2. Chaotic motion

What is chaos?

- In these lectures we study (simple) nonlinear deterministic models that can exhibit chaotic behavior

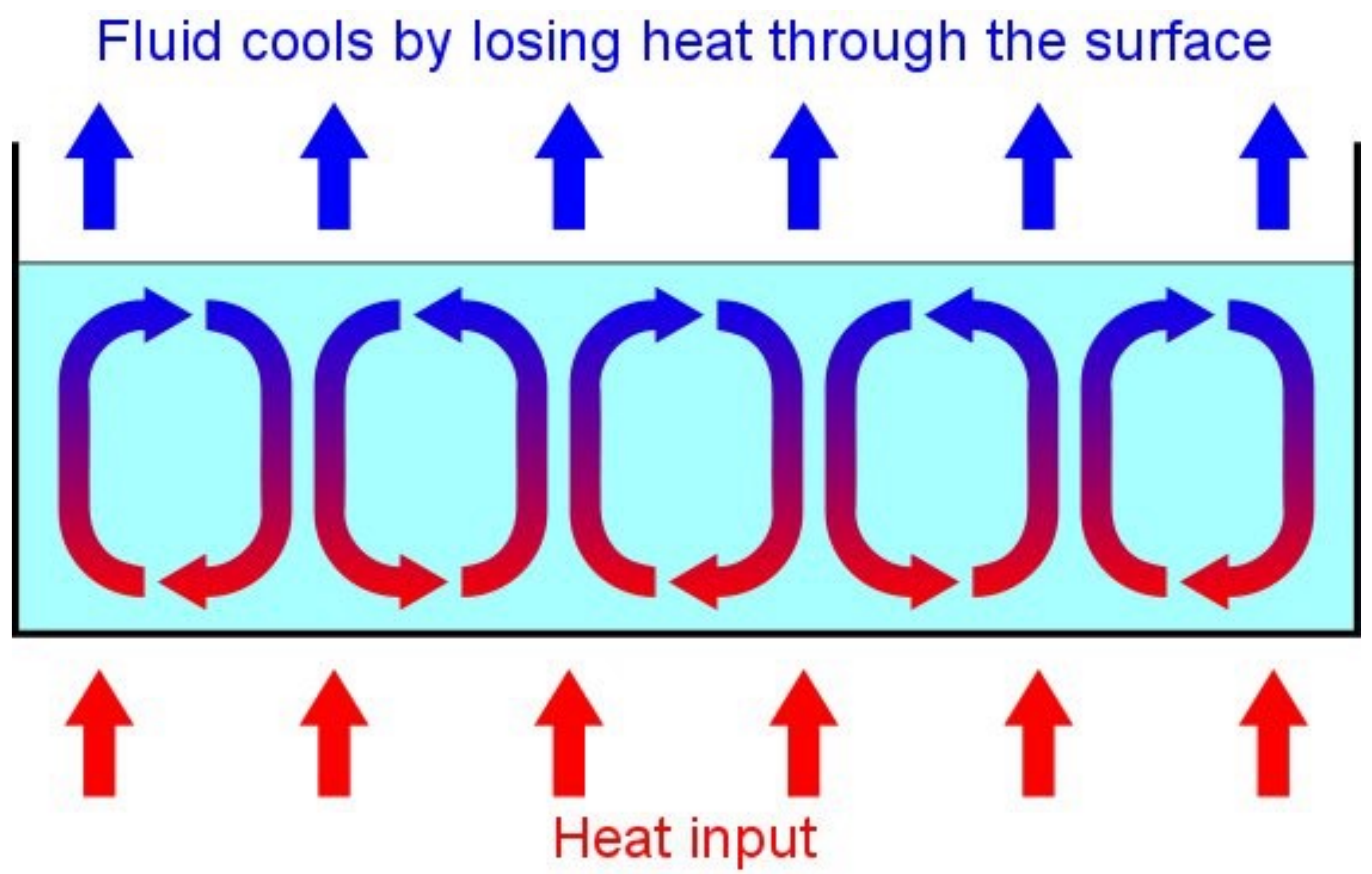
- The difference between chaos and noise?

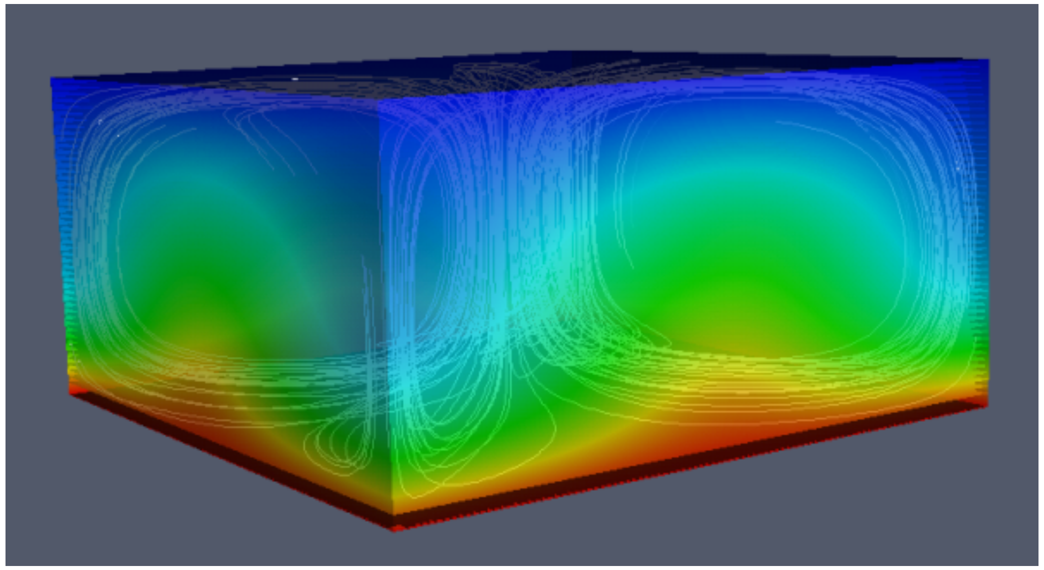
When do we encounter chaotic motion?

- “Top-down”:
  - You observe chaotic behavior.
    What rules govern this?
- “Bottom-up”:
  - You have a set of non-linear differential equations.
    Are the solutions chaotic or not?

