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Abstract

What follows is a broad-brush overview of the recent synergistic interactions between mathematics and theoretical physics of quantum field theory and string theory. The discussion is forward-looking, suggesting potentially useful and fruitful directions and problems, some old, some new, for further development of the subject. This paper is a much extended version of the Snowmass whitepaper on physical mathematics [1]. Version: May 13, 2024.

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1 General Remarks

At the dawn of the scientific revolution Galileo wrote that the book of nature is written in the language of mathematics [2, 3]. That insight continues to illuminate the pursuit of science to this day. As physics progresses in its quest to understand the most fundamental laws of nature the mathematics in use does not remain static. It is, rather, cumulative. Moreover, as stressed by both Einstein [4] and Dirac [5], it becomes more sophisticated and more abstract. The history of the relation of physics and mathematics teaches us that, very often, great advances in physics are accompanied by profound innovations in mathematics, and vice versa. The past fifty years have been a watershed moment in this venerable dialogue between physics and mathematics. Astonishing advances in physics have led to profound mathematical discoveries and conversely new areas of mathematics have been brought to bear on attempts to understand some of the most basic questions about nature. There is a community of scientists that places equal stress on both the applications to advances in mathematics along with the drive to a deeper understanding of nature. A good name for the enterprise pursued by this community is "physical mathematics." The subject continues to be very active and vibrant. Foundations are becoming firmer and some participants are thrusting into exciting new territory.

This broad overview is meant to accompany the Snowmass whitepaper on physical mathematics [1] We will attempt to indicate some of the promising areas for future research along these lines, while also recalling some older promising but recently untrodden paths. While many topics of interest are covered here the authors are keenly aware that there are a large number of important topics, quite relevant to current research in physical mathematics, which have been omitted. Some, but not all, of these are addressed in other Snowmass documents. And then, there are the unknown unknowns: It must be borne in mind that in research the greatest treasures are often unforeseen, and we expect that some of the future paths indicated here will lead to unexpected and unanticipated new directions.

There are several other overviews of physical mathematics, or physmatics, as it is sometimes called to which we draw the reader's attention. These include [6–15]. In addition two ongoing world-wide seminar series [16,17], as well as the annual international String-Math conferences, cover many topics discussed herein and give an excellent snapshot of some of the most interesting current research topics.

The plan of this paper is as follows: Section 2 is a discussion of Quantum Field Theory (QFT). It begins with a summary of attempts to give a general definition of what QFT is and covers some topics in topological QFT. It also covers new algebraic structures and new notions of symmetries in QFT, along with updates on resurgence. Complementing this, section 3 concerns some of the fundamental questions relating to string theory and M-theory, and their mathematical foundation. Much recent progress has been made in the study of anomalies, which are the topic of section 4, which addresses both anomalies in a general QFT setting, as well as anomaly cancellation in string theory. The quantum consistency of string theory vacua plays another central role in section 5, where we discuss some specific and concrete mathematical problems connected to general issues in quantum gravity such as holography as well as certain aspects of the so-called "swampland program". Section 6 touches on some of the many interactions between string theory, QFT, and certain aspects of number theory. Section 7 is an all-too-brief hint of some of the many and profound connections to condensed matter physics. Geometry and topology provide a large set of connections to QFT and string theory and are the topic of sections 8 and 9, where we discuss the relation between geometric invariants and QFTs, as well as the geometrization program of QFTs, and superconformal field theories (SCFTs), respectively. A former section of this panorama has evolved into a separate publication [18]. A partial list of topics to be covered in [18] is given in section 10.

2 Quantum Field Theory

The *lingua franca* of physical mathematics is quantum field theory. There are many dialects of quantum field theory, and altogether, quantum field theory is perhaps one of the most spectacularly successful frameworks for describing nature in the history of science. Yet many physicists—and most (all?) mathematicians—feel that we still lack a truly satisfactory and universally applicable definition. Therefore, it is a major open question to find an axiom system for quantum field theory that adequately captures the astonishing range of phenomena, both physical and mathematical, that it describes.

2.1 Approaches To Defining Quantum Field Theory

Axiom systems for quantum mechanics date from the late 1920s and 1930s. The *Dirac-von Neumann* axioms [19–21] start with a complex separable Hilbert space \mathscr{H} and define states and observables in terms of \mathscr{H} . Time evolution is a unitary 1-parameter group acting on \mathscr{H} generated by a self-adjoint operator, the Hamiltonian. An alternative set of axioms for algebraic quantum mechanics was put forth by von Neumann and Irving Segal [22]. Here one focuses on observables, which form a C^* -algebra \mathfrak{A} . States are then positive unital functionals $\mathfrak{A} \to \mathbb{C}$. These frameworks for quantum mechanics are mathematically rigorous, and in turn quantum mechanics was a main impetus for the development of much of this mathematics, such as the theory of operators.

Quantum field theory (QFT) has long resisted such a clean mathematical treatment. There are, of course, several traditional approaches in the physics literature. These include: perturbing away from free field theories, which can be defined via an action principle; and lattice theories, which are particularly well-suited to computations at strong coupling. Renormalization theory is a key ingredient in all of these frameworks. In this section we focus on mathematical axiom systems for QFT. We remark that each known approach to QFT has important limitations. For essays on general issues at the foundations of QFT, see [23].

The first attempts at an axiomatic formulation of QFT take place in flat Minkowski spacetime M. The Wightman axioms [24] postulate a unitary representation of the Poincaré group on a Hilbert space \mathscr{H} as well as a field map which assigns (possibly unbounded) operators on \mathscr{H} to test functions on M. Roughly, these axioms specialize to the Dirac-von Neumann axioms in case dim $\mathbb{M}=1$. The alternative Haag-Kastler axioms [25] for QFT on \mathbb{M} are in the spirit of algebraic quantum mechanics. They assign a C^* -algebra $\mathfrak{A}(U)$ (of observables) to each open subset $U \subset \mathbb{M}$ with compact closure, compatibly with the Poincaré group and causality. These axiom systems—though incomplete and not universally applicable—are the foundation and inspiration for much mathematical work in QFT.

Before we turn to modern axiom systems for QFT, we list some of the key issues one confronts in formulating axioms:

- 1. Some QFT's have no action principle and no obvious "fundamental" degrees of freedom.
- 2. Some QFT's have many action principles, with completely different "fundamental" degrees of freedom.
- 3. Even when there is an action principle, interacting QFT's with running coupling constants are not clearly and rigorously mathematically defined.
- 4. S-matrix amplitudes exhibit remarkable properties making startling connections to mathematical fields not traditionally related to physics. Moreover, they encode physical properties such as locality in highly nontrivial ways.
- 5. The universe of "theories" is *not* a disjoint union of string theory and QFT. From AdS/CFT we believe that some field theories are equivalent to string theories. On the other hand, there are other "theories," which are neither local quantum field theories nor traditional string theories (with gravity).
- 6. Many field theoretic phenomena have *geometrical reformulations*, reducing highly nontrivial facts of field theory to simple geometrical constructions. See section 9 for more discussion.
- 7. A fully local theory should associate physical quantities to subspaces of all codimension and have coherent gluing rules for gluing of such subspaces.
- 8. Even when there is an action principle, a QFT is not completely defined by an action and a list of local operators. One must include (higher categories of) defects of all dimensions, in accord with the previous point.

In the mid 1980's Graeme Segal introduced a novel axiom system for two-dimensional conformal field theory [26]. In this telling a field theory is a functorial assignment of a topological vector space to a circle and of a trace class (nuclear) linear map to a 2-dimensional bordism endowed with a conformal structure. In brief: a 2d conformal field theory is a linear representation of the bordism category of Riemann surfaces. As opposed to the Wightman axioms, which model QFT on flat Minkowski spacetime, the Segal axioms model Wick-rotated QFT on compact Riemannian manifolds. Significantly, while they were originally developed for two-dimensional conformal field theories, the Segal axioms are now understood to apply to quantum field theories in all dimensions. For example, they apply to topological field theory, where the explicit relationship to classical bordism was flagged by Michael Atiyah [27]. Within the past year Segal, together with Maxim Kontsevich, posted a preprint [28] which takes up the axiom system in the general QFT framework. Furthermore, they prove an analog of the Osterwalder-Schrader theorem: they construct a QFT on globally hyperbolic Lorentz manifolds from a QFT on Riemannian manifolds in which positivity of energy is encoded by having the theory defined on manifolds with certain complex metrics. The Segal axioms have provided an inroad into QFT for mathematicians of all stripes. Much remains to investigate, particularly analytic aspects, and this is a fertile ground for future investment. An interesting application of some observations of [28] to quantum gravity appeared in [29].

In the past decade Kevin Costello and Owen Gwilliam developed an axiom system for Wick-rotated QFT which is an analog of the Haag-Kastler axioms in flat Minkowski spacetime [30,31]. They define factorization algebras on smooth manifolds, modeled after the chiral algebras of Beilinson-Drinfeld [32]. The latter were motivated by two-dimensional conformal field theory, whereas the former are meant to apply to QFT in all dimensions. A factorization algebra \mathcal{F} on a smooth manifold M assigns a chain complex $\mathcal{F}(U)$ to each open subset $U \subset M$, and the assignment satisfies a multiplicative version of duals to the usual sheaf axioms (i.e., it satisfies a multiplicative cosheaf axiom). Many examples have been constructed using Costello's previous work on perturbation theory. Here is a very small quirky sample of applications of these ideas to physical theories: [33–39]. This framework has opened new mathematical avenues into modern aspects of QFT, such as the AdS/CFT correspondence. Again there is much for the future that is worthy of robust support.

For further discussion of axiomatic approaches to defining QFT, together with an ample set of references to the older literature, see the recent Snowmass whitepaper [40].

2.2 Topological Quantum Field Theory

From the mathematical point of view, the best established QFT's are the topological quantum field theories (TQFT). These have been put on a firm footing and capture some of the essential aspects of locality. One way of taking the idea of locality to its logical conclusion is the notion of a fully extended (a.k.a. "fully local") TQFT based on symmetric monoidal n-categories, where n is the dimension of spacetime in a physical theory [41–46]. In recent years there has been a vigorous development of this topic and mathematicians have extended these ideas in very sophisticated ways; a small sample is [47,48]. (There is a recent discussion of fully local non-topological field theories in [49].)

Although the subject has matured, and the language of TQFT is a commonly used and powerful tool in mathematics, there remain many important open problems in the subject. Many bordism categories of physical interest (e.g. including various kinds of tangential structures) remain relatively unexplored. Other extensions under exploration include equivariant, nonunitary, and nonsemisimple TQFT's. The study of defects and boundary conditions has much future potential. TQFT's have also proven to be quite powerful in addressing questions about anomalies in field theory and topological phases of matter, and we will comment further on that below.

Many important classification questions remain unanswered. The classification of deformation types of invertible TQFT's [50] gives some hope that further and more general classification questions are not completely out of reach. An example of a long-standing open question is the classification of 3d unitary TQFT's. An old conjecture [51] posits that they are all associated with 3d Chern-Simons-Witten theory, once one has chosen a suitable level and compact gauge group. (Closely related to this, the modular tensor categories (MTC's) of rational conformal field theories (RCFT's) should all be built by using the Wess-Zumino-Novikov-Witten model [52] using a small set of standard constructions.) There are recent interesting challenges to this conjecture [53–56] which should be further investigated. In particular, Teleman has shown that the Haagerup MTC cannot come from a Chern-Simons-Witten theory with compact gauge group [57]. It is important to bear in mind here the relevance of the notion of Witt equivalence [58, 59]. One can ask whether the Haagerup MTC's are Witt equivalent to the theory with a compact gauge group. ¹

Ironically, the first topological field theory to be discovered, Donaldson-Witten theory [62], does not obey the official definition of a TQFT. There are difficulties making sense of the instanton Floer theory on arbitrary 3-manifolds, and it is certainly not known how to define a fully extended TQFT corresponding to the Donaldson invariants. (It certainly cannot be unitary [63] or semisimple.) In a similar vein the GL-twisted $\mathcal{N}=4$ SYM [64] has been extensively developed and intensively studied, but it is not known how to define the topological theory on arbitrary four-manifolds. Similar remarks apply to three-dimensional Chern-Simons theories with noncompact gauge groups. These are examples of theories that are only partially defined, they have not been defined as fully extended (a.k.a. fully local) TQFTs. It is not known if a fully extended version exists, or if there are fundamental obstructions to such an extension. Clearly, there remain fundamental aspects to be understood even in some of the most basic examples of what physicists regard as topological quantum field theory.

There are other examples of such partially defined TQFT's where, in fact, we do know of deformations or regularizations that still lead to tractable theories. One good example is a two dimensional relative of Donaldson theory, the topological Yang-Mills theory in two dimensions. It has an infinite set of states, the torus partition function is divergent and only a restricted set of correlation functions makes sense. However, it has several well-known regularizations: the gauged G/G WZW theory [65–67], and a finite coupling two-dimensional Yang-Mills theory [68,69]. The latter is not, strictly speaking, a topological theory, as it depends on a choice of a measure, and the Wilson loop observables, although quite acceptable in the Yang-Mills theory, are not, strictly speaking, topological - they only depend on the areas of the components of the surface on the complement of the Wilson lines [70,71]. Another interesting "deformation" is q-deformed 2d YM and further generalizations which have shown up in a variety of contexts [72–76].

¹Witt equivalence has recently been finding physical applications of fusion and modular categories. See, for example [60] where it is used to discuss the existence of topological boundary conditions in three-dimensional topological theories or [61] where it is applied to the question of time-reversal symmetry in 3d Chern-Simons theory.

2.3 Algebraic Structures In Quantum Field Theory

2.3.1 Algebraic Structures Related To Operator Product Algebras

The notion of an operator product expansion indicates that algebraic structures of some sort play a fundamental role in quantum field theory. This notion has been formalized and generalized in many ingenious ways and the investigation of algebraic structures in QFT has been a dominant and enduring theme of research and promises to continue being an important topic.

The study of two-dimensional conformal field theory led to the mathematical study of vertex operator algebras (see e.g. [77–79]), a notion with a clear mathematical definition which nicely captures and makes precise constructions which had previously been used in the physics literature. Today, vertex operator algebra (VOA) theory is a well-established topic of mathematics but many interesting open problems remain in this field. There is still much to learn about the representation categories of VOA's. For example, one natural question is whether there is an analog of Tannaka-Krein reconstruction. In another direction, one of the beautiful recent results in traditional VOA theory is a complete classification of the holomorphic VOAs of central charge 24 [80–83]. Another major gap in our knowledge is any semblance of a classification of holomorphic vertex operator algebras of central charge greater than 24. This appears to be a very challenging problem, even more challenging than finding a useful classification of even unimodular lattices. Virtually nothing is known about what the general holomorphic CFT looks like. It is natural to speculate that there is an analog of the Smith-Minkowski-Siegel mass formula for holomorphic CFTs, but, so far as we know, no work on this idea has been published.

Remarkably, VOA's and structures similar to VOA's of various kinds have been discovered in a wide variety of new contexts in higher dimensional QFT. One generalization of the subject of vertex operator algebras is the entire framework of factorization algebras [30], as discussed in §2.1. (These generalize the E_d algebras relevant to TQFT.) An important open problem is to connect these ideas to more traditional ideas of quantum field theory based on Wightman and Haag-Kastler axioms. The work [84] discusses the relation between VOA and the Haag-Kastler axioms, and should provide a first step in this direction. It would also be important to understand better the relation to the recent work [28]. Another rich source of nontrivial algebraic structures are the supersymmetric theories discussed in the next section.

2.3.2 Algebraic Structures Associated With Supersymmetric Sectors And BPS States

Using supersymmetry and defects one can define important subsectors of the full operator product algebra of certain quantum field theories. This has been done with great success in the past ten years. Examples include the chiral algebras appearing in $d=4, \mathcal{N}=2$ supersymmetric field theories [85–87]. Another beautiful example are the algebraic structures associated with holomorphic sectors of supersymmetric theories [88,89], and with hybrid holomorphic-topological twists of supersymmetric theories with extended supersymmetry [72,90,91]. This is currently a very active direction of research with many potential applications [36,92–96].

Another class of algebraic structures are the generalizations of Gerstenhaber algebras that appear in TQFT's of cohomological type such as those studied in [97–99], leading to connections to BV algebras and homotopical algebras.

Closely related to the above are algebraic structures associated to BPS states. One early hint that spaces of BPS states should have an interesting algebra-like structure was inspired by the work of Nakajima [100, 101] and is described in [102]. This led to various proposals for algebraic structures associated to BPS states in [103–106]. One curious application of these ideas is a very suggestive approach to the important problem of finding a conceptual basis for understanding the role of genus zero groups in monstrous moonshine [107–109]. More recently, the subject of quiver algebras associated to BPS states on D-branes has just begun to be developed [110–115], giving a new physical interpretation of some old constructions of Nakajima and Kontsevich-Soibelman with many promising directions for generalizations and further development.

In a parallel development, an in-depth study of the mathematical structures of BPS states in twodimensional QFT with (2,2) supersymmetry has revealed a very rich mathematical structure similar to, but different from, topological string theory. This involves the 2-category of theories, interfaces, and boundary-condition-changing operators, and involves various constructions in homotopical algebra [116–120]. For yet another set of parallel developments see the remarks at the end of the section 2.9. This structure was developed in an effort to understand the "categorification of wall-crossing formulae." Indeed these papers have explained a categorical version of the 2d Cecotti-Vafa wall-crossing formula [121,122], and of Stokes' phenomenon in general. But the extension to four dimensions has remained elusive and is an important open problem. In [118] the "web-formalism" of [116,117] was reinterpreted in terms of polygons. This led to a higher-dimensional generalization in terms of polytopes. It might be very interesting to find a physical interpretation of the higher-dimensional version of the web-formalism developed in [118]. Many of the above developments were motivated - in part - by the desire to find a categorification of the full 2d4d wall-crossing formula. This goal remains elusive, and is surely a good direction for future research.

For a recent overview of L_{∞} structures in field theory see [123].

2.4 Classification And Deeper Understanding Of New Classes Of Supersymmetric Field Theories

A well-known and very important problem in physical mathematics is finding a satisfactory formulation of the six-dimensional superconformal field theories with (2,0) supersymmetry. Perhaps it should be formulated axiomatically (an attempt was made in section 6 of [124]). Or perhaps there is some formulation in terms of some local degrees of freedom, perhaps by making sense of a "nonabelian 2-form gauge potential." There are many attempts in the literature taking the latter approach. For two examples see [125–127] and [128–131]. String theory constructions, which led to the discovery of these theories [132–134], strongly indicate we know what the examples are: The basic components are made of "Abelian" theories, and nonabelian theories classifed by an ADE classification. Just assuming the theories exist has led to a host of very fruitful mathematical constructions shedding light on hyperkähler geometry, Hitchin systems, exact WKB theory, and low dimensional topology, among other things.

One important aspect of the 6d theories are the various supersymmetric defects they possess. Among these the half-supersymmetric codimension two defects are among the most important. Their basic classification has been investigated in, for examples, [135–137]. These defects have an interesting operator product structure whose "structure constants" are themselves four-dimensional quantum field theories [138]. Clearly they deserve continued study.

It would also be of great interest to understand the full range of possibilities and map out the geography of four-dimensional theories with $\mathcal{N}=2$ supersymmetry. An overview of the current status is provided in the Snowmass paper [139]. There are three broad classes of known theories:

- 1. One class of such theories have Lagrangians $\mathcal{N}=2$ based on semi-simple Lie groups and hypermultiplets. All such UV-complete theories are classified in [140].
- 2. Another class (which partly overlaps with the previous one) are the theories obtained as limits of ADE (0,2) superconformal theories on spacetimes of the form $C \times \mathbb{R}^4$ where C is a (possibly punctured) curve. These are the class \mathcal{S} theories [141–144]. (For some reviews see [124,145,146].)
- 3. Geometrically engineered theories and models engineered using brane constructions. For the current status of these see section 9.

