is because we'll always end up stuck at C. And, if we start at C, we're stuck at C. That being said, if we converted each of the directed edges of this graph into undirected edges, we get an undirected cycle which is not allowed.
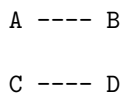
- The nodes must be connected.

  Consider the following graph:

```
          10
         /  \
        /    \
       5      15
      / \    / \
     3   7  13  20
```
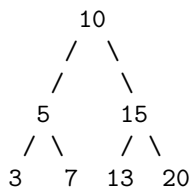
This is a graph; it has nodes and it has edges. There are no undirected cycles. Finally, all the nodes are connected (all nodes are connected to the other nodes in this tree in some way). Therefore, this is a tree.

Consider the following graph:

```
    A ---- B

    C ---- D
```

This is not a tree because nodes A and B are not connected to C and D.

4.3   Special Cases of Valid Trees

There are a few cases of valid trees that we should discuss.

- The empty ("null") tree. This tree has 0 nodes and 0 edges.

- A tree containing a single node. This has 1 node and 0 edges.
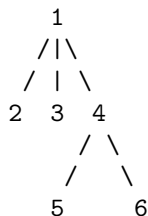
```
|----|
| 42 |
|----|
```

4.4   Rooted vs. Unrooted Trees

Now, we talk briefly about rooted vs. unrooted trees.

- A **rooted** tree is a tree with a hierarchical structure (there is some sense of direction from top to bottom). This looks something like:

```
        1
       /|\
      / | \
     2  3  4
          / \
         /   \
        5     6
```

With a rooted tree, we can consider some definitions.

- For some node, the **parent** of a node is the node that is directly connected above said node.
- For some node, the **child** (or **children**) of a node is/are the node(s) that are directly connected below said node.
- The **root** node is the node at the very top (and thus doesn't have a parent node). In the example above, node 1 is the root node.
- A node is considered to be a **leaf** node if it doesn't have any children. Nodes 2, 3, 5, 6 are considered leaves.
- A node is considered to be an **internal** node if it does have children. Nodes 1, 4 are considered internal nodes.

For example, consider node 4. This node's parent is node 1. This node has two children: node 5 and node 6.

- An **unrooted** tree is one where there is not a top-to-bottom hierarchical structure, but more of an inside-outward structure. This looks something like:

```
   2                6
    \              /
     \            /
      1 ------ 4
     /          \
    /            \
   3              5
```
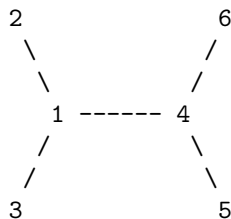
With an unrooted tree, we now have the following definitions:

- The **neighbors** of a node are nodes that are directly connected to said node. For example, node 1 has three neighbor (2, 3, 4). Node 4 also has three neighbor (1, 5, 6).
- A node is considered to be a **leaf** node if it has one neighbor.
- A node is considered to be an **internal** node if it has more than one neighbor.

4.5   Rooted Binary Trees

There are a lot of data structures that involve rooted binary trees. So, let's talk about them.

First and foremost, it's rooted (hence the name). This means:

- There is a root node.

- All of the edges have a downward hierarchical relationship.

Trees, in general, do not need to be binary. Any internal node can have any arbitrary number of children. *However*, for a **binary tree**, any node must have either 0, 1, or 2 child/children nodes.

To emphasize this point, consider the binary tree:

```
        1
       / \
      /   \
     2     3
          / \
         /   \
        4     5
```
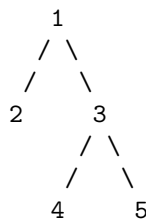
Here, we see that:

- Nodes 2, 4, 5 have 0 children.

- Node 1 has 2 children.

- Node 3 has 2 children.

This is a perfect binary tree because every node that's internal has exactly 2 children and every leaf node has exactly 0 children. *However*, a binary tree can support single-child relationships. For example, let's consider the same binary tree, with an additional node:

```
        1
       / \
      /   \
     2     3
    /     / \
   /     /   \
  6     4     5
```
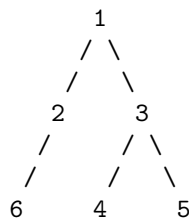
Here, we see that:

- Nodes 4, 5, 6 have 0 children.

- Node 1 has 2 children.

- Node 3 has 2 children.

- Node 2 has 1 child.

