FDM Demo: 2, 4, 6, and 8th Order FDM

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1 Problem Statement

Consider a constant-area cylinder undergoing heat diffusion. The ends of the rod have constant temperatures. Energy balance on a differential element of the cylinder gives

$$u''(x) = -\alpha^2 u(x) + u_a \tag{1}$$

where $u(x)$ represents the temperature of the cylinder as a function of x. Equation 1 is a homogeneous differential equation whose exact solution can be found as a boundary value problem. Assume the solution takes the form

$$u(x) = C \sinh(\alpha x) + D \cosh(\alpha x). \tag{2}$$

The boundary conditions for each case will be used to solve for the constants C and D. Equation 2 represents the general solution for the equation.

2 Analytical Solutions

2.1 Case 1

In case 1, we consider the ambient temperature to be 0° Celsius. The temperature at the left end of the cylinder is 0° as well. The temperature at the right end of the cylinder is 100°.

$$u(0) = 0 \tag{3}$$

$$u(L) = 100 \tag{4}$$

To solve for C and D employing boundary conditions from equations 3 and 4:

$$0 = C \sinh(\alpha \cdot 0) + D \cosh(\alpha \cdot 0) + 0$$
$$D = 0$$
$$100 = C \sinh(\alpha \cdot L) + D \cosh(\alpha \cdot L) + 0$$
$$100 = C \sinh(\alpha L)$$
$$C = \frac{100}{\sinh(\alpha L)}$$

Therefore, for Case 1,

$$u(x) = \frac{100}{\sinh(\alpha L)} \sinh(\alpha x) \tag{5}$$

2.2 Case 2

For the second case, the right end is still 100° C. The left end is insulated, so that no heat is lost from that end. The atmospheric temperature is still considered as $0°C$. Therefore, the boundary conditions are

$$u(L) = 100 \tag{6}$$

$$u'(0) = 0. \tag{7}$$

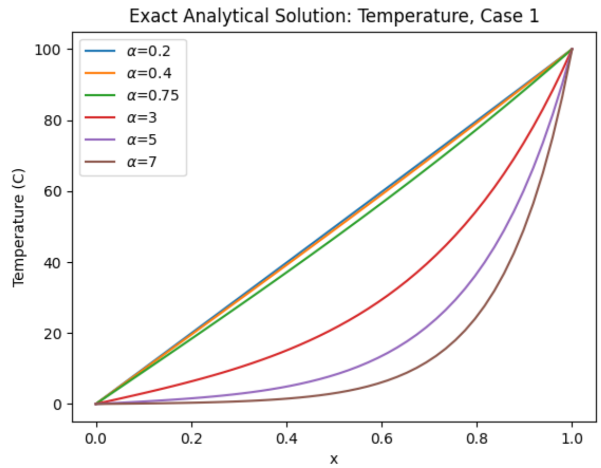

Figure 1: Exact Analytical Solution, Case 1

To solve for C and D we need the derivative of $u(x)$.

$$u'(x) = \alpha C \cosh(\alpha x) + \alpha D \sinh(\alpha x)$$

Then,

$$\begin{gathered} u'(0) = 0 = \alpha C \cosh(\alpha \cdot 0) + \alpha D \sinh(\alpha \cdot 0) \\ 0 = \alpha C \\ C = 0 \\ u(L) = 100 = C \sinh(\alpha \cdot L) + D \cosh(\alpha \cdot L) \\ 100 = D \cosh(\alpha L) \\ D = \frac{100}{\cosh(\alpha L)} \end{gathered}$$