Within each of these classes quite a lot is known. Some of the Lagrangian theories can be realized as the low energy limits of the theory on a stack of D3-branes probing an orbifold singularity, the quiver gauge theories [147]. The requirements of asymptotic freedom constrains the choices of gauge groups and matter multiplets of quiver theories. For example, if the gauge group is a product of (special) unitary groups, then the only choices of quivers are the Dynkin diagrams of simple simply laced Lie algebras or affine Lie algebras.² In the latter case the gauge group and matter content theory is uniquely determined up to a single integer N. For such theories, possibly mass deformed in an $\mathcal{N}=2$ supersymmetric way, the Seiberg-Witten geometry has been computed in [151] using the exact quantum field theory calculations performed using localization on a suitably (partly) compactified moduli space of quiver instantons. The result of the limit shape calculation identifies the Coulomb branch with a base of an algebraic integrable system whose phase space is a moduli space of either the ADE instantons of charge N on $\mathbb{R}^2 \times T^2$, or the ADE monopoles with Dirac singularities on $\mathbb{R}^2 \times S^1$.

²One can have also a more exotic possibility of finding a Lagrangian description for class \mathcal{S} theories which manifests only $\mathcal{N}=1$ supersymmetry (see [148–150] for some examples). We will say more about this in Section 9.

(Two pedagogical reviews of limit shapes are [152,153].) In these results the charges are determined by the ranks of gauge groups and the numbers of fundamental hypermultiplets. The results are in broad agreement with earlier conjectures based on string theory considerations [154–156], although some of the earlier conjectures require modification. The theories with eight supercharges, appropriately lifted to k+4 dimensions on a spacetime of geometry $\mathbb{R}^4 \times T^k$ lead to the moduli spaces of instantons (monopoles) on $\mathbb{R}^{2-k} \times T^{2+k}$ ($\mathbb{R}^{2-k} \times T^{1+k}$), respectively [151]. (In the lifted theories one can add Chern-Simons couplings and/or encounter new anomalies.)

One might wonder if the problem simplifies if one tries to classify theories with a large N limit using the AdS/CFT correspondence. The affine type quiver theories were previously studied, in the large N limit, in the context of the AdS/CFT correspondence, in [157, 158], while the large N type S-class theories were studied in [159]. (The holographic picture gives an interesting view on the renormalization group flow, which is sometimes possible, in the $\mathcal{N}=2$ case, to connect to the field theory computations [160]. This is a nice example of the geometrization of QFT discussed in more detail in section 9 below.)

2.5 Generalizations Of The Notion Of "Symmetry"

Many of the most important developments in 20th century physics, including relativity, quantum mechanics, and gauge theory make fundamental use of the notion of symmetry. In recent decades the notion of symmetry has been further enhanced and extended and applied in surprising new ways. Further generalizations of symmetry will surely be a theme for several years to come.

One example of an important new notion is that of "generalized p-form symmetries." In various forms these have been used for a long time. For example the use of "center symmetry" played an important role in the early studies of phases in nonabelian gauge theories [161, 162]. In [163] they played an important role in defining flux sectors in arbitrary Abelian gauge theories (including self-dual theories) based on generalized cohomology theories. In an important advance, "generalized p-form global symmetry" was formulated in terms of topological defect operators [164] where these defect operators were put to very good use. In the case of two-dimensional rational conformal field theories, defects as symmetries—and duality defects—appeared a decade earlier in [165]. A hint on generalized symmetry and topology also appeared in [166,167]. New insights into instanton effects, even in textbook examples of quantum mechanics were found and striking impliciations for QCD at $\theta = \pi$ were pointed out [168]. Since then there has been widespread use of such generalized symmetries. Combined with the theory of anomalies striking applications have been made to determine dynamics of nontrivial theories [169–176]. Motivated, in part, by these developments there has been some exploration into the notion of "two-group symmetries" [177,178] and these works clearly point the way to potentially interesting future developments.

Further developments along these lines have tried to generalize even further the notion of symmetry in terms of "categorical symmetries" and "noninvertible symmetries." Algebras of symmetries are common in both mathematics and quantum theory, and algebras include noninvertible elements. In mathematics various categorified generalizations of algebras are being developed. For example, tensor categories, for which there is a robust theory [179], are a once-categorified version of an algebra, and this higher algebra underlies much research on Reshetikhin-Turaev-Witten and Turaev-Viro models. There are also higher categorical versions, including " E_n -algebras," which enter topological field theory and are destined to find applications beyond the topological case. The whole subject and its application to physics is under active development. Certainly there is no settled definition of what "categorical symmetry" should mean in field theory. Roughly speaking, a categorical symmetry involves topological defects with sums of simple objects in their fusion rules (in a semisimple setting) and couplings of n-dimensional theories to n+1-dimensional topological field theories. A Simons Collaboration is devoted to exploring these ideas further [180]. Some of the recent developments are summarized in the Snowmass contribution [181]. From a mathematical point of view the emergence of categorical structures in QFT is particularly noteworthy, with most recent developments in this direction in [182–191]. An in depth discussion of generalized categorical symmetries in the context of QFTs, string theory and holography can be found in sections 2.9 and 9. Although not phrased as a "symmetry" very closely related ideas had been previously explored in [192–194].

In another direction, the papers [195–197] have explored deeply the role of symmetry in black hole physics, quantum gravity and the AdS/CFT correspondence. It would be interesting to combine the ideas of these papers with some of the emerging ideas on global categorical symmetry. Some early

developments in this direction can be found in [198] and in holography/string theory in [199,200]. See also section 2.9.

Another very important emerging set of ideas concerns asymptotic symmetries in gauge theories. These were, indeed the basis for an early version of the AdS/CFT conjecture [201] and recently they have been much studied in the context of BMS symmetries and the infrared structure of massless gauge theories [202]. Related ideas concerning a hypothetical "celestial conformal field theory" holographically dual to gravity in flat space have been discussed in a number of papers [203]. The existence of such a celestial conformal field theory would be quite remarkable since it is far from obvious why the Mellin transforms of scattering amplitudes should behave like the correlators of a local quantum field theory.

Finally, the Batalin-Vilkovisky formalism gives yet another, perhaps not unrelated, generalization of symmetry. The conventional Lie algebraic symmetry corresponds to a homomorphism \mathcal{V} from some Lie algebra \mathfrak{g} to the algebra Vect(M) of vector fields on some (super)manifold, e.g. a space of fields (more generally, the space of fields and anti-fields in the BV sense). The classical action S_0 , which is a function on M is invariant under the $\mathcal{V}(\mathfrak{g})$ -action. It means that it can be extended to a function S on $M \times \mathfrak{g}^* \oplus \Pi \mathfrak{g}$ in such a way, that the \mathfrak{g}^* dependence of S is linear ($\Pi \mathfrak{g}$ -dependence is at most quadratic), and S solves the classical master equation $\{S,S\}=0$. The extension of S_0 to S corresponds to the familiar BRST formalism (except that BV allows for both gauged and global symmetries to be treated in a similar way). The L_{∞} -symmetry is a non-linear formal generalization of this construction, where the requirement of linearity in anti-fields for ghosts is dropped. The need for such generalization is provided by the renormalization group, as integrating out the charged degrees of freedom, produces the above-mentioned non-linear terms, 3 . For related ideas see [204].

2.6 The Space Of Quantum Field Theories And The Renormalization Group

It has been noted on many occasions that one of the main barriers to a more productive dialogue between physicists and mathematicians is a rigorous understanding of scale transformations and the renormalization group. Some important progress - in the perturbative regime - was made in [205], which makes essential use of the Batalin-Vilkovisky formalism. Another, quite different approach to renormalization theory makes use of ideas of noncommutative geometry, and phrases renormalization in terms of a Birkhoff factorization problem in a suitable Hopf algebra [206–208].

In spite of these and other approaches, a full nonperturbative treatment remains open. Indeed, it is the subject of one of the Clay Millenium Problems.

Although we do not have a completely satisfactory and universal definition of what should be meant by a "quantum field theory," it might not be too early to ask what can reasonably be said about the full space of quantum field theories - although it must be admitted that there is no really rigorous definition of the "space of quantum field theories". Nevertheless, physicists have some intuitive pictures based on Wilsonian ideas and the crucial concept of renormalization group flow.

While understanding the space of 1d QFT's presents some interesting issues the first really non-trivial case is the case of two-dimensions. This has long been recognized as a very important question for fundamental physics [209,210]. Here there are well-known and profound results on renormalization group flow relating it to Einstein's equations [211,212]. In addition to this there is the remarkable Zamolodchikov c-theorem [213]. An important analog appears in the case of conformal field theories with a non-conformal boundary theory [214–216]. This progress grew out of an interesting approach to background-independent string field theory [215, 217–219]. In this approach one considers a two dimensional conformal field theory on a surface with boundary. The two-dimensional "bulk" is conformal but one allows boundary operators and boundary conditions that break the conformal invariance. The RG does not change the conformal field theory in the "bulk" but does alter the boundary conditions. The "space" of such theories (with a fixed conformal theory in the two-dimensional bulk) is an interesting example of a space of theories might be more tractable than the general case of all two-dimensional theories. It is possible one can say something about the space of components of such a "space of quantum field theories" using ideas from K-theory, an idea which has not been very deeply explored, although it was mentioned in [220].

In the mathematical realm a very interesting proposal for a space of 2d quantum field theories with (0,1) supersymmetry was made by Stolz and Teichner [221, 222]. The Stolz-Teichner program grew

 $^{^3\}mathrm{We}$ thank A. Losev for illuminating discussions on this point

out of attempts to extend the ideas of Graeme Segal on elliptic cohomology to a theory conjecturally equivalent to the theory of topological modular forms. The subject has recently gotten a big boost from several recent works applying it to interesting questions in quantum field theory [223–225] and anomaly cancellation in heterotic string theory [226–228]. There is clearly a lot more to be learned in this subject. For example, the generalized cohomology theory of topological modular forms (tmf) has a 576-fold periodicity, analogous to the 8-fold periodicity of Clifford algebras and real K-theory. A clear physical explanation of this remarkable aspect of tmf might be highly interesting.

Moving on to higher dimensions, there are conjectural analogs of the Zamolodchikov theorem, and there has been remarkable progress on this topic [229, 230]. Moreover it is related to questions of quantum information theory [231–235]. Some bold conjectures about the global nature of the space of conformal field theories were made by Seiberg in [236]. It remains a challenge to see of any of the above ideas can shed light on Seiberg's conjectures.

Some of the issues here are closely related to those touched on in the condensed matter theory section 7.

2.7 Fundamental Formulation Of Non-Lagrangian Superconformal Field Theories

As was mentioned above, an important long standing open problem is the formulation of the six-dimensional superconformal field theories with (2,0) supersymmetry. A key obstacle here is the proper understanding of the formulation of self-dual theories. Note that these include the important sub-case of chiral scalar fields in 1+1 dimensions. The subject has a long and extensive history.

Even the free self-dual Abelian theories - often dismissed as trivial - have not received a complete and rigorous formulation as fully local (aka fully extended) field theories. There is a long history of attempts to write action principles for these theories. See [237–241] and the many references therein. There is still no universally agreed-upon approach to defining these theories via an action principle. Moreover, even in the Abelian case there are many subtle topological issues some of which are discussed in [163,239,240,242–247]. Finding a complete and definitive treatment of self-dual Abelian gauge fields based on an arbitrary self-dual generalized cohomology theory remains an important open problem. These are free theories, so, although the topic is subtle and difficult, there is no insuperable obstacle to giving such a treatment. And there is really no excuse for not doing so. To illustrate the incompleteness of our understanding, we remark that the anomaly theory of the basic self-dual field is constructed in [245], where an integral lift of the middle Wu class is used. We inquire: What are the physical consequences of this choice? For example, dimensional reduction of the 6-dimensional self-dual field to 4 dimensions yields Maxwell theory, and we can ask how this choice manifests there.

For generic choices of compactification parameters the class \mathcal{S} theories [141–144] do not possess a known Lagrangian description. On the other hand, the semi-simple gauge group Lagrangians with manifest $\mathcal{N}=2$ supersymmetry and hypermultiplet matter, describing an interacting SCFT are classified in [140]. An interesting question is to understand whether Lagrangians manifesting less symmetry can be constructed which will flow to a generic $\mathcal{N}=2$ theory, i.e. a class \mathcal{S} theory. This question can be generalized to whether any conceivable SCFT has a description in terms of weakly coupled fields. The reader will find more on this question in section 9.

2.8 Nonperturbative Effects, Resurgence, Stokes Phenomena, And Exact WKB Methods

If QFT is understood in terms of the Feynman path integral, with Planck constant \hbar measuring the strength of the action in the exponential, the behavior of physical quantities as analytic functions of \hbar should be at least as complicated as that of an exponential integral. In a nice analytic situation the latter is a vector-valued function of parameters, the basis being given by the periods of the corresponding form over the submanifolds known as Lefschetz thimbles. A critical point of the action may be in the complex domain. The conventional perturbative expansion takes into account one, typically real, classical solution, i.e. the one that asymptotes to some number of in-coming and out-going particles. The sum of loop corrections almost never converges, e.g. by Dyson's argument [248].

However, this divergence is already present in finite dimensional exponential integrals, so one can try to isolate the divergence of the perturbative expansion of an exponential integral within that of a path integral. So far this wish was partly granted. One widely used technique is Borel resummation, which is a way to trade the factorially growing sum over diagrams for an exponential integral of a rational form. For some nice expository accounts of these ideas see [249–251].

The development of this idea leads to the notion of a trans-series and resurgence, see [252–256], and [257] for a modern review. One class of problems on which these methods were tested since the mid-1970's ⁴, are the spectra of the one-dimensional Schrödinger operators $-\partial_q^2 + U(q)$. The convenient choices of the potentials U(q) are polynomials in q (see [258] for a beautiful discussion).

The singularities of the Borel transform are, in finite dimensional examples, related to the critical points of the action, which, in turn, are telling us about the possible Lefschetz thimbles. To some extent this is confirmed in Lipatov's method of evaluating large order perturbative contributions by a saddle point method [259]. However, it has been argued [260] there are singularities on the Borel plane not attributed to the instantons of [259]. Very little is known about these singularities - which are sometimes referred to as renormalons. If a semiclassical approach is valid then it is possible that the missing critical points can be found in the properly understood complexification of the space of fields [261, 262]. The practitioners are widely spread across the spectrum of optimism on this issue. For example it is not clear if renormalons can be approached semiclassically, as has been stressed by M. Mariño.

The plethora of exact computations made available by localization in supersymmetric gauge theories and topological strings, supplemented with dualities, mapping \hbar to some geometric parameters, gives a huge playground for investigations of these ideas, including the ill-understood "instanton-fractionalization" [263, 264], sometimes used to explain the renormalons. (It would be desirable if the computations of [263] could be phrased in a controlled truncation to a quantum mechanical problem.) It is also of interest to consider non-supersymmetric, asymptotically free, integrable theories in two dimensions such as O(n) sigma models, the Gross-Neveu, model and so forth in this context. Recent results in this area [265] indicate that some standard lore about renormalons needs to be revised.

The standard WKB method can be enhanced to give exact results in nontrivial quantum systems using ideas explored some years ago in [258, 266]. Again, embedding quantum systems in the supersymmetric context e.g. via Bethe/gauge correspondence gains some mileage. For example, the exact quantization conditions [267] were related [268, 269] to equations [270] describing the vacua of the Ω -deformed four dimensional \mathcal{S} -class $\mathcal{N}=2$ theories, in the NRS coordinates⁵ on the moduli space $\mathcal{M}_{\mathbb{C}}$ of flat $SL(2,\mathbb{C})$ -connections, cf. [272].

In an independent line of development, the study of wall-crossing and the counting of BPS states led to a new mathematical object, known as *spectral networks* [273] which generalizes exact WKB theory and leads to explicit constructions of BPS degeneracies of class S theories, as well as Darboux coordinates on moduli spaces of flat connections. These Darboux coordinates sometimes coincide with Fock-Goncharov and other cluster coordinates. The canonical line bundle on the spectral curve admits a flat connection whose holonomy provides these coordinates, which are thereby closely related to (and generalize) Voros symbols. Spectral networks provide the data to give a converse construction, namely a construction of the corresponding nonabelian flat connection from a flat connection on the spectral cover. These aspects of exact WKB theory were studied in [142,273–278], and more generally in [279].

The results of this development (in particular through the "conformal limit and the relation to opers" [280,281]) have implications beyond that of supersymmetric field theory, leading to new insights in ordinary differential equations, such as the Schrödinger equation. See [282–284] for a recent discussion. In particular [282] has a useful review of several aspects of exact WKB analysis and spectral networks and the relation to opers. Very recently the Borel summability of the WKB series in many quantum problems has been rigorously established [285].

The relation between Abelian and Nonabelian connections given by spectral networks has a q-deformed analog [286,287] which has been beautifully developed and extended in the recent works [288, 289]. This subject should be further developed and should lead to useful insights about quantization of character varieties, connecting to the works [290–292] on quantum cluster varieties on the one hand, and the two [293, 294] and four dimensional [295] approaches to quantization of these spaces, on the other. There are also intriguing recent math results on the representation theory of Skein modules [296]

⁴In works of e.g. Balian-Bloch, Dingle, Leray, Sibuya, Zinn-Justin.

⁵Sometimes called the complexified Fenchel-Nielsen coordinates in the literature, the NRS atlas [271] of Darboux coordinates interpolates between Fenchel-Nielsen, Goldman, Kapovich-Milson, and Klyachko coordinate systems on various real slices and tropical limits of $\mathcal{M}_{\mathbb{C}}$. They can also be viewed as a generalized system of Fock-Goncharov coordinates, associated to laminations with closed orbits.

which deserve a physical explanation. ⁶

The work of [116,119] can be interpreted as a categorification of Stokes phenomenon: The so-called S-walls are Stokes' lines. A construction of categorified parallel transport of a flat connection was given and the categorified S-wall factors in section 7.6 of [116] are categorified Stokes' factors. This categorification of exact WKB theory should be developed further.

Considerations involving Stokes phenomena have also lead to striking results about 3-manifold topology in the context of complex Chern-Simons theory [251,298,299]. See section 8.2 below for more discussion about the relation to the topology of three-manifolds and links.

2.9 Defects

Traditional quantum field theory has been concerned with correlation functions of local operators. However, going back to the work of Wilson on lattice gauge theory it has been known that extended "operators" such as Wilson lines (the trace of the holonomy of a gauge field in a representation) play an important role in formulations and explorations of field theory. These "operators" are more properly understood as "defects," couplings of a quantum field theory to a lower-dimensional theory of fields localized on a submanifold of spacetime. Having phrased things that way, a host of generalizations emerge. Moreover, the concept of defects can be further broadened to include other local objects, and extended operators, defined by allowing singular behavior of fields in the neighborhood of specified subvarieites. In this way one can include disorder operators, 't Hooft lines and other generalizations such as surface defects, and higher dimensional defects. See [300] for one careful attempt to give a definition of supersymmetric defect lines ('t Hooft-Wilson lines) in $\mathcal{N}=4$ SYM. Likewise, a surface defect in a four-dimensional gauge theory with a gauge group G could be described as a coupling of a two dimensional sigma model having G as a global symmetry, to the gauge fields propagating in the ambient space, but pulled back to the surface. (But not all surface defects can be described in this way.) See [103, 301–307] for a small sampling of studies and reviews of surface defects.

Anton Kapustin, in his talk [308, §2] at the 2010 International Congress of Mathematicians, clarified the relationship between defects and extended field theory. We remark that extended field theory is most developed for topological field theories, as in section 2.2, but one can also contemplate extended general quantum field theories as well. Thus for an n-dimensional Wick-rotated field theory \mathscr{F} , the value $\mathscr{F}(Y)$ on a closed Riemannian (n-1)-manifold Y is a (topological) vector space, the value $\mathscr{F}(Z)$ on a closed Riemannian (n-2)-manifold Z is a linear category (presumably with topology), etc. Suppose X is an n-dimensional Riemannian manifold—a Wick-rotated spacetime—and $W \subset X$ is a 1dimensional submanifold, thought of as the worldline of a particle. The link of $W \subset X$ is a sphere S^{n-2} . and defects supported on W are labeled by objects in the category $\mathscr{F}(S^{n-2})$. More precisely, we should take a limit as the radius of the linking sphere shrinks to zero. These ideas extend to submanifolds in X of lower codimension, to which the theory \mathscr{F} associates higher categories. Furthermore, one can extend to submanifolds with boundaries and corners and other singularities. One important aspect to understand better is the data on which a defect depends. As a very simple example, the Wilson lines in 3d Chern-Simons theory depend on a framing of the normal bundle. This can be seen quite clearly in the quantum correlators of the Wilson line defects. The analogous data for other defects in most contexts has not been properly investigated.

