

1 Nonlinear Equations & Bisection Method (Section 3.1)

Let's consider the problem of finding zeros.

(Example.) Suppose we have the functions $\sin(x)$ and e^x , and suppose we want to find the values of x such that $\sin(x) - e^x$. Let

$$f(x) = \sin(x) - e^x$$

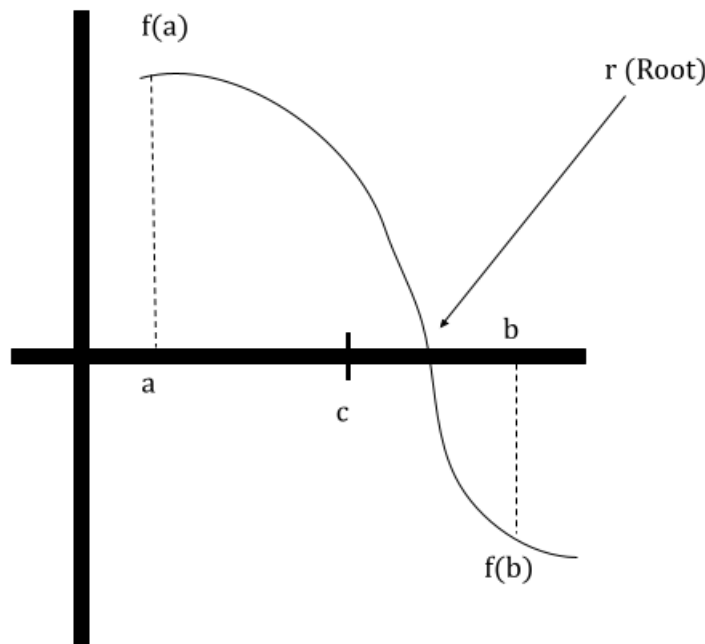
so that we can solve for $\sin(x) - e^x = 0$. We want to seek the “zero” (i.e., the root) of $f(x)$.

1.1 Bisection Method

The bisection method makes the assumption that $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous. We want to consider the interval $[a, b]$ so that

$$f(a) \cdot f(b) < 0.$$

This implies that either $f(a)$ or $f(b)$, but not both, are negative (i.e., $\text{sgn}(f(a)) \neq \text{sgn}(f(b))$).



Then, we can solve for

$$c = \frac{b + a}{2} = \frac{a + (b - a)}{2}.$$

c is the midpoint of a and b . Then, we let check

- If $f(a) \cdot f(c) < 0$, then we let $b \leftarrow c$ and start with the new interval $[a, b]$.
- Otherwise, $f(b) \cdot f(c) < 0$ and so we let $a \leftarrow c$ and start with the new interval $[a, b]$.

We keep repeating this until the interval becomes sufficiently small.

1.2 Algorithm Idea and Stopping Conditions

Let

- M be the maximum iterations,

- δ be the interval tolerance ($|b - a| < \delta$), and
- ϵ be the zero tolerance ($|f(c)| < \epsilon$).

The algorithm takes in five inputs: $(M, \delta, \epsilon, a, b)$, where a and b (such that $a \leq b$) represents the interval endpoints (i.e., $[a, b]$).

Algorithm 1 Bisection Method

```

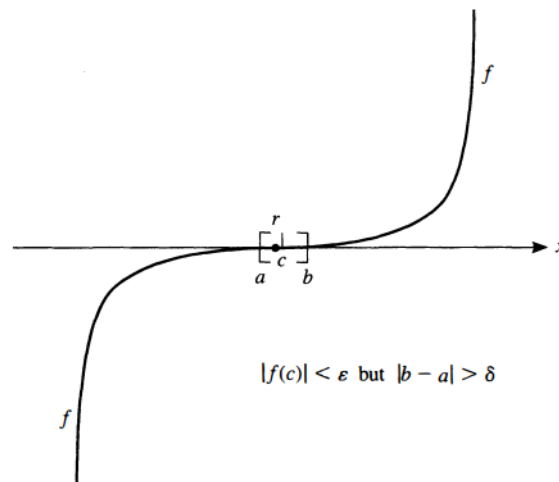
1: function BISECTION( $M, \delta, \epsilon, a, b$ )
2:    $u \leftarrow f(a)$ 
3:    $v \leftarrow f(b)$ 
4:    $e = b - a$ 
5:   if  $\text{sgn}(u) = \text{sgn}(v)$  then
6:     Error
7:   end if
8:   for  $k \leftarrow 1$  to  $M$  do
9:      $e \leftarrow e/2$ 
10:     $c \leftarrow a + e$ 
11:     $w \leftarrow f(c)$ 
12:    if  $|e| < \delta$  or  $|w| < \epsilon$  then
13:      Break
14:    end if
15:    if  $\text{sgn}(u) = \text{sgn}(v)$  then
16:       $a \leftarrow c$ 
17:       $u \leftarrow w$ 
18:    else
19:       $b \leftarrow c$ 
20:       $v \leftarrow w$ 
21:    end if
22:  end for
23:  return  $c$ 
24: end function