Therefore, for Case 2,

$$u(x) = \frac{100}{\cosh(\alpha L)} \cosh(\alpha x). \tag{8}$$

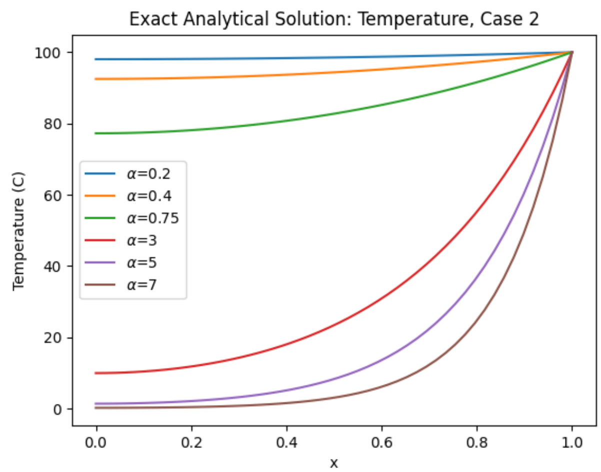

Figure 2: Exact Analytical Solution, Case 2

3 FDM: 2nd Order Scheme

3.1 Case 1

To begin, take the grid from $[0, L]$ and divide into n amount of segments. Let $x_i = i\Delta x$ and $i = 1, 2, \ldots, n$. Using the differential equation for the temperature, equation 1, at each nodal position (x_i), I use the Taylor series:

$$u(x_i \pm \Delta x) = u(x_i) \pm u'(x_i)\Delta x \pm u''(x_i)\frac{\Delta x^2}{2} + \ldots$$
$$u(x_i + \Delta x) - 2u(x_i) + u(x_i - \Delta x) = u''(x_i)\Delta x^2$$

Now I can approximate u and its second derivative used in equation 1:

$$-u(x_{i-1}) + (2 + \alpha^2\Delta x^2)u(x_i) - u(x_{i+1}) = 0 \text{ for } i = 1, \ldots, n-1. \tag{9}$$

In Equation 9, $\kappa = 2 + \alpha^2\Delta x^2$, and U_i represents the i^{th} approximate temperature:

$$-U_{i-1} + \kappa U_i - U_{i+1} = 0 \tag{10}$$

where $U_i \approx u(x_i)$. By solving Equation 10, I can approximate the solution for Equation 1 with measurable precision. The general solution is given with

$$\begin{bmatrix} \kappa & -1 & 0 & \ldots & 0 & 0 \\ -1 & \kappa & -1 & \ldots & 0 & 0 \\ 0 & -1 & \kappa & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \kappa & -1 & 0 \\ 0 & 0 & 0 & -1 & \kappa & -1 \\ 0 & 0 & 0 & 0 & -1 & \kappa \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ \vdots \\ U_{n-2} \\ U_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ -100 \end{bmatrix} \tag{11}$$

3.2 Case 2

Case 2 is formulated in a similar way to Case 1, with the exception that boundary conditions 6 and 7 are used. Therefore, I will skip to solving Equation 10 for these new BCs.

$$\begin{aligned} i = 0:\ & U_{-1} - (2 + \alpha^2 dx^2)U_0 + U_1 = 0 \\ i = 1:\ & U_0 - (2 + \alpha^2 dx^2)U_1 + U_2 = 0 \\ i = 2:\ & U_1 - (2 + \alpha^2 dx^2)U_2 + U_3 = 0 \\ i = 3:\ & U_2 - (2 + \alpha^2 dx^2)U_3 + U_4 = 0 \\ i = 4:\ & U_3 - (2 + \alpha^2 dx^2)U_4 + U_5 = 0 \\ & U_0' = 0 \\ & U_5 = 100. \end{aligned}$$

With these, we'll look at the first point, $i = 0$.

$$U_{-1} - \alpha^2 dx U_0 + U_1 = 0$$

There are two unknowns for this equation, U_{-1} and U_0. However, I have one more equation I can use:

$$U_i' = \frac{U_{i+1} - U_{i-1}}{2dx}$$

and this can be solved for U_{-1} like so:

$$U_{-1} = U_1 - 2dxU_0'.$$

The boundary conditions state that $U_0' = 0$, and the equation for $i = 0$ was given, so that

$$\begin{gathered} U_{-1} - (2 + \alpha^2 dx^2)U_0 + U_1 = 0 \\ U_1 - 2dxU_0' - (2 + \alpha^2 dx^2)U_0 \\ 2U_1 - (2 + \alpha^2 dx^2)U_0 = 0 \\ U_1 - \frac{\kappa}{2}U_0 = 0. \end{gathered}$$

