1 Moving to $\mathbb{R}^3$

For the most part, most of what we talked about in $\mathbb{R}^2$ applies here as well.

1.1 The Basics

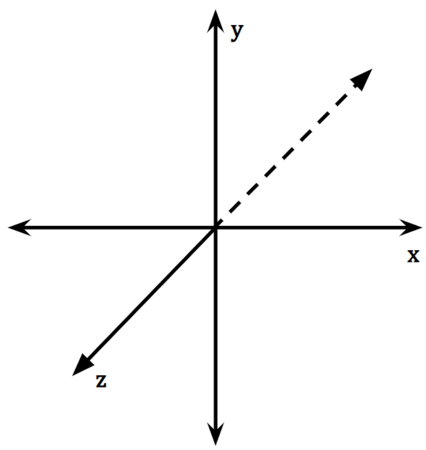
We first begin by talking about some of the basics in $\mathbb{R}^3$.

1.1.1 Basic Notation in $\mathbb{R}^3$

- **Points:** In $\mathbb{R}^3$, points are triples. They are still written in vector form like so:

$$\mathbf{x} = \langle x_1, x_2, x_3 \rangle = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

- **Axes:** In $\mathbb{R}^3$, the xyz-axes are in a different orientation than expected. In many other classes, y is facing towards us; <u>however</u>, in this course, y will be facing upwards while z is facing towards us.



As a side note, this is *still* a right-handed coordinate system. The cross product rule still uses the right-hand rule.

- **Standard Basis Vectors:** The standard basis vectors (unit vectors) in $\mathbb{R}^3$ are as follows:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

1.1.2 Transformations

In particular, linear transformations, translations, and affine transformations are **identical** to the definitions on $\mathbb{R}^2$. To see what we mean, consider the following:

- **Translations:** A translation is defined by

$$T_{\mathbf{u}}(\mathbf{x}) = \mathbf{x} + \mathbf{u},$$

where $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \mathbb{R}^3$.

- **Affine Transformation:** An affine transformation is defined by
$$A(\mathbf{x}) = B(\mathbf{x}) + \mathbf{u},$$
where B is linear.

Consider the following *scaling* transformations:

- **Uniform Scaling:** For some $\alpha \in \mathbb{R}$, uniform scaling just scales the vector $\mathbf{x}$ by a factor of α. So, we have
$$S_\alpha(\mathbf{x}) = \alpha\mathbf{x}.$$
- **General Scaling:** For some $\alpha, \beta, \gamma \in \mathbb{R}$, scaling a vector involves multiplying each of the constant terms by the corresponding terms in the vector. That is,
$$S_{\langle\alpha,\beta,\gamma\rangle}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_1 \\ \beta x_2 \\ \gamma x_3 \end{bmatrix}.$$

1.1.3 Rotations Around the Origin

Rotations in $\mathbb{R}^3$ are more complicated. Here, we denote $R_{\theta,\mathbf{u}}$ to be the rotation angle θ around axis $\mathbf{u}$, where $\mathbf{u} \neq \mathbf{0}$. The direction is given by the right-hand rule.

(Example.) Consider $R_{\frac{\pi}{2},\mathbf{i}}$, which is a 90 degree rotation (or $\pi/2$ radians) around the x-axis. How does $R_{\frac{\pi}{2},\mathbf{i}}$ act on the standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$?

> First, we're holding the x-axis fixed since we're rotating around the x-axis. Thus, this rotation will map $\mathbf{i}$ to itself. Next, we note that $\mathbf{j}$ will be mapped to $\mathbf{k}$. Likewise, $\mathbf{k}$ will be mapped to $-\mathbf{j}$.

1.1.4 Matrix Representation of Linear Transformations

Now, we'll talk about 3×3 matrix representations of linear transformations. These are more or less the same thing as in $\mathbb{R}^2$.

(Example.) $S_{\langle\alpha,\beta,\gamma\rangle}$ is represented by
$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}.$$

(Example.) Consider $R_{\frac{\pi}{2},\mathbf{k}}$, which is a 90 degree rotation around the z-axis. We note that:

- The $\mathbf{i}$ vector is mapped to the $\mathbf{j}$ vector.
- The $\mathbf{j}$ vector is mapped to the $-\mathbf{i}$ vector.
- The $\mathbf{k}$ vector is mapped to itself.

