1 Properties of Rings

We begin by talking about a few important properties.

1.1 Basic Rules of Multiplication

Theorem 1.1

For all $a \in R$, we have:

$$a0 = 0a = 0$$

Proof. We know that:

$$0a = (0+0)a = 0a + 0a$$

Subtracting both sides by $0a$ gives:

$$0 = 0a + (0a - 0a) \implies 0 = 0a$$

By symmetry, we can do the same for $0a$. Therefore, we are done. □

Theorem 1.2

For all $a, b \in R$, we have:

$$a(-b) = (-a)b = -(ab)$$

Proof. First, we have:

$$a(-b) + ab = a(-b+b) = a0 = 0$$

Now, if we add $-(ab)$ to both sides, we have:

$$a(-b) + ab + -(ab) = -(ab) \implies a(-b) = -(ab)$$

By symmetry, $(-a)b = -(ab)$ as well. □

Theorem 1.3

For all $a, b \in R$, we have:

$$(-a)(-b) = ab$$

Proof.

$$\begin{aligned}
&(-a)0 = 0 \\
&\iff (-a)(b + (-b)) = 0 \\
&\iff (-a)b + -a(-b) = 0 \\
&\iff -(ab) + -a(-b) = 0 \\
&\iff ab + -(ab) + -a(-b) = ab \\
&\iff -a(-b) = ab
\end{aligned}$$

So, we are done. □

Theorem 1.4

For all $a, b, c \in R$, we have:

$$a(b-c) = ab - ac \text{ and } (b-c)a = ba - ca$$

Proof.

$$\begin{aligned} a(b-c) &= ab + -(ac) \\ &= ab + (-a)c \\ &= ab - ac \end{aligned}$$

By symmetry, we can apply the other side as well. So, we are done. □

1.2 Rules of Multiplication with Unity Element

Theorem 1.5

For all $a \in R$ where R has a unity element 1, we have:

$$(-1)a = -a$$

Proof. Applying the theorem that we proved:

$$(-1)a = -(1a) = -a$$

So, we are done. □

Theorem 1.6

$$(-1)(-1) = 1$$

Proof. Applying the theorem that we proved:

$$(-1)(-1) = 1(1) = 1$$

So, we are done. □

1.3 Uniqueness of Unity and Inverses

Theorem 1.7

If a ring has a unity, it is unique. If a ring element has a multiplicative inverse, it is also unique.

Proof. We will prove both parts individually. Suppose R is a ring.

1. Suppose e and e' are unity elements in a ring R. Then, we know that:
   - $e = ee'$ since e' is a unity.
   - $e' = ee'$ since e is a unity.

   Therefore:

   $$e = ee' = e'$$

   Which means that the unity must be unique.

2. Suppose $a \in R$ and further suppose that x and y are both multiplicative inverses of a. Then:

   $$x = x1 = x(ay) = (xa)y = 1y = y$$

Therefore, $x = y$ and the two inverses are equal.

Therefore, we are done. $\square$

2 Subring

Recall that, with groups, we have objects called *subgroups*. The same thing applies here: with rings, we have objects called *subrings*.

Definition 2.1: Subring

A nonempty subset S of a ring R is a **subring** of R if S itself is a ring with the operations of R.

2.1 Subring Test

Theorem 2.1: Subring Test

A nonempty subset S of a ring R is a subring if S is closed under subtraction and multiplication; that is, if $a - b \in S$ and $ab \in S$ whenever $a, b \in S$.

2.2 Examples of Subrings

Below are some examples of subrings.

2.2.1 Example 1: Trivial Subring

The trivial subring $\{0\}$ is a subring of any ring R. This is because:

$$0(0) \in R \qquad 0 - 0 \in R$$

2.2.2 Example 2: Ring

Any ring R is a subring of itself. This is because for any $a, b \in R$, we know that $a - b = a + (-b) \in R$ and $ab \in R$.

2.2.3 Example 3: Integers

For any positive integer n, the set below is a subring of the integers $\mathbb{Z}$:

$$n\mathbb{Z} = \{0, \pm n, \pm 2n, \pm 3n, \dots\}$$

Take any $a, b \in \mathbb{Z}$. Then, suppose we have an and bn. We know that:

$$an - bn = (a - b)n \in \mathbb{Z}$$

$$an(bn) = anbn$$

Since $anb \in \mathbb{Z}$, it follows that $(anb)n \in n\mathbb{Z}$.