1 Graphics Pipeline, Linear & Affine Transformations in $\mathbb{R}^2$

1.1 Inverses of Linear and Affine Transformations

We begin with a definition.

Definition 1.1

Let A and B be transformations of $\mathbb{R}^2$. We say that B is the **inverse** of A, written $B = A^{-1}$, if:

1. $A \circ B = I$.
2. $B \circ A = I$.

1.1.1 Finding Inverses of Linear Transformations

How do we compute inverses? One way to do so is to do it by matrices. Let $A : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be represented by the matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then, A^{-1} is represented by M^{-1}, where

$$M^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \frac{1}{ad - bc}.$$

1.1.2 Finding Inverses of Affine Transformations

Let $A(\mathbf{x})$ be defined by

$$A(\mathbf{x}) = B(\mathbf{x}) + \mathbf{u},$$

where B is linear and $\mathbf{u} \in \mathbb{R}$ so that A is affine. How do we compute its inverse? Now, to find A^{-1}, we need to solve

$$\mathbf{y} = B(\mathbf{x}) + \mathbf{u}$$

to get $\mathbf{x}$ in terms of $\mathbf{y}$. Now, we have

$$\begin{aligned}
\mathbf{y} &= B(\mathbf{x}) + \mathbf{u} \\
&\implies B(\mathbf{x}) = \mathbf{y} - \mathbf{u} \\
&\implies \mathbf{x} = B^{-1}(\mathbf{y} - \mathbf{u}) \\
&\implies \mathbf{x} = B^{-1}(\mathbf{y}) - B^{-1}(\mathbf{u}) \\
&\implies \mathbf{x} = \underbrace{B^{-1}(\mathbf{y})}_{\text{Linear Part}} + \overbrace{(-B^{-1}(\mathbf{u}))}^{\text{Translation Part}}.
\end{aligned}$$

1.2 Homogeneous Coordinates

We can represent points in $\mathbb{R}^2$ with triples $\langle x, y, u \rangle$; this represents $\langle \frac{x}{u}, \frac{y}{u} \rangle$ for $u \neq 0$.

(Example.) Some examples of homogeneous coordinates for $\langle 2, 1 \rangle \in \mathbb{R}^2$ are:

- $\langle 2, 1, 1 \rangle$.
- $\langle 4, 2, 2 \rangle$.
- $\langle 8, 4, 4 \rangle$.
- $\langle -2, -1, -1 \rangle$.

If $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ and if

$$f(\mathbf{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x} + \begin{bmatrix} e \\ f \end{bmatrix},$$

then f – acting in homogeneous coordinates – is represented by the 3×3 matrix

$$\begin{bmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{bmatrix}.$$