Moon landing problem

dynamics:

$$\dot{h}(t) = v(t)$$

$$\dot{v}(t) = -g + \frac{\alpha(t)}{m(t)}$$

$\alpha \equiv$ thrust $\in [0,1]$

$$\dot{m} = -k\,\alpha(t)$$

mass

I.C.

$$h(0) = h_0$$
$$V(0) = V_0$$
$$m(0) = m_0$$

$$\min_{\alpha} \quad P(\alpha) = \int_0^{\tau} \alpha(t)\,dt \qquad h(\tau) = 0, \quad v(\tau) = 0$$

fixed end free time problem

Soln:

$$f = \begin{bmatrix} v \\ -g + \alpha/m \\ -k\alpha \end{bmatrix}, \quad r = \alpha, \quad C = 0$$

$$H = -r + \lambda^T f = -\alpha + \lambda_1 V + \lambda_2\left[-g + \frac{\alpha}{m}\right] + \lambda_3[-k\alpha]$$

$$\alpha^* = \max_\alpha H = \arg\max_\alpha \left[\underbrace{\left(-1 + \frac{\lambda_2}{m} - \lambda_3 k\right)}_{b}\alpha + \lambda_1 V - \lambda_2 g\right]$$

Bang Bang

$$\alpha^* = \begin{cases} 0 & b < 0 \\ 1 & b > 0 \end{cases}$$

show b changes monotonic

↳ b always increases or always decreases

Policity:

