

# 1 QR Decomposition of a Tall Matrix

Find the full QR decomposition of

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}.$$

- Step 1: First, we start with  $\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ , the first column of  $A$ . We want to map  $\vec{a}_1 \mapsto \|\vec{a}_1\|_2 \mathbf{e}_1$ , so we

have  $\|\vec{a}_1\|_2 = \sqrt{4} = 2$  and

$$\vec{a}_1 \mapsto 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Using the lemma discussed in lecture 10, we can define

$$\vec{v}_1 = \vec{a}_1 - \|\vec{a}_1\|_2 \mathbf{e}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\|\vec{v}_1\|_2 = 2$$

and so

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|_2} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Then, we have

$$Q_1 = I - 2\vec{u}_1\vec{u}_1^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

From this, it follows that

$$Q_1 A = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & -5 & 2 \end{bmatrix}.$$

- Step 2: We now look at the second column of  $Q_1 A$  (*not*  $A$ ). Note that this is<sup>1</sup>  $\vec{a}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -5 \end{bmatrix}$ , and

$\|\vec{a}_2\|_2 = \sqrt{25} = 5$ . So, mapping  $\vec{a}_2 \mapsto \|\vec{a}_2\|_2 \mathbf{e}_2$ , we have

$$\vec{a}_2 \mapsto 5 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \end{bmatrix}.$$

<sup>1</sup>As from lecture, we set the value in the first row and at that column to 0.

So, using the lemma again, we define

$$\vec{v}_2 = \vec{a}_2 - \|\vec{a}_2\|_2 \mathbf{e}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -5 \end{bmatrix} - \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 0 \\ -5 \end{bmatrix}$$

$$\|\vec{v}_2\|_2 = \sqrt{(-5)^2 + 5^2} = \sqrt{50}$$

and so

$$\vec{u}_2 = \frac{1}{\sqrt{50}} \begin{bmatrix} 0 \\ -5 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-5}{\sqrt{50}} \\ 0 \\ \frac{-5}{\sqrt{50}} \end{bmatrix}.$$

Then,

$$Q_2 = I - 2\vec{u}_2\vec{u}_2^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ \frac{-5}{\sqrt{50}} \\ 0 \\ \frac{-5}{\sqrt{50}} \end{bmatrix} \begin{bmatrix} 0 & \frac{-5}{\sqrt{50}} & 0 & \frac{-5}{\sqrt{50}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

From this, it follows that

$$Q_2(Q_1A) = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Notice how, in step 2, we found an upper-triangular matrix. Therefore, we have that

$$R = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{aligned} Q_2(Q_1A) &= R \\ \implies Q_1A &= Q_2^{-1}R \\ \implies A &= Q_1^{-1}Q_2^{-1}R \\ \implies A &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}^{-1} R \\ \implies A &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} R \\ \implies A &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} R. \end{aligned}$$

So,

$$Q = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

## 2 Reflector

Find a reflector  $Q$  that maps the vector  $\vec{x} = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 3 \\ 1 \end{bmatrix}$  to a multiple of the first column of the  $5 \times 5$  identity,  $\mathbf{e}_1$ . Compute  $Q$  by writing it as

$$Q = I - 2 \frac{\vec{u}\vec{u}^T}{\|\vec{u}\|_2^2}$$

for some appropriate  $\vec{u}$ , and write it as a completely assembled matrix.

We want to map  $\vec{x} \mapsto \|\vec{x}\|_2 \mathbf{e}_1$ . Notice how

$$\|\vec{x}\|_2 = \sqrt{3^2 + 4^2 + 1^2 + 3^2 + 1^2} = \sqrt{9 + 16 + 1 + 9 + 1} = \sqrt{36} = 6.$$

