1 Basic Concepts & Taylor's Theorem (Section 1.1)

Let $f(x) : \mathbb{R} \mapsto \mathbb{R}$ be a general function (typically nonlinear). We may also write $f([a,b]) : [a,b] \mapsto \mathbb{R}$ to denote a general function over an interval $[a,b]$. We also write $C^n(\mathbb{R})$ or $C^n([a,b])$ to denote the *classes* of n-times continuously differentiable functions. We write $C^0(\mathbb{R}) = C(\mathbb{R})$ to mean the class of only continuous functions.

(Example.) $f(x) = |x|$ is continuous but is not differentiable at $x = 0$. Thus, $f(x) = |x|$ is in $C^0(\mathbb{R})$.

$f(x) = e^x$ is in $C^\infty(\mathbb{R})$.

(Exercise.) Show that

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is once differentiable (but not twice).

First, we need to check that f is continuous. First, we know that

$$\lim_{x \mapsto \infty} x^2 \sin\left(\frac{1}{x}\right) = 0,$$

so $f(x)$ is continuous. Next,

(a) When $x \neq 0$, then $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ and

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

Thus, $f(x)$ is differentiable when $x \neq 0$.

(b) When $x = 0$, we note that

$$f'(0) = \lim_{x \mapsto 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \mapsto 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x} = \lim_{x \mapsto 0} x \sin\left(\frac{1}{x}\right) = 0.$$

Thus, $f(x)$ is differentiable when $x = 0$.

Therefore, $f(x)$ is differentiable. With this in mind, we know that

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

But, $\lim_{x \mapsto 0} f'(x)$ doesn't exist since $\lim_{x \mapsto 0^+} f'(x) \neq \lim_{x \mapsto 0^-} f'(x)$. Because differentiability implies continuity, $f'(x)$ is not differentiable.

(Exercise.) With $[a,b] = [1,3]$ and $f(x) = 3 - 2x + x^2$, find ξ in the Mean-Value Theorem: $f(b) - f(a) = f'(\xi)(b - a)$.

Recall that $\xi \in [a,b]$. We know from the problem that $f(b) - f(a) = f'(\xi)(b-a)$, and so rearranging the terms gives us

$$f'(\xi) = \frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(1)}{3 - 1} = \frac{6 - 2}{3 - 1} = \frac{4}{2} = 2.$$

We also know that

$$f'(x) = 2x - 2,$$

so we need to find $f'(\xi) = 2$. This gives us

$$2\xi - 2 = 2 \implies \xi = 2.$$

1.1 Taylor Series

Theorem 1.1: Taylor Series with Lagrange Remainder

If $f \in C^m([a,b])$, and if the derivative $f^{(m+1)}$ exists on the open interval (a,b), then for any points $x, c \in [a,b]$,

$$f(x) = \sum_{k=0}^{m} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_m(x),$$

where $E_m(x)$, the remainder (or error) term, is

$$E_m(x) = \frac{f^{(m+1)}(\xi)}{(m+1)!}(x-c)^{m+1},$$

where $c < \xi < x$ or $x < \xi < c$ depending on the values of x and c.

Remark: Note that $f^{(k)}(x)$ is the kth derivative of $f(x)$. So, given $f(x)$, you will need to find $f'(x)$, $f''(x)$, and so on, and then generalize these derivatives. See the below examples for more information.

(Example.) Suppose $f(x) = \ln(x)$ with interval $[a,b] = [1,10]$ and $c = e^1$. Let $|x - c| < 1$ (i.e., x is relatively close to c). Then,

