1 Generating Functions

We will first begin by covering the **discrete distributions**.

Recall that the mean $\mathbb{E}(X)$ and variance

$$\text{Var}(X) = \mathbb{E}(X^2) = [\mathbb{E}(X)]^2$$

of a random variable provides useful information about the "shape" of a distribution. Note that these quantities only involve the **first moment** $\mathbb{E}(X)$ and **second moment** $\mathbb{E}(X^2)$ of X. But, what about higher moments?

Definition 1.1

$\mu_n = \mathbb{E}(X^n)$ is called the nth moment of the X.

Recall that LotUS tells us how to compute this, using the PMF/PDF of X; in particular,

$$\mathbb{E}(X^n) = \sum_x x^n p(x)$$

and

$$\mathbb{E}(X^n) = \int_{-\infty}^{\infty} x^n f(x) dx.$$

Definition 1.2

The **moment generating function (MGF)** of X is the function

$$g(t) = \mathbb{E}(e^{tX}).$$

Remark: Note that this is the expectation of the random variable $Y = e^{tX}$.

Recall the Taylor Series of e^x is given by

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Hence,

$$e^{tX} = \sum_{n=0}^{\infty} \frac{X^n t^n}{n!}.$$

Therefore, by Linearity of Expectation, we have

$$g(t) = \mathbb{E}(e^{tX}) = \sum_{n=0}^{\infty} \mathbb{E}\left(\frac{X^n t^n}{n!}\right) = \sum_{n=0}^{\infty} \mathbb{E}(X^n)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{\mu_n t^n}{n!}.$$

Theorem 1.1

The MGF

$$g(t) = \mathbb{E}(e^{tX}) = \sum_{n=0}^{\infty} \frac{\mu_n t^n}{n!}.$$

Hence, for each $n \geq 1$, we have that

$$\frac{d^n}{dt^n} g(0) = \mu_n.$$

Remark: We note that the MGF here contains all the information about its moments, which in many cases is enough to uniquely determine a distribution.

(Example.) A Binomial(n,p) RV X has MGF

$$g(t) = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k q^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k q^{n-k} = (pe^t + q)^n,$$

where $q = 1 - p$.

If can be checked that $g'(0) = np$ and $g''(0) = n(n-1)p^2 + np$. Hence, as we already know, $\mathbb{E}(X) = np$. Additionally, we know that

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 \\
&= [n(n-1)p^2 + np] - (np)^2 \\
&= -np^2 + np \\
&= np(1-p) \\
&= npq.
\end{aligned}$$

Remark: There are some technical points to consider.

- Note that $g(0) = 1$.
- The function (series) $g(t)$ may not exist. That is, the series $g(t) = \mathbb{E}(e^{tX}) = \sum_{n=0}^{\infty} \frac{\mu_n t^n}{n!}$ may not converge in a neighborhood of 0.

Theorem 1.2

Suppose that X is a RV that only takes a finite number of possible values. That is,

$$\sum_{i=1}^{n} \mathbb{P}(X = x_i) = 1,$$

for some $x_1, \ldots, x_n$. Then, the distribution of X is uniquely determined by its MGF. That is, if some other RV Y has the same MGF, then it has the same distribution as X.

Theorem 1.3

If the MGFs $M_X(t)$ and $M_Y(t)$ exists, and for some $\epsilon > 0$, $M_X(t) = M_Y(t)$ for all $t \in (-\epsilon, \epsilon)$, then the CDFs $F_X(t) = F_Y(t)$. That is, X and Y have the same distribution.

Remark: That is, if (they might not) the MGFs exist and are equal in some (perhaps very small) open neighborhood of 0, then they have the same distribution.

1.1 Important Properties of the MGF

Here are some useful properties.

- Linearity:
$$g_{aX+b}(t) = e^{bt} g_X(at),$$
since $\mathbb{E}(e^{t(aX+b)}) = e^{tb}\mathbb{E}(e^{(at)X})$.

- Independence: If X and Y are independent, then
$$g_{X+Y}(t) = g_X(t) g_Y(t).$$
More generally, if $X_1, \ldots, X_n$ are independent, then the MGF of their sum
$$S_n = \sum_{i=1}^{n} X_i$$

is

$$g_{S_n}(t) = \prod_{i=1}^{n} g_{X_i}(t).$$

Hence, if $X_1, \ldots, X_n$ are IID with common MGF $g(t)$, then

$$g_{S_n}(t) = [g(t)]^n.$$

(Example.) A Bernoulli(p) RV has MGF

$$g(t) = \mathbb{E}(e^{tX}) = e^{t(1)}p + e^{t(0)}q = e^t p + q.$$

This is because a Bernoulli random variable can only take two values: 1 or 0. Since a Binomial is just a sum of these Bernoulli random variables, the MGF of a Binomial(n, p) has MGF

$$(e^t p + q)^n.$$