S1336 - Project 2

Erik Weilow

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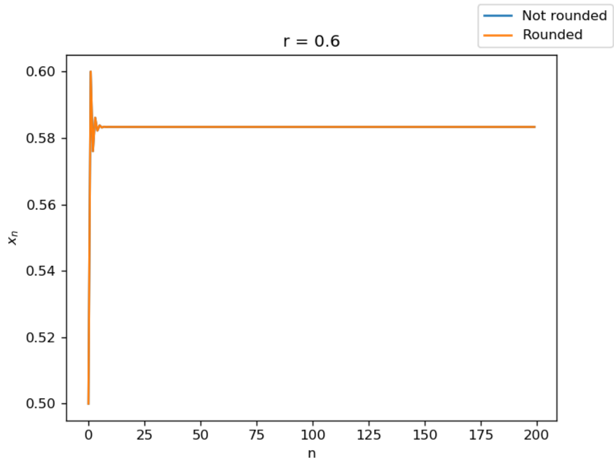
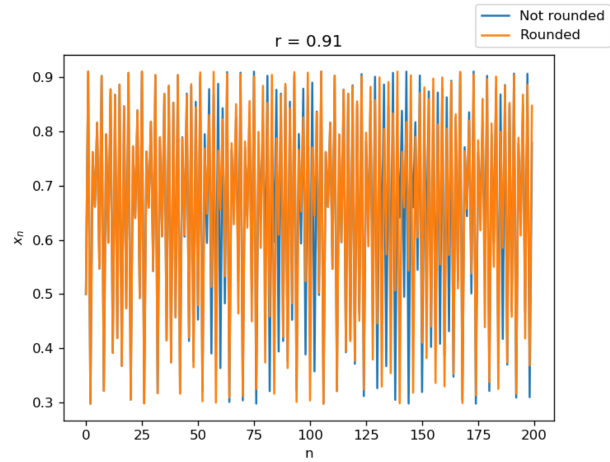
2.1

Effects of rounding errors





It can be concluded that when $r < r_\infty$, rounding doesn't have much of an effect on the system.

When $r > r_\infty$, rounding does however have an effect on the system.

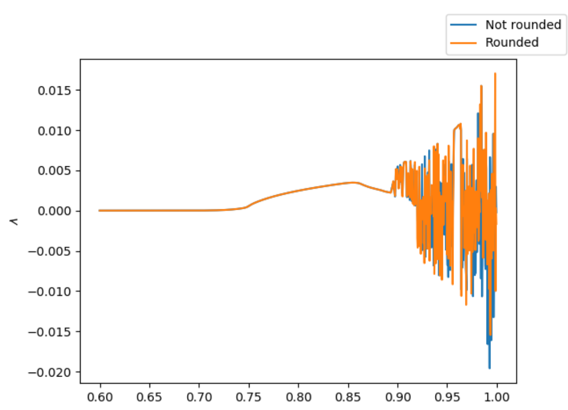
2.2

If settling for $n = 200$, we can use the following formula to calculate λ:

$$\lambda = \frac{1}{n} \sum_{i=0}^{i=n-1} \log \left| \frac{\Delta x_{i+1}}{\Delta x_i} \right|$$

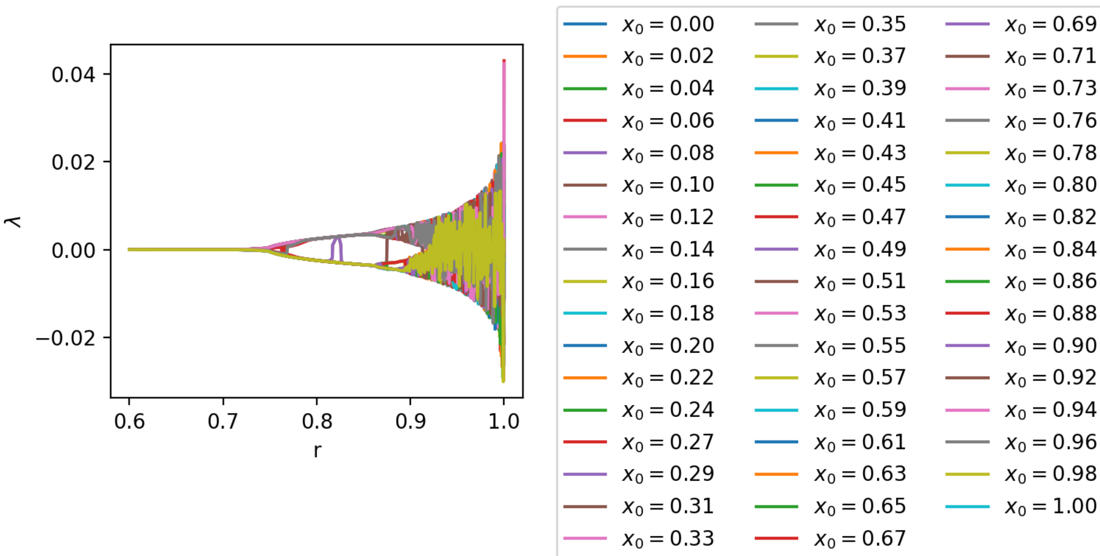
This can be done numerically by solving the system, for many values of r.

The Lyapunov exponent for a population model



We see that for $0.76 < r < r_\infty$, when the system isn't chaotic, the sign of λ is positive. Effects of rounding doesn't affect λ unless $r > r_\infty$.

The dependence on initial value



In this plot, we see that selecting different x_0 has an effect on the calculation of λ. For $r < 0.76$, it's clear that the system is non-chaotic since λ is strictly equal to 0. For approximately $0.76 < r < 0.87$, λ is either positive or negative depending on the initial value. It then takes on many values, incredibly dependent on x_0.