And this is still a valid binary tree (even though 2 only has one child.)
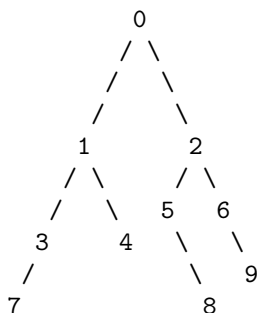
4.6 Tree Traversals

If we store our data in some tree data structure, we need a tree traversal algorithm in order to iterate through all nodes (and our data). In this class, we'll talk about the following tree traversal algorithms:

- **Preorder:** <u>V</u>isit, <u>L</u>eft, <u>R</u>ight.

- **In-Order:** Left, Visit, Right.

- **Postorder:** Left, Right, Visit.

- **Level-Order:** 1st Level (Left to Right), 2nd Level (Left to Right), . . .

We should note that preorder, in-order, and postorder traversals are examples of **depth first search** (DFS) whereas level-order traversal is an example of **breadth first search** (BFS). Regardless of the tree traversal algorithm we use, we will always start at the root.

4.6.1 Preorder Traversal (V, L, R)

Consider the following tree:

```
            0
           / \
          /   \
         /     \
        1       2
       / \     / \
      /   \   5   6
     3     4   \   \
    /           \   9
   7             8
```
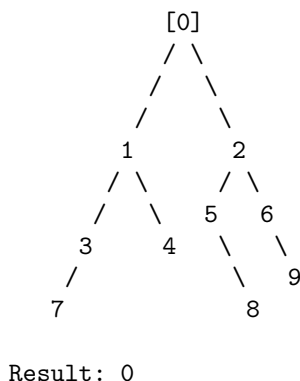
In a preorder traversal, we are guaranteed that for any given node, its ancestors were visited before the node itself. So, in the tree above, if we visit node 1, then we are guaranteed that node 0, its parent, has already been visited. If we visit node 3, then we are guaranteed that node 1 and node 0 have already been visited.

Preorder traversal has three steps:

- Visit the current node.

- Recurse to the left node.

- Recurse to the right node.

Going back to the example tree, let's run through the preorder traversal algorithm. Note that, in the tree, I'll denote `[n]` as saying that we are at node n and `(n)` as saying that we have already visited node n.

- We start at the root node (0).

```
           [0]
           / \
          /   \
         /     \
        1       2
       / \     / \
      /   \   5   6
     3     4   \   \
    /           \   9
   7             8

      Result: 0
```

- Now, we traverse to the left node (1).

```
            (0)
            / \
           /   \
          /     \
        [1]      2
        / \     / \
       /   \   5   6
      3     4   \   \
     /           \   9
    7             8
```
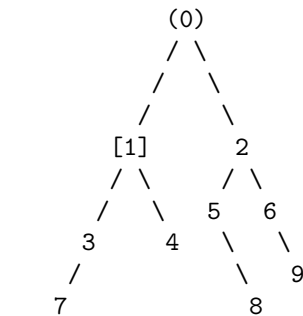
Result: 0, 1

- Now, we traverse to the left node (3).

```
            (0)
            / \
           /   \
          /     \
        (1)      2
        / \     / \
       /   \   5   6
     [3]    4   \   \
     /           \   9
    7             8
```
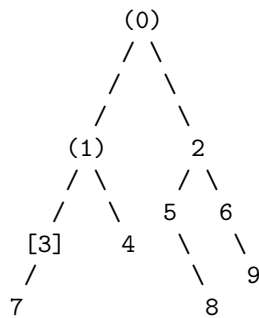
Result: 0, 1, 3

- Now, we traverse to the left node (7).

```
            (0)
            / \
           /   \
          /     \
        (1)      2
        / \     / \
       /   \   5   6
     (3)    4   \   \
     /           \   9
   [7]          [8]
```
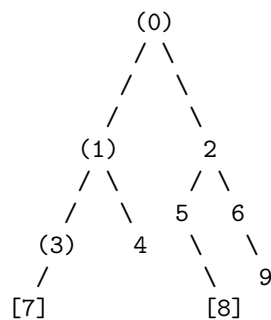
Result: 0, 1, 3, 7