So, we're mapping

$$\vec{x} \mapsto 6 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now, we can define

$$\vec{v} = \vec{x} - \|\vec{x}\|_2 \mathbf{e}_1 = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 1 \\ 3 \\ 1 \end{bmatrix}$$

$$\|\vec{v}\|_2 = 6.$$

So,

$$\vec{u} = \frac{1}{6} \begin{bmatrix} -3 \\ 4 \\ 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{2}{3} \\ \frac{1}{6} \\ \frac{1}{2} \\ \frac{1}{6} \end{bmatrix}.$$

Thus,

$$Q = I - 2\vec{u}\vec{u}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} -\frac{1}{2} \\ \frac{2}{3} \\ \frac{1}{6} \\ \frac{1}{2} \\ \frac{1}{6} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{2}{3} & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

Therefore, the answer is

$$\begin{bmatrix} \frac{1}{2} & \frac{2}{3} & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{9} & -\frac{2}{9} & -\frac{2}{9} & -\frac{2}{9} \\ \frac{1}{6} & -\frac{2}{9} & \frac{17}{18} & -\frac{1}{6} & -\frac{1}{18} \\ \frac{1}{2} & -\frac{2}{9} & -\frac{1}{6} & \frac{1}{2} & -\frac{1}{6} \\ \frac{1}{6} & -\frac{2}{9} & -\frac{1}{18} & -\frac{1}{6} & \frac{17}{18} \end{bmatrix}.$$

### 3 Least Squares

Consider the overdetermined system

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} [x] = \begin{bmatrix} 9 \\ 5 \end{bmatrix}.$$

- (a) Calculate a full QR decomposition of the coefficient matrix with the help of Householder reflectors.
- (b) Using the QR decomposition from part (a), calculate the least squares solution (the minimizer).
- (c) Calculate the norm of the residual with the help of  $Q$  (the minimum).

Here,  $A$  is a  $2 \times 1$  matrix and so  $n = 2$  and  $m = 1$ .

#### Part (A)

We want to find the QR decomposition of  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . There's only one column, which we'll call  $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . Note that  $\|\vec{a}\|_2 = \sqrt{1^2 + 1^2} = \sqrt{2}$  and so we can map  $\vec{a} \mapsto \|\vec{a}\|_2 \mathbf{e}_1$  by

$$\vec{a} \mapsto \sqrt{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}.$$

Using the lemma discussed in class, we define

$$\vec{v} = \vec{a} - \|\vec{a}\|_2 \mathbf{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - \sqrt{2} \\ 1 \end{bmatrix}$$

$$\|\vec{v}\| = \sqrt{(1 - \sqrt{2})^2 + 1^2} = \sqrt{4 - 2\sqrt{2}}$$

and so we have

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|_2} = \frac{1}{\sqrt{4 - 2\sqrt{2}}} \begin{bmatrix} 1 - \sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1 - \sqrt{2}}{\sqrt{4 - 2\sqrt{2}}} \\ \frac{1}{\sqrt{4 - 2\sqrt{2}}} \end{bmatrix}.$$

From there, we have

$$Q = I - 2\vec{u}\vec{u}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} \frac{1 - \sqrt{2}}{\sqrt{4 - 2\sqrt{2}}} \\ \frac{1}{\sqrt{4 - 2\sqrt{2}}} \end{bmatrix} \begin{bmatrix} \frac{1 - \sqrt{2}}{\sqrt{4 - 2\sqrt{2}}} & \frac{1}{\sqrt{4 - 2\sqrt{2}}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then,

$$QA = R = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}.$$

Therefore<sup>2</sup>,

$$A = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}}_Q \underbrace{\begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}}_R.$$

#### Part (B)

If  $\vec{y} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$ , then

$$Q^T \vec{y} = Q \vec{y} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 9 \\ 5 \end{bmatrix} = \begin{bmatrix} 7\sqrt{2} \\ 2\sqrt{2} \end{bmatrix}.$$

<sup>2</sup>Note that  $Q$  is orthogonal and also our householder reflector, so  $Q^T = Q^{-1} = Q$ .

