Some Comments on Physical Mathematics

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written version available at
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Part I:

Snapshots from the

Great Debate

over

Kepler

Galileo

the relation between

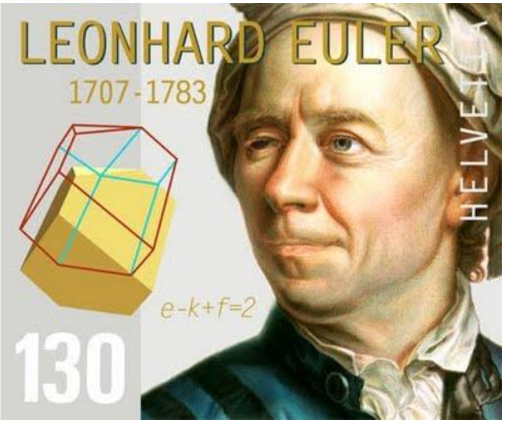
Mathematics and Physics

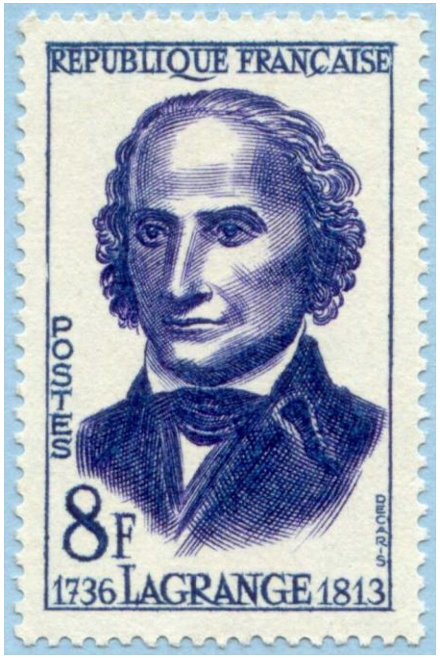
Newton

Leibniz

When did Natural Philosophers become either physicists or mathematicians?

Even around the turn of the 19th century …





But, we can read in volume 2 of the journal Nature ….

1869: Sylvester's Challenge

A pure mathematician speaks:

of physical philosophy; the one here in print," says Professor Sylvester, "is an attempted faint adumbration of the nature of mathematical science in the abstract. What is wanting (like a fourth sphere resting on three others in contact) to build up the ideal pyramid is a discourse on the relation of the two branches (mathematics and physics) to, and their action and reaction upon, one another—a magnificent theme, with which it is to be hoped that some future president of Section A will crown the edifice, and make the tetralogy (symbolisable by $A + A'$, A, A', AA') complete."

1870: Maxwell's Answer

An undoubted physicist responds,

SECTIONAL PROCEEDINGS

SECTION A.—*Mathematical and Physical Science.*—President, Prof. J. Clerk Maxwell, F.R.S.

The president delivered the following address :—

Maxwell recommends his somewhat-neglected dynamical theory of the electromagnetic field to the mathematical community:

phenomena must be studied in order to be appreciated. Another theory of electricity which I prefer denies action at a distance and attributes electric action to tensions and pressures in an all-pervading medium, these stresses being the same in kind with those familiar to engineers, and the medium being identical with that in which light is supposed to be propagated."

1900: Hilbert's 6th Problem

Mathematische Probleme.

Vortrag, gehalten auf dem internationalen Mathematiker-Kongreß zu Paris 1900.

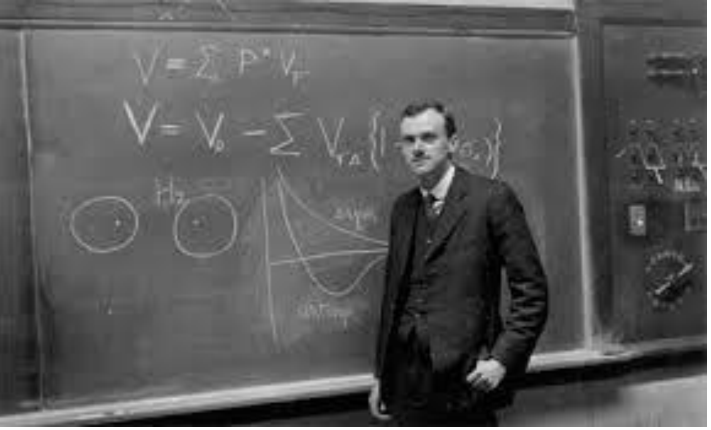
Von

D. Hilbert.

Wer von uns würde nicht gern den Schleier lüften, unter dem die Zukunft verborgen liegt, um einen Blick zu werfen auf die bevorstehenden Fortschritte unsrer Wissenschaft und in die Geheimnisse ihrer Entwickelung während der künftigen Jahrhunderte! Welche besonderen Ziele werden es sein, denen die füh-

