1 Gaussian Elimination and the LU Decomposition

We'll now consider the problem of *solving* a system of n linear equations in n unknowns

$$A\mathbf{x} = \mathbf{b}$$

by Gaussian elimination. Here, we'll assume that A is invertible[1] and, as usual, $n \times n$. No special properties of A are assumed (e.g., not triangular, not symmetric, not positive definite.).

Strategy: We want to transform $A\mathbf{x} = \mathbf{b}$ into an *equivalent system* $U\mathbf{x} = \mathbf{y}$ with U being an upper triangular matrix. Then, we can use back substitution to obtain the solution.

1.1 Elementary Transformations

The following transformations can be performed on a system of linear equations, which will not change the solution. Note that we'll assume the use of matrices to represent the problem.

1. Add a multiple of one row to another row.
$$R_i \mapsto R_i + cR_j,$$
where c is the multiple.
2. Interchange two rows (also known as pivoting).
$$R_i \leftrightarrow R_j$$
3. Multiply a row by a non-zero scalar.
$$R_i \mapsto cR_i,$$
where c is the non-zero scalar.

These transformations will be applied to a system of the form $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$.

1.2 Applying Elementary Operations

For now, we'll talk about Gaussian Elimination (GE) without row interchanges (pivoting). For now, we'll assume that $a_{11} \neq$. We want to convert all entries under a_{11} to 0.

1. First, let's get rid of a_{21}. We can use the operation
$$R_2 \mapsto R_2 - \frac{a_{21}}{a_{11}} R_1$$
to do just this:
$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}}_{\mathbf{b}} \xrightarrow[R_2 \mapsto R_2 - \frac{a_{21}}{a_{11}} R_1]{\text{Type (1) operation.}} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3 \\ \vdots \\ b_n \end{bmatrix}.$$
Note that, while we were able to get rid of a_{21}, the other entries in row 2 (a_{22}, a_{23}, and so on) were updated (hence $a_{22}^{(1)}$, $a_{23}^{(1)}$, and so on). We should also note that b_2 was updated, since each row operation affects the *entire* row, which means both the A and the $\mathbf{b}$.

[1] There's a unique solution $\mathbf{x}$, go find it!

2. Next, let's get rid of a_{31}. Very similarly to the previous step, we can do

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2n}^{(1)} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3 \\ \vdots \\ b_n \end{bmatrix} \xrightarrow[R_3 \mapsto R_3 - \frac{a_{31}}{a_{11}} R_1]{\text{Type (1) operation.}} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2n}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & \dots & a_{3n}^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(1)} \\ \vdots \\ b_n \end{bmatrix}$$

3. Suppose we want to get rid of a_{i1}. This is just

$$R_i \mapsto R_i - \frac{a_{i1}}{a_{11}} R_1$$

for $i = 3, \dots, n$. So, this gives us

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2n}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & \dots & a_{3n}^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(1)} & a_{n3}^{(1)} & \dots & a_{nn}^{(1)} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(1)} \\ \vdots \\ b_n^{(1)} \end{bmatrix}.$$

Next, we want to get rid of $a_{32}^{(1)}$, $a_{42}^{(1)}$, and so on[2]. Let's assume that $a_{22}^{(1)} \neq 0$.

1. Let's get rid of a_{32}. To do this, we'll use the operation

$$R_3 \mapsto R_3 - \frac{a_{32}^{(1)}}{a_{22}^{(1)}} R_2.$$

This gives us

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2n}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & \dots & a_{3n}^{(1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(1)} & a_{n3}^{(1)} & \dots & a_{nn}^{(1)} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(1)} \\ \vdots \\ b_n^{(1)} \end{bmatrix} \xrightarrow[R_3 \mapsto R_3 - \frac{a_{32}^{(1)}}{a_{22}^{(1)}} R_2]{\text{Type (1) operation.}} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2n}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \dots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \\ \vdots \\ b_n \end{bmatrix}.$$

2. Likewise, we can repeat this for a_{i2} using the operation

$$R_i \mapsto R_i - \frac{a_{i2}^{(1)}}{a_{22}^{(1)}} R_2$$

for $i = 4, \dots, n$. This gives us

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2n}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \dots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{n3}^{(2)} & \dots & a_{nn}^{(2)} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \\ \vdots \\ b_n^{(2)} \end{bmatrix}.$$

[2] We omit $a_{22}^{(1)}$ because, remember, our goal is to make our system into an upper-triangular system.