This goes right back into Equation 12 for Case 1 like so:

$$\begin{bmatrix} \frac{\kappa}{2} & -1 & 0 & \dots & 0 & 0 \\ -1 & \kappa & -1 & \dots & 0 & 0 \\ 0 & -1 & \kappa & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \kappa & -1 & 0 \\ 0 & 0 & 0 & -1 & \kappa & -1 \\ 0 & 0 & 0 & 0 & -1 & \kappa \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ \vdots \\ U_{n-2} \\ U_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 100 \end{bmatrix} \tag{12}$$

3.3 Approximation of Heat Loss

To approximate the heat loss at the end of the rod,

$$\dot{Q} = -kaU'(L) \tag{13}$$

where $U'(L)$ is found once again through a Taylor series expansion as:

$$U'(L) = \frac{\left(1 + \frac{dx^2\alpha^2}{2}\right) U_n - U_{n-1}}{dx} \tag{14}$$

3.4 Results

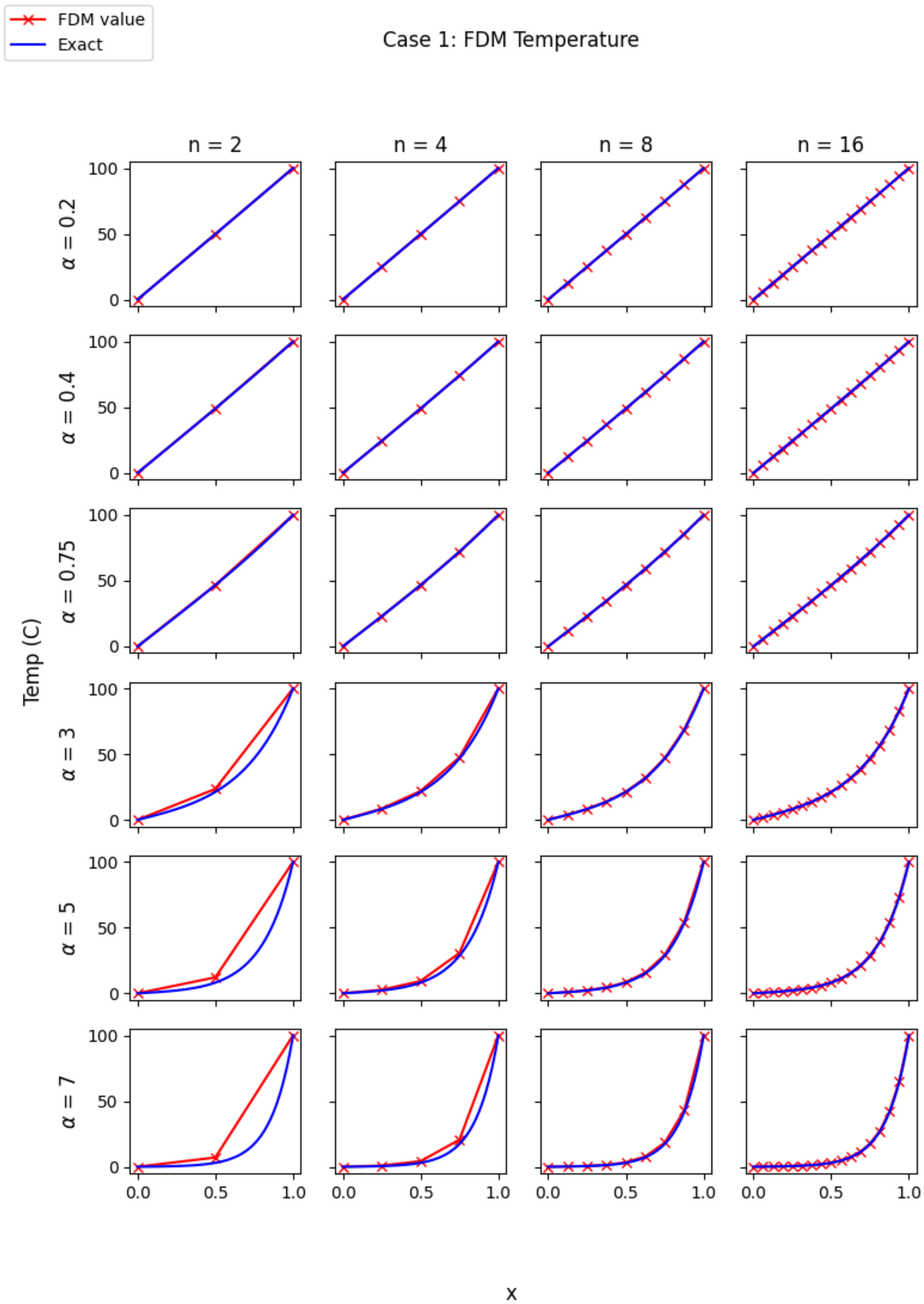

Figure 3: 2nd Order: Case 1

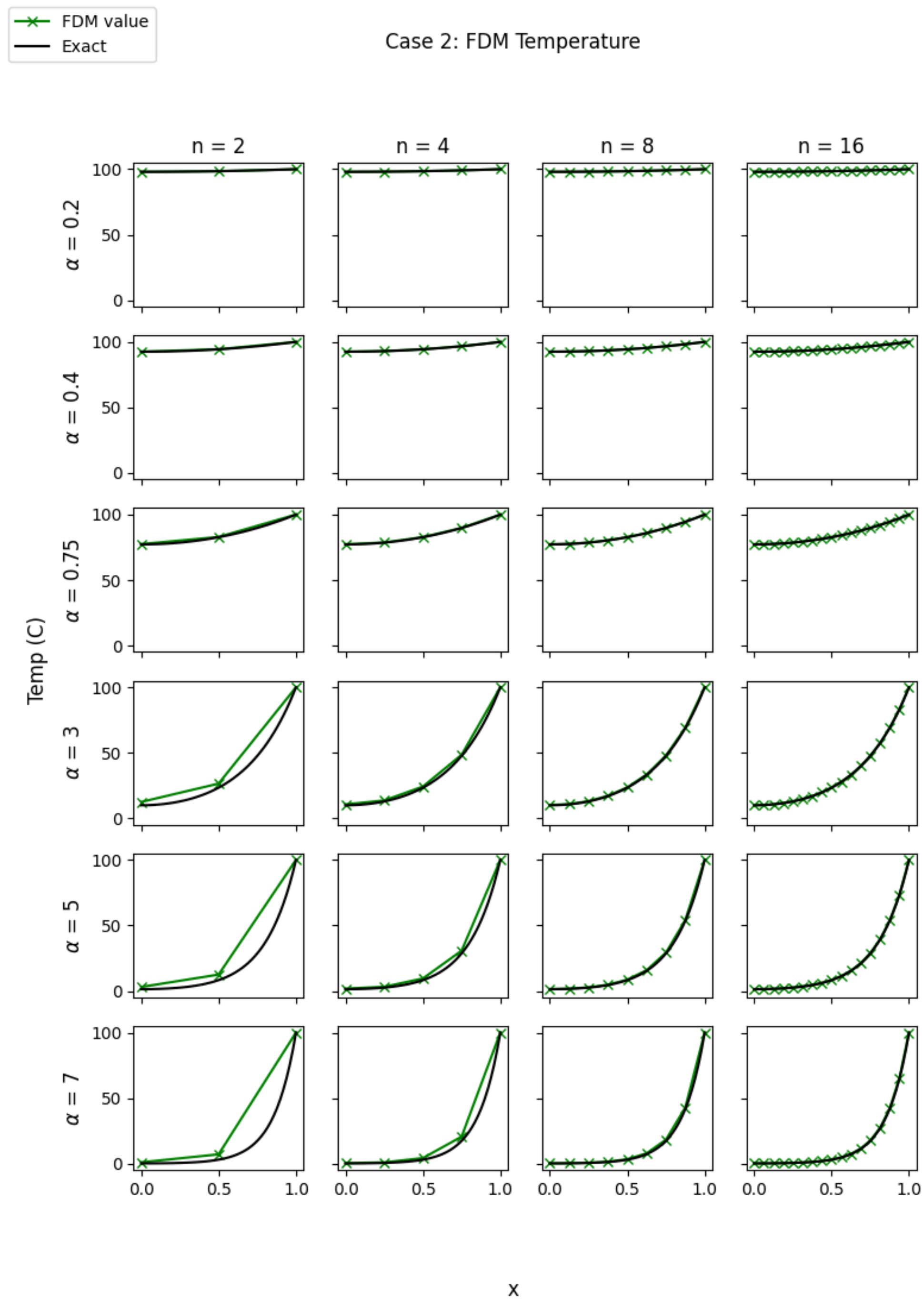

Figure 4: 2nd Order: Case 2

4 FDM: 4th Order Scheme

4.1 Case 1