6. Mathematische Behandlung der Axiome der Physik.

Durch die Untersuchungen über die Grundlagen der Geometrie wird uns die Aufgabe nahe gelegt, *nach diesem Vorbilde diejenigen physikalischen Disciplinen axiomatisch zu behandeln, in denen schon heute die Mathematik eine hervorragende Rolle spielt; dies sind in erster Linie die Wahrscheinlichkeitsrechnung und die Mechanik.*



1931: Dirac's Paper on Monopoles

Quantised Singularities in the Electromagnetic Field

P.A.M. Dirac

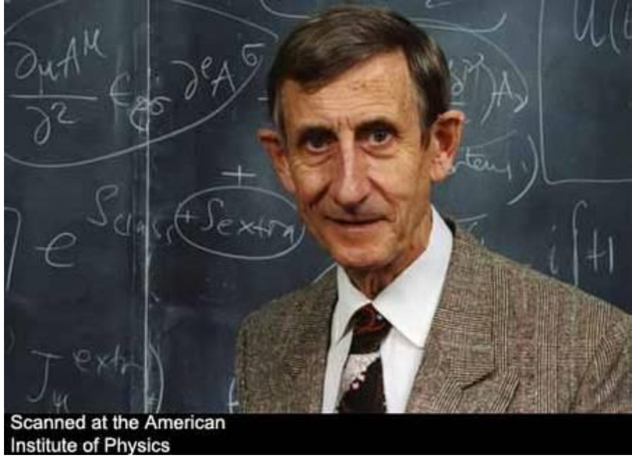
Received May 29, 1931

§ 1. *Introduction*

The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected. What, however, was not expected by the scientific workers

for the description of general facts of the physical world. It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of the mathematics rather than with a

1972: Dyson's Announcement



MISSED OPPORTUNITIES[1]

BY FREEMAN J. DYSON

> It is important for him who wants to discover not to confine himself to one chapter of science, but to keep in touch with various others.
>
> JACQUES HADAMARD

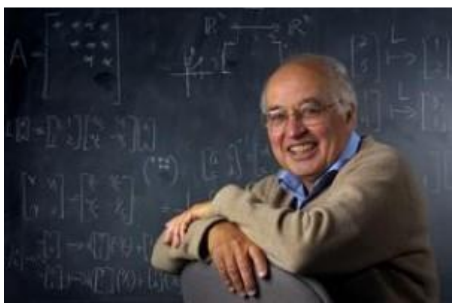
1. **Introduction.** The purpose of the Gibbs lectures is officially defined as "to enable the public and the academic community to become aware of the contribution that mathematics is making to present-day thinking and to modern civilization." This puts me in a difficult position. I happen to be a physicist who started life as a mathematician. As a working physicist, I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce. Discussing this divorce, the

Well, I am happy to report that Mathematics and Physics have remarried!

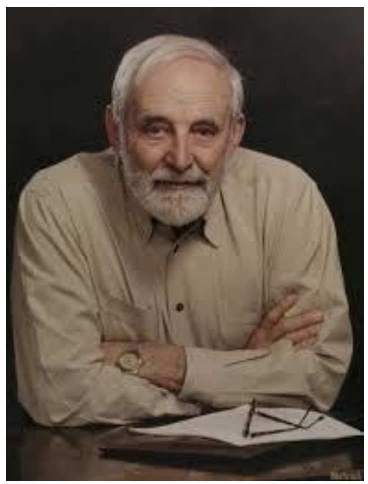
But, the relationship has altered somewhat…

A sea change began in the 1970's …..

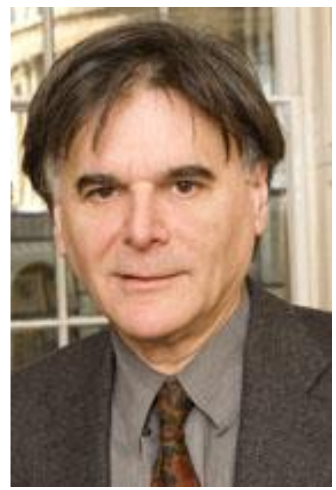
A number of great mathematicians got interested in the physics of gauge theory and string theory, among them,