- Example: butterfly effect in weather forecasting

Rayleigh-Bénard convection





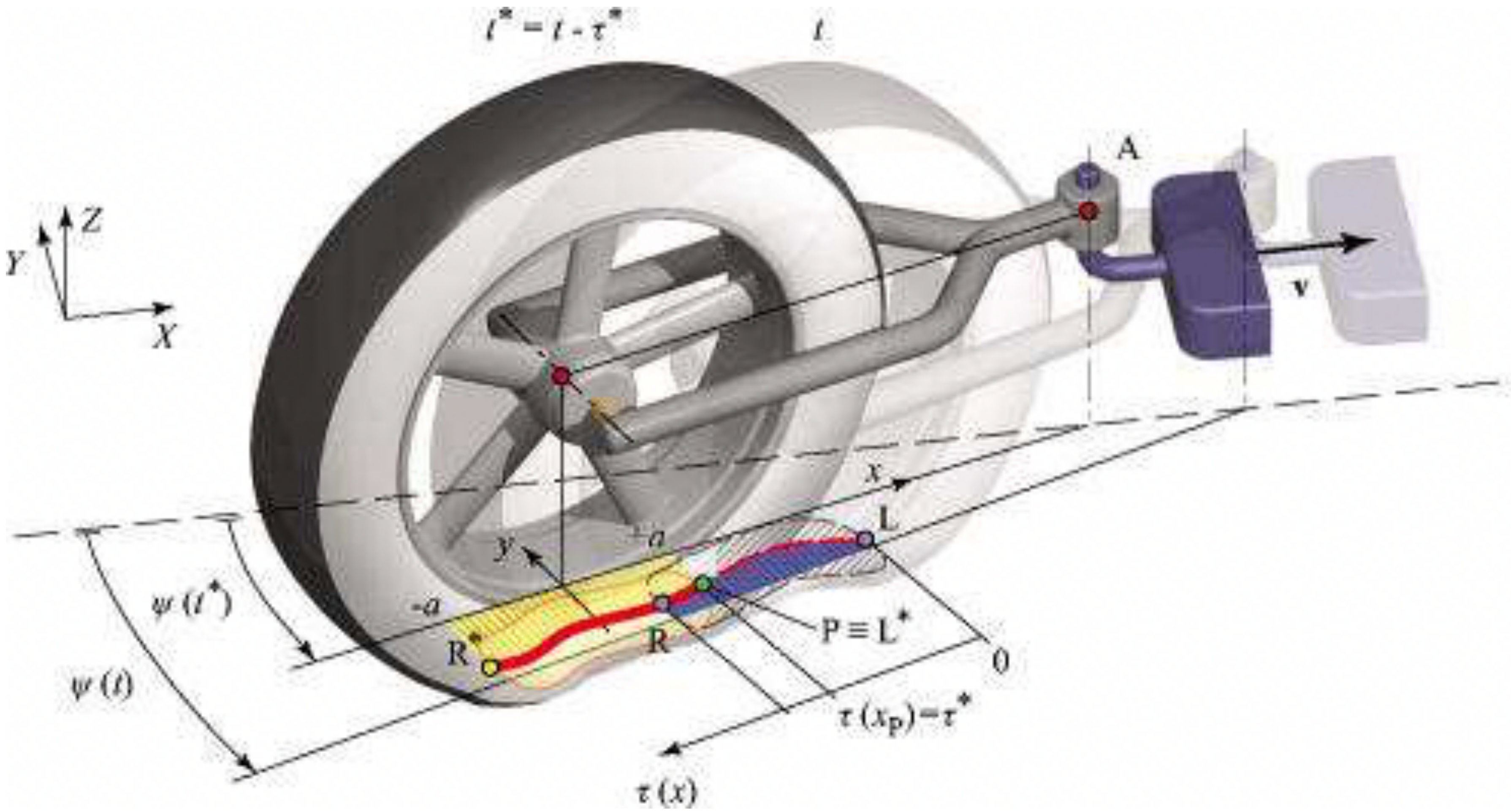
Chaos theory

- For a dynamic system to be chaotic:

1. it must be sensitive to initial conditions
2. it must be topologically mixing
3. it must have dense periodic orbits

Example: speed wobble

- The steering wheel(s) on a vehicle, e.g. plane, shopping cart, ...



Population dynamics

$$P_{n+1} = P_n(a - b\,P_n)$$

a_n : population in year
b_n : coefficient for natural growth
P_n : coefficient for reduction by overcrowding or spread of disease

