1 Midterm Review

Today is simply midterm review.

1.1 Page 257, Problem 29

List the distinct elements of $\mathbb{Z}[x]/\langle 3, x^2+1\rangle$.

Proof. First, note that

$$\langle 3, x^2+1\rangle = \{3f_1(x) + (x^2+1)f_2(x) \mid f_i(x) \in \mathbb{Z}[x]\}$$

Now, the elements of this ring is in the form

$$f(x) + \langle 3, x^2+1\rangle$$

An observation to see is that $f(x)$ must have degree less than 2. This is because

$$x^2+1 \in \langle x^2+1\rangle \implies x^2 + \langle x^2+1\rangle = -1 + \langle x^2+1\rangle$$

In other words, the coset corresponding to x^2 is the same as the coset corresponding to -1. For example, if we had x^3, this would correspond to $x^2x = (-1)x$ and x^4 would correspond to $x^2x^2 = (-1)(-1) = 1$ and so on. So, we have

$$\mathbb{Z}[x]/\langle 3, x^2+1\rangle = \left\{a + bx + \langle 3, x^2+1\rangle\right\}$$

Of course, we can't forget the 3 is a member of our ideal. So,

$$a = 3q_1 + r_1$$

$$b = 3q_2 + r_2$$

where $r_1, r_2 \in \{0, 1, 2\}$ so that

$$a + bx = 3q_1 + r_1 + (3q_2 + r_2)x = 3(q_1 + q_2x) + r_1 + r_2x$$

and so it follows that

$$a + bx + \langle 3, x^2+1\rangle = r_1 + r_2x + \langle 3, x^2+1\rangle$$

Therefore, our answer is

$$\mathbb{Z}[x]/\langle 3, x^2+1\rangle = \{r_1 + r_2 + \langle 3, x^2+1\rangle \mid r_1, r_2 \in \mathbb{Z}/3\mathbb{Z}\}$$

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