1 Fixed Point and Functional Iteration (Section 3.4)

Let F be a one-dimensional real-valued (and typically continuous) function. Then, the **functional iteration** is defined by

$$x_{m+1} = F(x_m), \quad m \geq 0.$$

Note that this generalizes the previous approaches that we've had; for example, we can represent Newton's method in this way.

(Example.) Consider Newton's method. We can write

$$x_{m+1} = \underbrace{x_m - \frac{f(x_m)}{f'(x_m)}}_{F(x_m)}.$$

If the limit, $\lim_{n\mapsto\infty} x_{m+1} = s$ exists, then

$$s = \lim_{m\mapsto\infty} x_{m+1} = \lim_{m\mapsto\infty} F(x_m) = F\left(\lim_{m\mapsto\infty} x_m\right) = F(s).$$

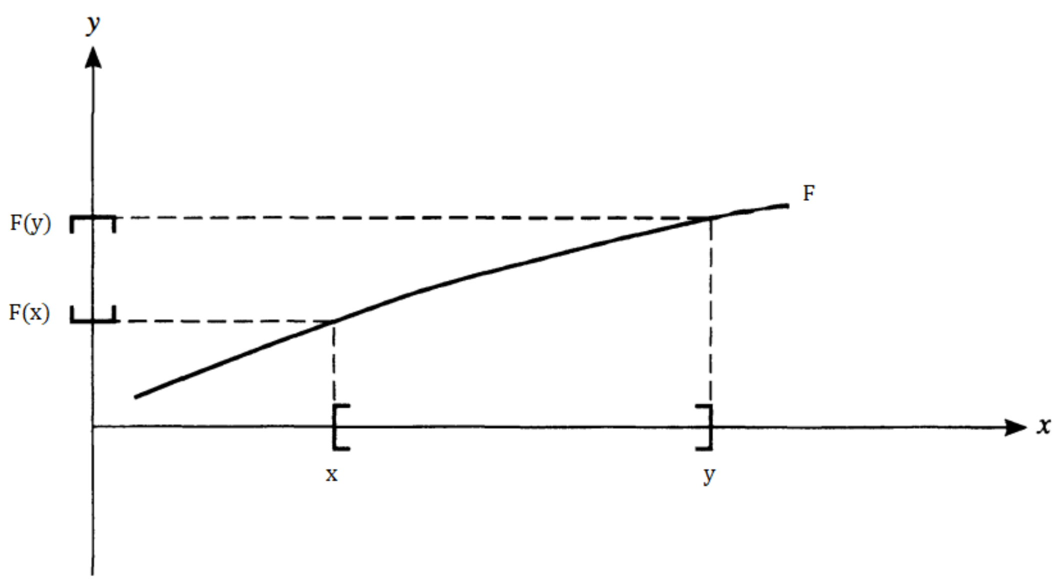
The fixed point, s, is defined by $\boxed{s = F(s)}$.

1.1 Contractive Mapping Property

Let F be a map on a closed set $C \subseteq \mathbb{R}$ into itself. For $0 < \lambda < 1$, we have

$$\boxed{|F(x) - F(y)| \leq \lambda |x - y|}$$

for $x, y \in C$. In other words, a mapping (or function) F is said to be **contractive** if there exists a λ such that the above is satisfied.



The interval between x and y is larger than the interval between $F(x)$ and $F(y)$ because $\lambda \leq 1$ (i.e., the intervals are being shrunk).

Theorem 1.1: Contractive Mapping Theorem

Let

- C be a closed subset of $\mathbb{R}$ (i.e., $C \subseteq \mathbb{R}$),
- F be a contractive mapping from C into C, and
- $x_0 \in C$ be a starting point.

Then, $x_{m+1} = F(x_m)$ converges to a unique fixed point s starting from x_0.

Proof. We'll prove both the convergence and uniqueness parts of the theorem.

Convergence: We have

$$\begin{aligned} |x_{m+1} - x_m| = |F(x_m) - F(x_{m-1})| &\leq \lambda |x_m - x_{m-1}| \\ &\leq \lambda^2 |x_{m-1} - x_{m-2}| \\ &\vdots \\ &\leq \lambda^m |x_1 - x_0|. \end{aligned}$$

Then,

$$\sum_{m=0}^{\infty} |x_{m+1} - x_m| \leq \sum_{m=0}^{\infty} \lambda^m |x_1 - x_0| = \frac{|x_1 - x_0|}{1 - \lambda}.$$

Thus,

$$\lim_{m \mapsto \infty} x_{m+1} = \lim_{m \mapsto \infty} x_m = s$$

converges. This proves the convergence part.