Rescaling: $P_n = (a/b)x_n$

Define: $r = a/4$

$$x_{n+1} = f(x) = 4r\,x_n(1 - x_n)$$

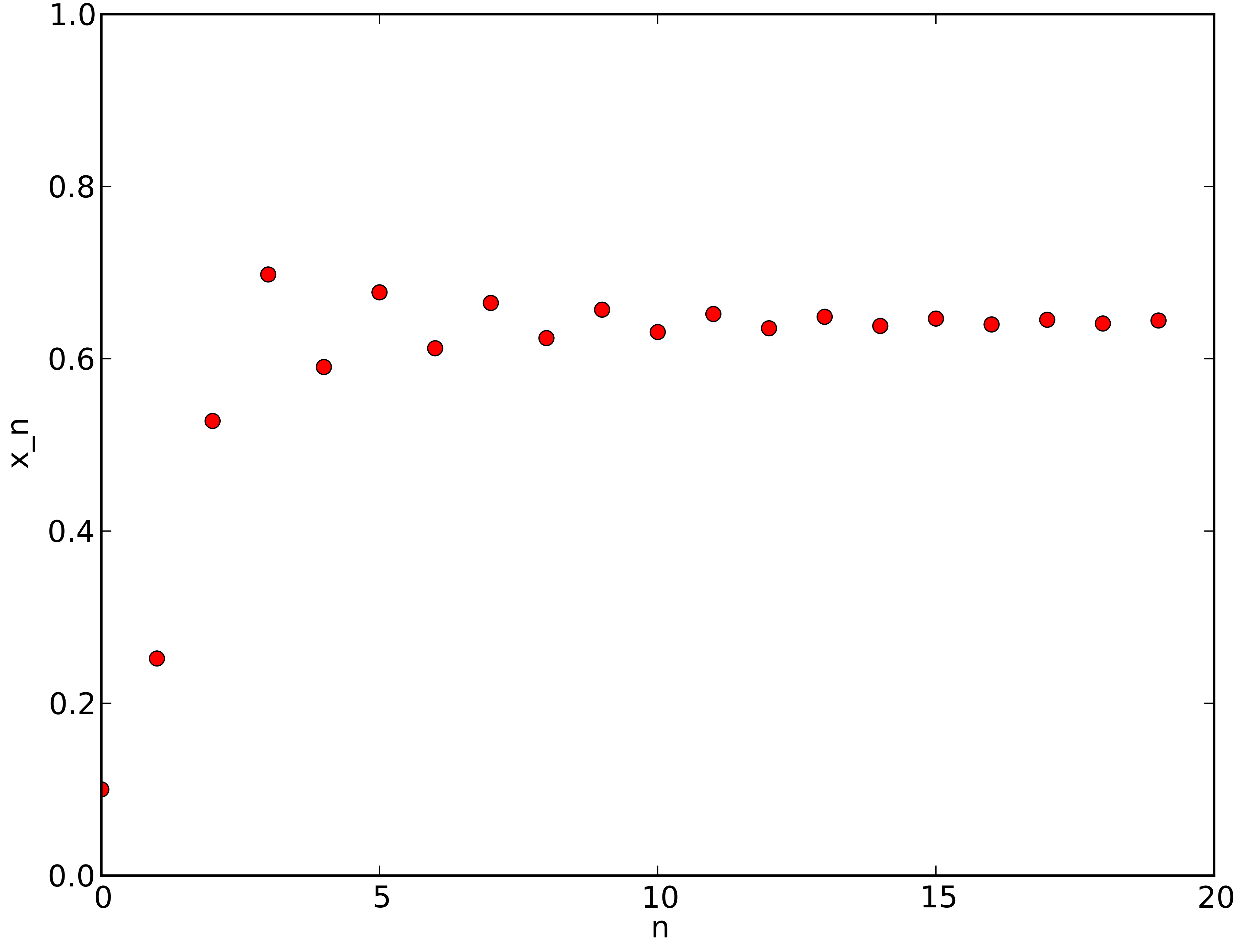
We impose $0 \le x \le 1\ ,\ \ 0 < r \le 1$

Logistic map

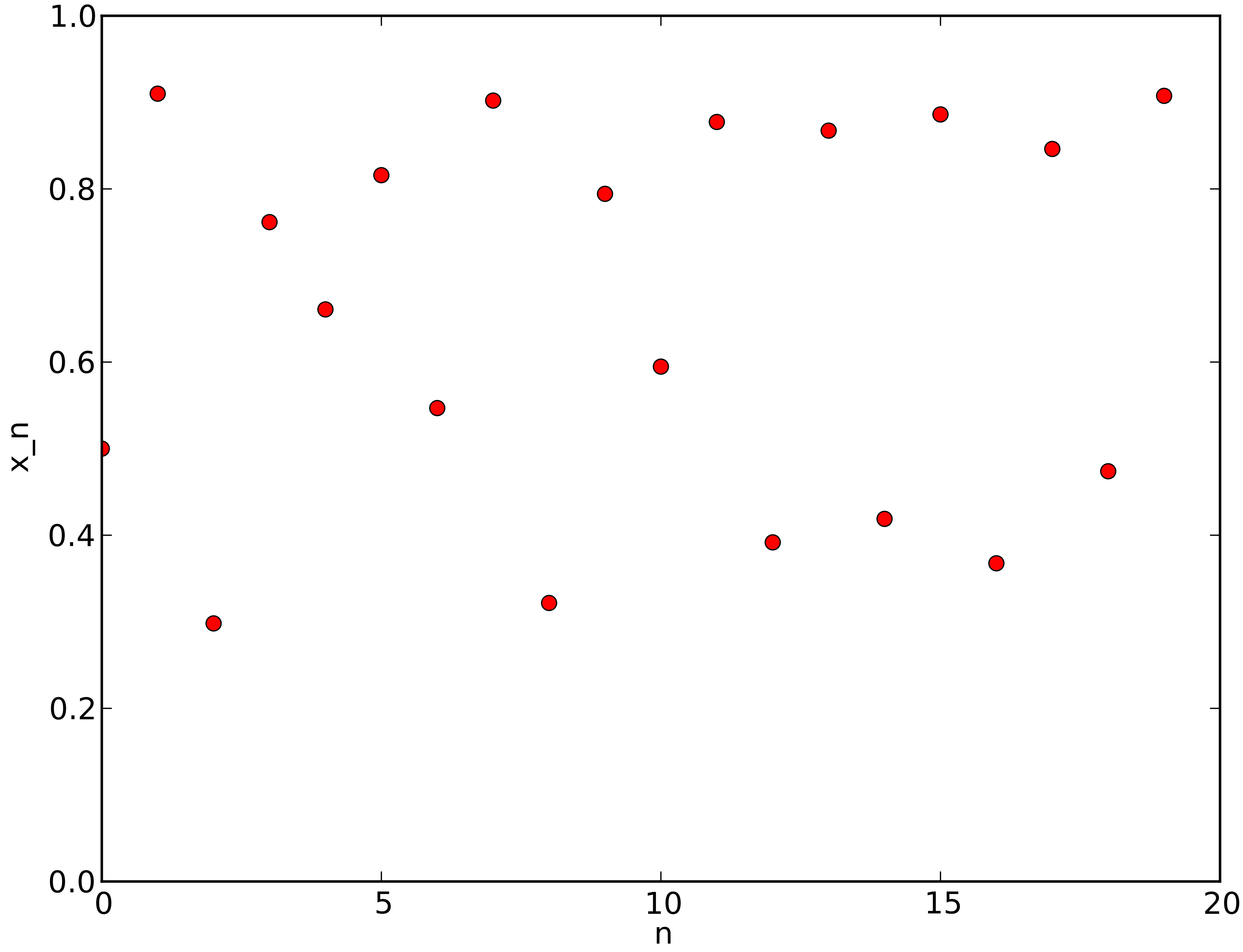
$$x_{n+1} = f(x) = 4r\,x_n(1 - x_n)$$

- $f(x)$ is a logistic map
- it maps any point in [0, 1] to another point in [0, 1]
- it is a deterministic prescription to find the future state given the present state of the system

