1 Vector Space

We now begin talking about **vector spaces**.

1.1 Definition of a Vector Space

Definition 1.1: Vector Space

A set V is said to be a **vector space** over a field F if V is an abelian group under addition (denoted by $+$) and, if for each $a \in F$ and $v \in V$, there is an element $av \in V$ such that the following conditions hold for all $a, b \in F$ and all $u, v \in V$.

1. $a(v+u) = av + au$
2. $(a+b)v = av + bv$
3. $a(bv) = (ab)v$
4. $1v = v$

Remarks:

- The members of a vector space are called *vectors*.
- The members of the field are called *scalars*.
- The operations that combine a scalar a and a vecotr v to form the vector av is called *scalar multiplication*.
- In linear algebra, we generally worked with vector spaces over $\mathbb{R}$. In this class, we will work with vector spaces over $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$, $\mathbb{F}_p$, $\mathbb{F}[x]/\langle f(x)\rangle$ (where $\langle f(x)\rangle$ is irreducible), $\mathbb{F}_3[i]$, and so on.

1.1.1 Example 1: Set of Matrices

Consider $M_2(\mathbb{R})$. Recall that this is defined by

$$M_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$$

If we take $2 \in \mathbb{R}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in M_2(\mathbb{R})$, then we have

$$2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} \in M_2(\mathbb{R})$$

1.1.2 Example 2: Polynomial Ring over a Field

Consider $\mathbb{F}_p[x]$. This is a vector space over $\mathbb{F}_p$ because:

- Addition of a vector space forms an abelian group.
- Scalar multiplication also works. If we think of $\mathbb{F}_p$ as the set of constant polynomials, then we can do something like
$$3(x^2 + x + 2) = 3x^2 + 3x + 6$$
where
$$3 \in \mathbb{F}_p \qquad 3x^2 + 3x + 6 \in \mathbb{F}_p[x]$$
- The four properties described in the definition are satisfied.

1.1.3 Example 3: An Important Example

Let E be a field, and let $F \subseteq E$ be a subfield. Then, E is a vector space over F.

For example, $\mathbb{C}$ is an $\mathbb{R}$-vector space. For some $a + bi \in \mathbb{C}$ and $r \in \mathbb{R}$, we can do

$$r(a + bi) = ra + rbi \in \mathbb{C}$$

Some other examples are

- $\mathbb{Q} \subseteq \mathbb{Q}[\sqrt{2}]$. Here, $\mathbb{Q}[\sqrt{2}]$ is a vector space over $\mathbb{Q}$.
- $\mathbb{F}_3 \subseteq \mathbb{F}_3[i]$. Here, $\mathbb{F}_3[i]$ is a vector space over $\mathbb{F}_3$.

1.2 Definition of a Vector Subspace

Definition 1.2: Vector Subspace

Let V be a vector space over a field F and let U be a subset of V. We say that U is a **subspace** of V if U is also a vector space over F under the (same) operations of V.

1.2.1 Example 1: Diagonal Matrices

The diagonal matrices

$$\left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\} \subseteq M_2(\mathbb{R})$$

is a subspace.

1.2.2 Example 2: Quadratic Polynomials

The set of quadratic polynomials

$$\{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{F}_p\} \subseteq \mathbb{F}_p[x]$$

is a subspace. A few notes to consider:

- This is a subspace because if we take two quadratic polynomials and add them together, we get a quadratic polynomial. If we multiply any quadratic polynomial by a constant, then it's still going to be quadratic (or even 0).
- This is *not* a subring because it's not closed under multiplication. For example, x^2 is in this set, but $x^2 x^2 = x^4$ is not.

1.2.3 Example 3: Span

Consider $\mathrm{Span}\{v_1, v_2, \ldots, v_n\} \subseteq V$. Recall that the span is the set of all linear combinations

$$\{a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \mid a_1, a_2, \ldots, a_n \in F\}$$

Note that the textbook writes this as $\langle v_1, v_2, \ldots, v_n \rangle$. In the context of vector spaces, this notation means the span.

If $\mathrm{Span}\{v_1, v_2, \ldots, v_n\} = V$, then we say that $\{v_1, v_2, \ldots, v_n\}$ spans V.

1.2.4 Example 4: Spanning Set

Consider the set of matrices

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

This is a spanning set of $M_2(\mathbb{R})$ because we can write these matrices as a linear combination

$$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

1.3 Linear Independence

Definition 1.3: Linearly Independent/Dependent

A set S of vectors is said to be **linearly dependent** over a field F if there are vectors $v_1, v_2, \ldots, v_n$ from S and elements $a_1, a_2, \ldots, a_n$ from F, *not all zero*, such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$$

A set of vectors that is not linearly dependent over F is called **linearly independent** over F.

Remarks:

- Put it another way, a set of vectors is linearly dependent over F if there is a nontrivial linear combination of them over F equal to 0.
- Essentially, this definition is about the redundancy in the representation of the span. For example, there might be two different linear combinations that give the same vector in the set. In this case, the vectors are linearly dependent.

1.3.1 Example 1: Linearly Independent Spanning Set

Consider the spanning set

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

This is linearly independent. If we take a linear combination and set it equal to zero, like so

$$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

then clearly $a = b = c = d = 0$ is the only solution.

1.3.2 Example 2: Linearly Dependent Spanning Set

Consider the following spanning set

$$\{x + 1, x^2, x^2 - 2, x\} \in \mathbb{F}_p[x]$$

This is linearly dependent because

$$(x^2 - 2) + (-1)x^2 + 2(x + 1) + (-2)x = 0$$