1 Distributions and Densities

1.1 CDF/PDF Transformations

We can use the distributions that we've talked about to build more *complex* distributions.

Theorem 1.1: CDF Transformation Theorem

Let X be a continuous random variable and Φ is a strictly monotone function. Then, the random variable

$$Y = \Phi(X)$$

has CDF

1. $F_Y(y) = F_X(\Phi^{-1}(y))$, if Φ is increasing.
2. $F_Y(y) = 1 - F_X(\Phi^{-1}(y))$, if Φ is decreasing.

Proof. Suppose that Φ is strictly increasing. Then, the inverse function Φ^{-1} exists and is increasing. Therefore,

$$\Phi(X) \leq y$$

if and only if

$$\Phi^{-1}(\Phi(X)) = X \leq \Phi^{-1}(y).$$

Hence,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(\Phi(X) \leq y) = \mathbb{P}(X \leq \Phi^{-1}(y)) = F_X(\Phi^{-1}(y)),$$

as claimed. The strictly decreasing case is similar. $\square$

Note that, by differentiating using the Calculus Chain Rule, we obtain the following corollary.

Corollary 1.1: PDF Transformation Theorem

Let X be a continuous random variable and Φ a strictly monotone function. Then, the random variable $Y = \Phi(X)$ has PDF

$$f_Y(y) = f_X(\Phi^{-1}(y)) \left| \frac{d}{dy} \Phi^{-1}(y) \right|.$$

Note that there is a transformation theorem in the case of discrete random variables, but it is much easier. In particular, the PMF of a random variable $Y = \Phi(X)$ is simply the function

$$p_Y(y) = \sum_{x : \Phi(x) = y} p_X(x).$$

(Example.) Let U be Uniform on $[0, 1]$. Then, consider the transformed version

$$V = 1 - U,$$

which is also uniform on $[0, 1]$. Note that

$$\Phi(u) = 1 - u$$

is decreasing, and $\Phi^{-1}(v) = 1 - v$. Therefore, by the CDF Transformation Theorem, we have

$$F_V(v) = 1 - F_U(\Phi^{-1}(v)) = 1 - F_U(1 - v) = 1 - (1 - v) = v.$$

Hence, V is Uniform on $[0, 1]$.

(Example.) Let X be a standard Normal$(0, 1)$. Consider

$$Y = X^2.$$

Recall that X has PDF

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

The function

$$\Phi(x) = x^2$$

is not either only increasing or only decreasing; rather, it is decreasing when $x < 0$ and increasing when $x > 0$. Therefore, we cannot apply the Transformation Theorem as is, but we can still apply the idea of its proof.

Note that

$$F_Y(y) = \mathbb{P}(X^2 \leq y) = \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}).$$

So, differentiating (to get the PDF), we find

$$f_Y(y) = \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}} = \frac{1}{\sqrt{2\pi y}} e^{-y/2}.$$

Note that this is known as the **Chi-squared** distribution.

(Example.) Select a random point (X, Y) in the two-dimensional plane $\mathbb{R}$, by selecting two independent standard Normal$(0, 1)$ random variables X and Y. Let

$$R = \sqrt{X^2 + Y^2}$$

be the distance from the origin $(0, 0)$. How is R distributed?

Note that X^2 and Y^2 both have Chi-Squared distribution; in particular,

$$f_{X^2}(s) = \frac{1}{\sqrt{2\pi s}} e^{-s/2}$$

for $s > 0$ and

$$f_{Y^2}(s) = \frac{1}{\sqrt{2\pi t}} e^{-t/2}$$

for $t > 0$. Since X and Y are independent, X^2 and Y^2 are also independent. So, if $R^2 = X^2 + Y^2 = r$, then we need $X^2 = s$ for some $s \geq 0$ and then $Y^2 = r - s \geq 0$. Hence,

$$f_{R^2}(r) = \int_0^r f_{X^2}(s) f_{Y^2}(r - s) ds = \frac{1}{2\pi} \int_0^r \frac{e^{-s/2}}{\sqrt{s}} \frac{e^{-(r-s)/2}}{\sqrt{r - s}} ds = \frac{e^{-r/2}}{2}.$$

Finally, we apply the Transformation Theorem one more time to obtain the PDF of $R = \sqrt{R^2}$; in particular, $f_R(r) = re^{-r^2/2}$ for $r > 0$. This is called the **Rayleigh** distribution.