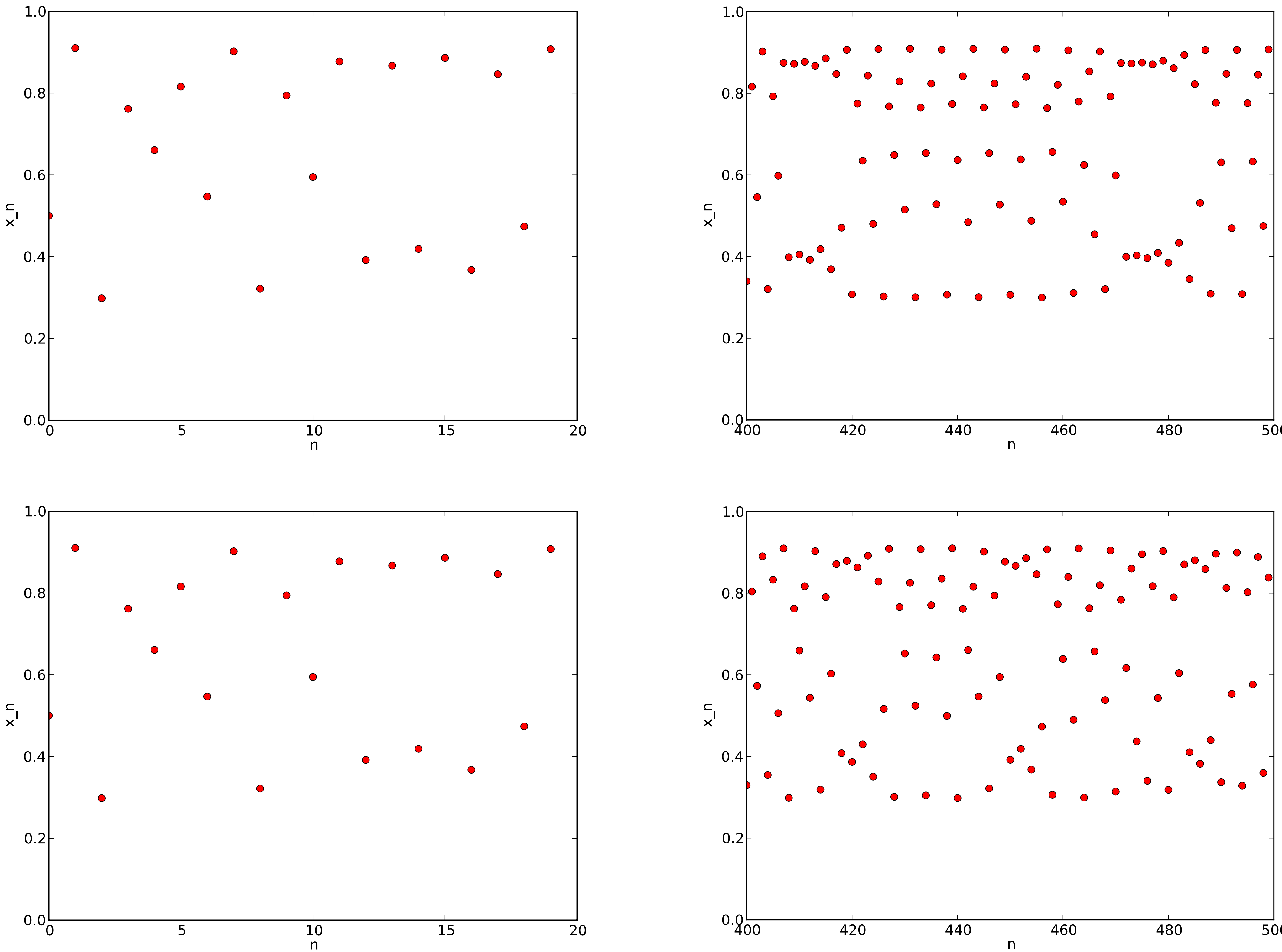
$r = 0.7, x_0 = 0.1$



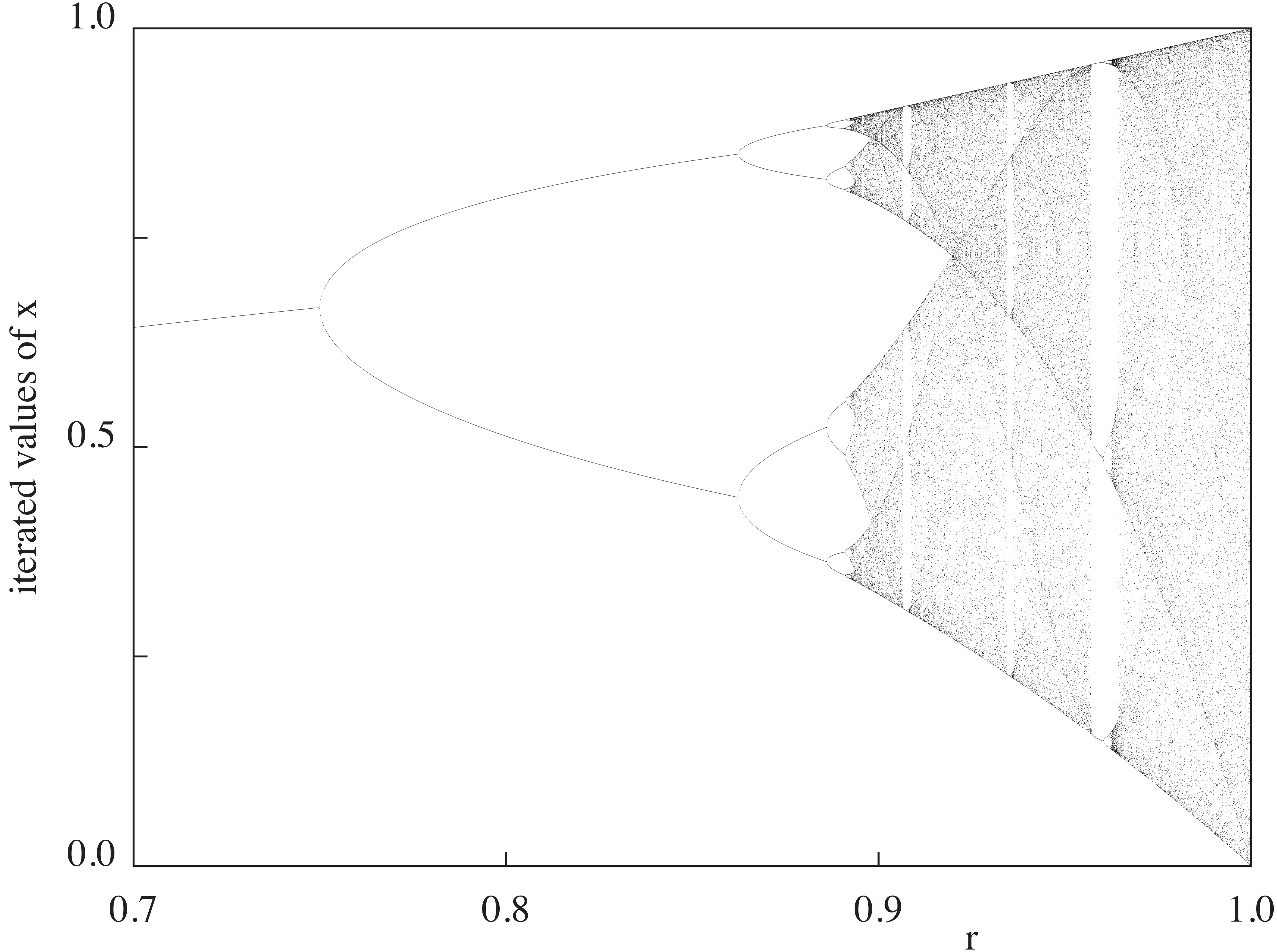
$r = 0.91, x_0 = 0.5$



r = 0.95, x_0 = 0.5 and x_0 = 0.5001



Bifurcation diagram



Logistic map