- Now, we would traverse to the left node. However, there's nothing there! So, we're done traversing left. Here, we can try traversing right. However, once again, there is nothing to traverse to. So, we're done with node 7. Let's move back to node 3.

```
            (0)
            / \
```

```
          /   \
         /     \
       (1)      2
       / \     / \
      /   \   5   6
   [(3)]   4   \   \
    /           \   9
  (7)           [8]
```
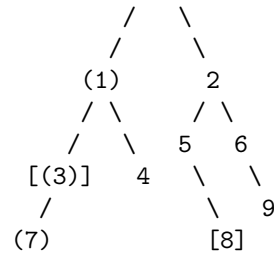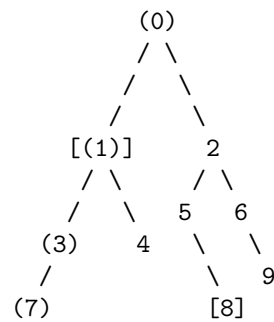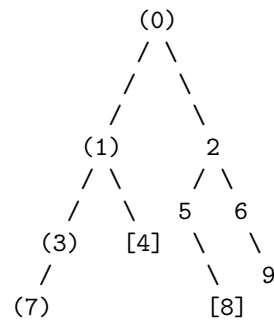
Result: 0, 1, 3, 7

- Now that we're back at node 3, let's try traversing right. However, there is no right node. So, we're done with 3 and we move back up to node 1.

```
            (0)
            / \
           /   \
          /     \
       [(1)]     2
        / \     / \
       /   \   5   6
     (3)    4   \   \
     /           \   9
   (7)           [8]
```

Result: 0, 1, 3, 7

- Now that we're back at node 1, let's try traversing right. In this case, we are able to, so we traverse to the right node (4).

```
            (0)
            / \
           /   \
          /     \
        (1)      2
        / \     / \
       /   \   5   6
     (3)  [4]  \   \
     /           \   9
   (7)           [8]
```

Result: 0, 1, 3, 7, 4

- At node 4, there is no left or right node. So, we're done with node 4. and we can move back up to node 1. Since we've already visited node 1 and its children, we can move back up to the root node, node 0.

```
         [(0)]
          / \
         /   \
        /     \
```

```
      (1)       2
      / \      / \
     /   \    5   6
   (3)   (4)  \    \
   /           \    9
 (7)          [8]
```
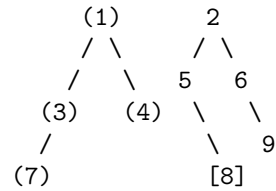
Result: 0, 1, 3, 7, 4

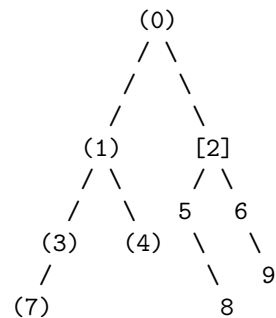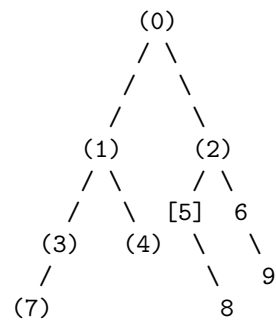- Now that we're at node 0, we can try to traverse right (since that's the only operation we can do). Since there are right nodes, we traverse to the right node (2).

```
            (0)
            / \
           /   \
          /     \
        (1)     [2]
        / \     / \
       /   \   5   6
     (3)   (4) \    \
     /          \    9
   (7)          8
```

Result: 0, 1, 3, 7, 4, 2

- Node 2 has a left and right child. Of course, we're going to traverse to the left node (5).

```
            (0)
            / \
           /   \
          /     \
        (1)     (2)
        / \     / \
       /   \  [5]  6
     (3)   (4) \    \
     /          \    9
   (7)          8
```

Result: 0, 1, 3, 7, 4, 2, 5

- Node 5 does not have a left child, but node 5 does have a right child, so we traverse to the right node (8).

```
            (0)
            / \
           /   \
          /     \
        (1)     (2)
        / \     / \
       /   \  (5)  6
     (3)   (4) \    \
```

```
        /           \   9
      (7)           [8]
```
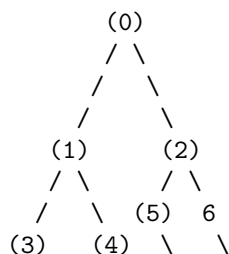
```
Result: 0, 1, 3, 7, 4, 2, 5, 8
```

