1 Ring Homomorphism

Ring homomorphism is very similar in nature to group homomorphisms. Here, a ring homomorphism preserves the ring operations.

Definition 1.1: Ring Homomorphism

A **ring homomorphism** φ from a ring R to a ring S is a mapping from R to S that preserves the ring operation. That is, for all $a, b \in R$:

$$\varphi(a+b) = \varphi(a) + \varphi(b) \qquad \varphi(ab) = \varphi(a)\varphi(b)$$

Remark: As is the case for groups, the operations on the left of the equal signs are those of R, while the operations on the right side are those of S.

Along with ring homomorphisms, there is also ring isomorphisms.

Definition 1.2: Ring Isomorphism

A **ring isomorphism** is a ring homomorphism that is both one-to-one and onto (i.e. bijective).

1.1 Properties of Ring Homomorphisms

Theorem 1.1

Let φ be a ring homomorphism from a ring R to a ring S, and let A be a subring of R and let B be an ideal of S.

1. For any $r \in R$ and any positive integer n, $\varphi(nr) = n\varphi(r)$ and $\varphi(r^n) = (\varphi(r))^n$.
2. $\varphi(A) = \{\varphi(a) \mid a \in A\}$ is a subring of S.
3. If A is an ideal and φ is onto S, then $\varphi(A)$ is an ideal.
4. $\varphi^{-1}(B) = \{r \in R \mid \varphi(r) \in B\}$ is an ideal of R.
5. If R is commutative, then $\varphi(R)$ is commutative.
6. If R has a unity 1, $S \neq \{0\}$, and φ is onto, then $\varphi(1)$ is the unity of S.
7. φ is an isomorphism if and only if φ is onto and $\ker(\varphi) = \{r \in R \mid \varphi(r) = 0\} = \{0\}$.
8. If φ is an isomorphism from R onto S, then φ^{-1} is an isomorphism from S onto R.

1.2 Examples of Ring Homomorphism

Here are some examples of ring homomorphisms.

1.2.1 Example 1: Integers and Modulo

Consider the mapping:

$$k \mapsto k \pmod{n}$$

This is a ring homomorphism from $\mathbb{Z}$ onto $\mathbb{Z}_n$, and is called the natural homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_n$.

1.2.2 Example 2: Complex Numbers

Consider the mapping:

$$a + bi \mapsto a - bi$$

This is a ring homomorphism from the complex numbers onto the complex numbers.

1.2.3 Example 3: Functions

Consider the ring of all polynomials with real coefficients $\mathbb{R}[x]$. Consider the mapping:

$$f(x) \mapsto f(1)$$

This is a ring homomorphism from $\mathbb{R}[x]$ onto $\mathbb{R}$.