

# 1 Projectors & Reflectors (3.2)

In this section, we'll talk about projectors and reflectors, something that's important for QR decomposition.

## 1.1 Projectors

### Definition 1.1: Projector

A **projector** is a matrix  $P$  with

$$P^2 = P.$$

### Definition 1.2: Orthoprojector

If  $P$  is a projector and also symmetric (i.e.,  $P = P^T$ ), then  $P$  is called an **orthoprojector**.

(Example.) Suppose  $\mathbf{u} \in \mathbb{R}^n$  is a unit vector (i.e.,  $\|\mathbf{u}\|_2 = 1$ ). Then,  $P = \mathbf{u} \cdot \mathbf{u}^T$  is an orthoprojector. That is,

$$P = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} = \begin{bmatrix} u_1^2 & u_1 u_2 & \dots & u_1 u_n \\ u_2 u_1 & u_2^2 & \dots & u_2 u_n \\ \vdots & \vdots & \ddots & \vdots \\ u_n u_1 & u_n u_2 & \dots & u_n^2 \end{bmatrix}.$$

To see why  $P$  here is an orthoprojector, we'll show that it satisfies some properties.

1. Definition of a projector.

$$P^2 = P \cdot P = (\mathbf{u} \cdot \mathbf{u}^T)(\mathbf{u} \cdot \mathbf{u}^T) = \mathbf{u}(\underbrace{\mathbf{u}^T \mathbf{u}}_1) \mathbf{u}^T = \mathbf{u} \mathbf{u}^T = P.$$

2. Definition of an orthoprojector.

$$P^T = (\mathbf{u} \mathbf{u}^T)^T = (\mathbf{u}^T)^T \mathbf{u}^T = \mathbf{u} \mathbf{u}^T = P.$$

There are some additional properties to know for this case.

- $P\mathbf{u} = \mathbf{u}$ :

$$P\mathbf{u} = (\mathbf{u} \mathbf{u}^T)\mathbf{u} = \mathbf{u}(\underbrace{\mathbf{u}^T \mathbf{u}}_1) = \mathbf{u}.$$

- If  $\mathbf{v} \perp \mathbf{u}$  (i.e.,  $\langle \mathbf{v}, \mathbf{u} \rangle = 0$ ), then  $P\mathbf{v} = \mathbf{0}$ .

$$P\mathbf{v} = (\mathbf{u} \mathbf{u}^T)\mathbf{v} = \mathbf{u}(\underbrace{\mathbf{u}^T \mathbf{v}}_0) = \mathbf{0}.$$

### Remarks:

- Note that if  $\mathbf{u} \in \mathbb{R}^{n \times 1}$ , then  $\mathbf{u}^T \in \mathbb{R}^{1 \times n}$  and so  $P$  will be an  $n \times n$  matrix.
- Note that  $\mathbf{u} \mathbf{u}^T \neq \mathbf{u}^T \mathbf{u}$ . In particular,  $\mathbf{u} \mathbf{u}^T$  is an  $n \times n$  matrix while  $\mathbf{u}^T \mathbf{u} = \langle \mathbf{u}, \mathbf{u} \rangle = \|\mathbf{u}\|_2^2$ .

## 1.2 Reflectors

Reflectors are built by *projectors*.

**Definition 1.3: Reflector**

For a unit vector  $\mathbf{u} \in \mathbb{R}^n$  (i.e.,  $\|\mathbf{u}\|_2 = 1$ ),  $Q = I - 2\mathbf{u}\mathbf{u}^T$  is called a (householder) **reflector**.

**Remarks:**

- We can rewrite the above with  $Q = I - 2P$ , where  $P = \mathbf{u}\mathbf{u}^T$  is a projector.
- If  $\mathbf{u}$  doesn't have unit norm, we can normalize it,

$$\frac{\mathbf{u}}{\|\mathbf{u}\|_2},$$

so that  $\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|_2} \right\|_2 = \frac{1}{\|\mathbf{u}\|_2} \|\mathbf{u}\|_2 = 1$  (note that  $\|\mathbf{u}\|_2$  is a scalar.) In this sense, we can write

$$Q = I - 2 \frac{\mathbf{u}}{\|\mathbf{u}\|_2} \frac{\mathbf{u}^T}{\|\mathbf{u}\|_2} = I - 2 \frac{\mathbf{u}\mathbf{u}^T}{\|\mathbf{u}\|_2^2}.$$

There are some properties of  $Q = I - 2\mathbf{u}\mathbf{u}^T$  (where  $\mathbf{u}$  is a unit vector) to know.

1.  $Q\mathbf{u} = -\mathbf{u}$ .

$$Q\mathbf{u} = (I - 2\mathbf{u}\mathbf{u}^T)\mathbf{u} = \mathbf{u} - 2\mathbf{u}\mathbf{u}^T\mathbf{u} = \mathbf{u} - 2\mathbf{u} = -\mathbf{u}.$$

2.  $Q\mathbf{v} = \mathbf{v}$  such that  $\mathbf{v} \perp \mathbf{u}$ .

$$Q\mathbf{v} = (I - 2\mathbf{u}\mathbf{u}^T)\mathbf{v} = \mathbf{v} - 2\underbrace{\mathbf{u}\mathbf{u}^T\mathbf{v}}_0 = \mathbf{v}.$$

3.  $Q^T = Q$ .

$$Q^T = (I - 2\mathbf{u}\mathbf{u}^T)^T = (I - 2P)^T = I - 2P^T = I - 2P = Q.$$

Here, note that  $I^T = I$ . Additionally, note that  $P^T = P$ .