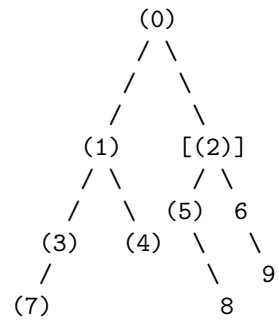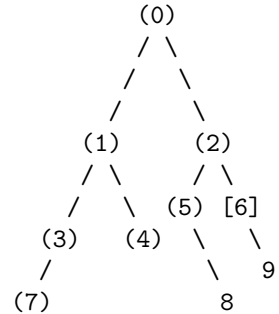- At node 8, we cannot traverse left or right. So, we're done and we traverse back to node 5; since we've done all operations possible on node 5 (we visited it, we tried to traverse left, and we tried to traverse right), we go back to node 2.
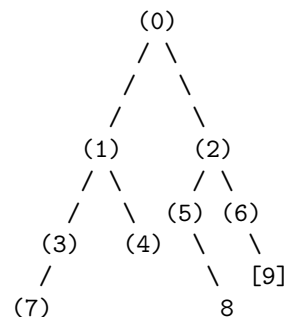
```
            (0)
            / \
           /   \
          /     \
        (1)     [(2)]
        / \     / \
       /   \  (5)  6
     (3)   (4) \    \
     /          \    9
   (7)           8
```

```
Result: 0, 1, 3, 7, 4, 2, 5, 8
```

- At node 2, we can only traverse to the right node. Since node 2 has a right child, we can traverse to the right node (6).

```
            (0)
            / \
           /   \
          /     \
        (1)     (2)
        / \     / \
       /   \  (5) [6]
     (3)   (4) \    \
     /          \    9
   (7)           8
```

```
Result: 0, 1, 3, 7, 4, 2, 5, 8, 6
```

- At node 6, we cannot traverse to the left node since there is no left child. But, there is a right child so we traverse to the right node (9).

```
            (0)
            / \
           /   \
          /     \
        (1)     (2)
        / \     / \
       /   \  (5) (6)
     (3)   (4) \    \
     /          \   [9]
   (7)           8
```

```
Result: 0, 1, 3, 7, 4, 2, 5, 8, 6, 9
```

- At this point, we traverse back to node 6. Since we're done with node 6, we traverse back to node 2. Since we're done with node 2, we traverse back to node 0 (root node). Since we're done with node 0, we're done! Our final result is:

```
0, 1, 3, 7, 4, 2, 5, 8, 6, 9
```

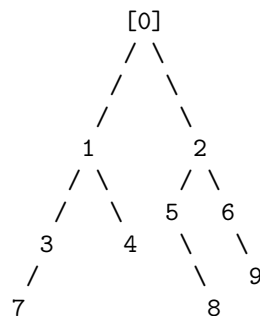### 4.6.2   In-order Traversal (L, V, R)

This algorithm works like so:

- Traverse down the left subtree.

- Once that's done, visit the current node.

- Then traverse down the right subtree.

Unlike the other traversal algorithms, an in-order traversal only really makes sense in the context of a binary tree. **Note that**, in the tree, I'll denote [n] as saying that we are at node n and (n) as saying that we *have been* at node n (but not necessarily visited it yet).

Consider the same tree example from the previous example.

- We start at node 0, the root node.

```
          [0]
          / \
         /   \
        /     \
       1       2
      / \     / \
     /   \   5   6
    3     4   \   \
   /           \   9
  7             8

Result:
```

- Because of in-order traversal, we immediately traverse to the left node. In this case, we go to node 1.
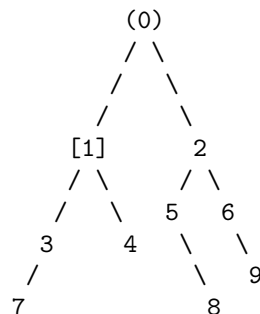
```
          (0)
          / \
         /   \
        /     \
      [1]      2
      / \     / \
     /   \   5   6
    3     4   \   \
   /           \   9
  7             8

Result:
```

