

1 Multivariable Interpolation

In this section, we'll focus on interpolation in multiple variables, in particular two variables. Our data is represented by

- the nodes,

$$\mathcal{N} = \{(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m)\}.$$

- the values associated with each node,

$$f(x_1, y_1), f(x_2, y_2), \dots, f(x_m, y_m).$$

To interpolate each point in \mathcal{N} , we seek an interpolation formula $F : \mathbb{R}^2 \mapsto \mathbb{R}$ so that

$$F(x_i, y_i) = f(x_i, y_i), \quad 1 \leq i \leq m.$$

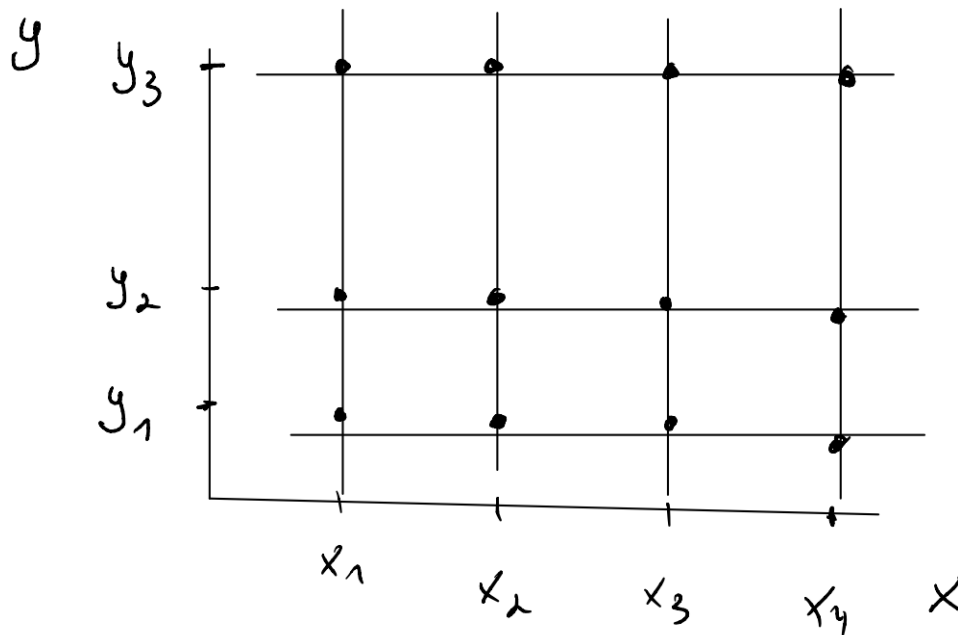
For arbitrary sets of nodes, this process is not always well-defined. So, we'll consider a special case.

1.1 Interpolating in the Special Case

Our special case deals with \mathcal{N} being a cartesian product, i.e.,

$$\mathcal{N} = \{(x_i, y_j) : 1 \leq i \leq p, 1 \leq j \leq q\}.$$

(Example.) If $p = 4$ and $q = 3$, we might have the points such that



We are interpolating along the vertical and horizontal lines here.

Remark: Points do not need to be equally spaced. In other words, it's not necessarily true that $y_2 - y_1 = y_1 - y_0$.

1.1.1 Cardinal Functions

Additionally, we only want to consider “cardinal” functions, i.e., functions $u_i(x)$ and $v_j(y)$ so that

$$u_i(x_j) = \delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \quad v_j(x_i) = \delta_{j,i} = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

for $1 \leq i, j \leq p$ and $1 \leq i, j \leq q$, respectively. We can construct such a cardinal function by

$$u_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^p \frac{x - x_j}{x_i - x_j}$$

$$v_j(y) = \prod_{\substack{i=1 \\ i \neq j}}^q \frac{y - y_i}{y_j - y_i}.$$

Let us define the set of vertical lines,

$$L_i = \{(x_i, y) : -\infty < y < \infty\}.$$

Then, we can define the function

$$(\bar{P}f)(x, y) = \sum_{i=1}^p f(x, y) u_i(x) = P_i(x, y), \quad P_i(x_i, y) = f(x_i, y).$$

Likewise, we have

$$(\bar{Q}f)(x, y) = \sum_{j=1}^q f(x, y_j) v_j(y).$$

There are two options for evaluating this:

- Option 1: Consider the tensor product¹,

$$\bar{P}\bar{Q} = (\hat{P}(\bar{Q}f))(x, y) = \sum_{i=1}^p \left(\sum_{j=1}^q f(x_i, y_j) v_j(y) \right) u_i(x) = \sum_{i=1}^p \sum_{j=1}^q f(x_i, y_j) v_j(y) u_i(x) = F(x, y).$$

This interpolates $f(x_i, y_j)$.

- Option 2: Consider the boolean sum. Then,

$$\begin{aligned} \bar{P} \oplus \bar{Q} &= \bar{P} + \bar{Q} - \bar{P}\bar{Q} \\ &= \sum_{i=1}^p f(x_i, y) u_i(x) + \sum_{j=1}^q f(x, y_j) v_j(y) - \sum_{i=1}^p \sum_{j=1}^q f(x_i, y_j) v_j(y) u_i(x) \\ &= F(x, y). \end{aligned}$$

This is also an interpolation.

¹Another way of writing this is $\bar{P} \otimes \bar{Q}$.