2.3

Lorenz attractor

To figure out which parts of space get attracted in to the basin of the Lorenz attractor, trajectories for different starting conditions randomly sampled on the interval

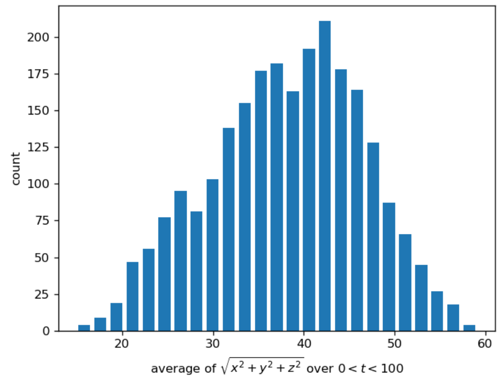
$$x(0) \in (-1000, 1000), \quad y(0) \in (-1000, 1000), \quad z(0) \in (-1000, 1000)$$

are computed.

The time averaged "radius" is calculated for each trajectory:

$$r = \frac{1}{n} \sum_{i=0}^{n} \sqrt{x(i\Delta t)^2 + y(i\Delta t)^2 + z(i\Delta t)^2}$$

25 CPU hours later, for a time step of $\Delta t = 0.00008$ and a time boundary of $0 < t < 150$, the following histogram can be plotted:

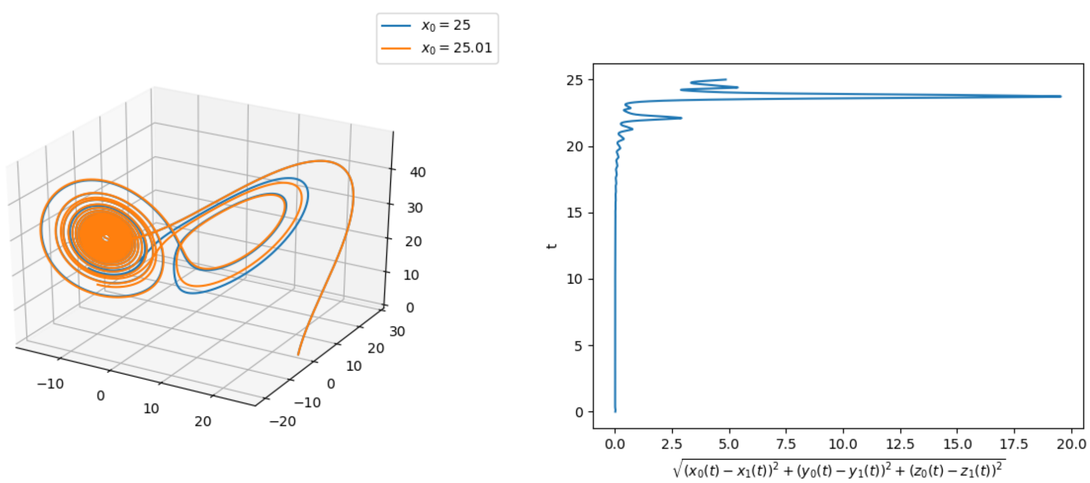


It shows that for all (roughly 2500) computed trajectories, they all average to within a distance 60 from $x = 0, y = 0, z = 0$. We can thus conclude that it's very unlikely that a solution doesn't get attracted into the basin.

It's also interesting to note that the distribution is starting to look like a normal distribution. This could be investigated further in the future and left as an excercise for the reader.

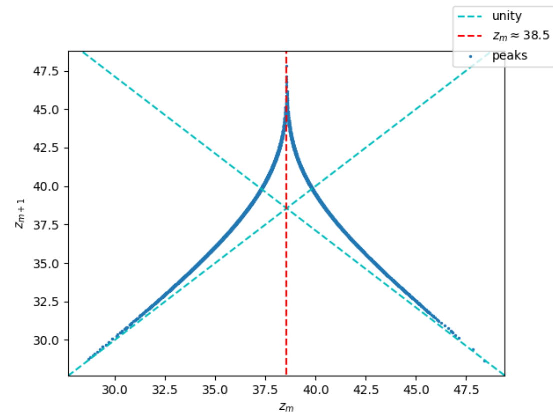
2.4

Sensitivity to initial values



The right plot shows the "distance" between the solutions at each time t. It can be seen that even for a small difference in initial value (0.01), the solutions differ much after some time.

2.5



We can have a look at the, seemingly symmetrical, plot of z_m vs z_{m+1}. This plot is generated for a time duration of $0 < T < 5000$ with a time step of $\Delta t = 0.0001$, so with a total of 50 million time steps.

We can clearly see that if we divide the z_{m+1} versus z_m curve into two regions, one to the left of $z_m \approx 38.5$ and one to the right, the magnitude of the slope of the curve is always greater than unity. It is also interesting to note that the map symmetrically maps z_m onto z_{m+1} around $z_m \approx 38.5$ (both $z_m = 35$ and $z_m = 42.5$ map to approximately $z_{m+1} = 36$). In reality, this plot lets us predict the next z_{m+1} for any given z_m, but not generally z_M where $M >> m$.

Now to answer why a slope magnitude greater than 1 indicates chaos. Qualitatively, I'm thinking that if we calculate the Lyapunov exponent for the plot (where we set $z_{m+1} = f(z_m)$) we'll have something like

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{i=n-1} \log|f'(z_m)|$$

But we've seen in the plot that $|f'(z_m)| \geq 1$, so

$$\log|f'(z_m)| \geq 0 \ \forall \ z_m$$

This means that $\lambda > 0$ for the z_m vs z_{m+1} plot and indicates chaos.