- Once again, we go to the left node. For the sake of saving space, we're going to condense two steps into one. First, we traverse node 3, and then we traverse node 7.
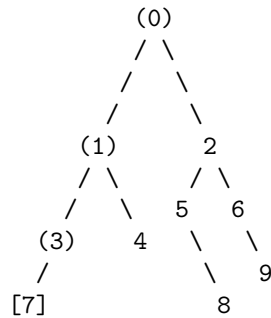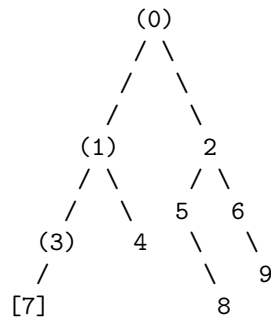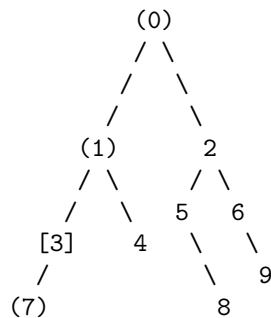
```
            (0)
            / \
           /   \
          /     \
        (1)      2
        / \     / \
       /   \   5   6
     (3)    4   \   \
     /           \   9
   [7]            8

   Result:
```

- We see that node 7 doesn't have a left child node! So, we *visit* this node.

```
            (0)
            / \
           /   \
          /     \
        (1)      2
        / \     / \
       /   \   5   6
     (3)    4   \   \
     /           \   9
   [7]            8

   Result: 7
```

Afterwards, we attempt to traverse to the right child node. However, there is no right child node associated with node 7, so we go back to the previous node (since we're done processing node 7).

```
            (0)
            / \
           /   \
          /     \
        (1)      2
        / \     / \
       /   \   5   6
     [3]    4   \   \
     /           \   9
   (7)            8

   Result: 7
```

- Now that we're back at node 3, we can formally *visit* it (since we're done going to the left child node).

```
            (0)
            / \
```

```
            /   \
           /     \
         (1)      2
         / \     / \
        /   \   5   6
      [3]    4   \   \
      /           \   9
    (7)            8
```
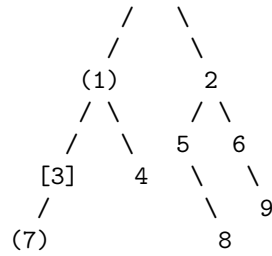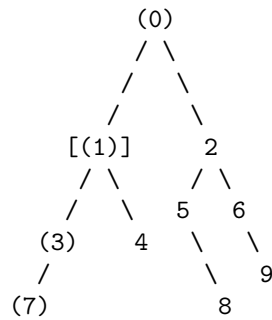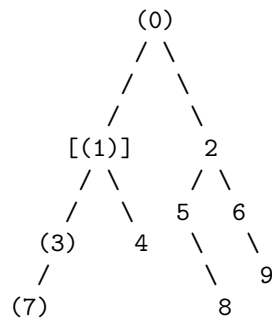
Result: 7, 3

After this, we can go back to the previous node: node 1.

```
             (0)
             / \
            /   \
           /     \
        [(1)]     2
         / \     / \
        /   \   5   6
      (3)    4   \   \
      /           \   9
    (7)            8
```
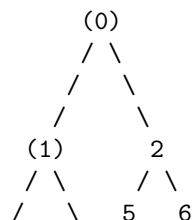
Result: 7, 3

- Since we've already visited node 1's left child node, we can now visit node 1 itself.

```
             (0)
             / \
            /   \
           /     \
        [(1)]     2
         / \     / \
        /   \   5   6
      (3)    4   \   \
      /           \   9
    (7)            8
```

Result: 7, 3, 1

Now, we attempt to traverse to node 1's right child node. Because node 1 *does* have a right child node, we can traverse to it, and so we traverse to node 4.

```
             (0)
             / \
            /   \
           /     \
         (1)      2
         / \     / \
        /   \   5   6
```

```
     (3)    [4]  \    \
      /            \   9
     (7)           8
```
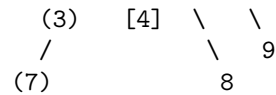
Result: 7, 3, 1

