1 Moving to $\mathbb{R}^3$

1.1 Rigid & Orientation-Preserving Transformations

1.1.1 Rigid, Orientation-Preserving Transformations in $\mathbb{R}^2$

Theorem 1.1

Let A be a rigid, orientation-preserving map on $\mathbb{R}^2$.

1. If $A(\mathbf{0}) = \mathbf{0}$, then A is a rotation R_θ for some θ.
2. If $A(\mathbf{u}) = \mathbf{u}$, then A is a generalized rotation $R_\theta^{\mathbf{u}}$.
3. *In general*, A is either a translation $T_{\mathbf{u}}$ or a generalized rotation $R_\theta^{\mathbf{u}}$ for some $\mathbf{u}$ (and some θ).

The proof for this theorem depends on this lemma:

Lemma 1.1

Suppose $\mathbf{x} \neq \mathbf{y}$. Then, A is uniquely determined by $\mathbf{u} = A(\mathbf{x})$ and $\mathbf{v} = A(\mathbf{y})$.

1.1.2 Euler's Theorem on Rotations in 3-Space

Theorem 1.2

Let $A : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be a linear, orientation-preserving and rigid map. Then, A is a rotation $R_{\theta,\mathbf{u}}$ for some θ, $\mathbf{u}$ where $\mathbf{u} \neq 0$.

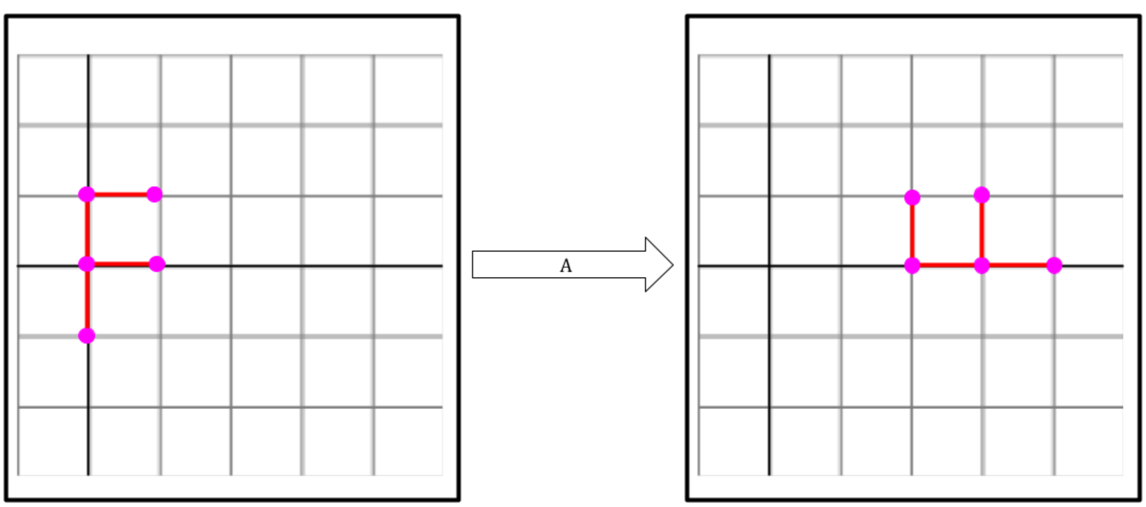
Lemma 1.2

Suppose $A(\mathbf{u}) = \mathbf{u}$ for some $\mathbf{u}$ such that $||\mathbf{u}|| = 1$. Then, A is a rotation $R_{\theta,\mathbf{u}}$ for some θ.

What if A is rigid, orientation-preserving, but $A(\mathbf{0}) \neq \mathbf{0}$? This is sometimes known as a glide rotation (or a screw motion).

1.1.3 Finding the Center of Generalized Rotation

Consider the following transformation:



Here, we have that $A(\mathbf{x}) = R_{90^\circ}(\mathbf{x}) + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$. In other words, $A = T_{\langle 3,0 \rangle} \circ R_{90^\circ}$. Our goal is to express A as a generalized rotation; that is, $A = R_\theta^{\mathbf{u}}$. It should be obvious that $\theta = 90^\circ$. Then, we need to find a $\mathbf{u}$ such that $A(\mathbf{u}) = \mathbf{u}$.

First, we want to choose a point $\mathbf{v}$ where $A(\mathbf{v}) \neq \mathbf{v}$. Let $\mathbf{w} = A(\mathbf{v})$.