In some situations, e.g. with topological defects, or with supersymmetric defects one can start to discuss their operator products. From the extended field theory point of view, these products are based on the E_k -algebra structure of the sphere S^{k-1} in the bordism category. Here again, the framing data is not usually mentioned, but the OPE of Wilson line defects does depend on such data [309]. The products of line defects in Chern-Simons theory are closely related to the fusion rules of anyons (and the framing data detects their spin). OPE's of line defects have also played an important role in the geometric Langlands program [64]. Recently there have been some investigations into the "operator product expansions" of surface defects [116] and interfaces [310], finding interesting relations to homotopy algebra and infinite-dimensional quantum algebras. It seems clear that there is a rich mathematical structure to be discovered here.

In the recent literature there has been much discussion of "non-invertible defects" and their relations to "non-invertible symmetries." Important examples of such noninvertible defects are defects which

⁶Note added for v3: A physical explanation, and extension, of some of the results of Bonahan and Wong is given in [297].

implement Kramers-Wannier-like dualities in two and higher dimensions. For some recent progress and the first construction of non-invertible symmetries in d>3 dimensional QFTs see [187,311,312]. Many of these are related to so-called condensation defects, which can be understood as gauging of a (higher-form) symmetry on a defect, as opposed to the full theory, [188,189,313–316]. The symmetry structure that emerges from these non-invertible defects should naturally fit into the structure of higher fusion categories (for fusion 2-categories see [317]) and has become a very interactive area of exchange between mathematics and physics, see [187–189,191] for some recent progress on formulating symmetry structures in terms of higher-categories.

In lower dimensions the categorical structure was exploited successfully and numerous physically interesting results can be derived by utilizing non-invertible symmetries [116,183,185,190,193,318–323]. In d=4 some examples have emerged studying e.g. constraints on pion decays from non-invertible symmetries [324,325], and the characterization of vacuum structure of 4d $\mathcal{N}=1$ SYM [199].

Some of the techniques here are related to old ideas of "condensation of anyons" [326–329], a topic which itself is closely related to simple current algebras [330] and extensions of chiral algebras [331]. This "condensation" is reminiscent of tachyon condensations of D-branes used construct D-branes of various dimensions with nontrivial torsion K-theoretic RR charge [332–340]. It is an interesting open issue whether ideas related to tachyon condensation, K theory, and stable non-supersymmetric branes can be related to the modern work on defect/anyon condensation. Some evidence for this was recently provided in the construction of non-invertible topological defects using branes in string theory [199], where the fusion is replicated by brane-dynamics such as the Myers effect.

Another potentially useful set of ideas in the theory of defects goes back to [105,341]. The space of supersymmetric ground states of a supersymmetric field theory in finite volume admits an action of a pencil of commutative associative rings of local operators, commuting with some supercharge. In addition, in Euclidean space-time, one can also act on the space of ground states by the non-commutative algebra of supersymmetric interfaces. Interestingly enough, compositions of "raising" and "lowering" interfaces may produce local protected operators, in the cohomology of some supercharge. The whole structure resembles a Lie algebra, or its Yangian, or quantum, deformation, as seen in the context of gauge theories with (4,4) supersymmetry in two dimensions. For the recent progress see [310]. These developments are partly based on the mathematical discoveries in [342, 343]. The computations [306] of expectation values of supersymmetric defects, relating them to local operators in the cohomology of some supercharge, reveal fascinating connections to both geometric [344] and analytic Langlands program [345–347] and two dimensional conformal field theory, e.g. [348].

3 String Theory And M-Theory

3.1 What Is The Definition Of String Theory And M-Theory?

We don't know.

This is a fundamental question on which relatively little work is currently being done, presumably because nobody has any good new ideas. Nevertheless, we should never lose sight of the fact that it is one of the central unanswered problems in fundamental theoretical physics. M(atrix) theory [349–352] and AdS/CFT [353] are profound insights and give partial answers, but they are tied to specific backgrounds, and do not give satisfactory definitions of M-Theory/String Theory. See the Snowmass whitepaper [354] discussing applications of the bootstrap philosophy to derive facts about string theory and [355–360] for progress towards deriving string theory from field theory following the ideas of holography.

One notable attempt to give a fundamental formulation of string theory is the subject of string field theory. A solid foundation was given in [361, 362] building on important previous work such as [363]. Unfortunately, the theory, as currently understood, is intractable. Important progress was made, nevertheless, with exact results on tachyon condensation [364, 365], an action for open superstring field theory using A_{∞} structure [366] and most recently distinct progress on D-instanton amplitude

computations in [367–369], which would probably benefit from the more precise matching to the instanton counting developed in the field theory context [370, 371].

3.2 Topological String Theory

We will be brief in this section because much more discussion can be found in [18]. Topological string theory is both a metaphor of the physical string theory, and a substructure of it. One central reason for studying topological string theory is that if we are ever going to make progress on the question of "what is string theory," it is reasonable to expect that the much simplified version of the question "what is topological string theory?" will be a more tractable and a useful first case study. Topological string theory was invented by Witten in [372, 373]. Some basic material for this subject is covered in [374, 375].

In the effort to understand topological string theory much attention has focused on the topological string partition function. In the A-model this is defined, as a formal (possibly asymptotic) series by a collection of functions \mathcal{F}_g on Calabi-Yau moduli space, which are themselves generating functions for Gromov-Witten invariants of genus g curves. This topological string partition function has been the focus of intense research. The famous work of Bershadsky, Cecotti, Ooguri, and Vafa [376] almost provides a recursive definition. In the compact case there is a crucial "holomorphic ambiguity," which has only been fixed up to $g \leq 51$ in some of the most favorable cases [377]. The open topological string has the form of Chern-Simons theory, perhaps with the addition of some Wilson loop observables [378].

The computation of \mathcal{F}_g for all g has essentially been solved by using techniques adapted from the theory of matrix models for the B-model [379,380] and by using the topological vertex for the A-model in the case of non-compact Calabi-Yau manifolds [381]. The status of non-perturbative definition of the topological string theory is discussed in [18]. Another review of the relevant issues can be found in [250].

A suggestion to define nonperturbative topological string theory based on Fredholm determinants associated to "quantum Riemann surfaces" was made in [382-385]. Curiously, similar ideas have recently been applied in the context of nonperturbative definitions of JT gravity [386]. Some important progress in the theory of quantum curves and the relation to topological string theory was made in [278,387–389]. The relation between the quantization of a classical system based on a spectral curve, and the topological string, a B model on a local Calabi-Yau which is a complexification of a handlebody whose boundary is that curve, or an A model on a local Calabi-Yau mirror, involves a version of a blow-up formula and is discussed in [18]. The topological string partition function is closely related to BPS state counting for compactification of strings on Calabi-Yau manifolds. Various generating functions of BPS state multiplicities, or twisted Witten indices of supersymmetric backgrounds of string and M-theory can often can be mapped to the free-energy of some version of a topological string, i.e. genus zero Gromov-Witten prepotential accounts for the one-loop effects of the bound states of instantons and W-bosons in M-theory on a local Calabi-Yau manifold, [390], while subjecting it to the constant graviphoton background on \mathbb{R}^4 reveals all-genus topological string partition function [391, 392]. These results predict nontrivial integrality properties of the Gromov-Witten invariants (the mathematical definition of those via stable maps only guarantees their rationality), which can be proven mathematically, using, curiously, p-adic analysis [393]. For physics justification of these results

The case of compact Calabi-Yau threefolds seems much more difficult and there are no examples where we can compute all the F_g , let alone give a nonperturbative definition. One potential way forward is through an intriguing connection to BPS statecounting known as the OSV conjecture (named after Ooguri-Strominger-Vafa). If properly understood, it might give a road to a nonperturbative definition of topological string theory using the physics of supersymmetric black holes [395]. The most thorough attempt to formulate and prove a sharp version of the OSV conjecture was made in [396], but a completion of the project requires the solution of a number of unsolved problems addressed in the conclusion of [396]. Some mathematical progress on these issues has been achieved in [397, 398].

One of the problems one must solve to use an OSV-like formula is an effective way of computing BPS degeneracies for compact Calabi-Yau manifolds. One important outcome of [396, 399–401] is an interesting formula for BPS degeneracies based on "attractor flow trees." The formula of [396, 399–401] has been refined and clarified in [402–404]. Recently there has been some rigorous mathematical work confirming the physical conjectures [405, 406]. Another notable approach to computing BPS degeneracies associated to non-compact Calabi-Yau manifolds uses the mapping to associated quivers

and employing fixed point formulae to obtain explicit results [407–411]. Other recent progress on the BPS spectrum for string compactification on non-compact Calabi-Yau manifolds can be found in [412–414]. A technique known as "exponential spectral networks" has been devised which might lead to a systematic way of computing BPS degeneracies on non-compact Calabi-Yau manifolds [415–418]. In spite of all this, finding the full BPS spectrum of the compactification on any compact Calabi-Yau other than (an orbifold of) a torus remains an important open problem.

3.3 String Perturbation Theory

Superstring perturbation theory remains a fundamentally important subject, central to the claims that string theory provides a truly UV finite theory of quantum gravity. In [419,420] Witten revisited, summarized, and clarified a great deal of work that had been done on superstring perturbation theory in the 1980's. In particular [421] demonstrated conclusively the important fact that supermoduli space is not split. Recent investigations [422,423] have raised the possibility of unexpected divergences in the type II string theory. These appear to be unphysical and it is important to clear up this point. It is important to incorporate B-fields in the consideration of summing over spin structure, perhaps using the methods of [424–426].

Most of the work on high-genus superstring perturbation theory has been done in the context of the "RNS string formulation." There is widespread belief that this formulation is not optimal - for example spacetime supersymmetry is not manifest (but it is there [427]). Related to this, the incorporation of nontrivial RR backgrounds, crucial to most forms of the AdS/CFT correspondence, is nontrivial. Subsequently, much work and progress has been made on alternative worldsheet formulations. See [428] for the state of the art on such alternative formulations, and [18] for further remarks. This remains an important research direction for the future.

In the context of the AdS/CFT correspondence, important advances using worldsheet perturbation techniques have been made, opening up a host of interesting future directions for research, and making contact with other current trends in thinking about quantum gravity. Some remarkable new results on the $SL(2,\mathbb{R})$ WZNW model have recently been achieved [359, 429, 430] pointing to a number of interesting directions in the application of string perturbation theory to the AdS/CFT correspondence.

3.4 LEET

A very important aspect of string compactification is the construction of low energy effective actions. From the mathematical viewpoint some of the main interest here are relations to hyperkähler and quaternionic kähler geometry. Some aspects of this are discussed in section 8 below, and other aspects, related to the theory of automorphic forms are discussed in section 6 below.

3.5 Noncommutative Spacetime

String theory clearly calls for some kind of emergent spacetime based on an algebraic structure. One time-honored idea is that spacetime is a noncommutative manifold, an idea which can be made fairly concrete in the context of string theory [431–434], following the earlier ideas [435–437]. One of the precursors of the interest to the realization of noncommutative geometry in string theory is the identification of the moduli space of torsion free sheaves with that of instantons on a noncommutative space [438]. The idea of an emergent spacetime from a more algebraic structure is part of the framework of both M(atrix) theory [349] and AdS/CFT [353]. A simple concrete baby example can be given in the case of 2d open-closed TQFT, which may be viewed as a string theory with 0-dimensional target space. From a semisimple open string Frobenius algebra one may recover the spacetime, along with the entire category of branes [439]. In the closely related subject of noncommutative field theory [431] many open problems remain. The extension of noncommutative field theory to include nontrivial background NS flux (background gerbe connection with nonzero curvature) is a very interesting problem which might involve operator algebra theory at a deeper level than thus far has been employed. One early attempt is the paper [334]. This subject has been somewhat neglected in recent years but there are surely many treasures left to uncover.

4 Anomalies

The subject of anomaly cancellation blossomed in the early-to-mid 1980's, in part driven by important advances in understanding the formal structure of perturbative anomalies [440] and by the renewed interest in string theory and higher dimensional supergravity theories. The main idea was to use anomalies as a systematic way to distinguish consistent from inconsistent field theories and supergravity theories [441, 442]. This period was crowned with the discovery of the renowned Green-Schwarz anomaly cancellation mechanism in 10-dimensional string theories [443].

4.1 Anomaly Cancellation In String Compactification

Much work has been done on anomaly cancellation in string theory and M-theory but only recently has a full consistency check of global anomaly cancellation in smooth M-theory backgrounds, not necessarily orientable (but endowed with some other tangential structures), been performed in [444], building on much previous work [445–449]. In [444] it is observed that there are two distinct trivializations of the anomaly, and it remains a puzzle to decide if one is more physically relevant than the other. The generalization of these results to include the presence of branes and singularities remains to be done. Many issues remain from the examination of "frozen singularities" in M-theory [450] and it would seem the time is ripe for a re-examination using the tools of equivariant differential cohomology, now that differential cohomology is becoming a more familiar tool for physicists.

In string theory, a full discussion of anomaly cancellation on the worldsheet in the presence of orientifolds remains to be done, and similarly a comprehensive discussion of global anomaly cancellation in the presence of branes and orientifolds has not been done. (Some significant unpublished work can be found in [451].)

In general, there is no proof that global anomalies are canceled in compactifications of string theory. For example, this is fairly nontrivial in compactifications of F-theory to 6d [452-457]. Some recent very intriguing work on anomaly cancellation in heterotic string theory can be found in [226-228]. These latter papers are particularly notable for their heavy use of the Stolz-Teichner conjecture relating the topology of the space of (0,1) supersymmetric 2d QFTs to topological modular forms.

Finally, an old and vexing issue, closely related to anomalies, is that of the proper quantization conditions for the various p-form gauge fields in M-theory and string theory. In the large distance weakcoupling limit it would appear that one proper model of the M-theory C-field is based on differential cohomology whose underlying generalized cohomology theory is singular cohomology [447]. On the other hand, in type II string theory the RR fields should be based on differential K-theory [163, 244, 339, 458–460]. The consistency of these two viewpoints is only partially understood [446, 461], and appears to rely on remarkable and subtle topological facts. As emphasized in [447] the M-theory Chern-Simons term is really a <u>cubic</u> refinement of the triple intersection product on differential H^4 and a full study of the implications of summing over torsion fluxes for this cubic refinement has not been carried out. (Some recent work on the cubic form in this context has appeared in [444, 462].) Moreover the compatibility of the use of twisted differential cohomology in IIB string theory with the expected S-duality has been a long-standing puzzle [446]. These old puzzles have recently been revived in a provocative paper [463] investigating duality symmetry anomalies in type IIB string theory; see also [464–469] for earlier work on duality anomalies in Type IIB. One possible way forward in addressing this old puzzle concerns reconciling the K-theoretic classification of RR charges in orientifolds [470] with Witten's discussion of S-duality for the baryon vertex in AdS/CFT [471].

4.1.1 Global Anomalies In 6d Supergravity Theories

It has long been recognized that not all LEETs are UV complete, or even internally consistent. This idea is at the heart of the 't Hooft anomaly matching conditions and the use of anomalies for separating "good" theories from "bad" theories. In modern parlance these conditions are related to "landscape" and "swampland" conjectures, and whether well-defined low energy theories with gravity can be derived from consistent string theory and/or M-theory compactification. One well-defined program for using anomalies to shed light on such issues is reviewed in [453]. Considerations of global anomalies lead to further constraints which have been explored in [246,455–457] but those papers leave some unresolved issues. For example, the discussion of [456,457] reduced global anomaly cancellation to the triviality of a certain 7-dimensional spin topological field theory. Roughly speaking, this theory is the difference

between the invertible anomaly theory of the 6d supergravity fields and the invertible theory where the Green-Schwarz counterterm is valued. It would be good to have an effective way to compute this theory given the defining data of the supergravity. Moreover, [456,457] identified new theta angles and torsion anomaly coefficients whose physical role has not been fully understood. Finally, it would be good to clarify which is the proper generalized cohomology theory underlying the differential cohomology used to formulate the self-dual fields in the context of 6d supergravity. For example, it is natural to wonder, given the experience from Appendix B of [460] whether the cleanest story would not come from using differential KO theory. This idea has not been seriously explored yet. It seems fair to say that a full and complete discussion of global anomaly cancellation in 6d supergravity remains to be given, although this is not a view shared by all experts in the field. Indeed, even in 8d supergravity important subtleties arise as discussed in [472].

4.2 Anomalies And Invertible Field Theories

The geometrical interpretation of anomalies originated in the mid 1980's as mathematicians such as Atiyah, Bismut, Freed, Quillen, and Singer formulated a theory of anomaly cancellation in terms of determinant line bundles. This has evolved into a modern interpretation [50, 473] using the concept of invertible field theories [448]. This modern interpretation is a generalization of the anomaly-inflow mechanism that goes back to [474, 475].

Briefly, field theories have an associative composition law with unit. Namely, two field theories can be "stacked" or tensored with no interaction. If $\mathscr{F}_1, \mathscr{F}_2$ are field theories, then the state space of $\mathscr{F}_1 \otimes \mathscr{F}_2$ on a spatial manifold Y is the tensor product $\mathscr{F}_1(Y) \otimes \mathscr{F}_2(Y)$. Correlation functions and evolution operators are similar tensor products. The unit theory 1 has 1-dimensional state space $\mathbb C$ on all spatial manifolds, and all correlation functions equal 1. A field theory α is invertible if there exists α' such that $\alpha \otimes \alpha'$ is isomorphic to 1. It follows that all state spaces of α are 1-dimensional, but they do not come with a basis element. Perhaps at first surprisingly, such theories may be nontrivial and furthermore have applications in physics beyond anomalies. For example, they capture short range entangled (SPT) phases in condensed matter theory, if one is willing to jump from discrete models to effective field theories. Invertibility takes us full circle from the bordism categories [476] of the Segal [26] and Atiyah [27] axiom systems for field theory to the classical bordism of Thom [477] and a modern variant [478]; see [50, 479]. Invertible theories are not necessarily topological. A good example of nontopological invertible field theories are those defined by an exponentiated eta invariant. These are related to parity anomalies [480–483] and are known as Dai-Freed theories. For two recent reviews summarizing the status of these theories see [484,485]. The general theory of nontopological invertible theories is under rapid development; a small sample of very recent papers is [486, 487].

This viewpoint has been a very flexible way of understanding anomalies but it is not clear that it extends to all anomalies, such as duality symmetry anomalies or conformal anomalies, so there is more to understand here. (For example, the conformal anomaly involves the Euler characteristic. It is not clear how to obtain this from the usual descent formalism two dimensions higher.) Moreover, as mentioned above there are new ideas involving extensions of the notion of symmetry to so-called "categorical symmetry", and these "symmetries" are also expected to have anomalies. This subject is in its infancy.

4.3 Anomalies And Dynamics

As first stressed by 't Hooft [488], anomalies provide powerful constraints on RG flow. These ideas were used to great advantage in advances in understanding the dynamics of supersymmetric gauge theories in the mid-1990's [489]. In the past few years they have been combined with the new ideas based on invertible topological field theories and generalized symmetries to discover new constraints on nontrivial RG flows in quantum field theories, see e.g. [168–176].

5 Mathematics Resulting From Holography And Quantum Gravity

Sometimes general conjectures about quantum gravity, and the role of string theory in quantum gravity, can motivate the discovery of new mathematical results. In this section we focus on some of these

5.1 General Quantum Gravity Conjectures With Precise Mathematical Consequences

The past few years have seen some heightened activity in deriving general properties of quantum gravity by studying generic properties of string compactifications and their global consistency conditions. Much of this originates in the weak gravity conjecture. Here we will focus on the connection to mathematics and refer to the reviews on this topic for a broader overview [453, 490–493]. In particular, we will focus on consequences in algebraic and differential geometry, especially in relation to moduli spaces of manifolds of special holonomy.

Sometimes the constrained structure of supergravity Lagrangians can lead to remarkable mathematical predictions. The reasoning is that the existence of a suitably covariantly constant spinor on a compact manifold implies a certain amount of unbroken supersymmetry when superstring theory or supergravity is compactified on that manifold. But then the scalar fields in the supergravity are related to moduli of the compactifying manifold. The constraints on how these scalar fields can appear in a supergravity Lagrangian have direct mathematical implications for the moduli space. The phenomenon of mirror symmetry was in fact anticipated by exactly this kind of reasoning [494–496]. Another nice example of this reasoning is [497] where the constraints of the LEET coming from $\mathcal{N}=4$ supergravity predicted the form of the moduli space of K3 surfaces with complexified Kähler class to be a double coset of O(4, 20), a fact later confirmed by precise mathematical analysis [498, 499].