$$f^{(1)}(x) = f'(x) = \frac{1}{x}.$$

$$f^{(2)}(x) = f''(x) = -\frac{1}{x^2}.$$

$$f^{(3)}(x) = f'''(x) = \frac{2}{x^3}.$$

$$f^{(4)}(x) = -2 \cdot 3\frac{1}{x^4}.$$

$$f^{(5)}(x) = 2 \cdot 3 \cdot 4\frac{1}{x^5}.$$

$$\vdots$$

$$f^{(k)}(x) = (-1)^{k-1}(k-1)!\frac{1}{x^k}$$

for $k = 1, 2, \ldots$. Then,

$$E_m(x) = \frac{1}{(m+1)!}f^{(m+1)}(\xi)(x-c)^{m+1}.$$

Using the value of $c = e^1$,

$$f^{(k)}(c) = (-1)^{k-1}(k-1)!\frac{1}{e^k}.$$

Combining everything, we end up with

$$\begin{aligned}
f(x) &= \sum_{k=0}^{m} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_m(x) && \text{General Taylor Series} \\
&= \frac{f^{(0)}(c)}{0!}(x-c)^0 + \sum_{k=1}^{m} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_m(x) && \text{Separate the first term in summation} \\
&= f(c) + \sum_{k=1}^{m} f^{(k)}(c)\frac{1}{k!}(x-c)^k + E_m(x) \\
&= f(c) + \sum_{k=1}^{m} \left((-1)^{k-1}(k-1)!\frac{1}{e^k}\right)\frac{1}{k!}(x-c)^k + E_m(x) \\
&= f(c) + \sum_{k=1}^{m} (-1)^{k-1}\frac{1}{e^k}\frac{1}{k}(x-c)^k + E_m(x) \\
&= f(e) + \sum_{k=1}^{m} (-1)^{k-1}\frac{1}{e^k}\frac{1}{k}(x-e)^k + E_m(x) && c = e \\
&= 1 + \sum_{k=1}^{m} (-1)^{k-1}\frac{(x-e)^k}{ke^k} + E_m(x)
\end{aligned}$$

How many terms in this approximation do we need in order for the error to be below a certain amount? In other words, what is the minimum m so that a Taylor expansion is accurate up to $\frac{1}{\alpha} \cdot 10^{-7}$? We have

$$|E_m(m)| \leq \frac{1}{\alpha} \cdot 10^{-7}.$$

We already computed the remainder, so

$$\left|\frac{1}{(m+1)} f^{(m+1)}(\xi)(x-e)^{m+1}\right| \leq \frac{1}{\alpha} \cdot 10^{-7}.$$

Using $|x - e| < 1$, we want to find m. In any case, $|\xi| < 1$.

(Exercise.) Consider the function $f(x) = \ln(x)$.

(a) Determine the Taylor series of $f(x)$ using Taylor's Theorem, with the interval $[a, b] = [1, 2]$ and $c = 1$.

From the previous example, we know that

$$f^{(k)}(x) = (-1)^{k-1}(k-1)!\frac{1}{x^k}.$$

Therefore,

$$f^{(k)}(c) = f^{(k)}(1) = (-1)^{k-1}(k-1)!.$$

So, using Taylor's Theorem, we have

$$\begin{aligned}
f(x) &= \sum_{k=0}^{m} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_m(x) \\
&= \sum_{k=1}^{m} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_m(x) \\
&= \sum_{k=1}^{m} \frac{(-1)^{k-1}(k-1)!}{k!}(x-c)^k + E_m(x) \\
&= \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k}(x-1)^k + E_m(x).
\end{aligned}$$

Note that the summation started at $k = 1$ because $f^{(0)}(1) = f(1) = \ln(1) = 0$. Thus,

$$\begin{aligned}
E_m(x) &= \frac{1}{(m+1)!}f^{(m+1)}(\xi)(x-1)^{m+1} \\
&= \frac{1}{(m+1)!}(-1)^{m+1-1}(m+1-1)!\frac{1}{\xi^{m+1}}(x-1)^{m+1} \\
&= (-1)^m \frac{1}{m+1}\frac{1}{\xi^{m+1}}(x-1)^{m+1}
\end{aligned}$$

Remember that $E_m(x)$ tells us how the polynomial approximation differs from $\ln(x)$. Note that this term is not a polynomial because ξ depents on x in a nonpolynomial way. In any case, writing out the polynomial formula (found in part (a)) for $\ln(x)$ gives us

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots + (-1)^{n-1}\frac{1}{m}(x-1)^m + E_m(x).$$

The error, then, is given by

$$|E_m(x)| = \frac{1}{m+1}\frac{1}{\xi^{m+1}}(x-1)^{m+1} < \frac{1}{m+1}(x-1)^{m+1}.$$

(b) How many terms in the series need to be used to compute $\ln(2)$ with accuracy of one part in 10^8?

With $x = 2$, we know that

$$f(2) = \ln(2) = \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} + E_m(2) = \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} + E_m(2),$$

and we know that $|E_m(2)| < \frac{1}{m+1}(2-1)^{m+1} = \frac{1}{m+1}$. Since $E_m(2)$ is the numerical error, to compute $\ln(2)$ with the desired accuracy, we need to find m such that $E_m(2) \leq 10^{-8}$. This means that we can solve the inequality for m:

$$|E_m(2)| < \frac{1}{m+1} \leq 10^{-8}.$$