M. Atiyah



R. Bott



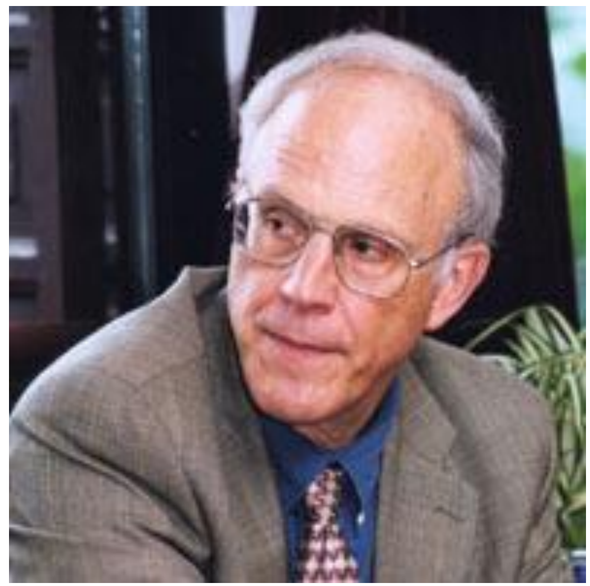
G. Segal

I. Singer

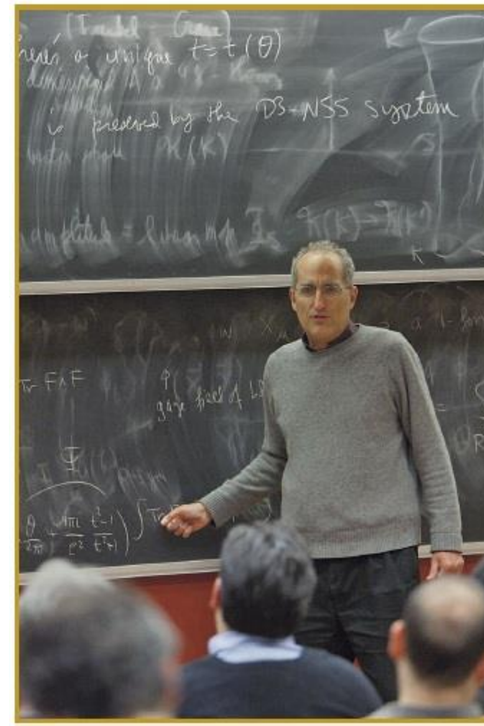
$$+\cdots$$

And at the same time a number of great physicists started producing results requiring increasing mathematical sophistication, among them

S. Coleman



D. Gross



E. Witten

$+ \cdots$

Physical Mathematics

With a great boost from string theory, after 40 years of intellectual ferment a new field has emerged with its own distinctive character, its own aims and values, its own standards of proof.

One of the guiding principles is certainly Hilbert's 6^{th} Problem (generously interpreted): Discover the ultimate foundations of physics.

As predicted by Dirac, this quest has led to ever more sophisticated mathematics…

But getting there is more than half the fun: If a physical insight leads to a new result in mathematics – that is considered a great success.

It is a success just as profound and notable as an experimental confirmation of a theoretical prediction.

The discovery of a new and powerful invariant of four-dimensional manifolds is a vindication just as satisfying as the prediction of a new particle.

Part II: Theories with Extended Supersymmetry

Part III: A Fruitful Collaboration

Part IV: Future Directions

Why Extended Supersymmetry ?

$$(Q_\alpha^1, Q_\alpha^2, \ldots, Q_\alpha^N)$$

1970's: Kinematics – construction of field multiplets and Lagrangians: Brilliant application of symmetry principles.

Around 1994 another great change began to take place:

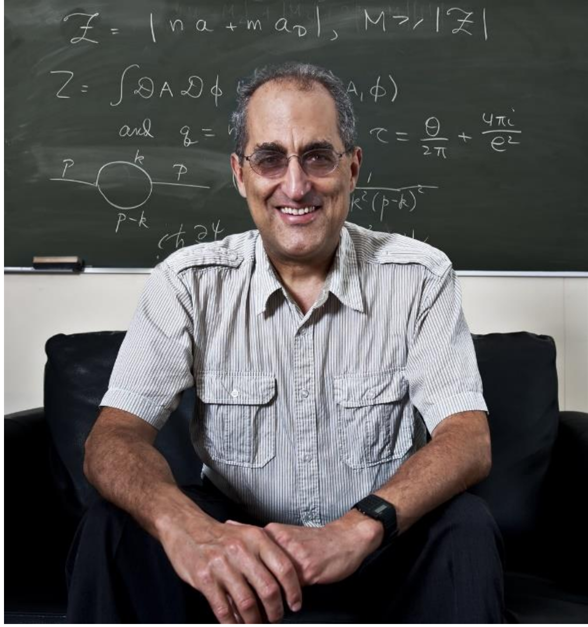
Nontrivial dynamics

Yet susceptible to exact results

Many physicists began to gather nontrivial evidence that these theories have strong-weak dualities

Work of Seiberg & Witten in 1994 made it clear that these theories are in the ``Goldilocks'' class

Seiberg-Witten Breakthrough



SU(2) gauge theory

$$A_\mu = \begin{pmatrix} Z_\mu & W_\mu^+ \\ W_\mu^- & -Z_\mu \end{pmatrix}$$

N=2 Susy → Higgs $\Phi = \begin{pmatrix} \varphi & \varphi^+ \\ \varphi^- & -\varphi \end{pmatrix}$

$$u = \langle \varphi \rangle \qquad SU(2) \rightarrow U(1)$$

At low energies physics is described by an N=2 version of Maxwell’s theory.

