1 Connectivity in Digraphs

How do we achieve a clean description of reachability in a directed graph?

- Note that reachability is no longer symmetric. That is, we can reach w from v but not the other way around.
- Can we maybe allow reachability in either direction?
- Maybe we can allow the ability to follow edges in either direction? However, this treats a digraph as an undirected graph.

1.1 Strongly Connected Components

Definition 1.1: Strongly Connected Components

In a directed graph G, two vertices v and w are in the same **strongly connected component** if v is reachable from w and w is reachable from v.

1.2 Equivalence Relation

Let $v \sim w$ if v is reachable from w and vice versa.

Proposition. *This is an equivalence relation. Namely:*

- $v \sim v$ *(v is reachable from itself).*
- $v \sim w \implies w \sim v$ *(relation is symmetric).*
- $u \sim v$ *and* $v \sim w \implies u \sim w$*.*

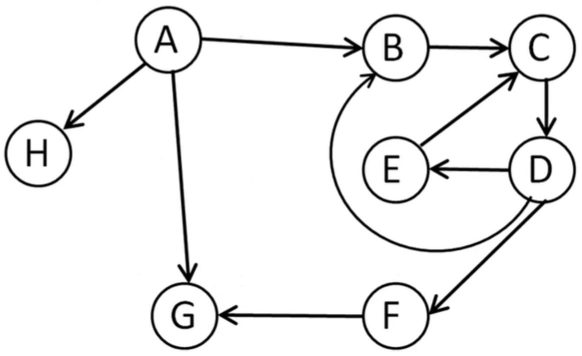
1.3 Relationship to Components

Essentially, when we have this equivalence relation, we can split a set into components (equivalence classes) so that $v \sim w$ if and only if v and w are in the same component.

Take any v, the set of all w so that $v \sim w$ is the component of v. Then, everything connects to everything else in this equivalence class and does not connect (both ways) to anything outside.

1.3.1 Example: SCCs

Consider the following graph:



How many strongly connected components does this graph have?

The answer is **5**. If we use the idea of equivalence classes above, then the equivalence classes are:

- B, C, D, E.
- A
- H
- G
- F

Here, we note that B, C, D, and E are all reachable from each other. However, these vertices and, for example, A are not reachable both ways.

1.4 Connectivity

Do strongly connected components completely describe connectivity in G?

- **No**. In directed cases, we can have an edge between strongly connected components. In the undirected case, connected components told you everything that there was to know. However, in the directed case, we can have these two strongly connected components consisting of just a single vertex v and w. However, just knowing these components doesn't tell you if there is an edge from v to w or vice versa. There could be an edge that could be extra information that we need to know.

```
[v] ---------> [w]
```

- We need extra information to describe how SCCs connect.

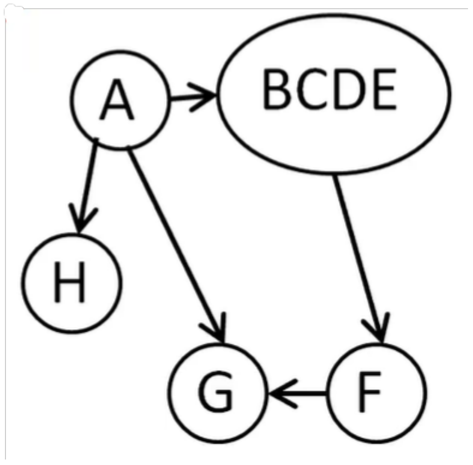
1.5 Metagraph

Definition 1.2: Metagraph

The **metagraph** of a directed graph G is a graph whose vertices are the strongly connected components of G, where there is an edge between C_1 and C_2 if and only if G has an edge between some vertex of C_1 and some vertex of C_2.

Remark: If you're given the strongly connected components and the metagraph of a graph G, then you can you can figure out connectivity within the full graph.

So, the corresponding metagraph of the above graph in the example would be:



1.5.1 Result

Theorem 1.1

The metagraph of any directed graph is a DAG.

Proof. Assume for the sake of contradiction that it is not a DAG. Then, the metagraph has a cycle. Now, let's suppose we have a bunch of strongly connected components; call these SCCs $C_1, C_2, \ldots, C_n$. Then, since the metagraph has a cycle, then the strongly connected components form a cycle. That is, we can go from C_1 to C_2 to C_n back to C_1. However, this implies that we have one giant strongly connected component since we can get from one C_i to another C_j $(i \neq j)$ and vice versa. $\square$

1.6 Computing SCCs

Given a directed graph G, compute the SCCs of G and its metagraph.

1.6.1 Easy Algorithm

- For each v, compute vertices reachable from v.
- Find pairs v, w so that v reachable from w and vice versa.
- For each v, the corresponding w's are in the SCC of v.

The runtime is $O(|V|(|V| + |E|))$.

1.6.2 Observation and Better Algorithm

Suppose that $SCC(v)$ is a sink in the metagraph.

- G has no edges from $SCC(v)$ to another SCC.
- We can run `explore(v)` to find all vertices reachable from v. This will contain all vertices in $SCC(v)$. But, it contains no other vertices.
- If v is in the SCC, then `explore(v)` finds exactly v's component.

With this observation, we consider the following strategy:

- Find v in a sink SCC of G.
- Run `explore(v)` to find the component C_1.
- Repeat process on $G \setminus C_1$.

The problem is, how do we find v that is in a sink?

Proposition. *Let C_1 and C_2 be SCCs of G with an edge from C_1 to C_2. If we run `DFS` on G, the largest postorder number of any vertex in C_1 will be larger than the largest postorder number in C_2.*

The reason why we care is because if v is the vertex with the largest postorder number, then:

- There is no edge to $SCC(V)$ from any other SCC.
- SCC is a <u>source</u> SCC.

However, we wanted a *sink* SCC. So, how do we relate these two?

- A sink is like a source, only with edges going in the opposite direction.

So, we define a reverse graph like so:

Definition 1.3: Reverse Graph

Given a directed graph G, the **reverse graph** of G (denoted G^R) is obtained by reversing the directions of all the edges of G.

Some properties of reverse graphs are:

- G and G^R have the same number of vertices and edges.
- $G = (G^R)^R$.
- G and G^R have the same SCCs.
- The sink SCCs of G are the source SCCs of G^R.
- The source SCCs of G are the sink SCCs of G^R.

The better algorithm we have is:

```
SCCs(G)
    Run DFS(G^R), record postorder
    Find v with largest v.post
    Set all vertices unvisited
    Run explore(v)
    Let C be the visited vertices
    Return SCCs(G - C) union {C}
```