- From node 4, we try to traverse to this node's left child. However, node 4 doesn't have a left child. So, we can actually visit node 4.
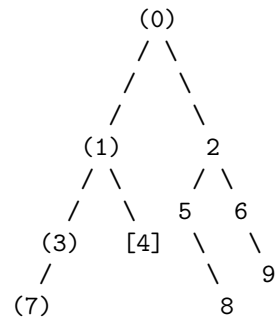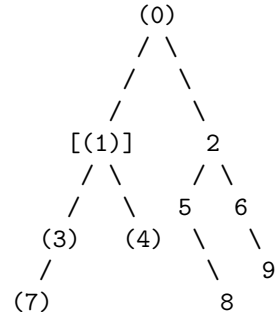
```
            (0)
            / \
           /   \
          /     \
        (1)      2
        / \     / \
       /   \   5   6
     (3)   [4] \    \
      /          \   9
     (7)          8
```
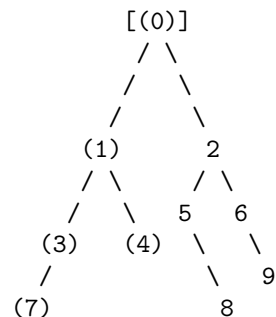
Result: 7, 3, 1, 4

Once we visit node 4, we try to traverse to this node's right child. Again, this node doesn't have a right child, so we traverse back to the parent node.

```
            (0)
            / \
           /   \
          /     \
       [(1)]     2
        / \     / \
       /   \   5   6
     (3)   (4) \    \
      /          \   9
     (7)          8
```
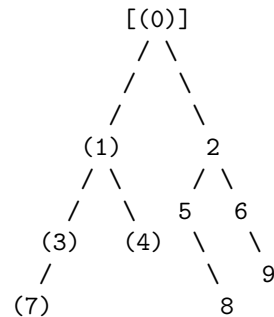
Result: 7, 3, 1, 4

- We're back at node 1, expect there's nothing to do at node 1 (since we're done with all possible operations). So, we go back to node 0 (the root node).
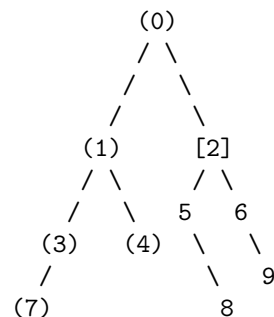
```
           [(0)]
            / \
           /   \
          /     \
        (1)      2
        / \     / \
       /   \   5   6
     (3)   (4) \    \
      /          \   9
     (7)          8
```

```
Result: 7, 3, 1, 4
```

Because we're done visiting the left child of the root node, we can now visit the node itself.

```
        [(0)]
         / \
        /   \
       /     \
     (1)      2
     / \     / \
    /   \   5   6
  (3)   (4) \    \
  /          \    9
(7)           8
```

```
Result: 7, 3, 1, 4, 0
```

At this point, we can traverse to the root node's right neighbor.

```
         (0)
         / \
        /   \
       /     \
     (1)     [2]
     / \     / \
    /   \   5   6
  (3)   (4) \    \
  /          \    9
(7)           8
```

```
Result: 7, 3, 1, 4, 0
```

- For the sake of conciseness, I'll omit the remaining steps. However, the result of in-order traversal is:

```
Result: 7, 3, 1, 4, 0, 5, 8, 2, 6, 9
```
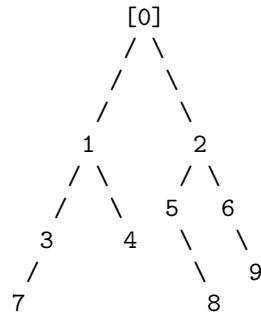
4.6.3   Postorder Traversal (L, R, V)

In postorder traversal, I'm guaranteed that, before I visit any of a node, that I have visited that node's descendents. The idea is as follows:

- Start by visiting the left nodes.

- Then, visit the right nodes.

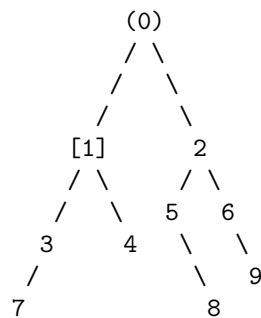- After those nodes are all visited, then visit the current

As usual, we're going to stick with the binary tree that we've used. **Note that**, in the tree, I'll denote [n] as saying that we are at node n and (n) as saying that we *have been* at node n (but not necessarily visited it yet).
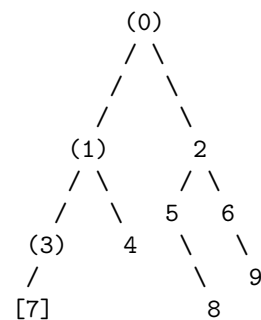
- Begin at the root node as always.

```
                    [0]
                    / \
                   /   \
                  /     \
                 1       2
                / \     / \
               /   \   5   6
              3     4   \   \
             /           \   9
            7             8
```