A more speculative aspect of quantum gravity concerns aspects of the "landscape" of possible string compactifications. One might ask what principles should prefer particular vacua, especially when they come in continuous families. One approach to this question is to posit that there should be a well-defined probability distribution on a suitable set of string vacua. For perturbative string compactifications there is a natural metric on the moduli spaces of string vacua - the Zamolodchikov metric [213]. It is tempting to use this metric to define a natural measure on string compactifications, but that will only make sense if the total volume is finite, a point first emphasized in [500]. In this way one is led to conjecture that, for example, the total volume of moduli spaces of Calabi-Yau three-folds is finite [500]. For moduli spaces with a fixed topological type the Zamolodchikov metric on the complex structure moduli space is the Weil-Peterson metric. Thus, the general quantum gravity conjecture leads to a very precise mathematical prediction. For moduli spaces with a fixed topological type this conjecture was shown to be correct in [501]. Analogous statements for, say, G_2 compactifications of M-theory remains interesting and open.

There are a number of general conjectures in quantum gravity which have generated a huge amount of interest in the past several years. Again, we refer to [490-493] for reviews. Some of these conjectures, when made suitably precise, can lead to impressive mathematical predictions. One example is the so-called "swampland distance conjecture" (SDC) of [502]. In rather general terms the conjecture states the following: A string compactification can have a moduli space M, which is parametrized by massless scalar fields. This moduli space can have a natural metric on it, and two points in M can be separated by an infinite distance. The conjecture states, that an infinite tower of states become exponentially light, when moving an infinite distance in the moduli space. More intuitively the conjecture can be rephrased as stating that an infinite distance limit either corresponds to a decompactification, or to a tensionless limit of a weakly-coupled string (which in turn furnishes the infinite tower of massless states). The latter refinement is sometimes called the emergent string conjecture (ESC) [503].

When the SDC is made mathematically precise in the context of Calabi-Yau compactification some very nontrivial mathematics emerges. For example tests of the conjecture using the spectrum of D-branes as a function of the complex structure moduli space makes detailed and precise use of the Schmid Nilpotent Orbit Theorem and the theory of degenerations of Hodge structures [504–508]. A key role in this analysis is that infinite distance in the Zamolodchikov-Weil-Peterson metric is characterized by infinite order monodromy of the periods. The ESC has motivated refined studies of the ways K3 surfaces can degenerate [509]. There is room to run in this subject: Mathematically the challenge is to find a characterization of the behavior at infinite distance limits for general Calabi-Yau three-fold geometries, as well as elliptically fibered Calabi-Yau manifolds of other complex dimensions.

A much harder version of these questions concern the mathematical predictions following from precise versions of the SDC in the context of other non-Calabi-Yau compactifications, e.g. for com-

pactifications on manifolds of exceptional holonomy. Some preliminary results for a special class of G_2 manifolds has appeared in [510]. Constructions of such exceptional holonomy manifolds are central to the Simons Collaboration on Special Holonomy [511], and will be discussed in section 9.2.

Some extensions of tests of the SDC to AdS compactifications have been studied in [512–514]. The AdS-distance conjecture implies in particular that in an AdS $\times X$ compactification of string theory, the AdS scale cannot be separated from the scale of the internal manifold X. In turn this implies bounds on the conformal dimension of the first non-trivial operator in the dual CFT. Mathematically this problem can be rephrased as implying a bound on the eigenvalue of the (scalar) Laplacian on X. Recently this was proven in [515] for a large class of holographic models where X is a Sasaki-Einstein five-manifold, and a conjecture was put forward, that such a bound should exist and be universal (only dimension dependent).

In the past year there has been exciting progress in the application of the theory of von Neumann algebras to questions about the entropy of black holes and de Sitter space [516–522]. Notably some nontrivial constructions in operator algebra theory relating type II and type III von Neumann algebras turn out to have natural physical interpretations in terms of black holes. It seems clear that this direction will lead to some interesting future progress

A completely different approach to landscape and swampland issues has been initiated in [523,524] and this is a potentially promising direction.

5.2 Holography

For well over 25 years the ideas of holography have been of central importance to the string theory and quantum gravity communities. Yet, in relation to its immense importance and conceptual depth, the direct connections between holography and pure mathematics have been relatively meager⁷. That situation has started to change.

A very significant advance in the theory of holography was recently made in [525–527], where the path integral of a form of two-dimensional quantum gravity was related to both matrix models and the mathematical work of M. Mirzakhani. (For some background on the relation to topological gravity see [528].) This work, combined with some older results known as the "factorization problem in AdS/CFT" [529,530], has in turn raised some key conceptual questions about the AdS/CFT correspondence, such as whether topological effects in quantum gravity (e.g. wormholes) require one to "average" over boundary quantum field theories. As mentioned above, moduli spaces of string theories come with a natural Zamolodchikov measure (provided the volume is finite), and the concept of Zamolodchikov measure can be extended to moduli spaces of superconformal field theories (provided the volume is finite). When this is done for conformal field theories of free bosons on a torus (a.k.a. Narain theories) the resulting ensembles bear some resemblance to three-dimensional Chern-Simons gravity [531, 532]. These recent papers have intriguing results, but puzzles remain [533] and it would be good to clarify the status of these theories and to understand them better. (As an example of one puzzle, it would be nice to clarify whether it is consistent, in a 3d theory of gravity, to sum over topologies which are just handlebodies.) A recent proposal for solving the ensemble-average puzzle has in fact led to new results in mathematics proper concerning (regularized) volumes of hyperbolic manifolds [534].

A related set of observations concern certain natural ensembles of two-dimensional (4,4) super-symmetric sigma models [536,537]. The main result here is that these can only have a weak-coupling gravity dual on a set of Zamolodchikov measure zero. Interestingly the papers [531,532] and [536,537] made nontrivial use of the work of C.L. Siegel in analytic number theory. It would be very interesting to understand if there are deeper relations between averaged QFT and analytic number theory. For example, it would be natural to expect that there are connections to Kudla-Millson theory [538]. Currently other ensemble averages of higher dimensional superconformal theories are under investigation with a view to possible holographic interpretations [539]. If the ensembles have string-theoretic interpretations as spaces of vacua one might expect the total Zamolodchikov volume again to be finite, but this remains to be seen in all but the simplest examples. This line of development might be a promising avenue for interesting new interactions between analytic number theory and QFT/ST.

 $^{^{7}}$ Connections between solutions to supergravity theories, and the geometry of Sasaki-Einstein manifolds has of course led to very important developments in mathematics, see section 9

⁸Another, very different, approach to these puzzles makes use of the notion of "non-invertible symmetries" mentioned above [535].

A curious related recent development, inspired by questions in quantum gravity, including the ensemble averaging question, are the "topological models of baby universes," investigated in [540–544]. These involve the well-defined problem of summing over bordisms with fixed boundaries in a topological field theory. The model of [544] was interpreted as supporting the view that holographic duality involves ensemble averages, but other interpretations are possible [541]. As far as we know there is no previous mathematical work in this direction and it presents a potentially interesting opportunity for some interactions between physicists and mathematicians. One important question is whether there are any analogs of these models in d > 2 dimensions. Naive generalizations will not work. It has been suggested by M. Kontsevich that it might be important to consider nonstandard ways of weighting the sum over topologies using the results of [545].

6 Interactions With Number Theory

There are several intriguing connections between string theory and supersymmetric QFT and number theory. Many potentially fertile seeds been planted, but have not yet blossomed, perhaps solely due to lack of cultivation. We survey a few of these developments here. There are a number of other intriguing relations to number theory that have arisen in the study of mathematical aspects of scattering amplitudes. These have been covered in a separate Snowmass document [546].

6.1 Automorphic Forms And Partition Functions

One of the primary connections to number theory is through automorphic functions and analytic number theory. An important historical example is the explanation of modularity in Monstrous Moonshine Conjectures via the relation of the Monster group to an orbifold conformal field theory [79,547–549] ⁹.

Partition functions - understood in very broad terms - can be automorphic for three (interrelated) reasons: They can be covariant under diffeomorphism symmetries, they can be covariant under duality symmetries such as S-duality, or they can be terms in low energy effective actions, whose duality symmetry implies automorphy of the functions entering the effective action. Often, but not always, these partition functions are associated with generating functions of invariants of BPS states. We will just list here a small set of examples of this phenomenon. It is not possible to give a complete list because the literature is quite vast.

- 1. Two dimensional CFT, especially RCFT, is a rich source of automorphic functions thanks to modular (or mapping class group) covariance of partition functions, conformal blocks, and correlation functions. These considerations continue to play an important role in the modern era. For example it is an important tool in investigations of spectral aspects of conformal field theory, where many of the questions are motivated by aspects of the AdS/CFT correspondence [551–554]. (A related set of ideas applies the conformal bootstrap to four-point functions and this has led to interesting results on the spectrum of the Laplacian on hyperbolic surfaces [555]. Note that the bootstrap equations follow from the mapping class group symmetry of the four-punctured sphere and are thus a form of modular invariance.) In a related set of developments modularity plays an important role in discussions of "extremal CFTs", i.e. CFTs with small numbers of low lying states [556-558]. Several important questions concerning existence of $\mathcal{N}=0$ extremal CFTs remain open and it would be extremely interesting to clarify them [557–562]. Higher dimensional theories involving general p-forms will illustrate similar phenomena. The higher dimensional generalizations are not well-studied. To choose but one example: the conformal blocks of self-dual six-dimensional theories are expected to be interesting automorphic objects for groups of disconnected diffeomorphisms of six-manifolds.
- 2. Terms in the action for the LEET of string compactifications with duality symmetries provide a rich source of examples. For example toroidal compactification of type II string theory naturally leads to Eisenstein series. For a review see [563]. If we combine this observation with the S-duality of type IIB string theory, then, in the hands of [564–567] nontrivial observations about some of the automorphic forms appearing in the Langlands program, and conjectures about

⁹The role of various kinds of Moonshine in physical mathematics is an important and rich subject in and of itself. We will not pursue it here because it is being addressed in a separate Snowmass document in [550].

constant terms, can be addressed. The book [568] has an extensive review of the relation of string theory scattering amplitudes and automorphic functions. These developments have also led to the development of the whole subject of modular graph functions (which are related to the integrands of expressions giving terms in the LEET). This is an active area of development [569–574]. The paper [569] points out an interesting connection to the work of Faltings and the Kawazumi-Zhang invariant.

- 3. Another notable use of the automorphy of the LEET is the T-duality symmetry (via heterotic/typeII duality) of the prepotential of type II theories compactified on K3-fibered Calabi-Yau manifolds. Study of these prepotentials led to a simple proof of the Borcherds lifts of automorphic forms [575] in what is a nice example of the math-physics dialogue. The work of [575] applied some standard techniques from string perturbation theory [576] in a fertile setting that explained the remarkable phenomenon of Borcherds lifting in a conceptual way. The method was generalized to the construction of automorphic forms for SO(p,q) in [577]. The resulting theory of singular theta lifts, and the related results of [578] have led to several ingenious extensions and generalizations in both the math and physics literature [569, 579–588], and have also been of use in physics-inspired investigation of four-manifold invariants, as discussed elsewhere in this paper.
- 4. The terms in the LEET mentioned above are often generating functions for invariants of BPS states. The automorphy of such BPS statecounting functions played extremely important roles in establishing fundamental string dualities [102,589–594] and in accounting for microstates of black holes [595]. The breakthrough paper [595] has led to a vast literature on BPS statecounting and automorphy, which continues to inspire and direct research to this day. Among the many modern issues resulting from BPS state counting the question of the automorphic nature of the generating functions of certain DT invariants of torsion sheaves on compact Calabi-Yau manifolds ("D4-D2-D0 counting functions") remains an important open one. Physical considerations suggest that the 2d (0, 2) elliptic genus associated with 5-branes should exhibit interesting automorphic properties [395,396,414,596–599]. Mathematically rigorous confirmation of some of these physical predictions have been made in the last several years and will probably be the source of interesting mathematical research into the future [600–606]. (For related, but slightly different appearances of modular forms in mathematically rigorous investigations of enumerative geometry of Calabi-Yau manifolds see [579,607,608].)
- 5. Another example of an important BPS counting function is the elliptic genus of $\mathcal{N}=2$ theories. In the context of AdS/CFT this led to the application of Rademacher expansions [609–614] ¹⁰. That in turn has led to important results in 3d quantum gravity [618] raising questions regarding unitarity, and the existence of a Hilbert space interpretation in 3d gravity, that have yet to be resolved. It also motivated some important developments in Moonshine. For example [619] observed an important connection between Rademacher summability and the renowned genus zero property of the Moonshine groups. (The relation between Rademacher summability and genus zero goes back at least to [616] and references therein.) In the past several years there have been extremely beautiful proofs of such "Fareytail expansions" of BPS counting functions via localization in the supergravity path integral, and the development of some of these localization tools have illuminated interesting issues in strongly coupled four-dimensional QFTs [620–624].
- 6. Partition functions of topologically twisted field theories are a natural source of automorphic functions thanks to nontrivial duality symmetries such as S-duality. A famous and paradigmatic example are the partition functions of Vafa-Witten twisted $\mathcal{N}=4$ SYM [102]. Extensions to higher rank go back to [625] and have been discussed in [599, 626–628]. The verification and extension of the physical predictions of have proven to be a rich source of inspirations in enumerative algebraic geometry and have led to important advances such as [629–632]. The partition functions of other S-dual symmetric quantum field theories will lead to similar counting functions for other enumerative invariants. See, for example, [633].

¹⁰The mathematics here is closely related to a mathematical subject known as Eichler cohomology [615–617].

6.2 Generalizations Of Automorphy

Mathematicians and physicists have been exploring various extensions of the notion of automorphic forms in the past few years. A number of promising directions have emerged that might well lead to important results in the future.

One of the most striking generalizations of modularity are the mock modular forms. See [634] for a recent review of some of the ways they have appeared in physics. A precise mathematical definition was codified in [635] but in fact many examples have been known since the work of Ramanujan, and, somewhat later, in the work of D. Zagier [636, 637]. Mock modularity has been appearing in many distinct ways in the literature on physical mathematics, and we will indicate some of the occurrences below. It would be desirable to understand if there is a common concept underpinning these various instances of mock modularity in physics. Perhaps related to this, modularity has a beautiful representation-theoretic interpretation in terms of $SL(2,\mathbb{R})$. It would be very desirable to have an analogous representation-theoretic basis for mock modularity, but such an interpretation is at present lacking.

Mock modular forms showed up in the study of automorphic properties and holomorphic anomalies of twisted supersymmetric gauge theory partition functions on four-manifolds [102,638]. More recently the appearance of mock modularity in this context has been understood more systematically in [633, 639–642] and indeed the study of some examples lead to open questions about mock modular forms [633].

Mock modularity also shows up naturally in the study of the Fareytail expansions of elliptic genera and holomorphic anomalies in the context of AdS/CFT [610]. Another, related, source is the appearance of these forms in BPS state counting functions for $\mathcal{N}=4$, d=4 supersymmetric black holes. This has been extensively investigated in the major work [643]. Similar considerations are expected to occur in $\mathcal{N}=2$, d=4 examples, but this has been much less investigated and is a natural direction for future research. For some relevant results see [414, 599, 644–646]. Another appearance of mock-modularity can be found in the effects of D3 instantons on low energy effective actions [647, 648].

Yet another source of mock modularity is non-compactness of the target space in a sigma model [649–657]. Recently a beautiful physical derivation of the APS index theorem based on path integrals made use of this phenomenon [658]. It should be possible to understand, conceptually, the appearance of mock modular objects in noncompact sigma models by applying similar considerations to the Dirac operator on loop space, to get an "APS theorem" for such Dirac operators, but this has not been done. The role of mock modularity in noncompact sigma models has recently played an unexpected and important role in the connection of topological modular forms to the classification of supersymmetric field theories [225].

Mock modular forms can be generalized to "mock modular forms of finite depth." A mock modular form - very roughly speaking - is a nonholomorphic function $f(\tau,\bar{\tau})$ on the upper half plane which has the property that $\bar{\partial}f$ is simply related to a holomorphic modular form. One can then use this idea to define iteratively a series of generalizations where a mock modular form of depth d is related, in an analogous fashion, to a mock modular form of depth (d-1), where holomorphic forms are of depth 0. Such functions have appeared in partition functions of twisted higher rank supersymmetric theories as well as in D-instanton expansions. They have been studied mathematically in, for examples, [659–661], which also make use of various generalizations of the error function. These ideas have been applied to the counting functions associated with M-theory black holes in [646] and to D-brane BPS counting functions in [662]. The paper [662] raises some unresolved puzzles related to the scaling BPS black hole solutions of [396]. (In general, the scaling solutions tend to present puzzles [396,410,411,663] and deserve to be understood better. A general condition for their existence can be found in [664].)

Another direction in which the notion of automorphy is being extended is to the domain of "quantum modular forms" and "false theta functions." At present, the precise definition "quantum modular form" by D. Zagier leaves open the possibility of several variations [665, 666]. It is possible that physical applications will guide the way to the "best" definition. One important recent source of quantum modular forms is via 3-manifold invariants associated to the partition functions of d=3 $\mathcal{N}=2$ field theories on plumbed 3-manifolds. These " \hat{Z} -invariants" have illustrated curious relations to mock modular forms and false theta functions [667–669]. A great deal needs to be clarified concerning these and related intriguing observations. Another source of "quantum modularity" in physics appears through the presence of MacMahon, and MacMahon-like functions in black hole microstate counting [412,670–672].

Yet another context in which modularity and its generalization play a role is the realm of supersymmetric partition functions/indices in various dimensions [673]. Computing such indices one can obtain explicit expressions which often exhibit interesting (mock/quasi/etc) modular behavior and its generalizations. For example, in the supersymmetric index in four dimensions [674–676] one can see hints of $SL(3,\mathbb{Z})$ modularity [677–681]. Specializing to $\mathcal{N}=2$ theories and to Schur indices [74,682] these hints of $SL(3,\mathbb{Z})$ modularity become hints of $SL(2,\mathbb{Z})$ modularity [683]. Some of these relations have been intensely studied: e.g. in the context of the relation of the index to chiral algebras [85] the Schur indices can be thought of as components of a vector valued modular form and are solutions to certain logarithmic modular differential equations [87]. There are also intriguing connections between these indices and the work of Kontsevich and Soibelman on wall crossing phenomena [684–687]. Moreover as the Schur indices are often given as integrals over Jacobi modular forms they can exhibit quasi and mock modular properties [688, 689] which would be very interesting to explore in further.

6.3 Geometric Langlands Program

The famous Langlands program in mathematics relates - very roughly speaking - arithmetic questions related to Galois groups and motives on the one hand and automorphic function theory (and harmonic analysis) associated with number fields, on the other. The Langlands program can be viewed as a generalization of class field theory (and is sometimes described as "non-abelian class field theory"). Some of the most important advances in number theory can be interpreted within the framework of this program, which, moreover, promises to unify many areas within number theory and perhaps even resolve some of the longstanding and outstanding questions within that field. When one replaces number fields by function fields of curves the questions can be stated more geometrically, and this has led mathematicians to formulate an analogous program, the renowned Geometric Langlands Program (GLP). When the curve C in question is over \mathbb{C} the GLP, in particular, connects [690, 691] the holomorphic quantization of Hitchin's moduli space $\mathcal{M}_H(C,G)$ for gauge group G and the classical holomorphic symplectic geometry of the moduli space of ^LG-local systems (flat connections, for physicists), see [692]. Moreover, Hitchin's space plays a role in Ngo's proof of the "fundamental lemma of the Langlands program" thus leading to one route from the modern work on the GLP back to the original Langlands program. Other ideas have been inspired by the connection to physics and have led to progress purely within the mathematical activity in this area. See [344,693–695] for introductory reviews on the Langlands program and its geometric cousin.