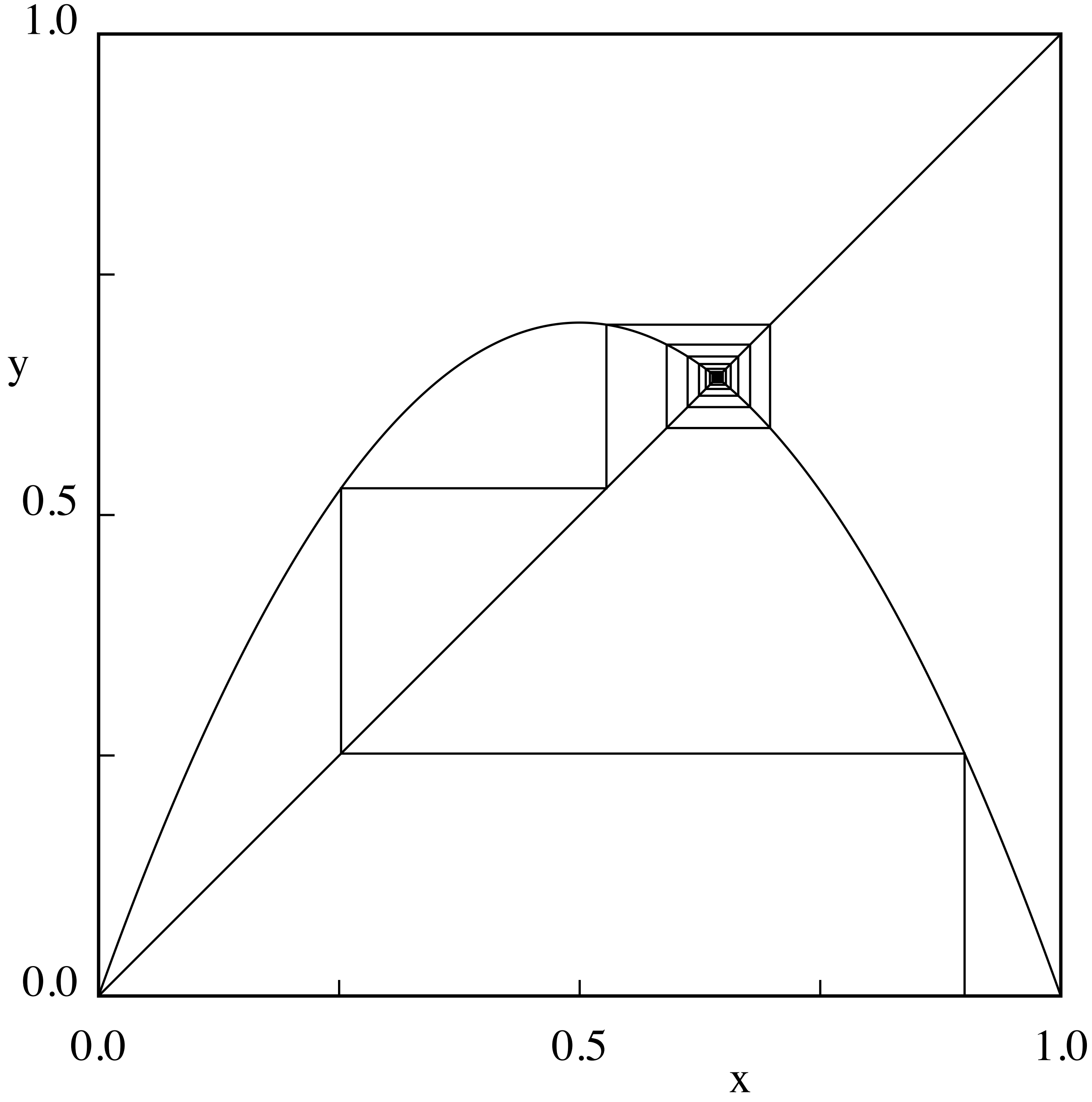


Figure 6.3: Graphical representation of the iteration of the logistic map (6.5) with $r = 0.7$ and $x_0 = 0.9$. Note that the graphical solution converges to the fixed point $x^* \approx 0.643$.