        Result:

- Let's now traverse to node 1, or node 0's left child node.

```
                    (0)
                    / \
                   /   \
                  /     \
                [1]      2
                / \     / \
               /   \   5   6
              3     4   \   \
             /           \   9
            7             8
```

        Result:

- From node 1, we can traverse to node 3 and then node 7.

```
                    (0)
                    / \
                   /   \
                  /     \
                 (1)     2
                 / \    / \
                /   \  5   6
              (3)    4  \   \
              /          \   9
            [7]           8
```
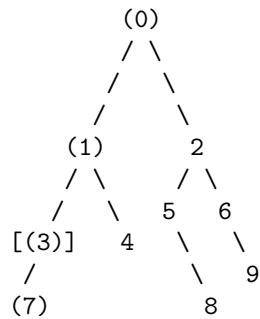
        Result:

At this point, since node 7 doesn't have any child nodes, we cannot traverse to node 7's left or right child nodes. Therefore, we can visit node 7.
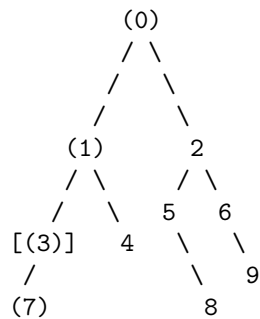
```
                    (0)
                    / \
                   /   \
                  /     \
                 (1)     2
```

```
        / \      / \
       /   \    5   6
     (3)    4    \   \
     /            \   9
   [7]            8
```
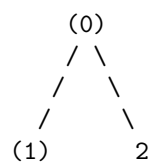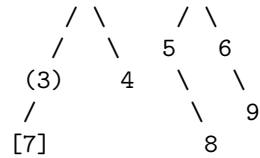
```
Result: 7
```

- Now, we can traverse back to node 3.

```
              (0)
              / \
             /   \
            /     \
          (1)      2
          / \     / \
         /   \   5   6
      [(3)]   4   \   \
       /           \   9
     (7)           8
```
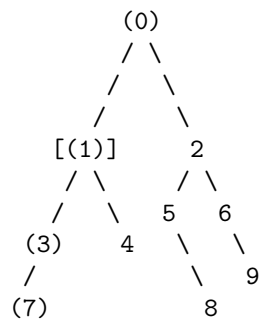
```
Result: 7
```

At node 3, we've already traversed to the left child node (7). Since node 3 doesn't have a right child node, we cannot traverse to that node. Therefore, we can visit node 3:

```
              (0)
              / \
             /   \
            /     \
          (1)      2
          / \     / \
         /   \   5   6
      [(3)]   4   \   \
       /           \   9
     (7)           8
```

```
Result: 7, 3
```

- Now, we can traverse to node 1.

```
              (0)
              / \
             /   \
            /     \
         [(1)]     2
          / \     / \
         /   \   5   6
      (3)    4   \   \
      /           \   9
    (7)           8
```

```
                    Result: 7, 3
```

At node 1, we now need to visit the right child node, so we do that.
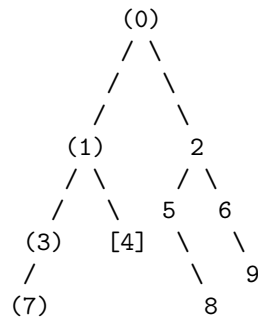
```
            (0)
            / \
           /   \
          /     \
        (1)      2
        / \     / \
       /   \   5   6
     (3)   [4] \    \
     /          \    9
    (7)          8

Result: 7, 3
```
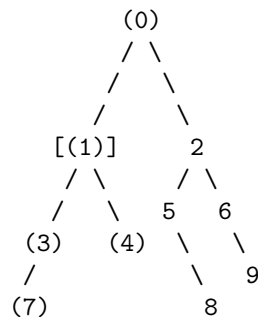
Since node 4 doesn't have a left or right child node, we cannot traverse to the left or right child node. So, we visit node 4.
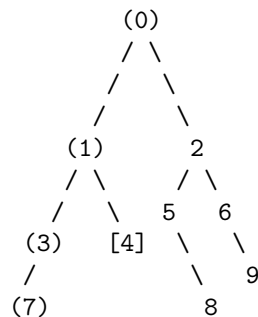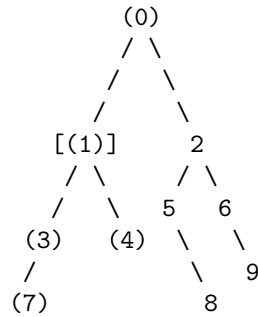
```
            (0)
            / \
           /   \
          /     \
        (1)      2
        / \     / \
       /   \   5   6
     (3)   [4] \    \
     /          \    9
    (7)          8

Result: 7, 3, 4
```

