# 1 Basic Concepts & Taylor's Theorem (Section 1.1)

Let $f(x) : \mathbb{R} \mapsto \mathbb{R}$ be a general function (typically nonlinear). We may also write $f([a,b]) : [a,b] \mapsto \mathbb{R}$ to denote a general function over an interval $[a,b]$. We also write $C^n(\mathbb{R})$ or $C^n([a,b])$ to denote the *classes* of $n$-times continuously differentiable functions. We write $C^0(\mathbb{R}) = C(\mathbb{R})$ to mean the class of only continuous functions.

(Example.) $f(x) = |x|$ is continuous but is not differentiable at $x = 0$. Thus, $f(x) = |x|$ is in $C^0(\mathbb{R})$.

$f(x) = e^x$ is in $C^\infty(\mathbb{R})$.

## 1.1 Taylor Series

**Theorem 1.1: Taylor Series with Lagrange Remainder**

Let $f \in C^m([a,b])$, with the derivative $f^{(m+1)}$ exists on the open interval $(a,b)$ and with values $x, c \in [a,b]$. Then,

$$f(x) = \sum_{k=0}^{m} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_m(\psi),$$

where $E_m(\psi)$ is the remainder (or error) term. We define

$$E_m(\psi) = \frac{f^{(m+1)}(\psi)}{(m+1)!}(x-c)^{(m+1)},$$

where $c < \psi < x$ or $x < \psi < c$ depending on the values of $x$ and $c$.

(Example.) Suppose $f(x) = \ln(x)$ with interval $[a,b] = [1,10]$ and $c = e^1$. Let $|x-c| < 1$ (i.e., $x$ is relatively close to $c$). Then,

$$f^{(1)}(x) = f'(x) = \frac{1}{x}.$$

$$f^{(2)}(x) = f''(x) = -\frac{1}{x^2}.$$

$$f^{(3)}(x) = f'''(x) = \frac{2}{x^3}.$$

$$f^{(4)}(x) = -2 \cdot 3 \frac{1}{x^4}.$$

$$f^{(5)}(x) = 2 \cdot 3 \cdot 4 \frac{1}{x^5}.$$

$$\vdots$$

$$f^{(k)}(x) = (-1)^{k-1}(k-1)! \frac{1}{x^k}$$

for $k = 1, 2, \ldots$. Then,

$$E_m(\psi) = \frac{1}{(m+1)!} f^{(m+1)}(\psi)(x-c)^{m+1}.$$

Using the value of $c = e^1$,

$$f^{(k)}(c) = (-1)^{k-1}(k-1)! \frac{1}{e^k}.$$

Combining everything, we end up with

$$\begin{aligned} f(x) &= \sum_{k=0}^{m} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_m(\psi) \\ &= 1 + \sum_{k=1}^{m} (-1)^{k-1} \frac{(k-1)!}{k!} \frac{1}{e^k} + E_m(\psi) \\ &= 1 + \sum_{k=1}^{m} (-1)^{k-1} \frac{1}{k} \frac{1}{e^x} (x-e)^k + \frac{1}{(m+1)} f^{(m+1)}(\psi)(x-e)^{m+1}. \end{aligned}$$

How many terms in this approximation do we need in order for the error to be below a certain amount? In other words, what is the minimum $m$ so that a Taylor expansion is accurate up to $\frac{1}{\alpha} \cdot 10^{-9}$? We have

$$|E_m(\psi)| \leq \frac{1}{\alpha} \cdot 10^{-9}.$$

We already computed the remainder, so

$$\left| \frac{1}{(m+1)} f^{(m+1)}(\psi)(x-e)^{m+1} \right| \leq \frac{1}{\alpha} \cdot 10^{-9}.$$

Using $|x-e| < 1$, we want to find $m$. Thus, $|\psi| < 1$.