Difference evolution

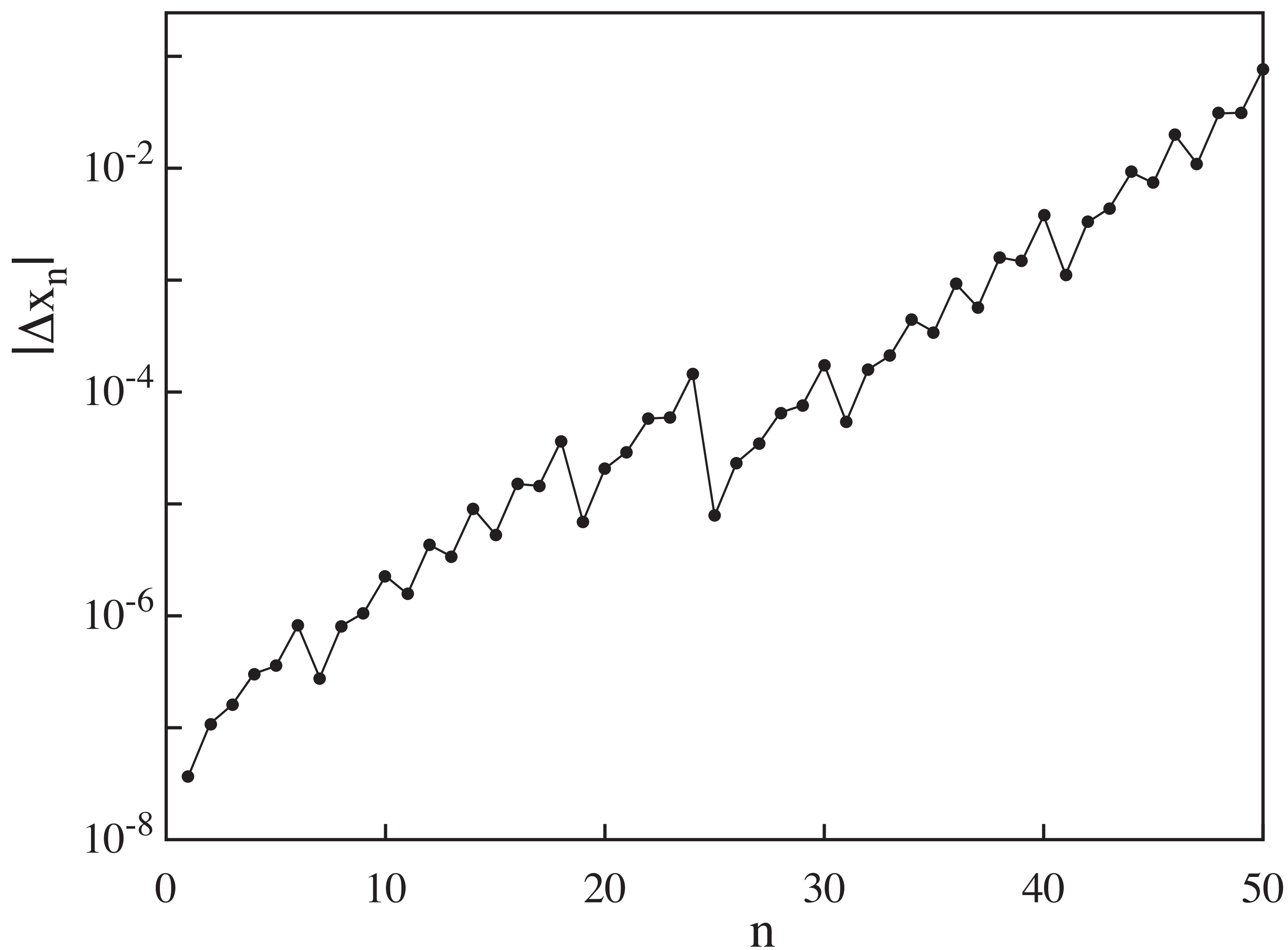


Figure 6.8: The evolution of the difference Δx_n between the trajectories of the logistic map at $r = 0.91$ for $x_0 = 0.5$ and $x_0 = 0.5001$. The separation between the two trajectories increases with n, the number of iterations, if n is not too large. (Note that $|\Delta x_1| \sim 10^{-8}$ and that the trend is not monotonic.)

Lyapunov exponent

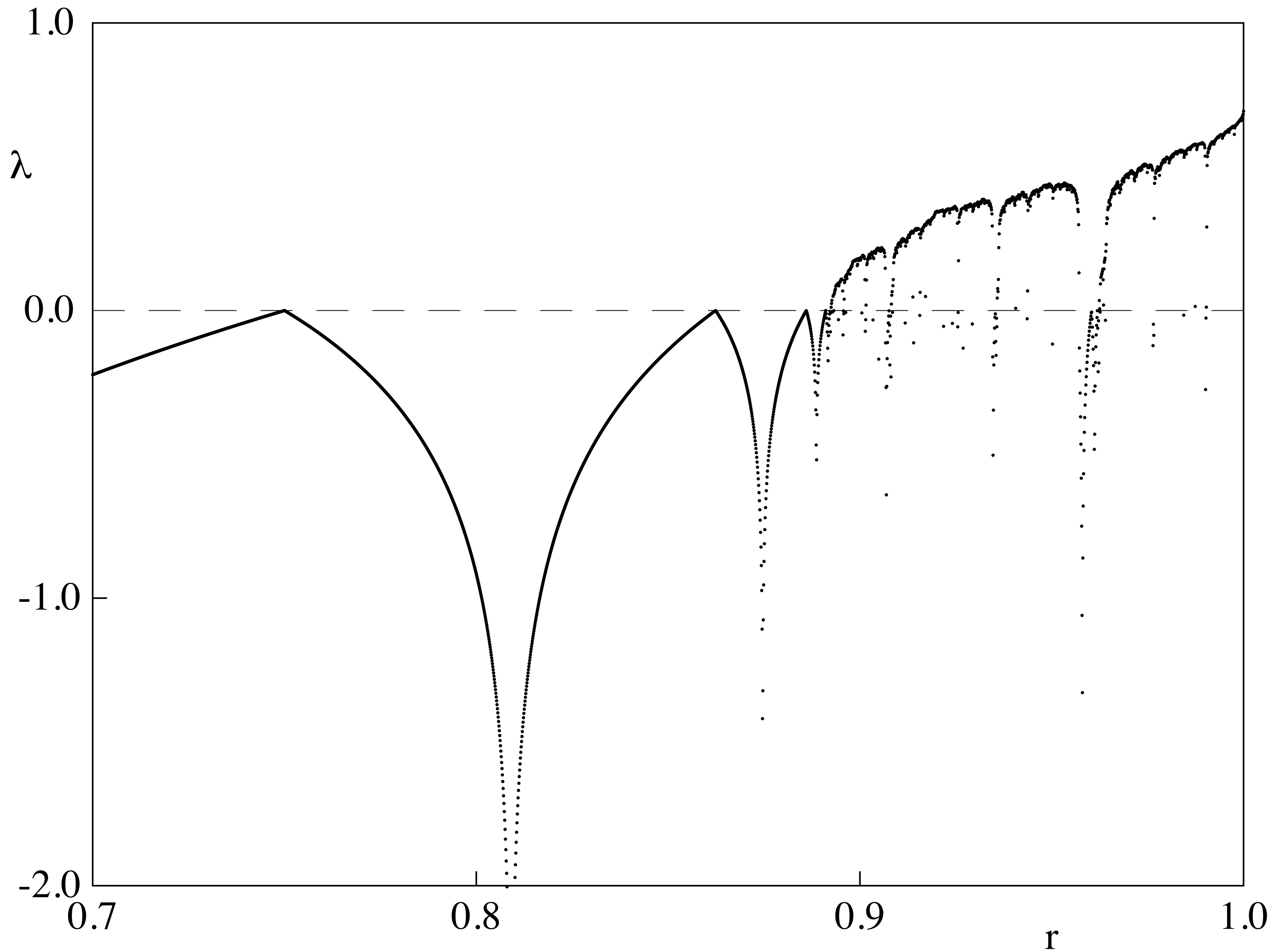
Lyapunov exponent λ: $$|\Delta x_n| = |\Delta x_0| e^{\lambda n} \tag{5}$$