It was understood early on that the GLP is intimately connected to two dimensional conformal field theory, more specifically to the theory of chiral algebras. This follows from the fundamental result of [696] realizing the center $Z(\widehat{\mathfrak{g}})$ of the universal enveloping algebra of a current algebra $\widehat{\mathfrak{g}}$ associated to a simple Lie algebra \mathfrak{g} at the critical level $k=-h^{\vee}$, as a limit of the associated W-algebra. In turn, W-algebras appear naturally in 2d CFT [697]. The elements of the center become the differential operators on $Bun_G(C)$ twisted by the square root of its canonical bundle $(\mathcal{M}_H(C,G))$ is a hyperkähler manifold, which in one of its complex structures is birational to $T^*Bun_G(C)$). There was a little setback, because unitarity of conformal field theories with current algebra symmetry requires k > 0. Despite this, other hints connecting GLP and physics were showing up, as described in [344]. In [698] a relation was found between the Bethe ansatz, Sklyanin separation of variables, and the Beilinson-Drinfeld view of GLP adapted to the case of genus zero curves with punctures, at least in the case $\mathfrak{g} = \mathfrak{sl}_2$. In [699] the separation of variables was interpreted in the language of D-branes wrapping various cycles in a hyperkähler manifold, as an example of mirror symmetry or T-duality. An example of a quantum integrable system associated with a genus one curve E, an elliptic Calogero-Moser (eCM) system of A_{N-1} -type, was studied in [700], while the classical construction was given in [701]. The full set of quantum integrals of motion of A_{N-1} eCM can be identified with the \mathcal{D} -module part of GLP adapted to the case of a genus one curve with one so-called minimal puncture. The dual side is represented by the variety L of opers, which in the present case are the order-N differential operators on the elliptic curve with a regular singularity at the puncture. The symbol of an oper is a meromorphic function on T^*E , which is a degree N polynomial along the cotangent directions, whose coefficients are meromorphic functions on E of specified poles at one point z=0 at E. The vanishing locus is the spectral curve Σ .

A year later this algebraic integrable system, in the very guise of [701], was proposed [702] as a candidate description of the Seiberg-Witten geometry of the four dimensional $\mathcal{N}=2^*$ with the gauge group SU(N). The elliptic curve determines the gauge coupling of the ultraviolet theory (which has

 $\mathcal{N}=4$ supersymmetry), on the one hand, the coupling constant is related in [701] to the residue of the Higgs field at the puncture, on the one hand, and to the mass of the adjoint hypermultiplet, on the other. Finally, the abovementioned spectral curve Σ is the Seiberg-Witten curve of the effective $\mathcal{N}=2^*$ theory.

Going back with the GLP story, the opers are, sometimes, called the quantum Seiberg-Witten curves. We will get to their meaning for the $\mathcal{N}=2^*$ theory (or its higher genus analogues) later. In general it is hard to understand the physics of the variety of opers within the realm of the conventional two dimensional conformal field theory with $\widehat{\mathfrak{g}}$ -symmetry with a notable exception of Liouville (or Toda) field theory, where the so-called BPZ differential equations, which are obeyed by the conformal blocks of correlators involving the so-called degenerate fields, do approach the opers on genus zero curves with punctures, in the $b \to 0$ or $b \to \infty$ limits (the large central charge, or quasiclassical, limit of Liouville), see [703] for review. The subsequent progress came from the physical realization of $\mathcal{M}_H(C,G)$ as an effective target space of a two dimensional sigma model. This is to be contrasted with the way the moduli space of (compact group) flat connections are related to the two dimensional conformal field theory on that same Riemann surface C. Given Riemann surfaces D and C, a twisted version of $\mathcal{N}=4$ super-Yang-Mills theory on $D\times C$, in the limit where C is much smaller than D becomes a sigma model with D as a worldsheet and $\mathcal{M}_H(C,G)$ as a target. Moreover, the Montonen-Olive S-duality of the $\mathcal{N}=4$ super-Yang-Mills is then related to T-duality in the sigma model [704, 705]. Then, some ten years later, a few more crucial ingredients became available: the generalized complex structures, the extension of the Fukaya category to hyperkähler manifolds [706], the identification of Hecke operators and Hecke eignesheaves of the GLP in terms of the actions of 't Hooft-Wilson lines on branes, and special branes on Hitchin moduli space, respectively. Thus, the connection of $\mathcal{N}=4$ super-Yang-Mills theory to GLP started to emerge [64] in a powerful way. In addition to the strong-weak coupling duality symmetries of supersymmetric gauge theories and sigma models, the GLP was thus connected to various geometric symmetries [64,703,707-713]. This led to important questions of the classification of supersymmetric boundary conditions of $\mathcal{N}=4$ SYM [714–716], which, in turn, plays an important role in the gauge-theoretic interpretation of knot homology, as described above. These insights also gave a large impetus to the development of the theory of supersymmetric defects [302], related to the ramified version of the GLP [303, 305, 708, 709]. In particular, the geometric Satake theorem has a natural interpretation in terms of extended observables, i.e. line operators [717–721] and surface defects. More generally, the ideas of the GLP, and its gauge-theoretic interpretation have played an important role in the development of geometric representation theory [722] of both finitedimensional and infinite-dimensional groups (such as loop groups). Also, thinking about the kind of quantization of $\mathcal{M}_H(C,G)$ involved in the GLP raised interesting questions in the general theory of quantization [270–272, 293–295, 345, 723, 724].

The insight into the GLP which follows from assuming the S-duality of the $\mathcal{N}=4$ super-Yang-Mills theory can be further strengthened (sometimes leading to the actual proofs) using the input from string theory. In particular, the realization of the $\mathcal{N}=4$ super-Yang-Mills as a six dimensional (2,0) superconformal theory compactified on an elliptic curve E of vanishing area, brings yet another tool in the game. Namely, take the (2,0) theory on $E \times C \times D$, but start with the compactification on C. This gives, at low energy, an $\mathcal{N}=2$ superconformal field theory in four dimensions, compactified on $E \times D$. Moreover, there is a deformation of the latter, the so-called Ω -deformation [370], which allows E to vary over D, possibly contracting at the boundaries and the corners of D, permitting for a smooth (partial) compactification to a manifold with torus isometry, while preserving some supersymmetry [295]. In this way some of the peculiar branes, discovered in [706] and used in [64], acquire (conjecturally) a natural geometric origin, as "no-boundary" condition in the geometry of the form cigar $\times \mathbb{R} \times S^1$, where the cigar is viewed as an S^1 fibration over \mathbb{R}_+ , degenerating over $0 \in \mathbb{R}_+$.

If, by some lucky guess, we can realize this $\mathcal{N}=2$ theory as a quiver gauge theory with unitary gauge group factors, localization [370,725] lets one perform some efficient exact calculations [306,307, 348,726–728], once the supersymmetry of Ω -background is identified with the rotation-equivariant de Rham differential acting on the space of gauge fields. In this way the ingredients of geometric Langlands correspondence are mapped to the expectation values of certain supersymmetric observables in $\mathcal{N}=2$ theory, such as surface defects of various kinds (the monodromy, or orbifold, or regular, defects, vortex string defects, Q-observables, etc.). Then, in case of genus 0 and 1 curves C with punctures, both the expectation values and the Dyson-Schwinger equations they obey, can be computed and hence mapped to the equations of [729,730], producing the opers and \mathcal{D} -modules generalizing those in [698].

In addition one finds an extension of the GLP away from the critical level. This is the so-called quantum Langlands correspondence, whose early incarnation is the relation between the $\widehat{\mathfrak{sl}}_2$ -current algebra conformal blocks and Virasoro conformal blocks with degenerate fields [731,732], and whose $k \to \infty$ limit gives the well-known Painlevé form of the Schlesinger equations [703,733].

Now, having the four dimensional $\mathcal{N}=2$ theory perspective on GLP, it is a natural next step to a q-deformed version of the latter, by lifting the theory to five dimensions [90,734,735] and compactifying on a circle with a twist, which in string theory terms corresponds to replacing the (2,0)-superconformal field theory by a little string theory [736].

The three-fold view on the six-dimensional theory on $D \times C \times E$ and other manifolds is very powerful, especially if supplemented by additional string dualities. These become operational once the six dimensional theory is related to the theory of NS fivebranes of IIA string theory. These ideas imply the BPS/CFT correspondence [737] between the conformal field theories (in the extended sense) in two dimensions (and their q-deformations) and supersymmetric sector of four dimensional theories (and their uplifts to higher dimensions). At the origin of this correspondence are the chiral tensor fields propagating in the six dimensional (2,0) theory [370], or their cousins living at the intersections of branes (which could be obtained by T-dualizing the cigar-shaped geometries subject to Ω -deformation [738], or by taking their S-duals [728]).

The typical localization computations in $\mathcal{N}=2$ gauge theories in four dimensions produce functions matching the conformal, or chiral, blocks of some two dimensional chiral algebras. The most celebrated example is the AGT correspondence [739], relating Liouville theory and class \mathcal{S} gauge theories of A_1 -type.

However, conformal blocks are not yet the physical observables of the two dimensional conformal field theory, one needs to couple chiral blocks with the anti-chiral blocks. For example, the 4-point function on a sphere can be expressed as a one-dimensional integral of a square of an absolute value of a conformal block, where the integration variable parametrizes the conformal dimension of an intermediate field [740]. This convolution can be mapped to a partition function of a class S theory corresponding to a 4-punctured sphere, which happens to be an SU(2) theory with $N_f = 4$ fundamental hypermuliplets, compactified on an ellipsoid [741]. The shape of the ellipsoid is determined by the b parameter of the Liouville theory, so that for b = 1 one gets the supersymmetric partition on a round four-sphere [742].

Thinking about the correlation functions of WZNW theory, analytically continued in the level k of the current algebra, presumably along the lines of [251], one naturally encounters questions of real analysis on Bun_G . At the critical level $k \to -h^{\vee}$ the correlation functions become the eigenfunctions of both sets of quantized Hitchin hamiltonians, twisted holomorphic and anti-holomorphic differential operators on Bun_G , acting on half-densities. The choice of an eigenvector, and the corresponding eigenvalue are determined by the choice of a classical limit of r = rk(G) fields, presumably the Tduals of the subset of Cartan-valued "free fields," the φ -fields of the Wakimoto realization [743–745]. Presumably [746], this relates the "analytic Langlands correspondence" of [345,347,724] to the bosonic instantonic theory, also known as the curved $\beta\gamma\bar{\beta}\bar{\gamma}$ -system [747–750] on the complete flag variety G/B, and to the higher rank generalizations of [272,751,752]. Finally, the main ingredient of both analytic and classical GLP, Hecke operators [346] (which commute with quantized Hitchin hamiltonians), show up in the $\mathcal{N}=2$ description of the theory, as the so-called Q-observables, the specific surface defects, which can be engineered using the folded instanton construction [753,754]. In this way explicit formulae for the eigenvalues of the Hecke operators of [345,346] can be obtained by localization. This example could become a specific application in mathematics of the six dimensional view on the GLP. For other applications see [720]. It is tempting to conjecture that the paper [755] also fits the six dimensional perspective.

The subject is clearly very deep and will surely be an inspiring source of much future progress.

6.4 Attractors And Arithmetic

The beautiful attractor mechanism for construction of supersymmetric black holes [756] leads to some interesting questions about arithmetic aspects of Calabi-Yau manifolds and special Hodge structures on these manifolds [757–759]. ¹¹ The attractor conjectures have recently received some attention by

¹¹A very beautiful generalization of the attractor mechanism based on defect lines was introduced in [760]. The ideas of this paper have not been developed very far, but would seem to be promising.

physicists and mathematicians. See [761] for an overview. An important distinction must be drawn between "rank one" and "rank two" attractors. Recently important evidence has been presented for the (not unexpected) result that rank one attractors are not arithmetic varieties [762]. Although the arguments in [762] are based on some (well-founded) conjectures, strictly speaking they do not apply to the three-dimensional case. Clearly there are loose ends to be tied up here. ¹²

The rank two attractors have especially nice properties and, thanks to standard conjectures in arithmetic geometry are almost certainly arithmetic. Recently some beautiful new examples of rank two attractors with curious properties have been discovered [763]. Interestingly, rank two attractors can be found by searching for suitable factorization of zeta functions. In examples these zeta functions are related to modular forms, but there is no known physical meaning of these modular forms.

Some interesting related work on aspects of complex multiplication in the context of RCFT can be found in [764], which takes note of connections between the weight two modular forms associated to L-functions of rational elliptic curves and traces in the R-sector of RCFTs

There are potential connections to the modular Calabi-Yau manifolds [765,766], but a clear physical interpretation has not been given. A potentially important contribution in this direction is the recent observation relating special flux compactifications to modular Calabi-Yau manifolds [767].

6.5 Other Directions Relating Number Theory And String Theory

We mention a number of other intriguing directions in which research is making new and potentially fruitful connections to number theory.

Some years ago [768] some interesting observations were made about the relation of $\mathcal{N}=1$ points in the Coulomb branch of d=4 $\mathcal{N}=2$ theories with Grothendiecks' *Dessin's d' Enfants*. This direction seems fascinating, but remains unexplored.

There are interesting ideas concerning a possible notion of "Hecke operators" on rational conformal field theories [769–771]. This direction is relatively unexplored. One of the main open problems, in our view, is to extend the observed Hecke action to modular tensor categories. Progress in this direction has been initiated in [771].

In a beautiful series of papers written over the past 20 years [763,772–776] P. Candelas and X. de la Ossa and collaborators have explored various arithmetic aspects of Calabi-Yau manifolds, including relations between point-counting and periods for Calabi-Yau manifolds over finite fields. Recent achievements of this program have included discoveries of remarkable properties of zeta functions of Calabi-Yau manifolds over finite fields [776] and intriguing new results relating attractor points to special values of L-functions [775].

There is a famous analogy between number fields and compact oriented three-manifolds, in which knots are analogous to primes. A nice exposition can be found in [777]. Starting with these ideas the research program [778–781] seeks to define an arithmetic version of Chern-Simons theory.

Special values of L-functions are of interest to mathematicians for a number of reasons. In a very interesting recent work [782] it is noted that in some d=4 $\mathcal{N}=2$ CY compactifications the $\mathcal{N}=2$ central charges of BPS states at special points can be written naturally in terms of special values of L-functions. It would be of interest to generalize these relations to a broader class of theories.

7 Interactions With Condensed Matter Physics

Although many of the techniques and motivations in physical mathematics have their origins in theoretical particle physics, the connections to condensed matter physics have been increasingly important. The use of topology in condensed matter physics goes back a long way (it can be traced back to a paper of Einstein's on semiclassical quantization [783,784]) and is now a major aspect of modern condensed matter physics. A proper discussion should include a least:

- 1. General theory of the Berry connection.
- 2. Relations to the quantum Hall effect and Chern-Simons theory.
- 3. Aspects of anyon physics.

¹²M. Elmi has informed us that there is numerical evidence that the generic rank 1 attractor point is not arithmetic.

- 4. Applications to quantum information theory and topological quantum computing, and their relation to fusion categories and modular tensor categories.
- 5. Topological insulators, topological superconductors, and topological aspects of band structure, including relations to (twisted, equivariant) K-theory and several distinct "10-fold ways."
- 6. Weyl semimetals.
- 7. Topological phases more broadly, including what is known about their classification, and how the classification is modified by conditions such as symmetries and/or short-range entanglement.
- 8. Relations of this classification program to the theory of anomalies.
- 9. Nonsupersymmetric dualities and higher dimensinal bosonization.
- 10. Emergent gauge fields.

Covering all this material properly would more than double the length of this panorama. Just a few of the very many relevant reviews include [785–796]. We will content ourselves with just a few of the most recent connections most relevant to other parts of this essay.

As we mentioned above, one of the most important open problems in physical mathematics, and a root cause of the occasional chasm separating the intuition of mathematicians and physicists, is the proper rigorous definition of quantum field theory with nontrivial RG flow. There is a significant body of mathematical work on quantum mechanical theories—from both condensed matter and quantum field theory—which incorporate energy and length scales. It is highly desirable to find closer connections of this work with the advances in the algebraic and topological sides of quantum theory.

It is often said that the best route to a rigorous definition of quantum field theory is to use a lattice regularization and find a careful definition of the continuum limit. An interesting new wrinkle on this old problem is the discovery of lattice models involving, for example, fractor phases. For one recent review, see [797]. The investigation of fractor phases has become an important research direction in condensed matter physics, as well as in high energy physics, because these lattice examples highlight how subtle taking the continuum limit can be. Recent work [798–801] has clarified the continuum interpretation of some of these models and has demonstrated that, from traditional viewpoints of quantum field theories, the models are somewhat exotic. Certainly, the considerations in these papers need to be further explored in several directions. For just one example, nontrivial interactions should be included - the above works are based entirely on quadratic actions. It is possible that a theory of "quantum field theory with foliations" is relevant to these fracton phases. An exploration along these lines can be found in [802–804] as well as in the related work [805] on topological defect networks. This is new and relatively unexplored territory. It would be desirable to extend these models to include more nontrivial foliations. Even in the framework of topological field theory it might be interesting to enhance the usual bordism categories to include foliations, or to explore whether, for example, the Godbillon-Vey class could serve as an interesting action principle. Note, that the abelian five dimensional Chern-Simons theory has the space of codimension two foliations with a flat connection on its leafs as a phase space [806]¹³. Also, a "Wick-rotated" Kodaira-Spencer theory describes the three dimensional foliations of a six dimensional manifold equipped with a volume form (this is an interpretation of Sect 2.1 of [807], referring to [376]), alternatively, one can also describe it by the "Wickrotated" version of the Polyakov-like formulation of the six dimensional Hitchin theory, obtained by replacing +6 by -6 in the Eq. (11) of [808]. There are also interesting relations between fracton phases and supersymmetric quiver theories [809,810] (as well as brane constructions [811]).

While fracton phases are currently popular we should not forget that even understanding rigorous formulations of continuum limits of lattice formulations of more standard theories remains an important open problem. Therefore it is important to be open to other novel ways to formulate theories with scale. For example, in [812] the two-dimensional Ising model is placed in the framework of extended topological field theory creating a potential new pathway to incorporate the renormalization group. Other new approaches to the problem of scale appear in [813,814].

¹³The theory with the action $\int A \wedge dA \wedge dA$ associates to a four manifold M^4 the phase space \mathcal{P}_{M^4} which is a quotient of the space of solutions to the Gauss law $F \wedge F = 0$ by the action of the group generated by the kernel of the form $\int_{M^4} F \wedge \delta A \wedge \delta A$, which is the semi-direct product of the gauge group and the group of diffeomorphisms of M^4 . The foliation is spanned by the kernels of F

One important question in this general set of ideas is when a topological phase must necessarily have gapless modes when formulated on a manifold with boundary. A fairly general result in this direction, in 2+1 dimensions, can be found in [60,815]. See also [173] for related obstructions to the existence of gapped boundaries in invertible theories. One potentially interesting direction for future research is to generalize these ideas and results other dimensions and to classify the various boundary (and defect) theories that can exist in the presence of topological phases. An alternative set of ideas addressing these kinds of problems can be found in [816].

A long-term project of Alexei Kitaev's has been the classification of short-range-entangled topological phases of matter in terms of a spectrum (in the sense of algebraic topology) in a generalized cohomology theory. This approach attempts to classify phases of matter not by classifying spaces of (local!) Hamiltonians but rather by examining the spaces of ground states of such local Hamiltonians. Kitaev has given precise definitions of short-range entangled states and a roadmap to their classification. A successful outcome of this program is likely to have a wide impact. ¹⁴

For some recent progress in the mathematical theory of SPT phases in quantum spin chains, see [820]. If one allows passage to continuum field theory, then short-range entangled states go over to *invertible* field theories [821], for which there is now a well-developed theory—at least in the topological case—in the framework of stable homotopy theory. This leads to a general and computable formula for the abelian group of invertible phases in all dimensions and for all symmetry types [50,822,823]. One can also use homotopical methods to compute groups of invertible phases on a fixed space or stack (space with symmetry) [824–826]. This includes *interacting* "crystalline phases", a subject of current interest in the condensed matter community; a very small sample of literature includes [827,828].

In the noninteracting case the classification of topological band insulators is fairly well-understood. It is based on K-theory [829–832], and, when crystalline symmetry is taken into account, twisted equivariant K-theory [833–836]. Some of the invariants implicit in twisted equivariant K-theory (obtained by localizing the K-theory) have been used to give extensive lists of materials which might possibly be crystalline topological insulators [837–840]. An interesting open question is whether there is topological information in the twisted equivariant K-theory which has not yet been taken into account in the classification schemes of [837–840].