A Manifold of Quantum Vacua

u is undetermined even at the quantum level
so there is a *manifold of quantum vacua*: $\mathcal{M}$

For SU(2) N=2 YM, *u* is just a complex number: $\mathcal{M} = \mathbb{C}$

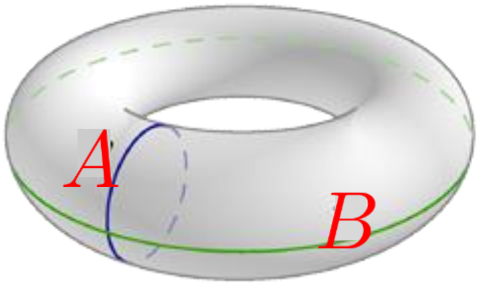
Low energy physics depends on *u*: The strength of Coulomb's law depends on *u*: The fine structure constant α depends on *u*.

Seiberg & Witten showed how to compute $\alpha(u)$ exactly in terms of two functions a(*u*) and $a_D(u)$

$$\alpha(u)^{-1} = \mathrm{Im}\left(\frac{da_D}{da}\right)$$

Elliptic Curves

$$y^2 = (z + u + z^{-1})$$



$$a(u) = \oint_A y \frac{dz}{z}$$

$$a_D(u) = \oint_B y \frac{dz}{z}$$

The theory of elliptic curves is a deep and wonderful subject in mathematics, and continues to be the focus of important research today.

For example, the proof of Fermat's Last Theorem in 1995 relied heavily on new results on elliptic curves.

Unreasonable Effectiveness

It is another example of Wigner's ``Unreasonable Effectiveness of Mathematics in the Natural Sciences''

But then, in the hands of Witten, the SW insight led to startling new results in mathematics – new four-manifold invariants.

Later, Moore & Witten explained in greater detail how these new invariants are related to the invariants of Donaldson.

And Marino, Moore, and Peradze used the existence of N=2 superconformal fixed points to predict new results in the topology of four-manifolds.

This is one good example (out of very many) of the Unreasonable Effectiveness of Physics in Mathematics .

The Threefold Promise of Seiberg-Witten Theory

Exact statements about the massless particles
and how they interact at low energies.

Exact statements about the spectrum of the Hamiltonian
for a subspace of the Hilbert space of the theory –
the so-called ``BPS subspace.''

Exact statements about path integrals with
extended observables inserted.

(``Line operators'' ,  ``surface operators'' , domain walls.)

What Are BPS States?

They are quantum states

(annihilated by some supersymmetry operators)

whose quantum properties are rigidly constrained by the supersymmetry algebra.

So we can make exact statements.

Mass Formula & Central Charge

Given a BPS state ψ there is an associated ``central charge'' $Z_\psi \in \mathbb{C}$

$$\mathrm{Mass}(\psi) = |Z_\psi|$$

$Z_\psi(u)$ turns out to be an interesting, computable function of $u \in \mathcal{M}$

Particle of (electric, magnetic) charge γ=(q,p) has

$$Z = Z_\gamma(u) = q a(u) + p a_D(u)$$

Finding the BPS Spectrum ?

$$M = |Z_{q,p}(u)| = |qa(u) + pa_D(u)|$$

Gives the mass a BPS particle of charge (q,p) *would have*, ***if*** it exists.

It does not tell us whether BPS particles of charge (q,p) do or do not exist!

The BPS spectrum of N=2 theories was only known for some isolated examples in the 1990's.

Around 2007 this began to change. Since then we have developed a much more systematic understanding of the BPS spectrum for many N=2 theories.

The key was a phenomenon known as ``wall-crossing''

Walls of Marginal Stability

BPS states can form boundstates, which are ***<u>themselves</u>*** BPS.