- Now, we can traverse back to node 1.

```
            (0)
            / \
           /   \
          /     \
       [(1)]     2
        / \     / \
       /   \   5   6
     (3)   (4) \    \
     /          \    9
    (7)          8

Result: 7, 3, 4
```
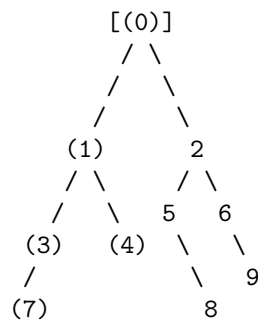
Since we already traversed to node 1's left and right child nodes (and thus node 1's left and right subtrees), we can visit node 1.
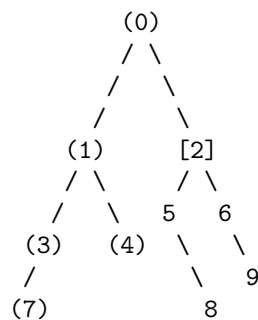
```
              (0)
              / \
             /   \
            /     \
         [(1)]     2
         / \      / \
        /   \    5   6
      (3)   (4)  \    \
      /           \    9
    (7)            8

       Result: 7, 3, 4, 1
```

- Now, we can traverse back to node 0.

```
             [(0)]
              / \
             /   \
            /     \
          (1)      2
          / \     / \
         /   \   5   6
       (3)   (4) \    \
       /           \   9
     (7)            8

       Result: 7, 3, 4, 1
```

Now that we're at node 0, we need to traverse to the right child node. So, we do that.

```
              (0)
              / \
             /   \
            /     \
          (1)     [2]
          / \     / \
         /   \   5   6
       (3)   (4) \    \
       /           \   9
     (7)            8

       Result: 7, 3, 4, 1
```

- We've now traversed to node 2. At this point, we need to traverse to node 2's left and right child nodes. To save some steps, we'll do it all in one go. Namely:

  - We traverse to node 5, and then node 8 (since node 5 doesn't have a left child node).
  - Because node 8 doesn't have a left or right child node, we visit node 8 and go back to node 5.
  - Because we've visited node 5's left and right child nodes (noting that node 5 doesn't even have a left child node), we can visit node 5 and then go back to node 2.
  - Now that we're at node 2, we can traverse to node 6, and then back to node 9.
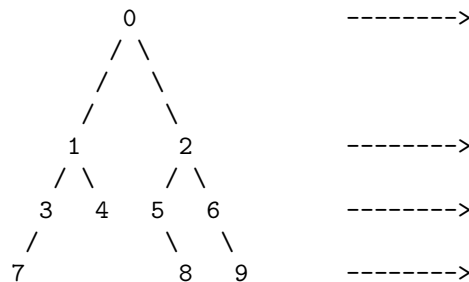
– At node 9, there is no left or right child node so we can visit node 9 and go back to node 6.

– At node 6, because there's no left child node and the right child node has already been visited, we can visit node 6 and go back to node 2.

– Since we've visited all of node 2's left and right child nodes, we can visit node 2 and go back to node 0, the root node.

– Since we've visited all of node 0's left and right child nodes, we can visit node 0. Thus, we're done.

- The final result, then, is:

```
Result: 7, 3, 4, 1, 8, 5, 9, 6, 2, 0
```

4.6.4   Level-Order Traversal

Unlike the other three algorithm, level-order traversal is an example of breadth first search.

The idea is relatively simple. We're traversing with respect to distance away from the root. In this sense, we're traversing like so:

```
      0                 ------->
     / \
    /   \
   /     \
  1       2             ------->
 / \     / \
3   4   5   6           ------->
/       \   \
7        8   9          ------->
```

So, essentially, if we traversed using level-order traversal, our result would be:

```
Result: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
```