4.  $\underbrace{Q^T = Q^{-1}}_{\text{Orthogonal}}$  and  $Q = Q^{-1}$  and  $Q^T Q = Q^2 = I$ .

$$Q^2 = QQ = (I - 2P)(I - 2P) = I - 2P - 2P + 4P^2 = I - 4P - 4P^2 = I - 4P + 4P = I.$$

**Lemma 1.1**

For any  $\mathbf{x} \in \mathbb{R}^n$  and  $\mathbf{y} \in \mathbb{R}^n$  such that

$$\mathbf{y} = [\|\mathbf{x}\|_2 \quad 0 \quad 0 \quad \dots \quad 0]^T,$$

define  $\mathbf{v} = \mathbf{x} - \mathbf{y}$  and  $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|_2}$ . Then,

$$Q = I - 2\mathbf{u}\mathbf{u}^T$$

is a reflector satisfying  $Q\mathbf{x} = \mathbf{y}$ .

**Remarks:**

- If  $\mathbf{x} = \mathbf{y}$ , then  $Q = I$ .
- Alternatively, if  $\mathbf{e}_1 = [1 \quad 0 \quad 0 \quad \dots \quad 0]^T$ , then

$$\mathbf{y} = \|\mathbf{x}\|_2 \mathbf{e}_1.$$

It should be noted that  $\mathbf{e}_2 = [0 \quad 1 \quad 0 \quad \dots \quad 0]$  and  $\mathbf{e}_n = [0 \quad 0 \quad 0 \quad \dots \quad 1]$ .

### 1.3 QR Decomposition (For the 3rd Time)

We will talk about reduced QR later; for now, we will focus on full QR. The idea is that, with QR, we'll do something like

$$Q_n \dots Q_2 Q_1 A \mapsto R.$$

The idea is that, starting from  $A$ , we can multiply the reflectors multiple times until we end up with  $R$ , which is an upper-triangular matrix. This is analogous to LU decomposition, where we did

$$L_n \dots L_2 L_1 A \mapsto U.$$

Now, for QR decomposition, given  $A \in \mathbb{R}^{n \times m}$  (our “tall” matrix), we want to find  $QR$ . We can rewrite  $A$  in column form,

$$A = [c_1 \ c_2 \ c_3 \ \dots \ c_i \ \dots \ c_m],$$

where  $c_i$  is the  $i$ th column for  $i = 1, 2, \dots, m$ . Recall that we want to derive  $R$ ; that is, we want an upper-triangular matrix. So, starting from the first column, we want to make all the entries under  $a_{11}$  0. We can use a reflector mapping  $Q_1$  to map the column,

$$c_1 \mapsto \|c_1\| \mathbf{e}_1$$

where  $\mathbf{e}_1 \in \mathbb{R}^n$ , so that we end up with

$$Q_1 A = \begin{bmatrix} \|c_1\| & * & * & \dots & * \\ 0 & * & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & \dots & * \end{bmatrix} = \begin{bmatrix} \|c_1\| & * & * & \dots & * \\ 0 & \underline{*} & \underline{*} & \dots & \underline{*} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \underline{*} & \underline{*} & \dots & \underline{*} \end{bmatrix}.$$

From the above matrix, we can represent the underlined stars as a new matrix:

$$\tilde{A} = \begin{bmatrix} \underline{*} & \underline{*} & \dots & \underline{*} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{*} & \underline{*} & \dots & \underline{*} \end{bmatrix} \in \mathbb{R}^{(n-1) \times (m-1)}.$$

So, if we have

$$\tilde{A} = [\tilde{c}_2 \ \tilde{c}_3 \ \dots \ \tilde{c}_m],$$

we want to define a reflector mapping

$$\tilde{Q}_2 : \tilde{c}_2 \mapsto \|\tilde{c}_2\| \tilde{\mathbf{e}}_1$$

where  $\tilde{\mathbf{e}}_1 \in \mathbb{R}^{n-1}$ . Now, define

$$Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{Q}_2 \end{bmatrix}$$

so that

$$Q_2 Q_1 A = \begin{bmatrix} \|c_1\| & * & * & \dots & * \\ 0 & \|\tilde{c}_2\| & * & \dots & * \\ 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & * & \dots & * \end{bmatrix} = \begin{bmatrix} \|c_1\| & * & * & \dots & * \\ 0 & \|\tilde{c}_2\| & * & \dots & * \\ 0 & 0 & \underline{*} & \dots & \underline{*} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \underline{*} & \dots & \underline{*} \end{bmatrix}.$$

From this, we can define

$$B = \begin{bmatrix} \underline{*} & \dots & \underline{*} \\ \vdots & \ddots & \vdots \\ \underline{*} & \dots & \underline{*} \end{bmatrix}.$$

Continuing this process, we should eventually end up with

$$Q_m \dots Q_1 A = \begin{bmatrix} \|c_1\| & * & * & \dots & * \\ 0 & \|\tilde{c}_2\| & * & \dots & * \\ 0 & 0 & \|\tilde{c}_3\| & \dots & * \\ \vdots & \vdots & \vdots & \ddots & * \\ 0 & 0 & 0 & \dots & \|\tilde{c}_m\| \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R.$$

Note that  $\tilde{Q}A = R \implies A = QR$ . Then, the question becomes: how do we define  $Q$ ?