If ψ_1 and ψ_2 are BPS states the 2-particle boundstate has

$$Z_{\psi_1 * \psi_2} = Z_{\psi_1} + Z_{\psi_2}$$

BE: $$|Z_{\psi_1} + Z_{\psi_2}| - (|Z_{\psi_1}| + |Z_{\psi_2}|)$$

$$MS(\psi_1, \psi_2) = \{u : Z_{\psi_1}(u) \parallel Z_{\psi_2}(u)\}$$

One real equation ⟹ ***<u>WALL</u>*** in $\mathcal{M}$

u

Boundstate Radius Formula

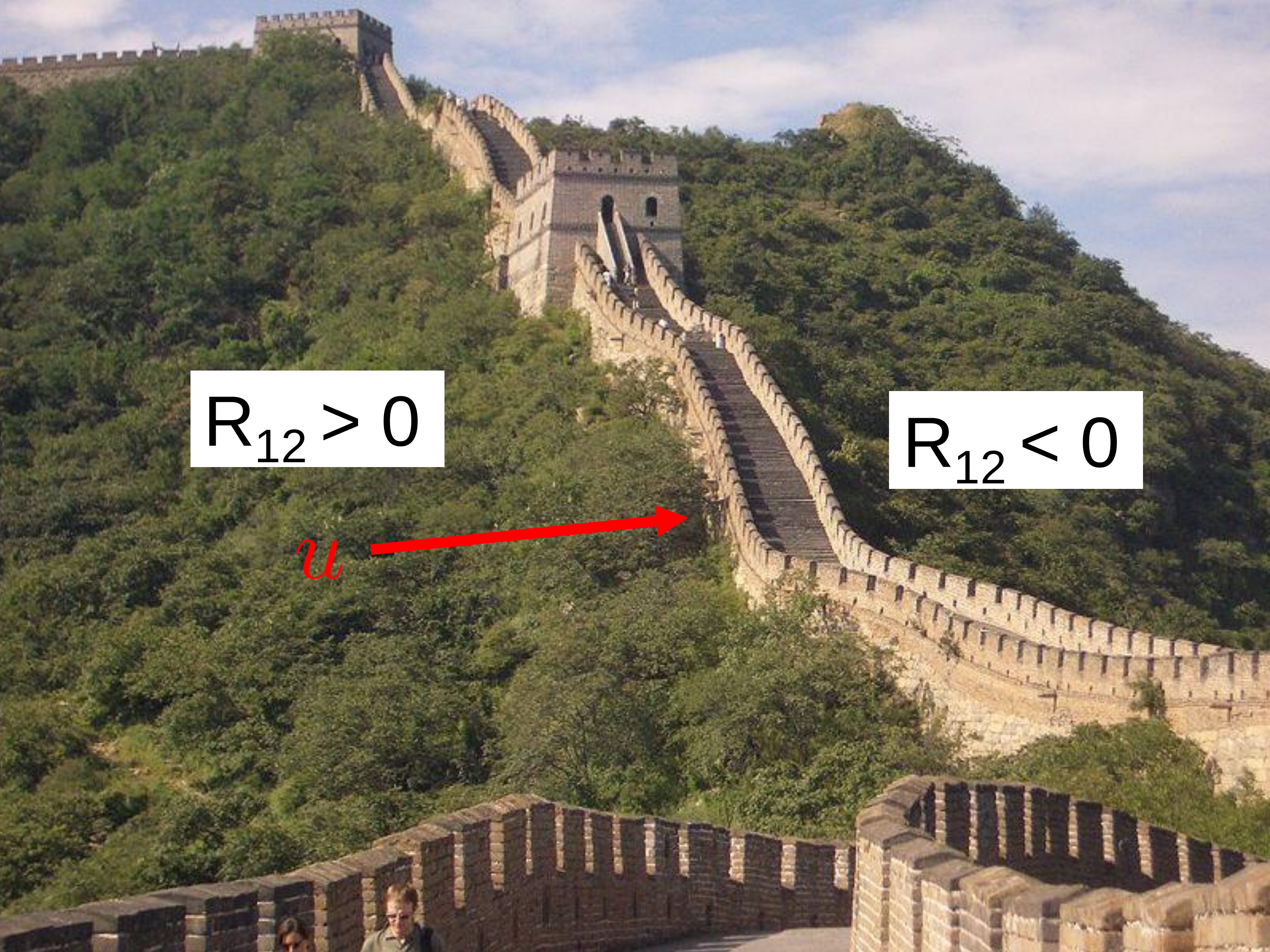
In a semiclassical picture the constituent BPS states are solitonic dyons of charges γ_1 and γ_2.

The electromagnetic forces cancel the scalar forces exactly at a magic boundstate radius.

