1 Singular Value Decomposition, Continued (4.1, 4.2)

(Continued from previous notes.)

1.1 Relationship to Norm and Condition Number

Recall that we defined the matrix 2-norm as

$$||A|| = \max_{\mathbf{x} \neq \mathbf{0}} \frac{||A\mathbf{x}||_2}{||\mathbf{x}||} = \sigma_1,$$

where σ_1 is the largest singular value. Note that this definition also makes sense for $A \in \mathbb{R}^{n \times m}$.

Theorem 1.1

$$||A||_2 = \sigma_1.$$

Since A and A^T have the same singular values, we have the following corollary.

Corollary 1.1

$$||A||_2 = ||A^T||_2.$$

Since A is nonsingular, A has full rank, i.e., rank n. A has n strictly positive singualr values, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$. Now,

$$A^{-1}Av_i = A^{-1}(\sigma_i u - i) \implies v_i = \sigma_i A^{-1} u_i \implies A^{-1}u_i = \frac{1}{\sigma_i} v_i,$$

so in particular we can map each σ like so:

$$\begin{array}{ccc} A & \qquad & A^{-1} \\ v_1 \xrightarrow[\sigma_1]{} u_1 & & u_1 \xrightarrow[\sigma_1^{-1}]{} v_1 \\ v_2 \xrightarrow[\sigma_2]{} u_2 & & u_2 \xrightarrow[\sigma_2^{-1}]{} v_2 \\ v_3 \xrightarrow[\sigma_3]{} u_3 & & u_3 \xrightarrow[\sigma_3^{-1}]{} v_3 \\ \vdots & & \vdots \\ v_n \xrightarrow[\sigma_n]{} u_n & & u_n \xrightarrow[\sigma_n^{-1}]{} v_n \end{array}$$

This tells us that the singular values of A^{-1} must be

$$\frac{1}{\sigma_n} \geq \frac{1}{\sigma_{n-1}} \geq \ldots \geq \frac{1}{\sigma_2} \geq \frac{1}{\sigma_1} > 0$$

such that

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & 0 & 0 \\ 0 & \frac{1}{\sigma_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \frac{1}{\sigma_n} \end{bmatrix}.$$

And, in particular,

$$||A^{-1}||_2 = \frac{1}{\sigma_n} \qquad ||A||_2 = \sigma_1.$$

Theorem 1.2

Let $A \in \mathbb{R}^{n \times n}$ be a nonsingular matrix with singular values $\sigma_1 \geq \ldots \geq \sigma_n > 0$. Then,

$$\kappa_2(A) = \frac{\sigma_1}{\sigma_n}.$$

1.2 More on SVD

Remember that there are two types of SVD:

- Full SVD.
$$A = U\Sigma V^T,$$
where A is $n \times m$, U is $n \times n$, Σ is $n \times m$, and V^T is $m \times m$. Here, $\text{rank}(A) = r \leq m$ and $n \geq m$.
- Reduced SVD
$$A = \hat{U}\hat{\Sigma}\hat{V}^T,$$
where A is $n \times m$, $\hat{U}$ is $n \times r$, $\hat{\Sigma}$ is $r \times r$, and $\hat{V}^T$ is $r \times m$.

In any case, we now know that

$$||A||_2 = \sigma_1 \qquad \kappa_2(A) = \frac{\sigma_1}{\sigma_n}.$$

1.3 Rank-1 Decomposition

Theorem 1.3

Let $A \in \mathbb{R}^{n \times m}$ be a nonzero matrix with rank r. Let $\sigma_1, \ldots, \sigma_r$ be the singular values of A, with associated right and left singular vectors $v_1, \ldots, v_r$ and $u_1, \ldots, u_r$, respectively. Then,

$$A = \sum_{j=1}^{r} \sigma_j u_j v_j^T,$$

where $u_j \in \mathbb{R}^n$, $v_j \in \mathbb{R}^m$, and $u_j v_j^T \in \mathbb{R}^{n \times m}$.

To see why this theorem works,

$$\begin{aligned}
A &= \hat{U}\hat{\Sigma}\hat{V}^T \\
&= \underbrace{\begin{bmatrix} u_1 & v_2 & \ldots & u_r \end{bmatrix}}_{\hat{U}} \underbrace{\begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sigma_r \end{bmatrix}}_{\hat{\Sigma}} \underbrace{\begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_r^T \end{bmatrix}}_{\hat{V}} \\
&= \begin{bmatrix} \sigma_1 u_1 & \sigma_2 u_2 & \ldots & \sigma_r u_r \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_r^T \end{bmatrix} \\
&= \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \ldots + \sigma_r u_r v_r^T \\
&= \sum_{i=1}^{r} \sigma_i u_i v_i^T.
\end{aligned}$$