

Iterative methods for solution of non-linear algebraic equations

Files for the present tutorial can be found using following links:

- [Header file `nonlin.h`](#)
- [Source file `nonlin.cpp`](#)

1. Introduction

During this tutorial we will learn how to use different methods for solution of the non-linear algebraic equations. As in practice it is impossible to find exact solution to this problem, we search for approximate solution only. To meet this goal, three different iterative numerical techniques will be introduced, namely: bi-section, secant and tangent methods. All of them require that the function, which root is the solution of the non-linear equation, is continuous.

2. Bi-section method

In the case of bi-section method we use the fact that if the root of the given, continuous function $f(x)$ is located inside of the interval, say $[a, b]$, the values of the continuous function $f(x)$ at the points a, b must then have opposite signs, hence $f(a) \cdot f(b) < 0$. This observation states so called Darboux theorem introduced by Jean Gaston Darboux (1842-1917). Using it we can solve non-linear algebraic equations, for example:

$$\cos(x) = x$$

In order to find its solution we will use function `bisec` defined inside `nonlin.h` and `nonlin.cpp` files. The header of this function is given below:

```
double bisec(double a, double b, double (*pf)(double x), double eps, int
```

The function `bisec` returns approximation of the root of the algebraic equation located in the interval $[a, b]$. To use function `bisec` the following parameters are required:

- **a** and **b** - the ends of the interval where the solution is searched

- **(*pf)** - pointer to the function containing algebraic equation to be solved
- **eps** - accuracy of the solution ε
- **iter** - pointer to the variable storing information about number of iterations required to reach solution with specified accuracy ε . If on both ends of the interval $[a, b]$ the evaluated values of the function $f(a), f(b)$ have the same signs, meaning $f(a) \cdot f(b) > 0$, then the pointer **iter** points to the value -1 and the function returns value 0.

Exercises

1. Write a function `double eqn(double x)` returning the value of $f(x)$ if the non-linear algebraic equation written in the form $f(x) = 0$
2. Read from the keyboard (initially define in the code) accuracy of the solution **eps** the values of the interval ends **a** and **b**. Remember $f(a) \cdot f(b) < 0$.
3. Write on the screen the root value and the number of iterations.
4. Rewrite the program in a way it solves your non-linear algebraic equation in a loop where accuracy is changing $\varepsilon = 2^{-20}, 2^{-19}, \dots, 2^{-3}$.
5. Make a diagram showing how number of iterations **iter** is changing with respect to selected accuracy **eps** (use logarithmic scale to show how accuracy is changing).

3. Methods of secants and/or tangents

As mentioned above the other methods that can be used to iteratively find a solution of non-linear algebraic equations are the method of tangents and the method of secants. In the case of the method of secants the following approximations of the solution are given by the intersection of the secant (defined on the two points) with abscissa (x- axis). In the next iteration, value of the function at point x_i is used in construction of the next secant. The iterative scheme in the method of secants has the following form:

$$x_{i+1} = x_i - f(x_i) \frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}$$

In the case of the method of tangents (also called Newton's method) the next approximation of the solution is obtained as the intersection of the tangent (defined at one point) with abscissa (x-axis). The iterative scheme for Newton's method, reads:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Exercises

1. Find solution to the equation defined above using both tangents and secants methods. Write on the screen exact number of iterations required to obtain solution with prescribed accuracy in: bi-section, secants and tangents methods.
2. Create the diagram showing dependence of the accuracy of the solution on the number of iterations (for each method, use logarithmic scale to depict accuracy)
3. Using three methods solve equation with parameter $\cos(x) = w \cdot x$ using $\varepsilon = 10^{-6}$. The value of the parameter w should change in a range $w = 0.5, 0.6, \dots, 15.0$.

4. Additional problem

The methods discussed herein, can also be used to find extrema of the functions. Try to find extremum of the function:

$$f(x) = (1 + x) \cdot \arctan(x)$$