1 Characteristic of a Ring

Consider the ring $\mathbb{Z}_3[i]$, with the elements:

$$\{0, 1, 2, i, 1+i, 2+i, 2i, 1+2i, 2+2i\}$$

For any element x in this ring, we have:

$$3x = x + x + x = 0$$

For example:

- $2i + 2i + 2i = 6i = 0i = 0$
- $(1+2i) + (1+2i) + (1+2i) = 3 + 6i = 0$
- And so on.

Similarly, in the ring $\{0, 3, 6, 9\} \subset \mathbb{Z}_{12}$, we have, for all x:

$$4x = x + x + x + x = 0$$

1.1 Characteristic of a Ring

Definition 1.1: Characteristic of a Ring

The **characteristic** of a ring R is the least positive integer n such that $nx = 0$ for all $x \in R$. If no such integer exists, we say that R has characteristic 0. The characteristic of R is denoted by $\operatorname{char} R$.

So, for example, the ring of integers $\mathbb{Z}$ has characteristic 0 and $\mathbb{Z}_n$ has characteristic n. For example, consider $\mathbb{Z}_3 = \{0, 1, 2\}$. Then, we know that:

$$3x = x + x + x = 0 \qquad \forall x$$

So the characteristic of $\mathbb{Z}_3$ is $\boxed{3}$. Now, consider $\mathbb{Z}_6$. We know that:

$$6x = x + x + x + x + x + x = 0 \qquad \forall x$$

So, its characteristic is $\boxed{6}$.

1.2 Characteristic of a Ring with Unity

Theorem 1.1: Characteristic of a Ring with Unity

Let R be a ring with unity 1. If 1 has infinite order under addition, then the characteristic of R is 0. If 1 has order n under addition, then the characteristic of R is n.

Proof. Suppose 1 has infinite order. Then, there is no positive integer n such that $n \cdot 1 = 0$, so R must have characteristic 0. Now, let's suppose that 1 does have additive order n. Then, we know that $n \cdot 1 = 0$ and n is the least positive integer with this property. So, for any $x \in R$, we have:

$$\begin{aligned} n \cdot x &= \overbrace{x + x + \cdots + x}^{n \text{ times}} \\ &= \overbrace{1x + 1x + \cdots + 1x}^{n \text{ times}} \\ &= (\overbrace{1 + 1 + \cdots + 1}^{n \text{ times}})x \\ &= (n \cdot 1)x = 0x = 0 \end{aligned}$$

So, R has characteristic n. $\square$

Theorem 1.2: Characteristic of an Integral Domain

The characteristic of an integral domain is 0 or prime.