After taking the logarithm: $$\lambda = \frac{1}{n} \log \left| \frac{\Delta x_n}{\Delta x_0} \right| \tag{6}$$

Rewriting: $$\lambda = \frac{1}{n} \sum_{i=0}^{i=n-1} \log \left| \frac{\Delta x_{i+1}}{\Delta x_i} \right| \tag{7}$$

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{i=n-1} \log |f'(x_i)| \tag{8}$$

Lyapunov exponent



Hamiltonian chaos

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z)$$

- Constants of motion
  - for time-independent systems: total energy, total momentum …
- Integrability
- More degrees of freedom than constants of motion: possibly chaotic

Double pendulum

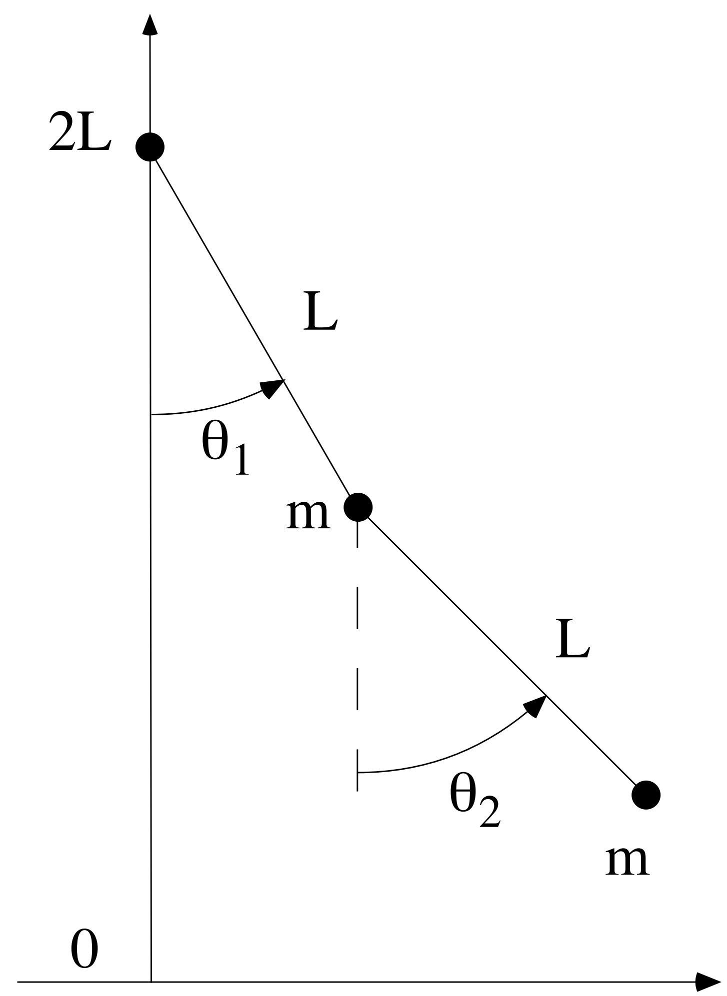


Figure 6.12: The double pendulum.