The formula was found by Frederik Denef.

$$R_{12}(u) = \langle \gamma_1, \gamma_2 \rangle \frac{|Z_{\gamma_1}(u) + Z_{\gamma_2}(u)|}{2\mathrm{Im}\left(Z_{\gamma_1}(u) Z_{\gamma_2}(u)^*\right)}$$

$$\langle \gamma_1, \gamma_2 \rangle = p_1 q_2 - p_2 q_1$$

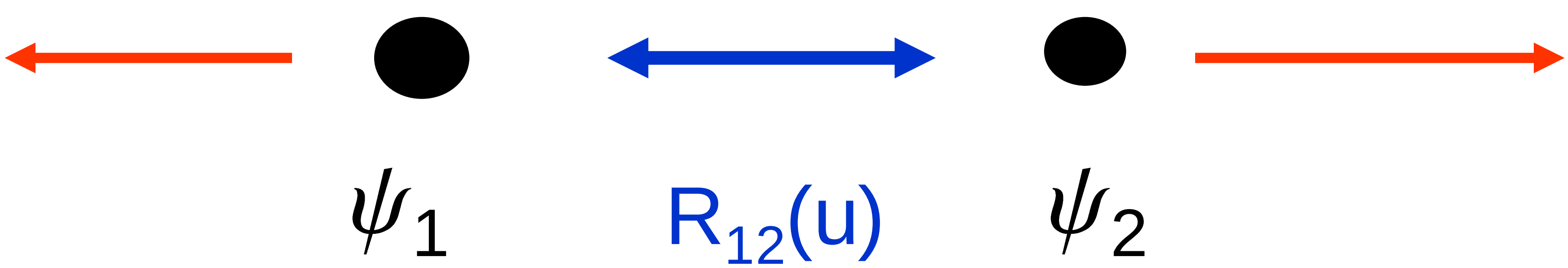


$R_{12} > 0$
$R_{12} < 0$
u

A Picture of Wall-Crossing

Lee & Yi; Kol; Mikhailov, Nekrasov & Sethi; Denef & Moore:

For u $\to$ MS(ψ_1,ψ_2) R$_{12}$(u) $\to \infty$



BPS Index

In general, understanding how the vector space of BPS states changes is too hard to start with.

An important simplification of the problem is to consider instead an ``index'' - a signed sum over the BPS states:

vector space integer

$$\mathcal{H}^{\mathrm{BPS}}_{(q,p)} \Longrightarrow \Omega_{(q,p)}$$

The BPS index Ω is typically *<u>independent</u>* of parameters and thus much more computable. However – because of wall-crossing it is only piecewise-continuous.

Denef-Moore derived formulae for $\Delta\Omega$ across wall for some boundstates, then, soon thereafter….

Kontsevich-Soibelman WCF

For (seemingly) different reasons K&S came up with a formula for the change of the index for all boundstates.

It used a new set of mathematical concepts:

$K_{(q,p)}$:   group elements in a certain nonabelian group

The phases of the central charges Z= a γ_1 + b γ_2 , with a,b nonnegative, change from cw to ccw when u passes through MS(γ_1 ,γ_2)

$$\prod_{\circlearrowleft} K_{(q,p)}^{\Omega^-(q,p)} = \prod_{\circlearrowright} K_{(q,p)}^{\Omega^+(q,p)}$$

Who ordered that !?

Part III

A Fruitful Collaboration

Davide Gaiotto (Perimeter)
Andy Neitzke (UT Austin)
and I set out to understand
where this comes from

(Two) physical derivations of the KSWCF

+ a nontrivial generalization of the KSWCF

2nd derivation: Above mechanism captures the essential physics of the KSWCF

Physics Results from GMN project

``New'' class of d=4 N=2 field theories (``class S'')

New relations of d=4, N=2 with integrable systems and CFT

Exact results for Wilson-'t Hooft (and other) line operators and surface operators

Math results from GMN project

1. New constructions of hyperkahler metrics

   and hyperholomorphic vector bundles

2. New perspectives & results on cluster algebras, cluster varieties and the higher Teichmuller theory of Fock and Goncharov

3. New construction of a combinatoric object associated to branched covers of Riemann surfaces, called a ``spectral network.''

Sample Result

Pure SU(2) N=2 gauge theory at finite temperature β:

Susy Wilson line wrapped on a Euclidean time circle of radius β

$$L_\zeta = \mathrm{Pexp} \int_0^\beta \left(\frac{\varphi}{2\zeta} + A + \frac{1}{2}\zeta\overline{\varphi} \right)$$

$$\langle \mathrm{Tr}_2 L_\zeta \rangle = \mathcal{Y}_e + \mathcal{Y}_e^{-1} + \mathcal{Y}_m$$