The connection between topological phases and TQFT's has inspired new efforts in the classification of TQFTs. Recently, 4d TQFTs were classified in [841–844], and most recently in [845], where it was shown that not all 4d TQFTs are necessarily gauge theories of Dijkgraaf-Witten (for a finite group, or higher-group) but can have a non-trivial so-called Majorana-layer, that obstructs mapping this to a gauge theory. 5d topological order was discussed furthermore in [846].

Another fruitful point of contact between condensed matter theory and high energy physics is the exploration of the notion of higher Berry curvatures and their relations to transport coefficients [847–851]. Among the many open questions these developments raise is the construction of a full differential cohomology class corresponding to the higher Berry curvatures. Another important problem is the generalization to include thermal transport coefficients.

Underlying many of the ideas about topological phases of matter is quantum information theory, a subject which has also been playing a central role in modern developments in quantum gravity. An interesting generalization of finite depth quantum circuits are the quantum cellular automata [852–854] which have interesting relations to C^* -algebras and index theory. It is natural to conjecture that they too will play an interesting role in fundamental physics.

8 Connections To Geometry And Low-Dimensional Topology

8.1 Two-Dimensions: Moduli Spaces Of Curves And Hurwitz Theory

The development of matrix models of two-dimensional gravity [855–857] (see [858] for a review) led to a physical proposal for computing intersection theory of Mumford-Morita-Miller classes on the moduli spaces of curves [859], a proposal which was spectacularly confirmed in [860]. See [861] for a review. A much less well-known example are the generalizations studied in [862], which remain largely unexplored (see, however, [863] for some developments). One interesting aspect of this work is that it uses an idea

¹⁴Michael Freedman has, for some time now, suggested that a useful mathematical framework for this program is coarse geometry and associated invariants, as developed by John Roe [817–819].

closely related to the definition of Bauer-Furuta invariants of four-manifolds [864]. ¹⁵

A different approach to the Witten conjecture was pursued in [868, 869] using Hurwitz moduli spaces. These moduli spaces also appear quite naturally in large N expansions of two-dimensional Yang-Mills theory [70,71,870,871]. More discussion can be found in [18]. Yet another approach to the Witten conjecture makes use of the ideas of geometric recursion [872,873]. See also [874] for appearance of Hurwitz number in topological string theory.

More recently some remarkable new observations about Hurwitz theory motivated by the AdS3/CFT2 correspondence have been observed in [875] (See also [876,877].). It would be quite interesting to know whether (and if so, how) these appearances of Hurwitz theory are related to each other.

8.2 Knots, Links, And Three-Manifold Invariants

In the past few decades there have been remarkable advances in the topology of 3-manifolds and knots. Some of these advances have been deeply related to explorations of quantum field theory. One of the most spectacular applications of quantum field theory to low dimensional topology is Witten's interpretation of the Jones polynomial in terms of Chern-Simons field theory [878] and the associated discovery of the WRT invariants of three-manifolds [879, 880]. The ramifications of these discoveries continue to the present day.

Many modern investigations of the relation between three-manifold topology and supersymmetric quantum field theory are ultimately related to the 6d (2,0) theory. Typically, one considers this theory on a product of two 3-manifolds, compactifies on the factors in different orders and learns something interesting. An important example of this general idea is provided by the study of theories T[M] associated to considering the 6d theory on a 3-manifold M [881].

One can study, for example, partition functions of theories T[M] on S^3 or more generally on lens spaces. Somewhat surprisingly, these are closely related to Chern-Simons theories with noncompact gauge groups such as complex gauge groups. See [882] for one example of such a computation. A great deal of work has been done on localization computations of supersymmetric quantum field theories on spaces admitting a torus action. For example, [72] rederived the Verlinde formula using the embedding of Chern-Simons theory into a twisted version of supersymmetric Yang-Mills theory. In [90, 883] the formalism applicable to lens spaces, $S^1 \times \Sigma$, and more general Seifert manifolds was developed. See [884] for a review. The techniques have been extended to larger classes of three-manifolds such as Seifert-fibered spaces [885–890], as well as to five dimensional theories [891].

Through the above route (and others) the study of the 6d (2,0) theory leads to the study of Chern-Simons theory for noncompact gauge groups, a subject which has been vigorously studied for decades now. A very incomplete set of references includes [251,892–895]. The subject continues to present important open problems. It cannot be a topological field theory in the strictest sense, because state spaces are infinite dimensional, and gluing is not obvious. Nevertheless, it is "almost" a TQFT. For some approaches to solving this problem see [889,896–898].

A very important example of a "non-compact group Chern-Simons theory" is the "Teichmuller TQFT." [896, 899–902]. In particular, the extremely beautiful paper of V. Mikhaylov [901] relates the Teichmuller TQFT to the 6d (2,0) theory, and formulates the hyperbolic volume conjecture of Andersen-Kashaev in terms of a conjecture on the nature of solutions of the Kapustin-Witten equations on $M \times \mathbb{R}_+$ with Nahm pole boundary conditions at the finite boundary. It was recently pointed out that this conjecture gives a natural explanation of some remarkable discoveries relating three-dimensional indices to Stokes matrices of three-dimensional complex Chern-Simons theory [299].

One concrete form of the partition function of a complex Chern-Simons theory on a 3-fold are the state-integral invariants, which can be defined for a subclass of 3-manifolds that admit simplicial decompositions with a so-called "positive angle structure." They take the form of a contour integral of products of factors, involving the quantum dilogarithm, and the precise contours are determined by the angle structure [892, 893, 903–909].

If, instead one considers the partition function on a product of a disk and a circle, with a suitable 2d QFT on the boundary, and when M has a presentation given by a "plumbing presentation" one is led to contour integral formulae for the "homological blocks" discussed in [898,910]. (Precursors to

¹⁵The general idea behind the Bauer-Furuta invariant can in principle be applied to many other topological field theories of cohomological type. This is largely unexplored territory. Some aspects have been explored in the context of Floer homotopy theory [865–867]. For a nice review, see the talk of C. Manolescu at the WHCGP, October 3, 2022.

¹⁶In unpublished work of A. Khan, D. Gaiotto, G. Moore and F. Yan

these blocks, sometimes called "holomorphic blocks" in this context were described in [911,912].) The derivation of these formulae could use further clarification - this is a good problem for the near-future. The resulting formulae are clearly of great interest. They give interesting q-series related to mock modular forms, resurgence and quantum modular forms [668]. Moreover, as shown in [911] they lead to many interesting identities and results on q-series.

Another source of three-manifold invariants based on twisted supersymmetric QFT are the Rozansky-Witten invariants associated to a choice of holomorphic symplectic manifold [913, 914]. (These are also known as B-twisted 3d $\mathcal{N}=4$ sigma models.) Recently interesting progress has been made on such theories associated with noncompact manifolds such as Coulomb branches of 3d $\mathcal{N}=4$ theories [915–917]. These investigations are part of a program to give a physical framework for the non-semisimple 3-manifold invariants discussed in [918–921]. ¹⁷

Yet another application of the 6d (2,0) theory is to link homology [927, 928]. One approach to a physical understanding of link homology is based on statecounting and the M5 brane [910,929–933]. A second (but closely related) approach to a physical basis for knot homology was initiated in [912,934]. The latter approach is intimately connected to the physical interpretation of the geometric Langlands program and the theory of supersymmetric boundary conditions in $\mathcal{N}=4$ SYM. This approach relies on Morse complexes based on the difficult Kapustin-Witten and Haydys-Witten equations ¹⁸ but under some conditions important simplifications can be achieved and the formulation of knot homology can be expressed in terms of two-dimensional Landau-Ginzburg theories [116,940,941]. A related, but distinct, program has been vigorously pursued by M. Aganagic [942,943] in work that makes interesting use [736,944,945] of the q-deformed Knizhnik-Zamolodchikov equations of [946].

In a rather different direction, an important source of progress in understanding the topology of 3-manifolds has been the study of Ricci flow [947–952] which itself is closely related to, the RG equations of nonlinear sigma models [211,212]. It is natural to wonder whether there can be useful technology transfer from the recently developed mathematics back to physics. See [953] for a very interesting recent development in this direction.

The theories T[M] for M a non-hyperbolic three-manifold flow to topological theories in the IR. It was recently conjectured that this provides an interesting, new way to classify both bosonic and fermionic phases of matter in 3d [954]. Fermionic phases arise by including refined global information in the dimensional reduction, which account for 1-form symmetries in the 3d theory [890]. This potentially provides a complementary, geometric, way to classify gapped phases, distinct from the approach using modular tensor categories (MTCs); see [179,793,955,956] for a discussion of bosonic topological order. Exploring this connection between non-hyperbolic three-manifolds and MTC seems an intriguing direction for the future.

8.3 QFT And Four-Manifold Invariants

One of the paradigmatic examples of physical mathematics is the interaction between supersymmetric field theory and four-manifold invariants. After the discovery of instanton solutions of Yang-Mills equations [957,958] the work of the Oxford school, led by M. Atiyah and N. Hitchin, culminated in the discovery of Donaldson invariants of smooth four-manifolds [959,960]. Motivated by questions posed by M. Atiyah, Witten produced a quantum field theory interpretation of the Donaldson invariants and in the process invented topological quantum field theory [62]. The desire to use this formulation to give an effective evaluation of Donaldson invariants was one of the motivations for the development of Seiberg-Witten theory [961,962] culminating in the introduction of Seiberg-Witten invariants [963], a development which revolutionized the study of four-manifolds [964]. Given this spectacular success one naturally wonders if further study of (topologically twisted) supersymmetric field theory can lead to further insights into the differential topology of four- (and three-) manifolds. In recent years there have been several developments along these lines.

¹⁷These works are in turn closely related to "logarithmic CFT," perhaps better called "non-semisimple conformal field theory." If one generalizes the representations of the Virasoro algebra provided by primary fields to reducible, but indecomposable, representations then the usual conformal Ward identities lead to correlators that involve logarithms. It is now recognized that there should be an important generalization of RCFT based on allowing such indecomposable representations and this is an active area of current research. For reviews see [922–926]. The subject is difficult. It is analogous to generalizing the representation theory of compact groups to the representation theory of noncompact groups. Nevertheless, it is now showing up in the physics of percolation, noncompact Chern-Simons theory, instantonic field theories [749,750], and the geometric Langlands program, so it is clearly important.

¹⁸Whose moduli spaces are beginning to be understood from a solid mathematical framework [935–939].

From the physical viewpoint the Seiberg-Witten theory can be used to give an evaluation of the Donaldson invariants via the study of a certain integral along the Coulomb branch, sometimes referred to as the "u-plane integral". This family of integrals was first studied in [638,965] and it leads to many remaining open problems. In the case of pure SU(2) theory the wall-crossing behavior of the Coulomb branch integral leads to a rather direct derivation of the famous "Witten conjecture" expressing the Donaldson invariants in terms of the Seiberg-Witten invariants for compact oriented four-manifolds with $b_2^+ > 1$ [638]. (One good aspect of this approach is that the four-manifold need not be toric, or even complex. But it does require the existence of an almost complex structure, as is standard in Donaldson theory.) The integral can be generalized to include other Lagrangian $\mathcal{N}=2,\ d=4$ theories [638, 965, 966]. The formulation of u-plane integrand for general d=4 $\mathcal{N}=2$ Lagrangian theories involves a crucial - but difficult - issue of determining "contact terms" in the presence of observables. The determination of these terms was described in [965] as a theory of deformations of special Kähler manifolds, yet to be developed. This remains an important open problem. As explained in [965] in the general Lagrangian case the path integral localizes to the moduli space of the nonabelian version of the Seiberg-Witten equations which, when localized further to the moduli space of instantons, involves densities different from those used to compute the Donaldson invariants. The developments described here are nicely reviewed in detail in the book [967].

The direct evaluation of u-plane integrals is an efficient way to use physics to derive the invariants for manifolds with $b_2^+ = 1$ such as \mathbb{CP}^2 and $S^2 \times S^2$. It turns out that an effective way to evaluate these integrals involves the use of modular and mock-modular forms. This was recognized early on for the special cases of \mathbb{CP}^2 and $S^2 \times S^2$ [638, 968–970], but the relation to mock modular forms is quite deep and quite general, as has been clarified recently in [633, 640–642]. The latter papers highlight as well the use of Jacobi-Maass forms. The technique has proven useful for the pure SU(2) theory and for the SU(2) $\mathcal{N}=2^*$ theory. In the case of SU(2) with fundamental flavors new important technical issues arise since the Coulomb branch is no longer a modular curve [971, 972].

The case of the SU(2) $\mathcal{N}=2^*$ theory presents a number of interesting variations on the above ideas [633]. First, it interpolates nicely between the Donaldson and Vafa-Witten invariants. Being superconformal it presents the added novelty that not only the integrand but the value of the Coulomb branch integral (as a function of the ultra-violet coupling) is a mock modular form. In fact, the evaluation on general $b_2^+=1$ manifolds for general spin-c structure presents some new open problems concerning mock Jacobi forms which have yet to be resolved [633]. Finally, and perhaps most interestingly, there are very close connections to a beautiful series of works on enumerative algebraic geometry by Göttsche, Kool, Nakajima, and Williams [973, 974].

The case of pure SU(N) theory was studied in [965, 966], and in [966] the wall-crossing technique was employed to give explicit formulae for the SU(N) Donaldson invariants in terms of Seiberg-Witten invariants. In principle the formulae can be generalized to all higher rank compact semisimple gauge groups. This has not been done, and it would be interesting to do so since it would lead to nontrivial predictions about Floer homology following the line of reasoning in [975, 976]. The mathematical challenges to verifying these physical predictions are formidable but nontrivial progress has been made [976, 977]. Generalizing the work of [633, 640–642] to the case of higher rank theories on $b_2^+ = 1$ manifolds is completely open, and presents an interesting challenge that will probably lead to new generalizations of the notion of automorphic functions. It would be very interesting to extend the relation of the results of [633] with [973, 974] to the higher rank case.

There are a number of other interesting and valuable generalizations of Coulomb branch integrals which should be pursued in the future, and we will describe some of these briefly.

One important generalization is to consider suitable couplings to supergravity to give field theoretic interpretations of Donaldson and Seiberg-Witten invariants associated with families of four-manifolds. This should connect to some old ideas of Donaldson concerning generalizations of the Donaldson invariants to give interesting equivariant cohomology classes on $\mathrm{BDiff}(X)$ of a four-manifold X [964,978]. There has been some limited mathematical investigation of this topic. See [979–986] for some examples. There has been some recent definite progress in formulating the relevant physical theories for addressing these problems [987] (which explores in great detail a suggestion made in [638]). Nevertheless, a great deal remains to be done. One key issue is finding a suitable generalizations of the 0-and 2-observables of Donaldson theory to the families case. It is natural to expect that there will be a family version of the "Witten conjecture" relating Donaldson and Seiberg-Witten invariants, but no concrete results on this have been published. Recently there have been some exciting developments on

the mathematical side concerning somewhat related problems (i.e. new results on the diffeomorphism groups of four-manifolds) [988–991]. It would be very interesting to see if these are connected to the invariants coming from SYM.

There should be new Coulomb branch integrals associated to 5d and 6d susy gauge theory, a topic which has been explored in [965,992–997]. Mathematically, the 5d invariants are related to "K-theoretic generalizations of Donaldson invariants." Roughly speaking, one includes an insertion of the \hat{A} genus in the intersection theory on the moduli space of instantons. Note, that on the physics side, for 4d/5d/6d theories with unitary gauge groups the relevant contact terms were computed in [151,370,998–1001], using several new ideas: Ω -deformation [370], noncommutative field theory [438], and localization. Very recently the Coulomb branch integral for such 5d invariants has been under very careful and intense scrutiny from several points of view [996,997] with a view to reproducing and generalizing the results of [994].

Moving on to the six-dimensional approach to the Donaldson invariants and their generalizations, many new issues arise. From one point of view, one would like to generalize the 4d theories from Lagrangian theories to all theories of class S. In [633], section 8, a number of open problems related to the formulation of general "u-plane integrals" for such theories were spelled out, and they will not be repeated here - but they constitute an interesting set of directions for future progress. Moreover, ever since the work of Vafa and Witten on S-duality of twisted $\mathcal{N}=4$ theory [102] a six dimensional perspective on four-manifold invariants has been pursued. In particular, a six dimensional lift of Donaldson theory has been considered in section 5.6 of [72]. A dual viewpoint emerges: two dimensional (0,2)-supersymmetric sigma models with instanton moduli spaces as targets on the one hand, and the double periodic version of Seiberg-Witten theory, on the other [734, 1002, 1003]. The first viewpoint upgrades Donaldson polynomials to the elliptic genus of instanton moduli space with some bundle of matter zero-modes, mathematically viewed as moduli space of Higgs sheaves on a complex surface [1004]. It is related to the attempts to find the four dimensional analogues of RCFT and the related "four-dimensional Verlinde formula" of [103].

The 6d viewpoint is also related to attempts to understand a field-theoretic origin of Nakajima algebras. The cohomology of the six dimensional counterpart of the Donaldson supercharge contains an infinite dimensional chiral algebra. (In this context it is worth recalling that elliptic genera of instanton moduli spaces are used in the attempts to build the second quantized string theory from Matrix theory, and in the calculations of black hole entropy [1005], suggesting further intriguing potential connections.)

There are several more recent, but closely related, developments. References [1006-1008] suggested that there might be new four-manifold invariants associated with the compactification of a 6d theory on a four-manifold. Roughly, one considers the tmf class of the resulting 2d susy QFT. However, it remains to be seen if the tmf class really captures new topological information beyond the homotopy type of the four-manifold. It is not obvious that the tmf class will depend on more than the homotopy type since the IR theory on the Coulomb branch depends only on the homotopy type of the four-manifold. Relations to vertex operator algebra theory, generalizing the profound and remarkable results of Nakajima relating the cohomology of moduli spaces of instantons to representations of affine Lie algebras, are also found in [1006-1008].

Finally, as has been stressed by Witten, one of the most promising directions for the discovery of new four-manifold invariants is to make good mathematical sense of a categorification of Vafa-Witten invariants.

8.4 Hyperkähler and Quaternionic Kähler Geometry

Hyperkähler metrics play an important role in investigations of moduli spaces of vacua of supersymmetric field theories. They have long been a source of fascination within mathematics. Indeed the hyperkähler quotient construction grew out of an extremely fruitful collaboration between physicists and mathematicians in [1009].

More recently, investigations into the BPS spectrum of field theory and their wall-crossing properties led to a new construction of hyperkähler metrics based on integral equations formally equivalent to those appearing in the thermodynamic Bethe ansatz (TBA) [1010]. The new technique might lead to useful exact results in hyperkähler geometry. There have been interesting rigorous mathematical [1011] and numerical [1012, 1013] checks of [1010], but some basic questions (like existence and uniqueness of solutions of the TBA equations) remain open. The techniques can, in principle, be extended to give exact solutions of Hitchin's equations on surfaces [1014] but this has been much less explored.

One of the initial hopes for the construction [1010] was the idea that one could use the construction to produce exact formulae for K3 metrics. For example, one could combine these ideas with the discussion of the D3 probe in F-theory as a deformation of a system consisting of four copies of the SU(2), $N_f=4$, theory [1015]. The main difficulty in this approach is that of determining the relevant spectral networks and BPS spectrum, since the D3 probe theory is not a standard quantum field theory. Recently this idea has been pushed further in the context of little string theory in [1016,1017]. It would be a major achievement of the mathematics-physics dialogue if these developments led to a tractable and exact formula for the metric on any smooth nondegenerate K3 surfaces.

There has been some effort to extend the new constructions of hyperkähler metrics to quaternionic Kähler metrics. The two are closely related by the Swann construction. These metrics are of importance in deriving the hypermultiplet moduli spaces of the supergravity LEET of four-dimensional $\mathcal{N}=2$ string compactifications. Progress has been made in [1018–1023] although obstacles remain because of the large growth of BPS black hole entropy as a function of charge, and the related issue of how to account for NS5-brane instanton contributions.

Recently there have been some very interesting developments concerning complex hyperkähler geometries associated with BPS states. These results shed light on some old work of D. Joyce on wall-crossing invariants and perhaps will lead to progress in the construction of quaternionic Kähler metrics [1024–1029].

