1 Gaussian Elimination with Pivoting (1.8)

In the previous section, we discussed Gaussian Elimination *without* row interchanges (i.e., pivoting). In this section, we will now permit row interchanges. Consider the following system,

$$\begin{bmatrix} 0 & 4 & 1 \\ 1 & 3 & 4 \\ 2 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

Notice that this matrix has several properties:

- It's invertible ($\det(A) \neq 0$), therefore a unique solution exists.
- However, using Gaussian Elimination *without* row interchanges would fail.

However, we can resolve this by switching two rows, e.g., R1 and R3, to get

$$\begin{bmatrix} 2 & 2 & 5 \\ 1 & 3 & 4 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}.$$

When deciding which two rows to switch, the rule is to use the row with the *first* entry containing the **largest absolute value**[1]. The idea is that we'll divide by first element in next step for Gaussian elimination.

1.1 Relation to the Permutation Matrix

Note taht a permutation matrix P is a matrix which only contains 0's and 1's such that each row and each column has exactly one entry equal to 1.

Example of Permutation Matrix	**<u>Not</u> a Permutation Matrix**
$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ This matrix has two 1's in the first row and last column.

How does the permutation matrix work in relation to row interchanges?

(Example.) Suppose we multiply a permutation matrix,

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

on the left.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 & 1 \\ 1 & 3 & 4 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5 \\ 1 & 3 & 4 \\ 0 & 4 & 1 \end{bmatrix}.$$

This corresponds to switching R1 and R3.

(Example.) Suppose we multiply the same permutation matrix as in the previous example on the right. Then,

$$\begin{bmatrix} 0 & 4 & 1 \\ 1 & 3 & 4 \\ 2 & 2 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 3 & 1 \\ 5 & 2 & 2 \end{bmatrix}.$$

[1] The reason why we choose the largest absolute value is because dividing by small numbers can cause errors, and these errors can accumulate.

Here, this corresponds to switching C1 and C3.

1.2 PLU Decomposition

Gaussian Elimination with pivoting leads to a decomposition of the form $PA = LU$, also known as *PLU decomposition*, where

- P is a permutation matrix,
- L is a lower-triangular matrix, and
- U is an upper-triangular matrix. The PLU exists if A is invertible.

Remarks:

- The PLU exists if A is invertible.
- In MATLAB, we can run `[P, L, U] = lu(A)`, where

$$P \cdot A = L \cdot U \implies A = P^T \cdot L \cdot U.$$

1.2.1 Finding PLU Decomposition

We'll find the PLU decomposition through an example.

(Example.) Suppose we want to find the PLU decomposition of

$$A = \begin{bmatrix} 0 & 4 & 1 \\ 1 & 3 & 4 \\ 2 & 2 & 5 \end{bmatrix}.$$

- Step 1: First, let's switch R1 and R3.

$$P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad P_1 A = \begin{bmatrix} 2 & 2 & 5 \\ 1 & 3 & 4 \\ 0 & 4 & 1 \end{bmatrix}.$$

- Step 2: Next, let's perform the operation $R_2 \mapsto R_2 - \frac{1}{2}R_1$.

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad L_1 P_1 A = L_1 \begin{bmatrix} 2 & 2 & 5 \\ 1 & 3 & 4 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5 \\ 0 & 2 & \frac{3}{2} \\ 0 & 4 & 1 \end{bmatrix}.$$

- Step 3: Next, notice that **4** is a pivot (the 2 in the second row is smaller than 4). let's switch R_2 and R_3.

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad P_2 L_1 P_1 A = \begin{bmatrix} 2 & 2 & 5 \\ 0 & 4 & 1 \\ 0 & 2 & \frac{3}{2} \end{bmatrix}.$$

- Step 4: We can now perform the operation $R_3 \mapsto R_3 - \frac{1}{2}R_2$.

$$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}, \qquad L_2 P_2 L_1 P_1 A = \begin{bmatrix} 2 & 2 & 5 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Here, we find our upper-triangular matrix

$$U = L_2P_2L_1P_1A = \begin{bmatrix} 2 & 2 & 5 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now that we found the upper-triangular matrix U, we need to find the decomposition. We don't know what the P or L matrices are. Notice that, with $L_2P_2L_1P_1$, we can do

$$P_2L_1 = \tilde{L_1}P_2.$$

Then,

$$L_2P_2L_1P_1 = \underbrace{L_2\tilde{L_1}}_{\tilde{L}}\overbrace{P_2P_1}^{P}.$$

Therefore, if

$$P_2L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & 1 & 0 \end{bmatrix},$$

then

$$\begin{aligned} P_2L_1 &= \tilde{L_1}P_2 \\ \implies \tilde{L_1} &= P_2L_1P_2^{-1} \\ \implies \tilde{L_1} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & 1 & 0 \end{bmatrix} P_2^{-1} \\ \implies \tilde{L_1} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\ \implies \tilde{L_1} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}. \end{aligned}$$

Therefore, going back to

$$L_2P_2L_1P_1 = \underbrace{L_2\tilde{L_1}}_{\tilde{L}}\overbrace{P_2P_1}^{P},$$

we have

$$\tilde{L}PA = U \implies PA = \tilde{L}^{-1}U.$$

From this, we find

$$\tilde{L}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}.$$

Finally, defining $L = \tilde{L}^{-1}$ gives us

$$PA = LU.$$