$$\mathcal{Y}_e(u, \zeta, \beta)$$

$$\mathcal{Y}_m(u, \zeta, \beta)$$

Exactly computable

$$\mathcal{Y}_e(u,\zeta,\beta) \quad \& \quad \mathcal{Y}_m(u,\zeta,\beta)$$

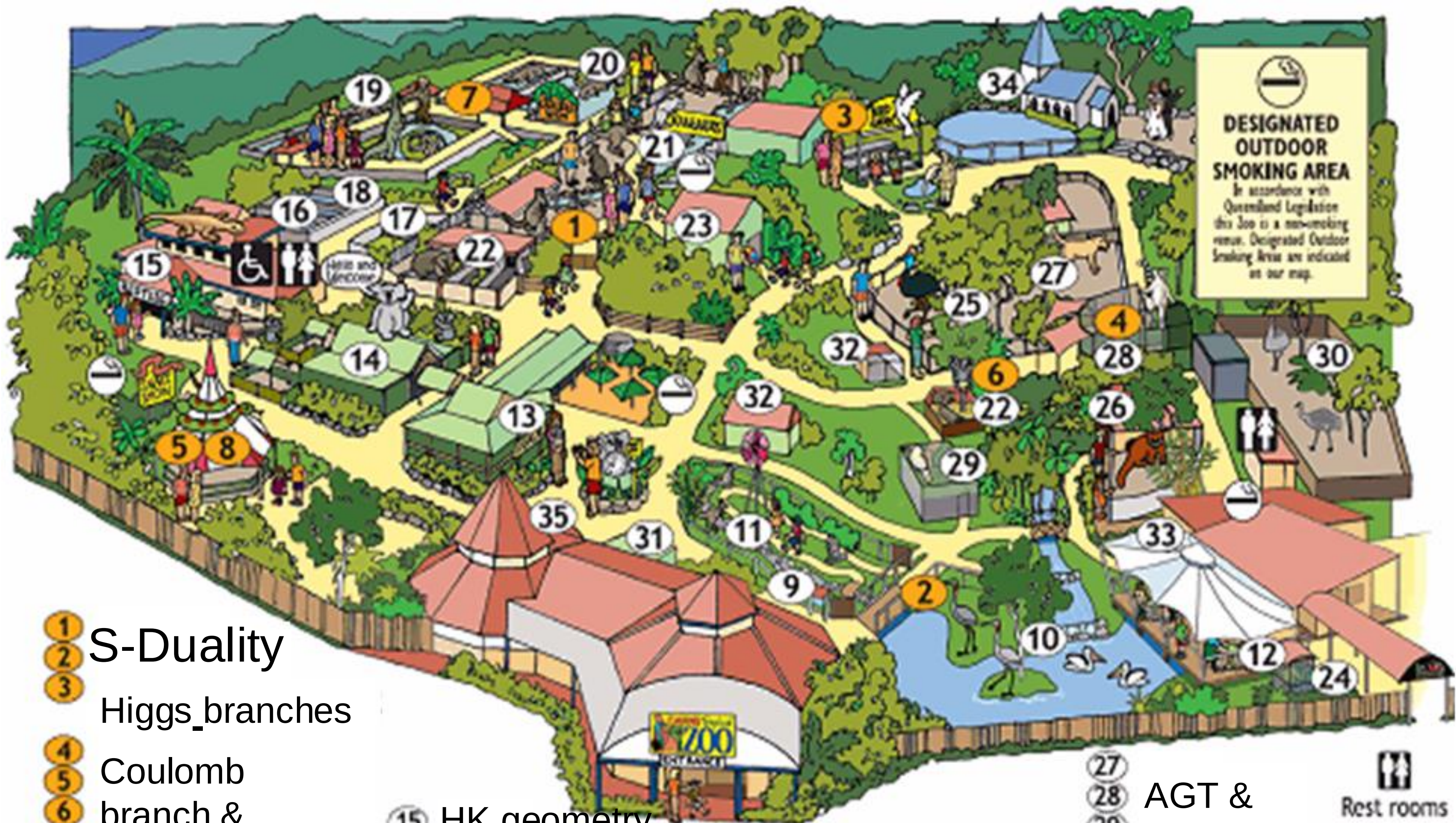
satisfy a system of integral equations formally identical to Zamolodchikov's Thermodynamic Bethe Ansatz

One of many relations to integrable systems…

+ Many connections to mathematics ….

They are special ``cluster coordinates'' on a manifold $\mathfrak{M}$ closely related to $\mathcal{M}$

Moreover, we can construct explicit solutions to Einstein's equations on $\mathfrak{M}$ using these functions.



1 2 3 S-Duality

Higgs_branches

4 5 6 7 8 Coulomb branch & Hitchin moduli

9 10 11 12 BPS states & wall-crossing

13 14 Line & Surface defects

15 HK geometry

16 17 18 19 Cluster algebras

20 Holographic duals

21 N=4 scattering

22 23 24 Ω-backgrounds, Nekrasov partition functions, Pestun localization.

