1 Newton's Method (Section 3.2)

Newton's Method is an efficient iterative method for solving nonlinear equations, assuming it works. Let r be a root of some function, and let x be an approximation to r. Then, our goal is to find an estimate of r, or $r = x_{m+1} = x_m + h$, where $x_{m+1}, x_m, h \in \mathbb{R}$. If f'' exists and is continuous, then by the Taylor series, we have

$$0 = f(r) = f(x_{m+1}) = f(x_m) + hf'(x_m) + \mathcal{O}(h^2).$$

Then, $h = \frac{f(x_m)}{f'(x_m)}$ will be part of an updating scheme. For a sufficiently small h (i.e., x is near r), then we can reasonably ignore the $\mathcal{O}(h^2)$ term.

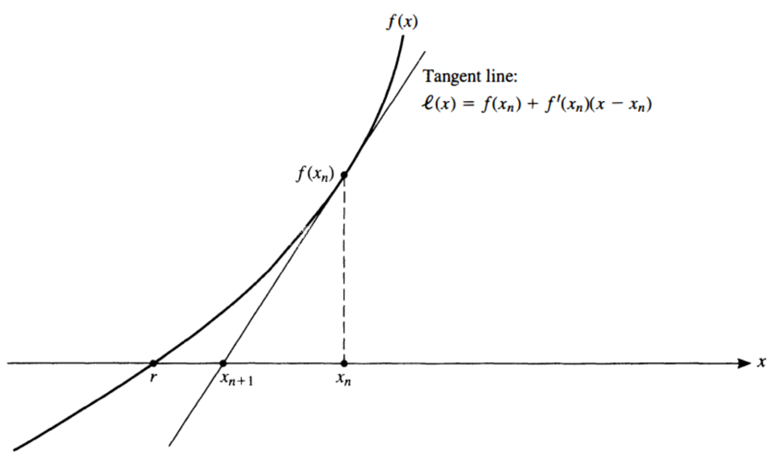
1.1 Newton Iteration in 1-Dimension

For $m = 0, 1, 2, \ldots$, we have

$$x_{m+1} = x_m - \frac{f(x_m)}{f'(x_m)}.$$

In other words, Newton's method begins with an estimate x_0 of r, and then defines inductively. If we let $x_{m+1} = x$, then the linearization at x_m is

$$f(x_{m+1}) = f(x) \approx f(x_m) + (x - x_m)f'(x_m) = \ell(x) = 0.$$



1.2 The Algorithm

Let

- M: the maximum number of iterations.
- δ: the step tolerance such that $|x_{m+1} - x_m| < \delta$.
- ϵ: the convergence tolerance $|f(x_{m+1})| < \epsilon$.

With a suitable x_0 being the starting point, the algorithm is as follows.

Algorithm 1 Newton's Algorithm

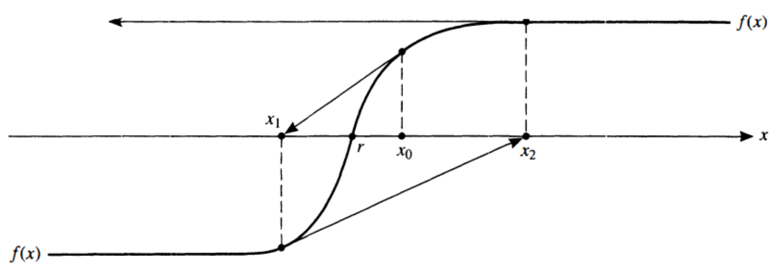
function NEWTON(x_0, M, δ, ϵ)
    $v \leftarrow f(x_0)$
    if $|v| < \epsilon$ **then**
        return
    end if
    for $k \leftarrow 1$ to M **do**
        $x_1 \leftarrow x_0 - \frac{v}{f'(x_0)}$
        $v \leftarrow f(x_1)$
        if $|x_1 - x_0| < \delta$ or $|f(x_1)| < \epsilon$ **then**
            break
        end if
        $x_0 \leftarrow x_1$
    end for
    return v
end function

1.3 Requirements

Defining the correct starting point x_0 is important. A bad starting point can result in nonconvergence.

The function itself also matters. Other problems include when $f'(x_0) = 0$ or $f'(x_0)$ is infinite.

(Example.) Consider the following function,



For this function, if $|x_0 - r| < |x_n - r|$, then $|x_m - r| < |x_{m-1} - r|$ and nonconvergence happens. In particular, the shape of the curve is such that for certain starting values, the sequence $[x_n]$ diverges.

1.4 Error Analysis

Let the error be defined by $e_m = x_m - r$. Assume that $f(r) = 0 \neq f'(r)$ and f'' is continuous. Then,

$$\begin{aligned} e_{m+1} &= x_{m+1} - r \\ &= \left(x_m - \frac{f(x_m)}{f'(x_m)} \right) - r \\ &= e_m - \frac{f(x_m)}{f'(x_m)} \\ &= \frac{e_m f'(x_m) - f(x_m)}{f'(x_m)}. \end{aligned}$$

We can now incorporate a Taylor expansion,

$$0 = f(r) = f(x_m - e_m) = f(x_m) - e_m f'(x_m) + \frac{1}{2} e_m^2 f''(\xi)$$

for some aribitrary ξ between x_m and r that makes the equation equal. Then,

$$\begin{aligned} -(f(x_m) - e_m f'(x_m)) &= \frac{1}{2} e_m^2 f''(\xi) \\ &\implies e_{m+1} = \frac{\frac{1}{2} e_m^2 f''(\xi_m)}{f'(x_m)} \\ &\implies e_{m+1} \approx C e_m^2, \end{aligned}$$

where C is a bound of $\frac{\frac{1}{2} f''(\xi_m)}{f'(x_m)}$. So, in Newton's method, we have a quadratic convergence so that $e_{m+1} \leq Ce_m^2$ or $|x_{m+1} - r| \leq C|x_m - r|^2$.

Remark: If f is $C^2(\mathbb{R})$ is increasing, is convex (i.e., $f''(x) > 0$ for all x), and has a zero, then Newton's Method converges to it from any starting point.

(Example.) Let $R > 0$ and $x = \sqrt{R}$. Then,

$$f(x) = x^2 - R = 0$$

and

$$f'(x) = 2x.$$

Then, the iteration corresponds to

$$x_{m+1} = x_m - \frac{f(x_m)}{f'(x_m)} = x_m - \frac{x_m^2 - R}{2x_m} = \frac{1}{2}\left(x_m + \frac{R}{x_m}\right).$$