```

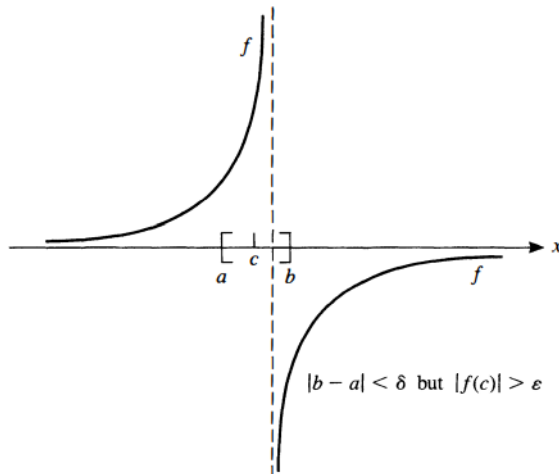
1.3 Tolerances

We introduce the tolerances, δ and ϵ , for robustness.

- We might have $|f(c)| < \epsilon$ but $|b - a| > \delta$.



- We also have $|b - a| < \delta$ and $|f(c)| > \epsilon$.



1.4 Error Analysis

Let $I_i = [a_i, b_i]$ be an interval. Then, we essentially have a sequence of intervals, $I_0, I_1, I_2, I_3, \dots$, where $a_0 \leq a_1 \leq a_2 \leq \dots \leq b_0$ and $b_0 \geq b_1 \geq b_2 \geq \dots \geq a_0$.

$$\begin{array}{ccccccc} a_{m+1} & & b_{m+1} & & & & \\ | & \text{---X---} & | & \text{-----} & | & & \\ a_m & & c_m & & b_m & & \end{array}$$

For $m \geq 0$, We know that $b_{m+1} - a_{m+1} = \frac{1}{2}(b_m - a_m)$, so applying it repeatedly gives us

$$\begin{aligned} b_m - a_m &= \frac{1}{2}(b_{m-1} - a_{m-1}) \\ &= \left(\frac{1}{2}\right)^m (b_0 - a_0) \end{aligned}$$

The sequence, b_m and a_m , are monotonic and convergent. Because we're constantly dividing the interval by 2, we know that

$$\lim_{m \rightarrow \infty} (b_m - a_m) = \lim_{m \rightarrow \infty} \left(\frac{1}{2}\right)^m (b_0 - a_0) = 0.$$

We also know that, as a and b are the left and right endpoints of this interval, both a and b will converge to the root of the equation, r , i.e.,

$$\lim_{m \rightarrow \infty} a_m = \lim_{m \rightarrow \infty} b_m = r.$$

Then,

$$\lim_{m \rightarrow \infty} f(a_m)f(b_m) \leq 0.$$

This yields

$$f(r)f(r) \leq 0 \implies f(r) = 0.$$

Therefore,

$$\begin{array}{ccccccc} | & \text{-----r---} & | & \text{-----} & | & & \\ a_m & & c_m & & b_m & & \end{array}$$

In this sense, if at a certain stage in the process we have the interval I_n and the process is now stopped, the root is certain to lie in this interval. However, the best estimate of the root at this stage is not a_n or b_n , but the midpoint of the interval, c_n . In this sense, the error is then bounded by

$$|r - c_m| \leq \frac{1}{2}|b_m - a_m|.$$

This gives us the following theorem.

Theorem 1.1: Bisection Method

If $I_i = [a_i, b_i]$ is an interval and I_0, I_1, \dots, I_n denote the intervals in the bisection method, then the limits $\lim_{n \rightarrow \infty} a_n$ and $\lim_{n \rightarrow \infty} b_n$ exist, are equal, and represent a zero of f . If $r = \lim_{n \rightarrow \infty} c_n$ and $c_n = \frac{1}{2}(a_n + b_n)$, then

$$|r - c_n| \leq \frac{1}{2}|b_n - a_n| = 2^{-(n+1)}|b_0 - a_0|.$$

(Exercise.) Suppose $[a_0, b_0] = [50, 63]$. How many steps needs to be done using the bisection method to compute a root with relative accuracy of 10^{-12} ?

We want to seek $|r - c_m|/|r| \leq 10^{-12}$. This means that

$$|r - c_m|/50 \leq 10^{-12}.$$

(Note that we asked for the relative error.) The sufficient condition is

$$|r - c_m|/50 \leq \left(\frac{1}{2}\right)^{m+1} (b_0 - a_0)/50 \leq 10^{-12}.$$

From the theorem, we have

$$2^{-(n+1)}(13/50) \leq 10^{-12}.$$

From there, we can solve for n .