25 26 $Z(S^3 \times S^1)$ Scfml indx

27 28 29 AGT & Toda CFT

30 31 32 Quantum Integrable

33 34 35 Domain walls & 3D Chern-Simons

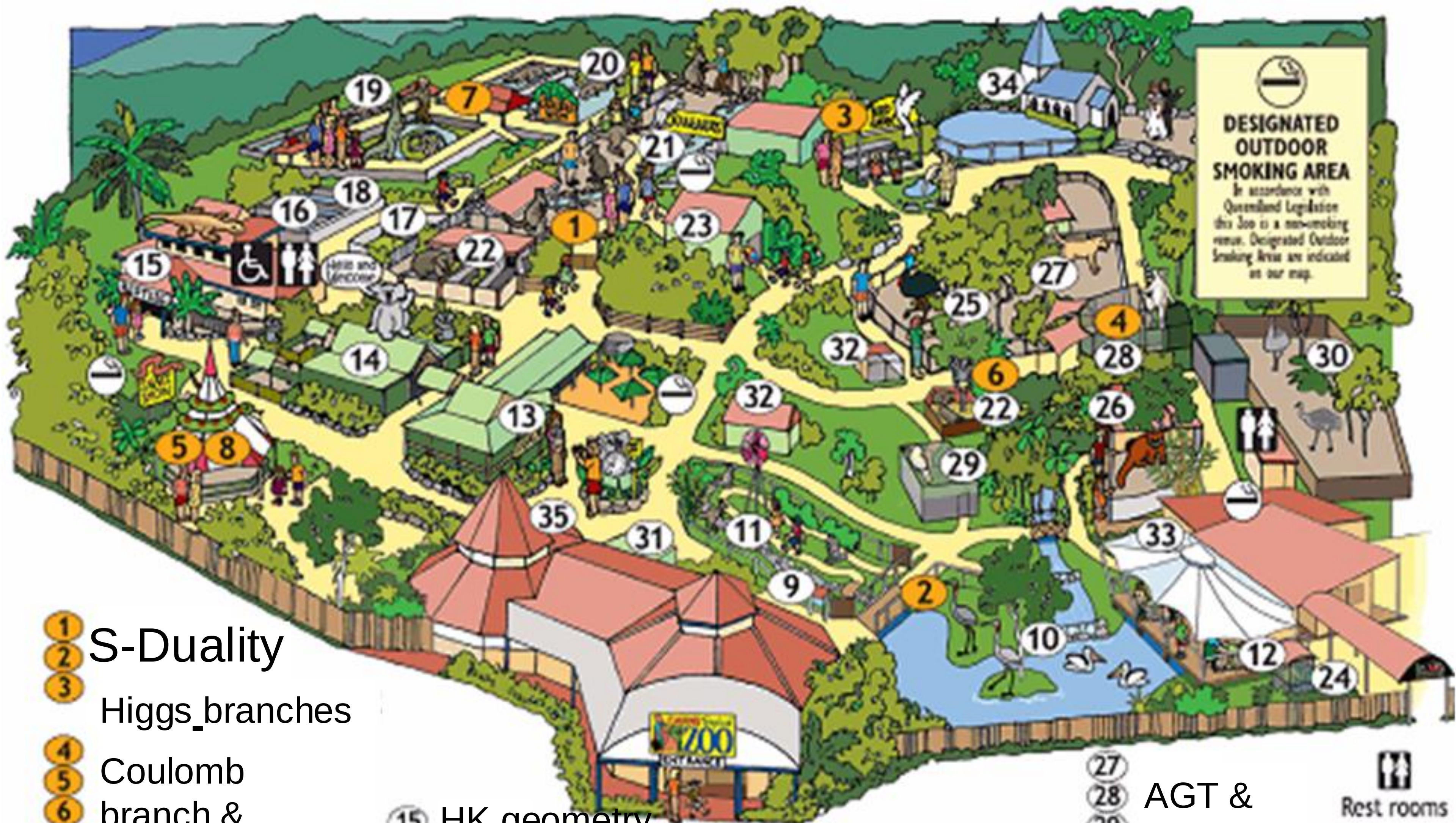
Part IV: Future Directions

Future Directions - A

Many good directions exist within the subject of theories with extended susy:

Go back to the original problem: Really understand the actual space of BPS states, not just the index. This requires new techniques and methods.

``CATEGORIFICATION''



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Future Directions - B

Study of the six-dimensional (2,0) theories

Claim, based on string theory constructions:

There is a family of stable field theories, S[$\mathfrak{g}$] with six-dimensional (2,0) superconformal symmetry. (Witten; Strominger; Seiberg).

Characterizing properties of these hypothetical theories can be deduced from their relation to string theory and M-theory.

But, these theories have not been constructed -- even by physical standards --


M5

Future Directions - C

The modern version of Hilbert's 6^{th} problem:

Find the defining axioms for M-theory.

The community seems to have put the problem aside –
at least temporarily.

But Physical Mathematics will surely play an important role in its resolution.

Physical Mathematics will be an important fixture of the intellectual landscape for some time to come.