1 Complementation

Theorem 1.1: Complementation

If A is a regular language over $\{0,1\}^*$, then so is its complement.

Remarks:

- This is essentially the same thing as saying that the class of regular languages is closed under complementation.
- How do we apply this? Let A be a regular language. Then, there is a DFA $M = (Q, \Sigma, \delta, q_0, F)$ such that $L(M) = A$. We want to build a DFA M' whose language is $\overline{A}$. Define:

$$M' = (Q, \Sigma, \delta, q_0, Q \setminus F)$$

Proposition. *M' accepts A^c.*

Proof. Because M accepts A, we define A to be:

$$A = \{w \mid M \text{ accepts } w\} = \{w \mid \delta^*(q_0, w) \in F\}$$

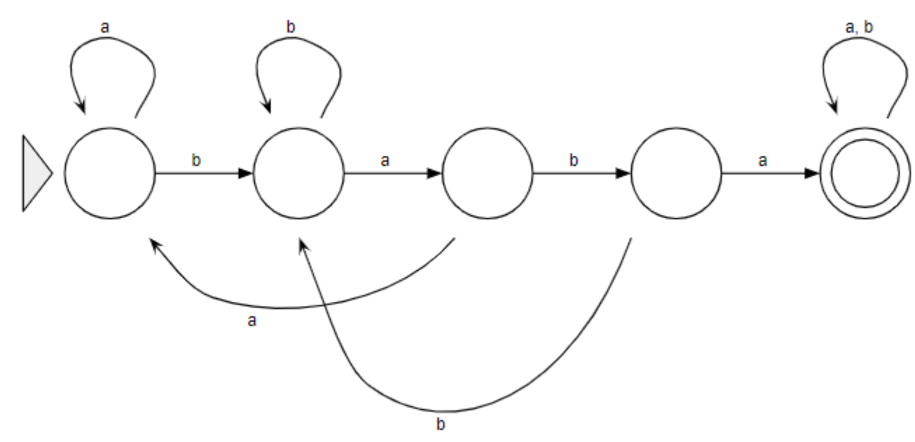
Recall that $\delta^*(q, w)$ is the state reached from q after reading the word w. Taking the complement of A, we have:

$$A^c = \{w \mid w \notin A\} = \{w \mid \delta^*(q_0, w) \notin F\} = \{w \mid \delta^*(q_0, w) \in Q \setminus F\}$$

So, M' accepts A^c. $\square$

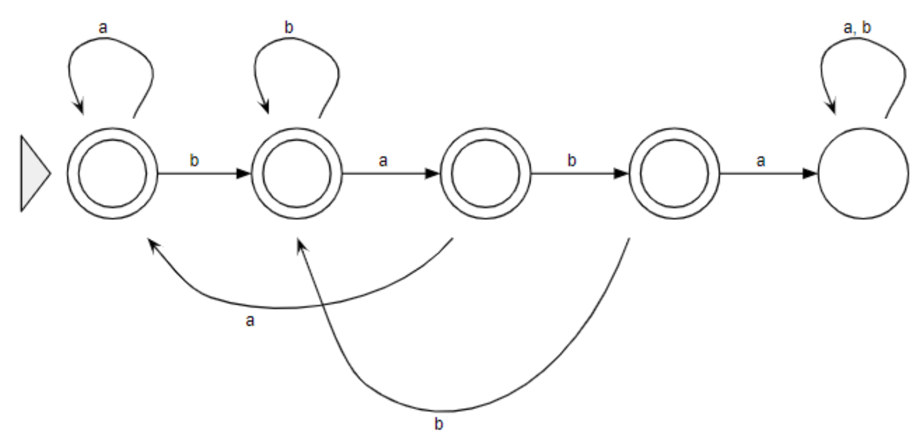
1.1 Example 1: Building DFA

Construct a DFA that recognizes $\{w \mid w \text{ contains the substring baba}\}$.



1.2 Example 2: Building DFA

Construct a DFA that recognizes $\{w \mid w \text{ doesn't contain the substring baba}\}$.



2 The Regular Operations

In arithmetic, the basic objects are numbers and the tools are operations for manipulating them (e.g. $+$ or $\times$). In the theory of computation, the objects are languages and the tools include operations designed for manipulating them. We call these **regular operations**.

Definition 2.1

Let A and B be languages. We define the regular operations **union**, **concatenation**, and **star** as follows:

- **Union:** $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
- **Concatenation:** $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$
- **(Kleene) Star:** $A^* = \{x_1 x_2 \dots x_k \mid k \geq 0 \text{ and each } x_i \in A\}$

Remarks:

- The union operation simply takes all strings in both A and B and puts them together into one language.
- The concatenation operation attaches a string from A in front of a string from B in *all possible ways to get the strings in the new language.*
- The star operation attaches any number of strings in A together to get a string in the new language. Note that *any number* includes 0, so the empty string ϵ is always in A^*.

Note that we can prove the union operation today, but we cannot prove the concatenation or star operators until later.

2.1 Union

Theorem 2.1

The class of regular languages over a fixed alphabet Σ is closed under the union operator.

Essentially, we want to show that if we have two regular languages A and B, then the union of them must also be regular.

2.1.1 Goal and Strategy

If M_1 is the DFA for A and M_2 is the DFA for B, we want to show that there is a DFA that recognizes $A \cup B$:

- The goal is to build a DFA that recognizes $A \cup B$.
- The strategy is to use DFAs that recognize each of A and B.

2.1.2 Basic Sketch

Proof. We want to show that M accepts w if M_1 accepts w *or* M_2 accepts w. Let A and B be any two regular languages over Σ. Given:

$$M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1) \qquad L(M_1) = A$$

$$M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2) \qquad L(M_2) = B$$

We want to show that $A \cup B$ is regular. The idea is to run these two DFAs M_1 and M_2 in parallel. So, we define:

$$M = (Q_1 \times Q_2, \Sigma, \delta, (q_1, q_2), F)$$

Where, for $r \in Q_1$, $s \in Q_2$, and $x \in \Sigma$, we define:

$$\delta((r, s), x) = (\delta_1(r, x), \delta_2(s, x))$$

$$F = \{(r, s) \mid r \in F_1 \text{ or } s \in F_2\}$$

Note that it is not $\{(r, s) \mid r \in F_1 \text{ and } s \in F_2\}$ because this would be under intersection. Likewise, it is not $F_1 \times F_2$ because it is also intersection.

(And so on...) □

2.2 Intersection

How would you prove that the class of regular languages is closed under intersection? The diea is that:

$$A \cap B = (A^c \cup B^c)^c$$

We've already shown that the union is closed and so is its complement.

2.2.1 Payoff

Consider the set:

$$\{w \mid w \text{ contains neither the substrings aba nor baab}\}$$

Is this a regular set?

We know that:

$$A = \{w \mid w \text{ contains aba as a substring}\}$$

$$B = \{w \mid w \text{ contains baab as a substring}\}$$

From which we know:

$$\overline{A} \cap \overline{B} = \overline{A \cup B}$$

3 General Proof Strategy/Structure

Theorem 3.1

For any L over Σ, if L is regular, then (the result of some operation on L) is also regular.

The proof template is as follows:

- **Given:** Name variables for sets, DFAs assumed to exist.
- **WTS:** State goal and outline plan.
- **Construction:** Use objects previously defined and new tools to work towards the goal. Give formal definition and explain.
- **Correctness:** Prove that construction works.
- **Conclusion:** Recap what you've proved.