We can continue this procedure until we produce an upper-triangular matrix. This upper-triangular matrix will look like

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2n}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \dots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}.$$

At the very end, we can use back substitution on the upper-triangular matrix.

Remark: We are dividing by a_{11}, $a_{22}^{(1)}$, $a_{33}^{(2)}$, and so on for a general matrix. In reality, *some of these entries could be 0.*

1.3 LU Decomposition

Theorem 1.1: LU Decomposition

Let A be an $n \times n$ matrix whose leading principal submatrices are all nonsingular. Then, A can be decomposed in exactly one way into a product

$$A = LU,$$

such that L is unit lower-triangular and U is upper triangular.

In other words,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \ell_{21} & 1 & 0 & \dots & 0 \\ \ell_{31} & \ell_{32} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \ell_{n1} & \ell_{n2} & \ell_{n3} & \dots & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & u_{22} & u_{23} & \dots & u_{2n} \\ 0 & 0 & u_{33} & \dots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & u_{nn} \end{bmatrix}.$$

(Exercise: LU Decomposition.) Let

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 1 & 2 & 6 \\ 3 & 5 & 7 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$.

$$\begin{bmatrix}1&3&4\\1&2&6\\3&5&7\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}\xrightarrow[R_2\mapsto R_2-R_1]{\text{Operation (1)}}\begin{bmatrix}1&3&4\\0&-1&2\\3&5&7\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}$$
$$\xrightarrow[R_3\mapsto R_3-3R_1]{\text{Operation (1)}}\begin{bmatrix}1&3&4\\0&-1&2\\0&-4&-5\end{bmatrix}\begin{bmatrix}1\\0\\-2\end{bmatrix}$$
$$\xrightarrow[R_3\mapsto R_3-4R_2]{\text{Operation (1)}}\begin{bmatrix}1&3&4\\0&-1&2\\0&0&-13\end{bmatrix}\begin{bmatrix}1\\0\\-2\end{bmatrix}$$

Notice that our matrix is in upper-triangular form. From here, we just need to solve

$$\begin{cases}-13x_3=-2\\-1x_2+2x_3=0\\x_1+3x_2+4x_3=1\end{cases},$$

which can be done via backwards substitution.

We performed three operations to get to the upper-triangular matrix. Each operation corresponds to a lower-triangular matrix. In particular, if we write out

$$\tilde{L_1}\tilde{L_2}\tilde{L_3}A=U,$$

where each of the L_i represents a lower-triangular matrix and corresponds to an operation, then $\tilde{L_3}$ represents the first operation, $R_2\mapsto R_2-R_1$, or $\tilde{L_3}=\begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}$. Likewise, $\tilde{L_2}$ represents the second operation, $R_3\mapsto R_3-3R_1$, or $\tilde{L_2}=\begin{bmatrix}1&0&0\\0&1&0\\-3&0&1\end{bmatrix}$. Finally, $\tilde{L_1}$ represents the third operation, $R_3\mapsto R_3-4R_2$, or $\tilde{L_1}=\begin{bmatrix}1&0&0\\0&1&0\\0&-4&1\end{bmatrix}$. Notice how the positioning of the numbers corresponds to the entry that we tried to eliminate in A. For example, in L_2, we put -3 in position L_{21} because we eliminated the 3 from position A_{21} in the second operation. Therefore,

$$\begin{bmatrix}1&0&0\\0&1&0\\0&-4&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-3&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-1&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&3&4\\1&2&6\\3&5&7\end{bmatrix}=\begin{bmatrix}1&3&4\\0&-1&2\\0&0&-13\end{bmatrix}$$

In any case, we know that

$$\underbrace{\tilde{L_1}\tilde{L_2}\tilde{L_3}}_{\tilde{L}}A=U.$$

Then,

$$\tilde{L}A=U\implies A=\tilde{L}^{-1}U=LU$$

where

$$L=\tilde{L}.$$

Remark: To see how each operation corresponds to a lower-triangular matrix, consider $R_2\mapsto R_2+5R_1$:

$$\begin{bmatrix}1&0&0\\5&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\5a_{11}+a_{21}&5a_{12}+a_{22}&5a_{13}+a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}.$$

Likewise, consider $R_3 \mapsto R_3 + 10R_2$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 10a_{21} + a_{31} & 10a_{22} + a_{32} & a_{23} + a_{33} \end{bmatrix}.$$