Double pendulum Poincaré plot

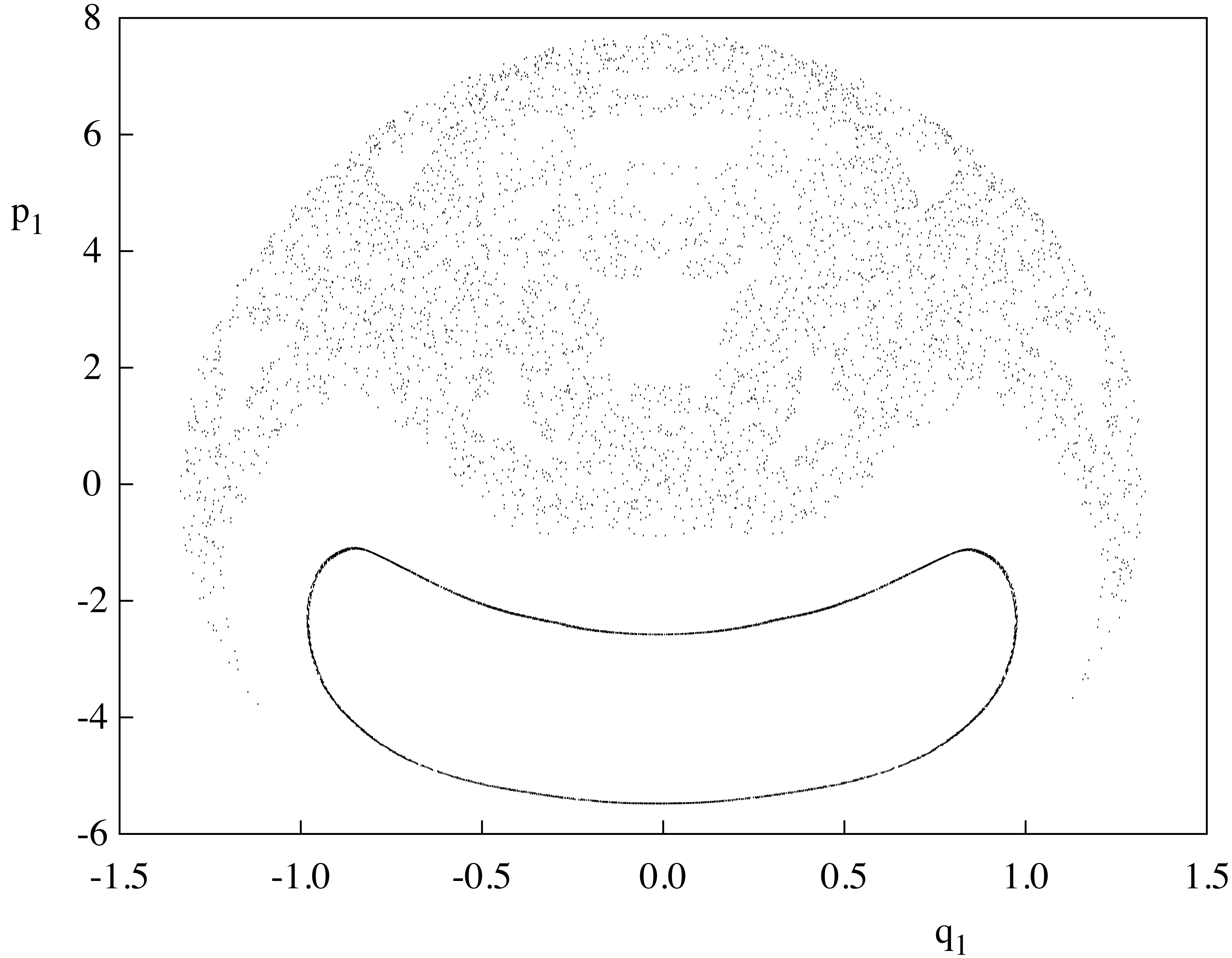


Figure 6.13: Poincaré plot for the double pendulum with p_1 plotted versus q_1 for $q_2 = 0$ and $p_2 > 0$. Two sets of initial conditions, $(q_1, q_2, p_1) = (0, 0, 0)$ and $(1.1, 0, 0)$ respectively, were used to create the plot. The initial value of the coordinate p_2 is found from (6.52) by requiring that $E = 15$.

The Lorenz model

$$\frac{dx}{dt} = -\sigma x + \sigma y \tag{10}$$

$$\frac{dy}{dt} = -xz + rx - y \tag{11}$$

$$\frac{dz}{dt} = xy - bz \tag{12}$$

- x: fluid flow velocity of circulation around the cell
- y: temperature difference between rising and falling regions
- z: difference in temperature profile between top and bottom from normal equilibrium profile
- Dimensionless parameters σ, r, b

References

- Wikipedia has a good page and a link to the Lorenz attractor:

  http://en.wikipedia.org/wiki/Chaos_theory

Projects: Simple population dynamics

2.1) Consider the population model $x_{n+1} = 4\,r x_n(1 - x_n)$. For $r > r_\infty = 0.892486417967\ldots$ this system exhibits chaotic behavior. The accuracy of floating point numbers retained on a computer is finite. Choose r=0.91 and x_0=0.5 and compute 200 iterations. See what changes when you round x to 6 digits after the decimal at every step (`round(x,6)` in python). Do you find the same discrepancy for $r < r_\infty$?

2.2) Calculate the Lyapunov exponent using formula (8) for some values of r between 0.76 and 1.0, start summing at i=20 to skip the initial transient. Does λ depend on x_0? What is the sign of λ if the system is not chaotic? What is the effect of the rounding of question 1) on λ?

Projects: Lorenz model

2.3) Use a Runge-Kutta integrator (RK4) to obtain numerical solutions of the Lorenz equations. Explore the basin of the attractor with σ=10, b=8/3 and r=28, i.e. starting from which parts of space do you end up in the attractor

2.4) Determine qualitatively the sensitivity to initial conditions. Start two point very close to each other and watch their trajectories.

2.5) Let z_m denote the value of z where z is a relative maximum for the mth time. You can determine the value of z_m by finding the average of the two value of z where the right hand of (12) changes sign from positive to negative. Plot z_{m+1} versus z_m and describe what you find. This procedure is one way that a continuous system can be mapped onto a discrete map. What is the slope of the z_{m+1} versus z_m curve? Is its magnitude always greater than unity? If so, then this behavior is an indication of chaos. Why? (more difficult question)

Note that is it easier to see when plotting the point without lines.

Projects: A spinning magnet

Note: Optional sub-project, because of the short preparation time

2.6) Consider a compass needle that is free to rotate in a periodically reversing magnetic field which is perpendicular to the axis of the needle. The equation of motion of the needle is given by:

$$\frac{d^2\phi}{d\,t^2} = -\frac{\mu}{I} B_o \cos \omega t \; \sin \phi$$

where ϕ is the angle of the needle wrt a fixed axis along the field, μ is the magnetic moment of the needle, I is the moment of inertia amd B_0 and ω are the amplitude and the angular frequency of the field. Choose an appropriate numerical method for solving the equation and plot the Poincaré map (ϕ,ϕ') at times $t = 2\pi\, n/\omega$ (n=0,1,2,. . .). Verify that if the parameter $\lambda = \sqrt{2B_o\mu/I/\omega^2} > 1$, then the motion of the needle is chaotic. (Note: there is no template, write your own program)