I can further extend the results from the 2nd order scheme to a 4th order scheme, to make the calculations more accurate. To construct the fourth order FDM, the process is identical to the second order, taking the Taylor series to the 4th order:

$$u(x_i+\Delta x) = u(x_i) + u'(x_i)\Delta x + u''(x_i)\frac{\Delta x^2}{2} + u'''(x_i)\frac{\Delta x^3}{6} + u''''(x_i)\frac{\Delta x^4}{24}$$
$$u(x_i-\Delta x) = u(x_i) - u'(x_i)\Delta x + u''(x_i)\frac{\Delta x^2}{2} - u'''(x_i)\frac{\Delta x^3}{6} + u''''(x_i)\frac{\Delta x^4}{24}$$

Therefore, to add these two equations together to get $u(x_i-\Delta x)+u(x_i+\Delta x)$:

$$u(x_i-\Delta x)+u(x_i+\Delta x) = 2u(x_i) + u''(x_i)\Delta x^2 + u''''(x_i)\frac{\Delta x^4}{12} \tag{15}$$

Next, I want to substitute these terms into the original differential equation, equation 1.

$$u_{i-1}+u_{i+1}-2u_i = u_i''\Delta x^2 + u_i''''\frac{\Delta x^4}{12}$$
$$\frac{u_{i-1}+u_{i+1}-2u_i}{\Delta x^2} = u_i'' + u_i''''\frac{\Delta x^2}{12}$$
$$\text{since } u_i'''' = \alpha^2 u_i''$$
