1 Expected Value and Variance

1.1 Conditional Expectation: Discrete Case

Recall that if X is a discrete random variable with PMF p, and B is an event with $\mathbb{P}(B) > 0$, then

$$p(x|B) = \frac{p(x)}{\mathbb{P}(B)}$$

is a probability distribution on B. This is the PMF of the random variable X, given B.

Definition 1.1

Let X be a random variable with PMF p. Suppose that $\mathbb{P}(B) > 0$. Then, the conditional expectation of X given B is

$$\mathbb{E}(X|B) = \sum_x xp(x|B).$$

1.1.1 Law of Total Expectation

Just like how there is a Law of Total Probability, there is also a Law of Total Expectation.

Theorem 1.1: Law of Total Expectation

Let X be a random variable on sample space Ω. Suppose that $B_1, \ldots, B_n$ is a partition Ω. Then,

$$\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}(X|B_i)\mathbb{P}(B_i).$$

This is useful because, often, $\mathbb{E}(X)$ is sometimes difficult to find directly. However, if we condition on a well-chosen B_i, then it becomes manageable.

(Example.) In the gambling game "craps," a player makes a bet and then rolls a pair of dice. If the sum is 7 or 11, the player wins. If it is 2, 3, or 12, the player loses. If the sum is any other number s, the player continues to roll until either another s (they win) or 7 (they lose) occurs (7 is lucky the first time). Now, let R be the number of rolls in a single game of craps.

1. Find $\mathbb{E}(R)$.

1. By the Law of Total Expectation, we have

$$\mathbb{E}(R) = \sum_{x=2}^{12} \mathbb{E}(R|X = x)\mathbb{P}(X = x),$$

where X is the initial sum. Note that if

$$x \in \{2, 3, 7, 11, 12\},$$

then

$$\mathbb{E}(R|X = x) = 1$$

since the game is immediately over if we get one of those numbers. In particular,

- There is 1 way to get a 2 (11).
- There are 2 ways to get a 3 (12, 21).
- There are 6 ways to get a 7 (16, 61, 25, 52, 43, 34).
- There are 2 ways to get a 11 (56, 65).
- There is 1 way to get a 12 (66).

Hence,

$$\sum_{x\in\{2,3,7,11,12\}} \mathbb{E}(R|X = x)\mathbb{P}(X = x) = \frac{12}{36}.$$

Now, if

$$x \in \{4, 5, 6, 8, 9, 10\},$$

then by a similar argument to the one above, we have

$$\sum_{x\in\{2,3,7,11,12\}} \mathbb{E}(R|X = x)\mathbb{P}(X = x) = \frac{24}{36}.$$

1.2 Conditional Expectation: Continuous Case

The situation is similar in the continuous case, but we instead have a conditional PDF

$$f(x|B) = \frac{f(x)}{\mathbb{P}(B)}$$

and the conditional expectation is given by

$$\mathbb{E}(X|B) = \int_{-\infty}^{\infty} xp(x|B).$$