Uniqueness: Suppose we have two fixed points s_1 and s_2, and $0 < \lambda < 1$. Then,

$$|F(s_1) - F(s_2)| \leq \lambda |s_1 - s_2|.$$

But, if s_1 and s_2 are two fixed points, then we have $F(s_1) = s_1$ and $F(s_2) = s_2$. So,

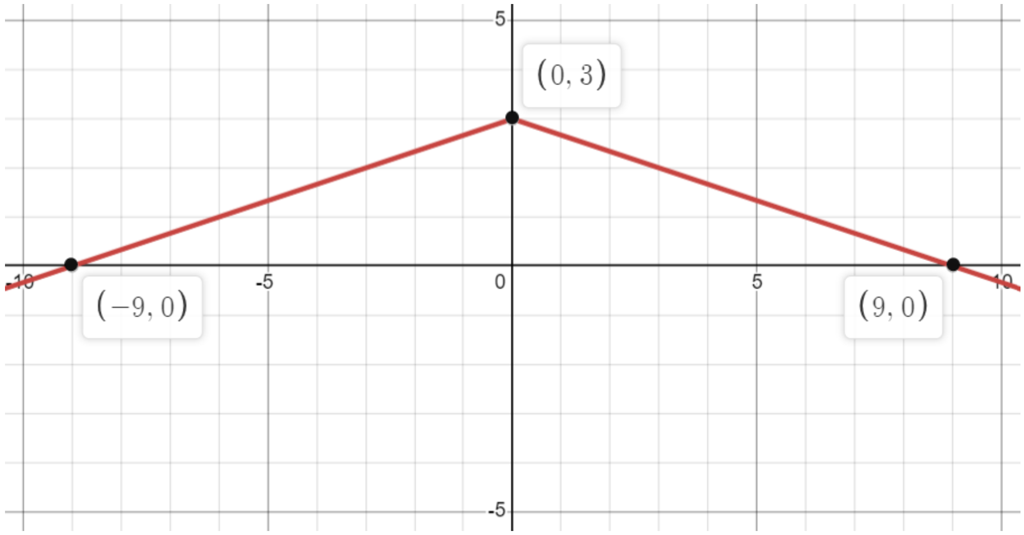
$$|s_1 - s_2| \leq \lambda |s_1 - s_2|.$$

Note that this can only hold if $s_1 = s_2$, as $\lambda = 1$ but remember that $\lambda < 1$.

Then, we're done. □

(Example.) Suppose we want to prove convergence for $x_{m+1} = 3 - \frac{1}{3}|x_m|$, with $x_0 = -15$ and $m \geq 0$.

Let $F(x) = 3 - \frac{1}{3}|x|$. If we plot $F(x)$, we get



Notice how F maps real values back to real values. Note that $C = \mathbb{R}$ and $x_0 \in C$. Now, we want to check the contraction property. If $x, y \in C$, then

$$|F(x) - F(y)| = \left|\left(3 - \frac{1}{3}|x|\right) - \left(3 - \frac{1}{3}|y|\right)\right| = \frac{1}{3}\left||x| - |y|\right|.$$

Applying the triangle inequality, we have

$$\frac{1}{3}\left||x| - |y|\right| \leq \frac{1}{3}|x - y|.$$

So, if we set $\lambda = \frac{1}{3} < 1$, then we've shown the contractive property.

Now, we want to think about the fixed point case. Since x_{m+1} converges, we have a fixed point. The fixed point is defined by

$$s = F(s).$$

So, $s = 3 - \frac{1}{3}|s|$. Then, solving for s gives us the fixed points. Since we have absolute values, we have two cases to consider.

- $s = 3 - \frac{1}{3}s$ if $s > 0$.
$$\frac{4}{3}s = 3 \implies s = \frac{9}{4}.$$
- $s = 3 + \frac{1}{3}s$ if $s < 0$.
$$\frac{2}{3}s = 3 \implies s = \frac{9}{2}.$$

Notice how we have 2 values of s. We have two equations above, but notice how the second equation cannot be true as the equation is only value if $s < 0$, but we have $s = \frac{9}{2} > 0$. Thus, $s = \frac{9}{4} \in C$ is our fixed point.

(Example.) Suppose we have

$$F(x) = 4 + \frac{1}{3}\sin(2x),$$

and let $C = \left[\frac{11}{3}, \frac{13}{3}\right]$. To consider the contraction property for this function, we can apply the Mean-

Value Theorem. Let $x, y \in C$. Then,

$$\begin{aligned}
|F(x) - F(y)| &= \left| \left(4 + \frac{1}{3}\sin(2x)\right) - \left(4 + \frac{1}{3}\sin(2y)\right) \right| \\
&= \frac{1}{3}|\sin(2x) - \sin(2y)| \\
&= \frac{1}{3}|2\cos(2\xi)(x-y)| && \text{Mean-Value Theorem, See Remark} \\
&\leq \frac{2}{3}\left|x - y\right|.
\end{aligned}$$

From there, it's clear that $\lambda = \frac{2}{3} < 1$.

Note that we can write a program to compute this fixed point based on this simple algorithm.

Algorithm 1 Computing Fixed Point

```
x ← 4
M ← 20
for k ← 1 to M do
    x ← 4 + (1/3) sin(2x)
end for
```

Here, x will contain the approximate fixed point.

Remark: Recall that the Mean Value Theorm is defined by

$$f(b) - f(a) = f'(\xi)(b-a).$$

Notice how, in particular, $f(x) = \sin(2x)$ in our example above. Then, when we look at $f'(\xi) = 2\cos(2\xi)$, we find that the maximum possible value this function can return is 2, thus how we were able to conclude that

$$\frac{1}{3}|2\cos(2\xi)(x-y)| \leq \frac{2}{3}\left|x-y\right|.$$