Finally, the physics of BPS states makes nontrivial predictions about the cohomology and differential geometry of certain hyperkähler manifolds. Examples of the relevant manifolds include moduli spaces of magnetic monopoles on \mathbb{R}^3 in Yang-Mills-Higgs theory. A remarkable paper of A. Sen used the predictions of S-duality of $\mathcal{N}=4$ SYM to derive nontrivial conjectures about the existence of self-dual normalizable harmonic forms on monopole moduli space [1030]. Historically, this paper provided a large impetus for the fundamental use of strong-weak duality symmetries in investigations of supersymmetric field theory and string theory. The essential technique used in [1030] is the collective coordinate description of BPS states in weakly coupled field theories. Some of Sen's predictions were rigorously proven in [1031]. When the technique of collective coordinate quantization is applied to describe BPS states in general $\mathcal{N}=2$, d=4 field theories physical statements about BPS states translate into nontrivial generalizations of Sen's predictions and can be phrased as general predictions about the L^2 kernels of Dirac operators coupled to hyperholomorphic bundles over monopole moduli space [1032,1033]. In some analogous situations the physical predictions based on BPS states about the delicate questions regarding the kernels of Dirac operators on noncompact spaces have been rigorously verified [1034].

The topology - specifically the homology and cohomology - of Hitchin moduli spaces and character varieties has been the subject of much work beginning with a study of SYZ-type [1035] mirror symmetry and its relation to Langlands duality [1036]. In another direction, a very general conjecture known as the "P=W" conjecture has been extensively studied in [1037–1049]. The results can be interpreted very nicely using BPS states associated with string theory compactification on local Calabi-Yau manifolds [1050–1053].

9 Geometrization Of Quantum Field Theory

Geometric engineering of quantum field theory (QFT) broadly refers to the various methods used to define QFTs from a certain geometric scheme. This provides a definition of QFTs beyond the standard paradigm of Lagrangian theories and gives access to strongly-coupled regimes. Broadly speaking, the following approaches fall under the header of geometric engineering:

- 1. **Geometric engineering:** The original framework that establishes the paradigm of geometric engineering is the construction of QFTs by decoupling the low-dimensional non-gravitational degrees of freedom of string or M-theory on a non-compact (and not necessarily smooth) space X. Some constructions of this type can be thought of as the limits of large volume of the standard string compactifications on X, where gravity has been decoupled. The requirement of supersymmetry in the resulting QFT usually implies that X has reduced holonomy.
- 2. Branes: QFTs can be geometrically defined by also studying the low energy dynamics of branes and various non-perturbative states of string and M-theory. These can be obtained in diverse

backgrounds which can be flat, have reduced holonomy, be singular, and/or be supported by

- 3. $\mathbf{QFT}_d \to \mathbf{QFT}_{d-n}$: Another powerful scheme of geometric engineering is to obtain QFTs in d-dimensions by reducing higher-dimensional field theories on manifolds Σ , possibly with boundaries. In this paradigm, the QFT in d dimensions is defined by the higher-d theory, the geometry and topology of Σ , and the specific data for the reduction such as choice of topological twist, fluxes for flavor symmetry, and boundary conditions on Σ for the higher-d theory. This approach is complemented by the previous methods when the higher-d theory admits a geometric definition. The reduction of the theory has a dual geometric picture where specific brane setups are wrapped on Σ .
- 4. **Holography:** QFTs can also be defined geometrically (though maybe the terminology of geometric engineering is less standard in this context) by constructing $AdS_{d+1} \times M$ solutions in string and M-theory supergravity. The reduction of string and M-theory on M provides a supergravity theory on AdS_{d+1} which then defines a conformal field theory in d dimensions by the AdS/CFT dictionary. This method is intimately related to the previous one since the AdS solutions often describe the near-horizon limit of brane configurations.

Many of the above geometric realizations are inter-connected. Numerous theories have descriptions in several of the above constructions, which are related by string dualities. An example is the 4d $\mathcal{N}=4$ Super-Yang Mills theory with gauge algebra $\mathfrak{su}(N)$: it has a constuction in terms of Type IIB on $\mathbb{C}^2/\mathbb{Z}_N \times T^2$, which can be thought of as a non-compact K3-surface times a torus; as theory on a stack of N D3-branes (after decoupling the center of mass mode), as the 6d (2,0) compactified on a T^2 , and finally it has as holographic description in terms of the Type IIB supergravity solution $AdS_5 \times S^5$. Different realizations can provide at times better control of certain aspects. E.g. some symmetries may be manifest in some descriptions and not in others.

The holographic setting provides a unique laboratory to explore strongly coupled QFTs via weakly coupled gravitational theories. Indeed, this has been most fruitful in cases where the QFTs have large number of degrees of freedom where the dual gravitational theory is in its classical regimes. There is a very close connection between brane-engineering and holography in terms of the underlying geometies. The prime example here are the Sasaki-Einstein manifolds X_5 that appear in the constuction of AdS₅ duals to 4d $\mathcal{N}=1$ SCFTs. In turn these SCFTs can be constucted by probing the singular Calabi-Yau three-folds, that are the cones over the Sasaki-Einstein manifolds X_5 [1054].

Geometric engineering provides new perspectives in the study of QFTs. First it often provides a way to study both weakly-coupled and strongly-coupled QFTs using one geometric framework. It provides a natural setting to extract and explore topological data (such as the generalized symmetries) and invariant quantities of RG-flows associated to QFTs such as 't Hooft anomalies. Geometric structures of QFTs such as moduli space of vacua, conformal manifolds, duality relations, and RG flows are also naturally realizable in geometric engineering. This paradigm also provides a way to make precise a classification program for QFTs and sharpens the notion of spaces of QFTs.

This is often aided by the fact that the QFT classification program can be mapped to a geometric one, where it finds a mathematically well-defined formulation. These can be for example geometries X with reduced holonomy and certain singularities, geometries M from solutions of supergravity equations or choices of boundary conditions and topological twists on compact manifolds Σ . The interconnection of QFTs and geometry is also a point of contact between the mathematics and physics communities in the seminar series [16].

9.1 Geometric Classification of Superconformal Field Theories

The canonical setup of geometric engineering will preserve some supersymmetry, which in the simplest case implies that X has reduced holonomy. Examples of this are Calabi-Yau n-folds as well as exceptional holonomy G_2 and Spin(7) manifolds. The use of geometrically engineering QFTs is three-fold: 1. it allows the construction of strongly-coupled QFTs, 2. important properties such as symmetries and their 't Hooft anomalies are encoded in the string theory construction, 3. the geometrization enables a classification program.

In the recent past, the utility of this approach was realized in the context of a geometric classification of 6d superconformal field theories (SCFTs) [1055–1057], and geometric exploration of 5d SCFTs

starting with [1058–1065]. SCFTs in dimensions 5 and 6 are automatically strongly-coupled, UV fixed points [1066, 1067], and not accessible with perturbative QFT methods. It is the existence of such strongly-coupled SCFTs, which is predicted by string theory.

9.1.1 6d SCFTs

6d SCFTs have a geometric engineering realization in Type IIB string theory. The theories with $\mathcal{N}=(2,0)$ supersymmetry are classified by an ADE gauge algebra, and have a construction from $\mathbb{C}^2/\Gamma_{ADE}$, where $\Gamma_{ADE}\subset SU(2)$ is a finite subgroup of ADE type. Note, that these SCFTs are not absolute theories, see section 4.3. The absolute theories in 6d with maximal supersymmetry are tabulated in [1068].

In contrast, the classification of 6d (1,0) theories is geometrically much more intricate: these theories have a realization in terms of F-theory compactifications on elliptically fibered Calabi-Yau three-folds. F-theory is Type IIB string theory, where the axio-dilaton $\tau = c_0 + 1/g_s$ is not constant, but non-trivially fibered over the type IIB compactification space. For elliptic Calabi-Yau threefolds the base is a Kähler surface B_2 (and only for trivial fibrations is the IIB compactification space Ricciflat). In the simplest instance when the elliptic fibration has a zero-section, it can be written in Weierstrass form $y^2 = x^3 + fx + g$, where f and g are sections of the canonical bundle of B_2 . The elliptic fibration develops singularities, which follow (at least in codimension one in the base) the Kodaira classification of singular fibers, which is characterized in terms of the vanishing order along a discriminant component z=0: $(\operatorname{ord}_z f, \operatorname{ord}_z g, \operatorname{ord}_z \Delta) < (4,6,12)$, where $\Delta=4f^3+27g^2$. Here z is a local coordinate on the base B_2 . SCFTs arise when the vanishing orders satisfy $\geq (4,6,12)$, and are so-called non-minimal fibers. The classification result in [1055-1057] states that the base geometries for 6d SCFTs are $B_2 = \mathbb{C}^2/\Gamma$ where now $\Gamma \subset U(2)$. For a review see [1069]. Resolving these singularities in the base, replaces B_2 with \tilde{B}_2 , which is resolved by a collection of rational curves. This characterizes the tensor branch of the 6d SCFT, i.e. the moduli space that is characterized by giving vevs to scalars in the tensor multiplet. The Kodaira type of the singular fiber above compact (non-compact) curve in the base, maps in F-theory to the gauge (flavour symmetry) group of the 6d theory. Additional codimension two singularities occur when two components of the discriminant intersect. These correspond in the 6d theory to hypermultiplet matter.

Frozen Phase of F-theory. In the realm of 6d SCFTs there are in addition so-called frozen singularities [450, 1070, 1071], which correspond to adding $O7^+$ -planes into the F-theory compactification. One of the main challenges in the 6d classification program is a first principle understanding of these frozen phases, the anomaly cancellation and dual M-theory description. To provide a precise definition of the frozen phase of F-theory one would need to determine what corresponds to a "frozen" F-theory background, when no dual IIA description exists. This is an important outstanding problem, which plays a central role in establishing a mathematically complete classification of 6d SCFTs.

9.1.2 5d SCFTs

A closely related question is the classification program of 5d $\mathcal{N}=1$ SCFTs. Again, these are strongly-coupled fixed points, whose existence is argued using string theory [1066, 1067, 1072, 1073]. The conjectured classification of 5d SCFTs is in terms of canonical singularities: i.e. non-compact Calabi-Yau three-folds X, with singularities that admit a resolution $\pi: \tilde{X} \to X$, such that the canonical classes $K_{\tilde{X}} = \pi^{-1}K_X + \sum_i a_i D_i$, where $a_i \geq 0$, where D_i are the exceptional divisors of the blowup. For $a_i = 0$ these admit so-called crepant resolutions. When $a_i > 0$ the singularity is terminal. Canonical singularities also admit complex structure deformations. There are two types of moduli spaces for 5d SCFTs: the Coulomb branch and Higgs branch. The latter has the structure of a hyper-Kähler cone.

The precise dictionary for canonical singularities to 5d SCFTs is extremely well developed and understood for the case when crepant relations exist. The resolved geometry describes the Coulomb branch of the 5d SCFT: this is an abelian gauge group with matter and Chern-Simons couplings. At special subloci of the Coulomb branch there can be non-abelian gauge theory descriptions – but not all theories have such a Lagrangian description. When the geometries have terminal singularities, the interpretation of the 5d SCFTs is not fully understood, although recent progress in understanding their deformations (and thereby the Higgs branch of the associated theories), has been made [1074–1076].

A classification program of 5d SCFTs was initiated using various approaches. The most systematic thus far is starting with the geometry underlying 6d SCFTs, and applying geometric deformations to these geometries. In particular M-theory on elliptically fibered Calabi-Yau three-folds gives rise to 5d KK-theories, which UV complete in 6d [1058]. Starting from these theories a systematic exploration can be carried out [1060–1065], including salient features such as the symmetries and dualities of such theories. It remains a challenging geometric question to prove that these constructions from 6d result in a full classification of 5d SCFTs.

Future Directions. Mathematically canonical singularities for 3-folds are in theory understood, based on the Mori minimal model program in algebraic geometry (for an introduction see [1077]). For 3-folds this is developed in much detail to the extent that an algorithm exists, how to perform crepant and small resolutions, and identify terminal singularities. The issue however is that the explicit classification of canonical three-fold singularities does not exist (as in a list, or a concrete realization of all such singularities). Clearly, the classification of 5d SCFTs provides some possible approach: starting with the gluing of compact surfaces and collapsing these to singularities as in [1059]. However this approach has not resulted in a characterization of the Calabi-Yau singularities, nor does it incorporate terminal singularities. Combining the abstract insights from the minimal model program, combined with the concrete construction of 5d SCFT from collections of compact surfaces, should be a very promising starting point to provide a concrete classification of canonical singularities. Note that for n-fold canonical singularities with n > 3, the minimal model program is far less explicit than for 3-folds, so that further insights from geometric engineering of QFTs (e.g. 3d theories or 4d theories for elliptic Calabi-Yau four-folds) could provide some guidance.

An important geometric question is the precise connection between canonical singularities and the quantum Higgs branch of the associated 5d SCFT. What is the interpretation of terminal singularities? Some aspects of this are discussed in [1074, 1078]. Another important question is the inter-relation of SCFTs accross dimensions: e.g. it is not known whether all 5d SCFTs be obtained from 6d SCFTs, using a circle-reduction and deformations (such as flavor Wilson lines, automorphism twists). Likewise an even more challenging question is the classification of 4d SCFTs with 8 or 4 supercharges.

Developing a unified framework – e.g. mapping geometric realizations to brane-constructions, and vice versa – will be essential in developing a complete classification of 5d SCFTs. Currently the descriptions have a large overlap, but there are theories, which evade constructions in one of the two frameworks. In particular, toric Calabi-Yau three-folds are dual to certain 5-brane brane-webs. However, webs where multiple 5-branes can end on a single 7-brane – so-called "white dots" [1079] – are not understood in the geometry. Physically this is again interlinked by Higgsing. Mathematically this will lead to a generalized notion of toric geometry, as anticipated in [1080, 1081].

There are numerous constructions for 4d SCFTs and many approaches to their characterization and classification have been put forward. Here we focus on the geometric engineering constructions. We discuss in section 9.3 constructions starting from higher dimensional QFTs, such as class \mathcal{S} . Using geometric engineering they can be obtained in Type IIB string theory on canonical singularities – i.e. the same spaces that underlie the construction of 5d SCFTs in M-theory [1082–1086]. See [146] for a recent pedagogical overview.

9.1.3 Moduli Spaces of Theories with 8 Supercharges

A beautiful connection between theories with 8 supercharges has emerged and made precise in the last few years, which interconnects the theories by dimensional reduction (with added benefits), and the relation between their moduli spaces. Progress in the recent years has shown that the Higgs branches of higher-dimensional theories (5d, 4d, in particular) can be characterized in terms of an auxiliary 3d theory – the magnetic quiver theory [1087, 1088].

The Higgs branch of a 5d SCFT is related to the versal deformations of the Calabi-Yau singularity that engineers the theory in M-theory. However the Higgs branch receives quantum corrections. For isolated toric or hypersurface singularities [1074] the deformation theory is well-defined, and in some instances the full quantum Higgs branch can be determined. In general this is however not understood, in particular for non-isolated singularities. Using an alternative description of 5d SCFTs, applicable for a large class of theories albeit not all, based on 5-brane webs, an algorithm for determining the Higgs branches using auxiliary 3d quiver gauge theories [1088, 1089]. These so-called magnetic quivers have 3d $\mathcal{N}=4$ supersymmetry and conjecturally, their Coulomb branch gives rise to the Higgs branch

of the 5d SCFT. The Coulomb branch of 3d $\mathcal{N}=4$ quivers can be given some characterization by various 3d partition functions (see e.g. [884,886,1090,1091]) with the Hilbert series being the simplest and most useful example. The expression for the Coulomb branch Hilbert series first obtained in [1092] is often called the monopole formula and for Lagrangian theories it can be obtained as a limit of the supersymmetric index [1093]. The Coulomb branches were also studied using the algebra of protected local operators in by Bullimore, Dimofte, Gaiotto [1094], and from more mathematical perspective in the works of Braverman, Finkelstein and Nakajima [1095, 1096]. This links the geometric engineering program of SCFT to developments in 3d mirror symmetry and symplectic singularities.

Deriving the Higgs branch from a geometric description, and proving the conjecture connecting it to the Coulomb branch of the 3d magnetic quiver is one of the outstanding question in this area. This program of developing the quantum moduli space (including its global stratification structure into Higgs and Coulomb branches) will develop in parallel to the classification of theories with 8 supercharges.

9.1.4 Generalized Symmetries and Anomaly Theories

Not only does geometric engineering construct QFTs, it also allows the computation of physical properties of the theories. This is of particular utility when these theories are strongly-coupled and non-Lagrangian. Generalized global symmetries, higher-group symmetries and their anomalies, as discussed in section 4, can be derived from the geometry.

Global symmetries are encoded in particular relative homology classes of the space X, relative to its boundary ∂X : non-compact (relative) q-cycles in $H_q(X, \partial X, \mathbb{Z})$, can be wrapped by branes, and give rise to defect operators. The defect operators modulo screening by local operators is defined as the set of such relative cycles, modulo screening branes wrapping compact q-cycles, which can be obtained by wrapping branes and their magnetic duals [1097–1107].

Anomalies for generalized higher-form symmetries are formulated in terms of the background fields: for q-form symmetries these are $B \in H_{q+1}(M_d, \Gamma^{(q)})$, where M is the spacetime. In M/string theory these are realized from supergravity form-fields expanded on the relative cohomology cycles of the compactification space X. The symmetry topological field theory is the d+1 dimensional TQFT, which upon imposing boundary conditions for the background fields, gives rise to the anomaly theory of the d-dimensional QFT. More generally anomalies for discrete symmetries require a refinement to generalized cohomology theories: the background fields for discrete symmetries descend from torsion cycles, which can be incorporated by treating the string/M-theory fields as cocycles in a differential cohomology theory [245,1108]. See [163,460] for introductions to differential cohomology aimed at physicists. In M-theory it appears that the differential cohomology theory has an underlying quantization based on $(w_4$ -shifted) singular cohomology theory, but in type II string theory the relevant cohomology theory appears to be differential K-theory [163, 240, 244, 460, 1109]. This differential cohomology approach to anomaly theories and symmetry TFTs was developed for higher-form symmetries in M-theory compactifications in [1110]: The reduction on ∂X results in the SymTFT, which after choosing boundary conditions, gives rise to the anomaly theory for 0- and 1-form symmetries.

The computation of anomaly theories can receive several contributions. The canonical ones come from reducing the topological interactions of string or M-theory. Another contribution comes from Stückelberg mechanisms that exist from the reduction of string or M-theory. In addition to providing new anomaly terms, these mechanisms also reduce the continuous symmetries from string or M-theory to discrete symmetries. The existence of such mechanisms occur when the various q-form gauge symmetries in string or M-theory fail to admit equivariant extensions with respect to the isometry group of the manifold where the theory is being reduced on. A systematic study of anomaly theories for continuous symmetries and the various obstructions that can lead to Stückelberg mechanisms have been initiated in [1111–1117].

Clearly the field of generalized symmetries, higher-groups and categorical symmetries has great potential for more applications in the context of geometric engineering. Higher-group symmetries can be computed from the geometry, but anomalies for these should also fall into the framework of symmetry TFTs and should arise from string theory. Non-invertible symmetry in higher dimensions have recently been constructed within string theory [199,200,1118,1119], and play a role in the context of the swampland program [1120].

9.2 Manifolds with Exceptional Holonomy

Special holonomy manifolds and calibrated geometry have played a central role in string compactification and in the study and application of D-branes and flux compactifications ever since Calabi-Yau manifolds were recognized as being of great importance in supersymmetric string compactification [1121]. In particular, Calabi-Yau manifolds and holomorphic curves and bundles have been objects of central interest, and have been at the focus of developments in enumerative algebraic geometry. Indeed, the entire subject of mirror symmetry - currently an important part of modern mathematical research grew out of these developments.

It is known from the Berger classification that there are fascinating exceptional cases of special holonomy manifolds in seven and eight dimensions with holonomy groups G_2 and Spin(7), respectively. Moreover, such manifolds admit natural generalizations of the instanton equations. Construction of such manifolds and instantons, and of their moduli spaces present serious challenges to differential geometers and analysts, but would be highly desirable in physics.