Since  $m = 1$ , we have

$$\hat{c} = [7\sqrt{2}] \quad \hat{R} = [\sqrt{2}].$$

The idea is to solve  $\hat{R}\vec{x} = \hat{c}$ , so

$$[\sqrt{2}] [x] = [7\sqrt{2}].$$

This gives us  $\vec{x} = [7]$ , the minimizer.

### Part (C)

From part (b), we know that  $Q^T \vec{y} = \begin{bmatrix} 7\sqrt{2} \\ 2\sqrt{2} \end{bmatrix}$ . Recall that if  $\hat{c}$  consisted of the first  $m$  elements, then  $\hat{d}$  will consist of the remaining elements. So,

$$\hat{d} = [2\sqrt{2}]$$

and so

$$\|\hat{d}\|_2 = \sqrt{(2\sqrt{2})^2} = \sqrt{8} = 2\sqrt{2},$$

the minimum.

### Remarks

In lecture 9, we wrote

$$Q^T \vec{y} = \begin{bmatrix} \hat{c} \\ \hat{d} \end{bmatrix},$$

where

$$\hat{c} = Q^T \vec{y}(1:m) \quad \hat{d} = Q^T \vec{y}(m+1:).$$

Likewise,  $\hat{R} = R(1:m, 1:m)$ .

## 4 Another QR Decomposition & Least Squares

Find the full QR decomposition for  $A$  and use it to find the minimizer  $\vec{x}$  and the minimum value that solves the least squares problem  $\min_{x \in \mathbb{R}} \|\vec{b} - A\vec{x}\|_2$  with

$$A = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}.$$

Note that  $n = 3$  and  $m = 1$ .

### QR Decomposition

Let  $\vec{a}$  be the first (and only) column of  $A$ . We know that

$$\|\vec{a}\|_2 = \sqrt{2},$$

so we're mapping  $\vec{a} \mapsto \sqrt{2}\mathbf{e}_1$ ; that is,

$$\vec{a} \mapsto \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix}.$$

Then,

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - \sqrt{2} \\ 0 \\ 1 \end{bmatrix}$$

$$\|\vec{v}\|_2 = \sqrt{(1 - \sqrt{2})^2 + 1^2} = \sqrt{4 - 2\sqrt{2}}.$$

Then,

$$\vec{u} = \frac{1}{\sqrt{4 - 2\sqrt{2}}} \begin{bmatrix} 1 - \sqrt{2} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1 - \sqrt{2}}{\sqrt{4 - 2\sqrt{2}}} \\ 0 \\ \frac{1}{\sqrt{4 - 2\sqrt{2}}} \end{bmatrix}.$$

From there,

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} \frac{1 - \sqrt{2}}{\sqrt{4 - 2\sqrt{2}}} \\ 0 \\ \frac{1}{\sqrt{4 - 2\sqrt{2}}} \end{bmatrix} \begin{bmatrix} \frac{1 - \sqrt{2}}{\sqrt{4 - 2\sqrt{2}}} & 0 & \frac{1}{\sqrt{4 - 2\sqrt{2}}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Then,

$$QA = R = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \end{bmatrix}.$$

### Minimizer

Note that

$$Q^T \vec{b} = Q \vec{b} = \begin{bmatrix} \sqrt{2} \\ 2 \\ \sqrt{2} \end{bmatrix}.$$

Since  $m = 1$ , we have

$$\hat{c} = Q^T \vec{b}(1 : m) = [\sqrt{2}] \quad \hat{R} = [\sqrt{2}]$$

so

$$\hat{R}\vec{x} = \hat{c} \implies [\sqrt{2}] \vec{x} = [\sqrt{2}]$$

and thus  $\hat{x} = [1]$ .

**Minimum**

Likewise,

$$\hat{d} = Q^T \hat{b}(m+1 :) = \begin{bmatrix} 2 \\ \sqrt{2} \end{bmatrix}.$$

So,

$$\|\hat{d}\|_2 = \sqrt{2^2 + (\sqrt{2})^2} = \sqrt{4+2} = \sqrt{6}.$$