**Note:** A lot of lecture 15 is actually in Lecture 14's notes since most of the lecture continues on Lecture 14.

# 1 Singular Value Decomposition (4.1)

The singular value decomposition, known as **SVD**, is a matrix decomposition (similar to eigenvector, eigenvalues, but less restrictive). SVD is used for

- low rank approximation (imaging).
- least squares when rank is not full.

**Theorem 1.1: SVD Theorem**

Let $A \in \mathbb{R}^{n\times m}$, with $A \neq 0$ and assume $n \geq m$ with $\text{rank}(A) = r \leq m$. Then, there exists orthogonal matrices $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$ and positive numbers $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ such that

$$A = U\Sigma V^T$$

with

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \ddots & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \sigma_r & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \in \mathbb{R}^{n\times m}.$$

This is called a full SVD[1] Here, $\sigma_1, \sigma_2, \ldots, \sigma_r$ are called the *singular values*.
**Remarks:**

- Notice that $A = U\Sigma V^T \implies AV = U\Sigma V^T V = U\Sigma$. If you compare this to eigenvectors and eigenvalues, you will notice that $AV = V\Lambda$.
- The SVD is not unique. Instead of $U$, we can try $-U$; likewise, instead of $V$, we can use $-V$.

[1] Later, we will introduced a reduced SVD.