Enumerative problems in exceptional holonomy spaces could quite possibly see substatial progress. One such question is related to the counting of associative three-manifolds in manifolds of G_2 holonomy and the related problem of counting of G_2 instantons [1122–1124], and a clearer understanding of their moduli spaces. The main mathematical challenge is that associative three-cycles can degenerate, by either forming singularities or splitting. A consequence of this phenomenon is that the signed count of associative cycles is not an invariant on the moduli of special holonomy metrics. Several proposals have been put forward to deal with this challenge [1123, 1124]. Complementing this, there is an Mtheory proposal of interpreting the superpotential of the 4d QFT, which is obtained by compatifying M-theory on a G_2 -holobnomy manifold, as the counting function of associative cycles based on the observation that the effective superpotential obtains non-perturbative contributions from M2-branes wrapping associative three-cycles [1125]. An open problem raised by [1125] is the computation of the contributions of non-isolated associative cycles. The formula for isolated cycles involves the Ray-Singer torsion. Since torsion is most naturally interpreted as a measure it is reasonable to conjecture that the formula of [1125] extends to non-isolated cycles, but this remains to be demonstrated from M-theory. There is evidence for the existence of infinitely many associative cycles in certain twisted connected sum G_2 -manifolds [1126–1128]. A Simons Collaboration on Special Holonomy in Geometry, Analysis, and Physics [511] has focused on problems in this general area and has been making excellent progress in the multitude of nontrivial problems presented by these geometric structures. See [1129] for a relatively recent review of some of the mathematical progress that has been made.

Geometric engineering with non-compact G_2 holonomy spaces has a relatively long-standing tradition starting with [1130]. M-theory compactification on G_2 holonomy manifolds gives rise to 4d $\mathcal{N}=1$ theories. In particular, such models have interesting particle physics applications: starting with the construction of pure Super-Yang-Mills theories [1131], and supersymmetric Standard-Model type models [1132]. The basic dictionary between gauge theory and geometry is as follows: codimension 4 singularities give rise to gauge groups (4d $\mathcal{N}=1$ vectors), matter in chiral multiplets arises from codimension 7 singularities, whereas non-chiral matter comes from codimension 6. The mathematical challenge, which to this moment is unsolved, is the construction of singular G_2 holonomy manifolds, which have codimension 4 and 7 singularities, thus giving rise to 4d gauge theories – coupled to gravity – which model (semi-)realistic models in 4d.

In the past few years, much progress in the context of the Simons Collaboration on special holonomy [511] has been made. Two areas in which there has been progress are addressing more global issues and the study of compact geometries with G_2 holonomy. These new compact geometries go beyond the class of torus orbiolds known as Joyce orbifolds [1133,1134]. The results in [1135] and unpublished work by Kovalev, suggested an infinite class of compact G_2 holonomy manifolds, which admit a fibration by K3-surfaces over three-manifolds (mostly S^3 and quotients thereof).

The geometries found in [1135] have already had numerous implications within physics: in [1136] models with conjectural codimension 4 and 6 singularities were constructed as singular limits of these geometries. Such K3-fibered G_2 holonomy compactifications have dual descriptions in string theory, F-theory and heterotic strings, where the generation of chiral matter is well-understood, and could guide the mathematical efforts in the construction of compact G_2 -geometries with chiral matter in 4d [1136–1140].

Another interface between geometry and physics arises in the context of M-theory/Type IIA limits: M-theory compactified on a circle gives rise to Type IIA string theory [1141]. Therefore, whenever

a G_2 manifold admits a circle-fibration, there should be a relation of the M-theory compactification to a Type IIA compactification. In general the IIA compactification will include D6-branes (which geometrize once uplifted to M-theory). A large class of new such G_2 geometries was uncovered in [1142], motivated by string theory, and their physics was discussed in [1143].

Mathematically the most important challenge remains the construction of compact but singular G_2 -manifolds, which have singularities both in codimension 4 and 7. Physically, these correspond to realizing gauge degrees of freedom (codimension 4 singularities e.g. of ADE-type) and chiral matter in 4d (codimension 7). Note that the singular limits of twisted connected sum G_2 -manifolds, which have been studied in the physics literature, have codimension 4 and 6 singularities, which only realized gauge groups and non-chiral matter, and cannot be deformed within this class of G_2 -manifolds to realize codimension 7 singularities that correspond to chiral matter [1138]¹⁹.

For Calabi-Yau manifolds, Yau's proof of the Calabi conjecture provides a simple criterion for SU(N) holonomy spaces. In the realm of G_2 and Spin(7) holonomy, no such theorem (or even conjecture) exists. This makes identifying geometries with exceptional holonomy particularly challenging to construct and identify, requiring highly advanced differential geometric and analytic tools.

Spin(7) compactifications in M-theory result in 3d $\mathcal{N}=1$ supersymmetry and in heterotic string theory with 2d $\mathcal{N}=(1,0)$ spacetime supersymmetry. The case of one supercharge is a particularly challenging subset of SQFTs, and furthering that study of Spin(7) holonomy can shed light on these QFTs. Most constructions (both in geometry and string theory) are obtained by considering antiholomorphic involutions on Calabi-Yau four-folds due to Joyce [1144], with several recent applications in terms of connected sum constructions in string theory [1145, 1146]. The main challenge is the resolution of singularities that arise in these constructions, in a way that is compatible with the Spin(7) structure.

9.3 Dualities Across Dimensions

In lower space-time dimensions, $D \leq 4$, one can construct weakly coupled field theories using standard Lagrangian techniques which describe interacting CFTs. In particular one can start from a free theory deformed by relevant couplings and flow to an interacting CFT in the IR, or, in certain cases in D=4, deform the free theory by an exactly marginal coupling and describe an SCFT without any RG flow involved. One can thus study CFTs, and SCFTs in particular, in $D \leq 4$, using a variety of purely field theoretic techniques. In cases when one has some supersymmetry, e.q. four supercharges in D=4, much non trivial physics in the IR can be directly and indirectly deduced from the Lagrangian descriptions. One very nice example is the notion of IR duality where two or more different weakly coupled theories flowing to the same theory in the IR [1147]. Another example is conformal duality where two or more weakly coupled (or partially weakly coupled) theories are related by tuning continuously an exactly marginal coupling [141,144,149,1148-1150]. A third example is the emergence of symmetry (flavor symmetry and supersymmetry) in the IR [148,1151–1156]. The list goes on. When studied directly in $D \leq 4$ these strong coupling phenomena can be hard to understand or predict, but new insight can be obtained from considering higher dimensional theories and employing dimensional reduction. For example using known IR dualities in D=4 one can place the relevant theories on a circle, which can be viewed as a supersymmetric relevant deformation, flow to low energies where the theory is effectively three dimensional and deduce IR dualities in D=3 and adding another circle one can go to D=2 [1157–1166]. In this manner many previously discovered dualities (e.g. [1167–1172]) in D < 4 can be related to D = 4 dualities [1173–1177]. However, what about the dualities (and other phenomena) in D=4?

An answer lies in connecting the lower dimensional QFTs to higher dimensional ones. One can start in D=6, the highest dimension admitting interacting supersymmetric SCFTs [1178] and turn on geometric deformations. In D=6 there are no interesting supersymmetric relevant deformations [1179]. However, we can deform a D=6 theory in an interesting way by placing it on a compact surface. We already have discussed going to D=5 by placing the D=6 models on a circle. This procedure can be generalized to placing the D=6 SCFTs on higher dimensional surfaces. For example, we can flow to D=4 by utilizing a Riemann surface. Such surfaces in general are curved and thus to preserve some supersymmetry we need to perform some twisting procedure. This procedure can be used to predict

¹⁹ Although one can get deformations that result in codimension 7 singularities, these always appear pair-wise and correspond to non-chiral matter.

existence of huge classes of D=4 SCFTs (and using three dimensional and four dimensional manifolds one can generalize to predicting existence of D=3 and D=2 SCFTs). One can read numerous properties of the lower dimensional theories directly from geometry. This includes, symmetries (zero form and higher form), anomalies, number and charged of relevant and marginal supersymmetric deformations, and in many cases duality relations. The canonical example of such a construction is generating $D = 4 \mathcal{N} = 2$ SCFTs starting with a (2,0) theory [141–144]: this is known as class \mathcal{S} . This has a vast generalization that lead to constructions of $D=4~\mathcal{N}=1$ SCFTs in [1180–1183]. The same techniques from (2,0) SCFTs can be applied also to studying compactifications of (1,0) SCFTs to generate even larger classes of $\mathcal{N}=1$ SCFTs in D=4 [1163, 1184–1191]. One can try to connect the explicit field theoretic constructions of theories in $D \leq 4$ and compactifications of D > 4 SCFTs. For example one can seek for D=4 field theoretic descriptions of the SCFTs obtained by compactifying D=6 SCFTs on a surface. Whether this can always be done and if not what are the obstructions, are important open problems. If one does find such a connection between compactifications and D=4Lagrangians one can speak about an across dimensions IR duality: we have a D=6 SCFT deformed by geometry flowing to a D=4 SCFT deformed by a relevant coupling flowing to the same SQFT. Once such a dictionary between D = 4 QFTs and D = 6 QFTs is established many of the otherwise sporadic lower dimensional understandings can be explained and predicted using the geometry explicitly present in the construction. A similar program could be pursued starting from D=5 SCFTs and flowing to D = 3 [1192, 1193].

More generally we can start from different D=6 SCFTs and deform them by different geometries. In general we would flow to different lower dimensional SQFTs. However in some cases we can end up with same lower dimensional models. In this case we can speak of D=6 IR dualities. Explaining and understanding such dualities probably should be done by going back to the geometric construction of D=6 SCFTs themselves in string/M/F-theory. Some examples of D=6 IR dualities can be found in [149, 1186, 1194–1196].

This line of research leads to many conceptual questions in SQFTs as well as to a lot of interesting physical mathematics. Let us list some of the interesting questions one can address.

- 1. Lagrangians vs no-Lagrangians: Can any lower dimensional SCFT geometrically constructed starting with higher dimensional one have a lower dimensional IR (or conformal) dual? Many of the theories obtained in compactifications are often called non-Lagragian paying tribute to the fact that a Lagrangian dual is not known at a certain point in time. So this question can be phrased as whether non-Lagrangian theories in lower dimensions truly exist or not? For example, a canonical instance of a "non-Lagrangian" theory, the Minahan-Nemeschansky E_6 $\mathcal{N}=2$ SCFT [1197], has several Lagrangian $\mathcal{N}=1$ constructions [150, 1198, 1199]. When one does find Lagrangian duals often the UV weakly coupled theories exhibit less symmetry, and in some cases supersymmetry, than the IR fixed point (See references above). To find across dimension dualities one thus needs to give up insisting on having all the symmetries of the IR manifest. In some cases a D=4 Lagrangian dual is not known, however constructions exist which start from a vanilla Lagrangian in the UV but then tune the parameters (couplings or flow to the IR) so that certain symmetries emerge and are gauged [1184, 1185, 1189, 1199–1202] (See also [1195] for a D=5 example). Thus one can wonder whether restricting to such constructions one can construct any conceivable lower dimensional SCFT. The same question can be raised regarding the higher dimensional SCFTs. We can try to construct D=6 or D=5 SQFTs giving up some symmetry and in particular space-time symmetry. For example one can consider D=4quiver gauge theories in the limit the size of the quiver is large and obtain D=6 SCFTs in a process called dimensional deconstruction [1203–1205].
- 2. Geometric origin of lower dimensional SCFTs: Do all lower dimensional SCFTs have a geometric construction starting with higher dimensional supersymmetric ones? The geometric construction starts with a (1,0) SCFT in D=6 which has eight supercharges so the question can be phrased as whether any supersymmetric SCFT (or even non-supersymmetric CFT) in lower dimensions is related to a theory with eight supercharges.
- 3. Structure of the space of lower dimensional QFTs: SQFTs in lower dimensions are interconnected by relevant and exactly marginal deformations. What is the structure of this space? How much of it is determined by higher dimensional geometry? When the D=4 SCFTs are obtained by direct dimensional reduction of D=6 SCFTs one can generally identify the

relevant and the marginal deformations with the KK reduction of the stress-tensor (leading to exactly marginal deformations related to complex structure moduli) and the conserved currents of the D=6 SCFT [1180, 1201, 1206, 1207]. An example of an interesting question here which received little attention is of studying and classifying $\mathcal{N}=1$ conformal dualities in D=4, that is weakly coupled cusps of D=4 conformal manifolds [1150]. Similarly, the structure of conformal manifolds in D=3 seserves much more intense investigation, see e.g. [1208–1210].

- 4. Structure of moduli spaces: A related question is the geometry of the moduli spaces of the SCFTs. The moduli spaces of theories with eight supercharges have deep connections with mathematical physics and physical mathematics [370,962]. One approach to this subject, which has been vigorously pursued in recent years, is to classify the possible geometries of D=4 $\mathcal{N}=2$ Coulomb branches and relate this to classification of D=4 $\mathcal{N}=2$ SCFTs [1211–1217]. One can also study correlation functions of Coulomb branch operators in D=4 $\mathcal{N}=2$ using localization methods leading to beautiful results [1218]. Yet another fruitful direction is to try to understand the moduli spaces of the strongly coupled SCFTs in D=5 and D=6 (again with eight supercharges) using various group theoretical methods (the so called magnetic quivers) [1080, 1087, 1219, 1220], see above. Finally, one can also consider the interplay between moduli space flows in higher dimensions, compactifications, and lower dimensional moduli spaces [1221]. This leads to interesting interconnections between geometric engineering of lower dimensional SCFTs and the deeper structure of moduli spaces. In particular studying the interplay between flows and moduli spaces has already led to derivations of across dimensional dualities [1198,1200] and studying this direction further could lead to many novel insights.
- 5. Mirror symmetry in lower dimensions and physics in D=4: Mirror symmetry in D=3 and D=2 is an extremely rich laboratory for deriving and studying results in physical mathematics. We understand well string theoretic and brane origins of many instances of such dualities: this structure is tightly tied to eight supercharges which for example allows for a clean distinction between Higgs and Coulomb branches. However, one can ask whether these dualities can also be understood purely using field theoretic constructions starting in D=6. In particular, we can try to find D=4 dualities which lead upon dimensional reduction to mirror symmetry. This necessarily leads to constructions with lower amounts of symmetry and supersymmetry. In fact some of the canonical cases of mirror symmetry in D=3 were recently imbedded in $\mathcal{N}=1$ D=4 dynamics [1163–1165]. What is the mathematical implication of such an embedding?
- 6. Supersymmetric quantities: Supersymmetry allows us to compute various quantities exactly with no regard to the values of coupling or the scale of RG flow. Often, if not always, such quantities can be related to counting problems of protected operators: various types of supersymmetric indices and limits thereof. This leads to deep relations between the theory of special functions and supersymmetric physics. For example, various dualities lead to non-trivial identities of elliptic-hypergeometric integrals and limits thereof, Painlevé tau-functions, etc. A set of pairs of concrete examples of the intimate relation between the mathematical works and the supersymmetric physics is [1147, 1222], [1223, 1224], [1163, 1225], [1226, 1227], and [1228, 1229]. Many times these quantities can be related to topological, conformal, and integrable QFTs in even lower dimensions (not necessarily supersymmetric). The canonical, concrete, and most studied example of those relations, generally called the BPS/CFT correspondence [302, 737], is the AGT correspondence [739] (See also [1230–1233]), relating specific four dimensional quantum field theories with $\mathcal{N}=2$ supersymmetry and specific two dimensional conformal field theories. See [1234] for a comprehensive review and list of references. Another example is the association of an integrable quantum mechanical system to supersymmetric theories with two dimensional $\mathcal{N}=2$ super-Poincare invariance, the Bethe/gauge correspondence [341], including those originating as four or six dimensional theories [1184, 1190, 1223, 1224, 1229, 1235–1241]. In the geometric program of relating compactifications of six dimensional theories to four dimensional ones the quantum mechanical integrable models appear for example as operators acting on supersymmetric indices [674–676] of the four dimensional theories introducing surface defects in to the index computation [1235]. For a given six dimensional theory one then obtains one, or more [1190] related, integrable models. The problem of classifying D=6 SQFTs thus can be tied to studying properties of spaces of integrable quantum mechanical models. In addition to

the above, the supersymmetric quantities often have surprising features appearance of which is not always understood, see e.g. [1242].

9.4 Holography And Classification of QFTs

Another fruitful direction of research that complements the geometric engineering methods above is to study the space of AdS solutions in string or M-theory which are related to reduction of higher-d theories on various manifolds similar to class S. For example, the classification of holographic duals of $\mathcal{N}=1$ SCFTs in D=4 in M-theory corresponds to characterising the space of supergravity solution in D=11 of the form $AdS_5 \times M_6$ [1243]. In the case of theories that are dual to reductions of 6D SCFTs on Riemann surfaces, the supergravity equation reduces to a single Monge-Ampère equation whose solution space governs the allowed choices of M_6 and therefore the space of such SCFTs [1244,1245]. These equations generalize the $SU(\infty)$ Toda system that governs the holographic duals of $\mathcal{N}=2$ SCFTs [159,1246]. Similarly the classification problem for the holographic duals of $\mathcal{N}=1$ SCFTs from massive type IIA corresponds to solving a pair of Monge-Ampère equations [1247].

Of course, the very Calabi-Yau metric is a solution to a (complex) Monge-Ampére equation on its Kähler potential, and the toric Calabi-Yau metrics are similarly described by the solutions of real Monge-Ampére equations in three dimensions.

The compactification program of SCFTs can be studied in holography by considering near horizon limits of branes systems wrapped on a compact manifold. The canonical examples, which have lead to much interesting progress, are obtained from geometric duals of the topological twist, as initiated in [1248]. There must exist a wider class of constructions beyond the paradigm of the topological twist where branes wrap non-constant curvature manifolds. A systematic classification of what is possible subject to the constraint of supersymmetry is lacking. Indeed this is an active arena of research. Recent important progress in these directions come from branes wrapping "spindles" (manifolds with orbifold fixed points) [1249–1252] and discs with interesting holonomies at their boundary [1253,1254]. The latter methods have lead to the construction of holographic duals of Argyres-Douglas theories. For further developments see [1255–1259].

In general the classification of BPS equations that describe the AdS duals of systems of branes and their reductions on various manifolds have been obtained from reducing supergravity with G-structure. This has been the program of Gauntlett, Martelli, Sparks, Waldram, Kim over a large body of literature [1054, 1260–1267]. (Another approach uses pure spinors. See for example [1268] for Poincaré compactification.) With the classification of possible BPS structures from supergravity, It is an important question to understand the solution space of the PDEs and their associated AdS geometries. One of the important successes in this program is the classification of holographic six dimensional SCFTs by the explicit realization of all AdS_7 solutions of supergravity in [1269] and their characterization in [1270]. A similar classification program for five dimensional holographic SCFTs in type II supergravity theories was initiated in [1271] where the basic PDEs that govern all AdS_6 solutions are established. An important objective will be to understand the full solution space of such PDEs.

Significant steps have been made in understanding and characterizing the AdS_6 solutions in type IIB and their field theory duals in [1272,1273]. More recently in [1274–1277] a systematic construction of a large family of AdS_6 solutions is obtained, these are dual to (p,q) brane engineering of five dimensional SCFTs.

Another interesting direction in holography is a systematic characterization of Holographic RG flows [1278] by the study of geometric flows of metrics. Flow equations are obtained in supergravity where they interpolate between $AdS \times \Sigma$. Here different Σ 's identify different CFT fixed point. The metric flow can encode various aspects of deformability of CFTs and provide a mathematical framework for exploring the space of CFTs. In particular important results about uniformatization of metric flows in the context of class $\mathcal S$ are described in [1279].

As importantly, using holographic RG flows one can study various aspects of extremalization principles that can fix volumes of cycles in fixed backgrounds, and determine existence conditions of specific metrics in a given Kähler class [1280–1282]. Such principles can be used to identify dual counterparts of field theory phenomena such as a-maximization in four dimensions [1283] and c-extremization in two dimensions [1284, 1285].

10 Some Important Topics In Physical Mathematics Not Covered Here

We are painfully aware that there are numerous exciting developments, and indeed entire sub-fields of physical mathematics which have not been covered in the above review. Some, but not all, of these omitted topics were covered in a previous section 10 of this review. That section has evolved into a separate publication [18]. Among the topics to be covered in [18] are

- 1. Relations to the higher algebra of operads, BV algebras, etc.
- 2. Further development of topological string theory and string field theory
- 3. Open-closed string duality
- 4. Topological Recursion and non-perturbative Dyson-Schwinger
- 5. Twisted, topological, and hybrid holography
- 6. Numerous connections to integrability, including 4d CS, Bethe/gauge and ODE/IM correspondences, QQ-systems and qq-characters.
- 7. Emergent and hidden hydrodynamics

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