

1 Fundamental Shortest Paths Formula

For any vertex w that isn't the source, $w \neq s$,

$$\text{dist}(w) = \min_{(v,w) \in E} \text{dist}(v) + \ell(v, w)$$

We can use a system of equations to solve for the distances. When $\ell \geq 0$, Dijkstra gives an order to solve in.

1.1 Algorithm Idea

Instead of finding the shortest paths, which may not exist, we instead find the shortest paths of length at most k . So, for $w \neq s$, we have:

$$\text{dist}_k(w) = \min_{(v,w) \in E} \text{dist}_{k-1}(v) + \ell(v, w)$$

1.2 Algorithm

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Bellman-Ford( $G, s, l$ )
   $\text{dist}_{\{0\}}(v) = \text{infinity}$  for all  $v$ 
   $\text{dist}_{\{0\}}(s) = 0$ 
  For  $k = 1$  to  $n$ 
    For  $w$  in  $V$ 
       $\text{dist}_{\{k\}}(w) = \min(\text{dist}_{\{k-1\}}(v) + l(v, w))$ 
   $\text{dist}_{\{k\}}(s) = \min(\text{dist}_{\{k\}}(s), 0)$ 

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1.3 Analysis

Proposition. If $n \geq |V| - 1$ and if G has no negative weight cycles, then for all v ,

$$\text{dist}(v) = \text{dist}_n(v)$$

In particular:

- If there is a negative weight cycle, there is probably no shortest path.
- If not, we only need to run our algorithm for $|V|$ rounds, for a final runtime of $\mathcal{O}(|V||E|)$.

1.4 Detecting Negative Cycles

If there are no negative weight cycles, Bellman-Ford computes shortest paths (and they might not exist otherwise). How do we know whether or not there are any?

1.4.1 Cycle Detection

Proposition. For any $n \geq |V| - 1$, there are no negative weight cycles reachable from s if and only if, for every $v \in V$:

$$\text{dist}_n(v) = \text{dist}_{n+1}(v)$$

1.5 Potential Function

Let

$$\ell'(v, w) = \ell(v, w) - d(v) + d(w) \geq 0$$

Imagine someone lending you $d(w)$ when you arrive at w , but then you have to pay it back when you leave.