Therefore, the matrix representation of this rotation is given by
$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

More generally, if A is a linear transformation, we let $\mathbf{u} = A(\mathbf{i})$, $\mathbf{v} = A(\mathbf{j})$, and $\mathbf{w} = A(\mathbf{k})$. Then, A is represented by the 3×3 matrix
$$M = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix} = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}.$$

1.1.5 Homogeneous Coordinates & Matrix Representations of Affine Transformations

We define the four-tuple $\langle x, y, z, w\rangle$ to be a **homogeneous** representation of $\langle \frac{x}{w}, \frac{y}{w}, \frac{z}{w}\rangle \in \mathbb{R}^3$, where $w \neq 0$.

Let us now suppose that $A(\mathbf{x}) = B(\mathbf{x}) + \mathbf{t}$, where B is a linear transformation and $\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} \in \mathbb{R}^3$, so that A is affine. Suppose that B is a 3×3 matrix representation

$$M = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} \end{bmatrix}.$$

Then, the 4×4 matrix

$$N = \begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & t_1 \\ m_{2,1} & m_{2,2} & m_{2,3} & t_2 \\ m_{3,1} & m_{3,2} & m_{3,3} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

represents the affine transformation A.

1.2 Rigid & Orientation-Preserving Transformations

We first begin by talking about these in *both* $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition 1.1: Rigid

A transformation A is **rigid** if the following conditions hold:

- It preserves distances between points; that is, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ (or $\mathbb{R}^3$),
$$||A(\mathbf{x}) - A(\mathbf{y})|| = ||\mathbf{x} - \mathbf{y}||.$$
- It preserves angles. To see what we mean here, if we have two vectors $\mathbf{u}$ and a vector $\mathbf{v}$, both rooted at some point, then suppose these two vectors form an angle θ. Then, the idea is that $A(\mathbf{u})$ and $A(\mathbf{v})$ also has the same angle θ.

Remark: We say that $||\mathbf{u}||$, the magnitude (also called *norm* or *length*), is equal to:

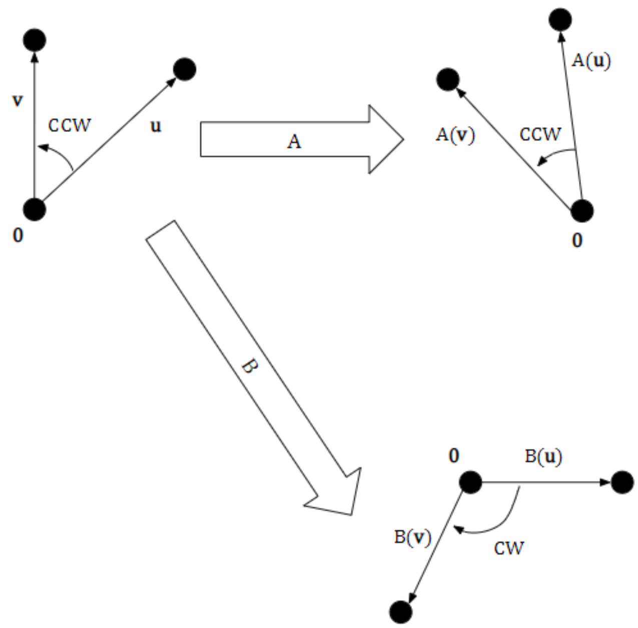
- $\sqrt{u_1^2 + u_2^2}$ in $\mathbb{R}^2$ if $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.
- $\sqrt{u_1^2 + u_2^2 + u_3^3}$ in $\mathbb{R}^3$ if $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$.

1.2.1 Orientation-Preserving in $\mathbb{R}^2$

Definition 1.2: Orientation-Preserving in $\mathbb{R}^2$

In $\mathbb{R}^2$, an affine transformation A is **orientation-preserving** if it preserves the direction of angles.

Consider the following figure, where A is an orientation-preserving transformation and B is not an orientation-preserving transformation (sometimes known as *orientation-reversing*).



In particular, rotations are orientation-preserving whereas reflections are not orientation-preserving.

1.2.2 Orientation-Preserving in $\mathbb{R}^3$

Informally, in $\mathbb{R}^3$, an affine transformation A is **orientation-preserving** if it preserves the "right-hand" rule.

Theorem 1.1

Let M, a 3×3 matrix, represent the linear transformation A. Then, A is orientation-preserving if and only if $\det(M) > 0$.