1 Basis Spline (Section 6.5)

The idea is that we'll have *limitless* knots. So, for the knots t_i such that $i \in \mathbb{Z}$ and

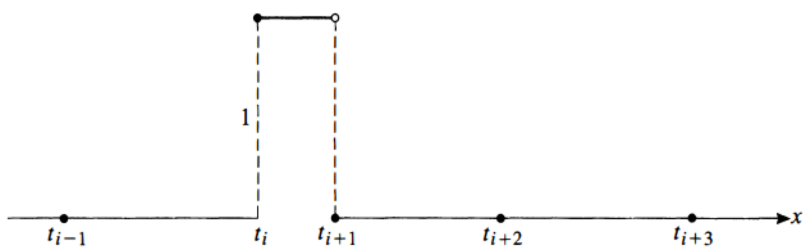
$$\ldots < t_i < t_{i+1} < \ldots,$$

$B_i^k(x)$ is a degree k, ith B-Spline.

1.1 0th Degree B-Spline

(Example.) For $k = 0$, we have

$$B_i^0(x) = \begin{cases} 1 & t_i \le x < t_{i+1} \\ 0 & \text{otherwise} \end{cases}.$$



Let's now consider the following sequence of 0th degree B-splines,

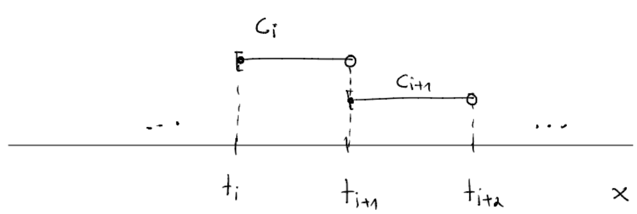
$$\{B_i^0, i \in \mathbb{Z}\}.$$

Some properties of this sequence include

- **Support** ($B_i^0(x) \neq 0$) is $[t_i, t_{i+1})$.
- $B_i^0 \ge 0$ for all possible x and all possible i.
- Continuous from right.
- For all x, $\sum_{i=-\infty}^{\infty} B_i^0(x) = 1$.

For given knots, B_i^0 is a basis for all degree zero splines.

(Example.) Let $S(x) = c_i$ for $t_i \le x < t_{i+1}$ and $i \in \mathbb{Z}$, we have



Then, we can say that

$$S(x) = \sum_{i=-\infty}^{\infty} c_i B_i^0(x).$$

1.2 Higher Degree B-Splines

We can make use of the following recursion to construct higher degree B-splines:

$$B_i^k(x) = \left(\frac{x - t_i}{t_{i+k} - t_i}\right) B_i^{k-1}(x) + \left(\frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}}\right) B_{i+1}^{k-1}(x)$$

Additionally, if we write the constant term as

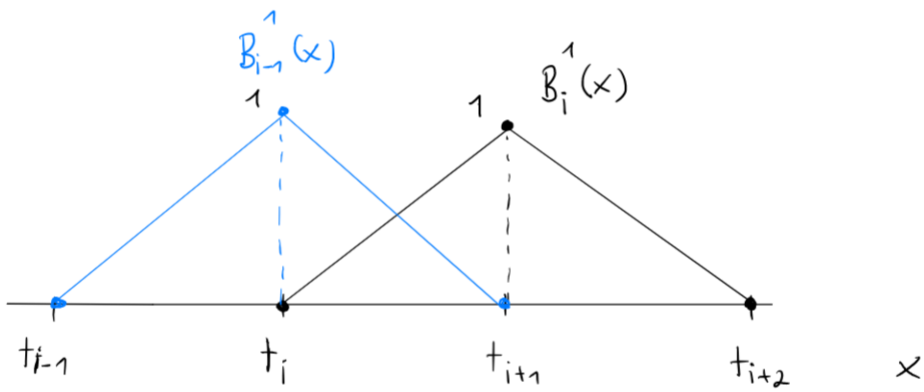
$$V_i^k(x) = \frac{x - t_i}{t_{i+k} - t_i},$$

then we can rewrite $B_i^k(x)$ as

$$B_i^k = V_i^k B_i^{k-1} + (1 - V_{i+1}^k) B_{i+1}^{k-1}.$$

(Example.) To find the degree 1 B-spline, we can write

$$B_i^1 = V_i^1 B_i^0 + (1 - V_{i+1}^1) B_{i+1}^0 = \begin{cases} 0 & x < t_i \text{ or } x \geq t_{i+2} \\ V_i^1 & t_i \leq x < t_{i+1} \\ (1 - V_{i+1}^1) & t_{i+1} \leq x < t_{i+2} \end{cases}.$$



What are some properties of the degree 1 B-spline?

- Support: $x \in (t_i, t_{i+2})$.
- $B_i^1(x) \geq 0$ for all i and x.
- Continuous and differentiable except at the knots themselves (i.e, t_i, t_{i+1}, t_{i+2}).
- For all x,

$$\sum_{i=\infty}^{\infty} B_i^1(x) = 1.$$

In particular, for any x, we can find an interval $t_j \leq x < t_{j+1}$. Then, B_{j-1}^1 and B_j^1 are the only non-zero B-splines:

$$B_{j-1}^1(x) = \frac{t_{j+k} - x}{t_{j+k} - t_j} = 1 - V_j^1.$$

$$B_j^k = \frac{x - t_j}{x_{j+k} - t_j} = V_j^1.$$

$$B_{j-1}^1 + B_j^1 = (1 - V_j^1) + V_j^1 = 1.$$

1.3 Algorithm to Generate Higher Degree B-Spline

Using the recursion defined above, in particular with t being defined as

$$t = \{t_i, t_{i+1}, t_{i+2}, \ldots, t_{i+1+k}\},$$

we have the following algorithm:

Algorithm 1 Higher Degree B-Spline

function BSPLINE(x, k, i, t)
  if $1 \leq k$ **then**
    $V_i \leftarrow \frac{x - t_i}{t_{i+k} - t_i}$
    $V_{i+1} \leftarrow \frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}}$
    $B_i^k = V_i \cdot \text{BSpline}(x, k-1, i, t) + (1 - V_{i+1}) \cdot \text{BSpline}(x, k-1, i+1, t)$
  else ▷ If $k = 0$ (base case)
    if $t_i \leq x$ and $x < t_{i+1}$ **then**
      $B_i^k \leftarrow 1$
    else
      $B_i^k \leftarrow 0$
    end if
  end if
end function