1 Chapter 5: Combinatorics

1.1 Combinations

Suppose we have a set A of size $|A| = n$. How many subsets of A are there? How many of these are of size k?

Definition 1.1

The number of ways to choose k elements from a set of n distinguishable objects is denoted by $\binom{n}{k}$.

Remarks:

- If $|A| = n$, then there are $\binom{n}{k}$ subsets of $S \subset A$ of size $|S| = k$, since each subset corresponds to a way of choosing k elements from the set of n elements.
- The number $\binom{n}{k}$ is also known as the **binomial coefficient**.

Theorem 1.1

For $0 \leq k \leq n$, we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Remarks:

- Recall that $0! = 1$.
- Recall that $\frac{n!}{(n-k)!} = (n)_k$.
- So, we can say that $\binom{n}{k} = \frac{(n)_k}{k!}$.

One thing to note is that the relationship

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

gives us a recursive method of computing binomial coefficients. This is known as the famous **Pascal's Triangle**. One thing to notice is that

$$\frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!}$$

and so it follows that

$$\binom{n}{k} = \binom{n}{n-k}.$$

(Example.) Why does a Four of a Kind beat a Full House in Poker?

- Recall that there are 52 cards in a deck. In Poker, you get 5 cards.
- A Four of a Kind means that you get 4 of one of the kind of cards and one other card (e.g. all 4 aces and a 2).
- A Full House is when you get 3 of one kind and 2 of the other (e.g. 3 aces and 2 3's.).

How many ways can we get a Full House?

- We need to get two types of cards. There are $13 \cdot 12$ ways to do this.
- Now, we need to make the full house. For one of the cards, there are 4 different suits, of which we want 3 suits. Thus, $\binom{4}{3}$.
- For the other card, there are again 4 different suits, of which we want 2 suits. Thus, $\binom{4}{2}$.

This gives us

$$13 \cdot 12 \cdot \binom{4}{3}\binom{4}{2}.$$

How many ways can we get a Four of a Kind?

- There are 13 ways to choose the first card (that we need 4 of one kind for).
- There are now 48 cards left. We just need to pick one card to be our extra card.

This gives us

$$13 \cdot 48.$$

The probability of a Full House is given by

$$\frac{13 \cdot 12 \cdot \binom{4}{3}\binom{4}{2}}{\binom{52}{5}} \approx 0.14\%.$$

The probability of a Four of a Kind is given by

$$\frac{13 \cdot 48}{\binom{52}{5}} \approx 0.024\%.$$

For both of these cases, the $\binom{52}{5}$ came from the fact that we get 5 cards from a deck of 52.

1.2 Binomial Distribution

There is an important probability distribution related to the binomial coefficients, called the **Binomial Distribution**. But, we first need to describe the concept of a **Bernoulli trial**.

Definition 1.2

A **Bernoulli trial** is a simple experiment that is either a *success* or *failure*. More specifically, it is a discrete random variable that either takes the value 1 (success) or 0 (failure).

Moreover, a Bernoulli(p) trial is one in which the probability of success is p. Hence, its PDF is

$$\mathbb{P}(X = 1) = p$$

and

$$\mathbb{P}(X = 0) = 1 - p.$$

For example, flipping "Tails" when tossing a fair coin is a Bernoulli(1/2) trial.

We haven't defined what independence means in probability, but informally, a series of events $E_1, \ldots, E_n$ are independent if their outcomes "do not affect" each other. For example, when we flip a coin, whatever we get on the first flip won't affect what we get on the second flip. So, in this case, we have

$$\mathbb{P}\left(\bigcap_{i=1}^{n} E_i\right) = \prod_{i=1}^{n} \mathbb{P}(E_i).$$

In other words, the probability that they all occur is just the product of the individual probabilities.

Definition 1.3

Let $n \geq 1$ be an integer and $P \in [0, 1]$. Let N be the number of "successes" in a series of n independent Bernoulli(p) trials. Then, we say that N has the Binomial(n, p) distribution.

Its PMF[1] is given by

$$\mathbb{P}(N = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for $0 \leq k \leq n$ and $\mathbb{P}(N = x) = 0$ otherwise. To see why this is the case,

- There are $\binom{n}{k}$ ways to choose which k of the n trials will be successful. Each of these k trials will be a success with probability p, and each of the remaining $n - k$ trials will be a failure with probability $1 - p$.
- Since these trials are independent, we can multiply everything together: each of the $\binom{n}{k}$ outcomes that would cause $N - k$ to have probability $p^k(1-p)^{n-k}$.

How do we see that this is a legitimate probability distribution? In other words, how do we know that this all adds up to 1?

Theorem 1.2: Binomial Theorem

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

With this in mind, we have

$$\sum_{k=0}^{n} \mathbb{P}(N = k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1^n = 1.$$

[1]Probability Mass Function