1 Expected Value and Variance

1.1 Examples of Finding Expected Value and Variance

(Example.) If X is Bernoulli(p), then $\mu = p$ and $\sigma^2 = pq$, where $q = 1 - p$. Then,

$$\mathbb{E}(X) = 1 \cdot p + 0 \cdot q = p.$$

Note that X^2 has the same distribution as X, since X only takes the values $0 = 0^2$ and $1 = 1^2$. Hence, $\mathbb{E}(X^2) = \mathbb{E}(X) = p$. Hence,

$$\text{Var}(X) = p - p^2 = p(1 - p) = pq.$$

(Example.) If X is Binomial(n, p), then it is the sum of n independent Bernoulli(p) trials. By the Linearity of Expectation, we know that

$$\mathbb{E}(X) = np.$$

Since the trials are *independent*, we have that

$$\text{Var}(X) = npq.$$

Note that this is the special case of the following fact.

Theorem 1.1

Suppose that $X_1, \dots, X_n$ are IID with mean μ and variance σ^2. Then, their sum $S_n = \sum_{k=1}^n X_k$ has mean $\mathbb{E}(S_n) = n\mu$ and variance $\text{Var}(S_n) = n\sigma^2$.

(Example.) If X is Geometric(p), then $\mathbb{E}(X) = \frac{1}{p}$. So, if p is really small, then we should expect to wait a while before our first success; likewise, if p is large, then we may not need to wait long before our first success. This is intuitive; in particular, the probability of success is p, so we should expect about 1 success in every p trials. But, intuition aside, there are several ways to compute this.

- Approach 1.

$$\mathbb{E}(X) = \sum_{k=1}^{\infty} kpq^{k-1} = p \sum_{k=1}^{\infty} kq^{k-1}.$$

Recall that this is a *geometric* random variable, so we will use the geometric series; in particular,

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$$

and

$$\sum_{k=1}^{\infty} kq^{k-1} = \frac{d}{dq} \sum_{k=0}^{\infty} q^k.$$

Hence,

$$\mathbb{E}(X) = p \frac{d}{dq} \frac{1}{1-q} = p \frac{1}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}.$$

Similarly, we can show that $\text{Var}(X) = \frac{q}{p^2}$.

- Approach 2.

Let X be the number of trials until the first success. Then, we have

$$\mathbb{E}(X) = 1p + (1 + \mathbb{E}(X))q.$$

Solving for $\mathbb{E}(X)$ gives us the desired solution.

(Example.) A Poisson(λ) has mean and variance $\mu = \sigma^2 = \lambda$. So,

$$\mathbb{E}(X) = \sum_{k=0}^{\infty} ke^{-\lambda}\frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} e^{-\infty}\frac{\lambda^k}{(k-1)!} = \lambda\sum_{k=0}^{\infty} e^{-\lambda}\underbrace{\frac{\lambda^k}{k!}}_{1} = \lambda.$$

Similarly, you can show that $\mathbb{E}(X^2) = \lambda(1+\lambda)$ so that $\text{Var}(X) = \lambda(1+\lambda) - \lambda^2 = \lambda$.

(Example.) An Exponential(λ) has $\mu = \frac{1}{\lambda}$ and $\sigma^2 = \frac{1}{\lambda^2}$. For this computation, the theorem following this example will be useful.

If X is Exponential(λ), then it is non-negative and $\mathbb{P}(X > x) = e^{-\lambda x}$. Hence,

$$\mathbb{E}(X) = \int_0^{\infty} e^{-\lambda x}dx = \frac{1}{\lambda}.$$

Theorem 1.2: Expectation Tail Sum for Non-Negative Random Variables

If X is a non-negative random variable (i.e., $\mathbb{P}(X \geq 0) = 1$), then

1. If X is discrete, then
$$\mathbb{E}(X) = \sum_{k=0}^{\infty} \mathbb{P}(X > k).$$
2. If X is continuous, then
$$\mathbb{E}(X) = \int_0^{\infty} \mathbb{P}(X > x)dx.$$

Proof. (Discrete.) Just note that

$$\begin{aligned}\mathbb{E}(X) &= \sum_{k=0}^{\infty} kp(k) \\ &= p(1) + (p(2)+p(2)) + (p(3)+p(3)+p(3)) + \dots \\ &= (p(1)+p(2)+p(3)+\dots) + (p(2)+p(3)+\dots) + (p(3)+\dots) + \dots \\ &= \mathbb{P}(X > 0) + \mathbb{P}(X > 1) + \mathbb{P}(X > 2) + \dots\end{aligned}$$

Hence, we're done. □

(Example.) If X is Normal(μ, σ^2), then, indeed, μ is the mean and σ^2 is its variance. To see why this is the case, see lecture slides.

(Example.) If X is Cauchy, then $\mathbb{E}(X)$ does not exist. Recall that a (standard) Cauchy has PDF

$$f(x) = \frac{1}{\pi(1+x^2)}.$$

Note that

$$\int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)} dx$$

diverges, since

$$\int_0^{\infty} \frac{x}{1+x^2} dx = \infty.$$