$$\alpha^* = 0 \quad t \in [0, t^*]$$
$$\alpha^* = 1 \quad t \in [t^*, \tau]$$

Because B is monotonic,

you can turn on engine & then let it drop

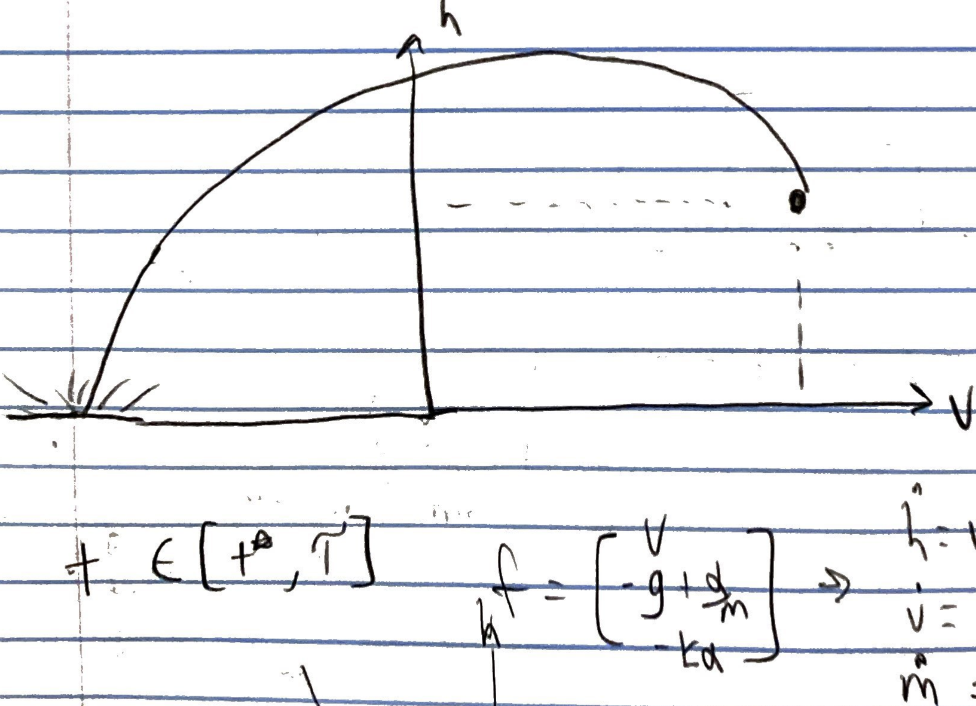
cannot result in $v(z)=0$

or

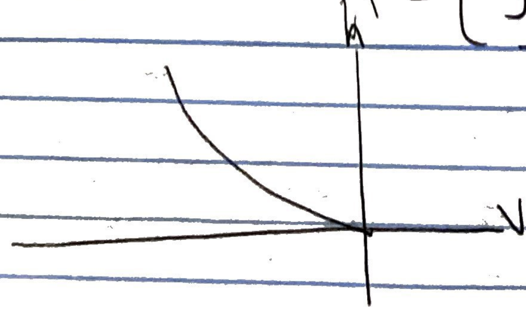
let it drop & then turn on engine

must choose this option

$t \in [0, t^*]$ $f = \begin{bmatrix} v \\ g \\ 0 \end{bmatrix} \rightarrow$ $\dot{h} = v$, $\dot{v} = -g$, $\dot{m} = 0$



$t \in [t^*, T]$ $f = \begin{bmatrix} v \\ -g + \frac{a}{m} \\ -ka \end{bmatrix} \rightarrow$ $\dot{h} = v$, $\dot{v} = -g + \frac{1}{m}$, $\dot{m} = -k$

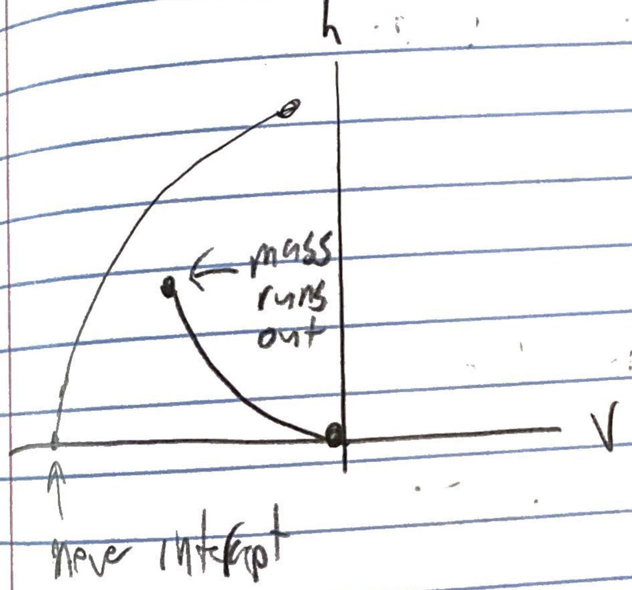


t^* when 2 curves intercept

2 infeasible points

- a run out of fuel
- $\tilde{T} < t^*$

make sure lines intercept



Proof that b is monotonic:

$$\frac{d\left(1+\lambda_2/m - \lambda_3 k\right)\alpha}{d\alpha} = \underbrace{1+\lambda_2/m - \lambda_3 k}$$

this is a constant therefore the function is monotonic because the derivative never changes sign derivation of dynamics:

$t \in [0, t^*]$ $\frac{dv}{dt} = -g \rightarrow dv = -g\,dt$

$V = -gt + V_0$

$\dot{h} = V \rightarrow dh = dt(-gt + V_0)$

① $h = -\frac{1}{2}gt^2 + V_0 t + h_0$

$t \in [t^*, T]$

$\dot{m} = -k \rightarrow m = -kt + m_0$

$\dot{V} = -g + \frac{1}{m} = -g + \frac{1}{-kt + m_0}$

$V = -gt + V_0 - \frac{1}{k}\ln|m_0 - kt|$

$\dot{h} = -gt + V_0 - \frac{1}{k}\ln|m - kt|$

② $h = -\frac{1}{2}gt^2 + V_0 t - \frac{1}{k^2}\left[m_0 \ln(m_0 - kt) - kt\ln(m_0 - kt) - m_0 + tk\right] + h_0$

① = ②

$$\boxed{t^* = \frac{m_0}{k}}$$

So the optimal control policy is to keep your engines off until you're at t^*, at which point you turn your engines fully on until $t = T$, at which point you have landed.