Math 20D Notes

Introduction to Differential Equations

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1 Introduction to Differential Equations

To motivate our definition of a differential equation, we first begin by talking about what an algebraic equation is. When we are given an algebraic equation like:

$$x^2 + 5x - 6 = 0$$

Our goal is to find the value of x. Here, x is the unknown. We need to find a solution (i.e. a number) for x such that the equation is satisfied.

In a differential equation, we are essentially doing the same thing. Given a differential equation, we need to find the unknown function that satisfies it. For instance, if we are given:

$$f'(x) - x = 0$$

Then $f'(x)$ is the unknown; that is, we need to find the function that satisfies this.

Of course, we aren't just limited to $f'(x)$. In fact, we may see the second, third, fourth derivative, and so on; essentially, given a differential equation, we need to find the unknown function involving derivatives of any order.

1.1 Independent vs. Dependent Variables

If a differential equation involvs the derivative of one variable with respect to another, then the former variable is called a **dependent variable** and the latter is called an **independent variable**.
Let's suppose we have the following differential equation:

$$\frac{dy}{dx} - x = 0$$

- y is the dependent variable.
- x is the independent variable.

If we wrote the above differential equation like so:

$$y' - x = 0$$

Then again:

- y is the dependent variable.
- x is the independent variable.

1.2 Classifying Differential Equations

There are three different criterias for classifying differential equations.

1.2.1 Linear vs. Non-Linear Differential Equation

A linear equation means that there is **no** power (or, more specifically, a power of 1) on any of the dependent variables. For example, the following differential equation is linear:

$$y' - x = 0$$

It should be noted that having y, y', y'', etc. (essentially, any derivative) does not automatically make a differential equation non-linear. The following differential equation is also linear:

$$2y + 3y' - xy'' = 0$$

It doesn't matter what the power of any dependent terms is; so, you can have terms like $x^{15}y''$ in a differential equation and that (alone) would make it linear.

A differential equation is non-linear if:

- Any of the dependent variables have a power that isn't 1. This makes the following differential equations non-linear:

$$2\boxed{y^2} + 3y' - xy'' = 0$$

$$\boxed{\sqrt{y-3}} + y' = x$$

- The dependent variables are being multiplied with each other. For example, y'' by itself is fine but $y'y''$ is not. This makes the following differential equation non-linear:

$$2y + 3\boxed{y'y''} - xy'' = 0$$

Keep in mind that we can multiply the derivatives with some power of the independent variables

1.2.2 Homogeneous/Non-Homogeneous

A homogeneous differential equation is one where there are no **lone** independent terms.

1.2.3 Order of Equation

The order of a differential equation depends on the highest derivative **alone**. For example, the following differential equation has an order of 2:

$$y'' - x = 0$$

The following differential equations have an order of 3:

$$y''' - y' = 1$$

$$(y''')^3 - y' = 10$$

Here, it doesn't matter what the power of any y term is.