$$\frac{u_{i-1}+u_{i+1}-2u_i}{\Delta x^2} = u_i'' + u_i''\alpha^2\frac{\Delta x^2}{12} = u_i''(1+\alpha^2\frac{\Delta x^2}{12})$$

For brevity $\beta = \alpha^2\frac{\Delta x^2}{12}$ and the second derivative can be solved for as

$$u_i'' = \frac{u_{i-1}+u_{i+1}-2u_i}{\Delta x^2}\frac{1}{1+\beta} \tag{16}$$

Using the mathematical approximation

$$1-\epsilon \approx \frac{1}{1+\epsilon} \tag{17}$$
$$u_i'' = (1-\beta)\frac{u_{i-1}+u_{i+1}-2u_i}{\Delta x^2} \tag{18}$$

If I put this term back into equation 1,

$$-(1-\beta)\frac{u_{i-1}+u_{i+1}-2u_i}{\Delta x^2} + \alpha^2 u_i = 0$$

or,

$$-(1-\beta)u_{i-1} - (2+10\beta)u_i - (1-\beta)u_{i+1} = 0 \tag{19}$$

4.2 Case 2

4.3 Approximation of Heat Loss

To approximate the heat loss at the end of the rod,

$$\dot{Q} = -kaU'(L) \tag{20}$$

where $U'(L)$ is found once again through a Taylor series expansion as:

$$U'(L) = \frac{\left(1 + \frac{dx^2\alpha^2}{2} + \frac{dx^4\alpha^4}{24}\right)U_n - U_{n-1}}{dx + \frac{dx^3\alpha^s}{6}} \tag{21}$$

4.4 Results

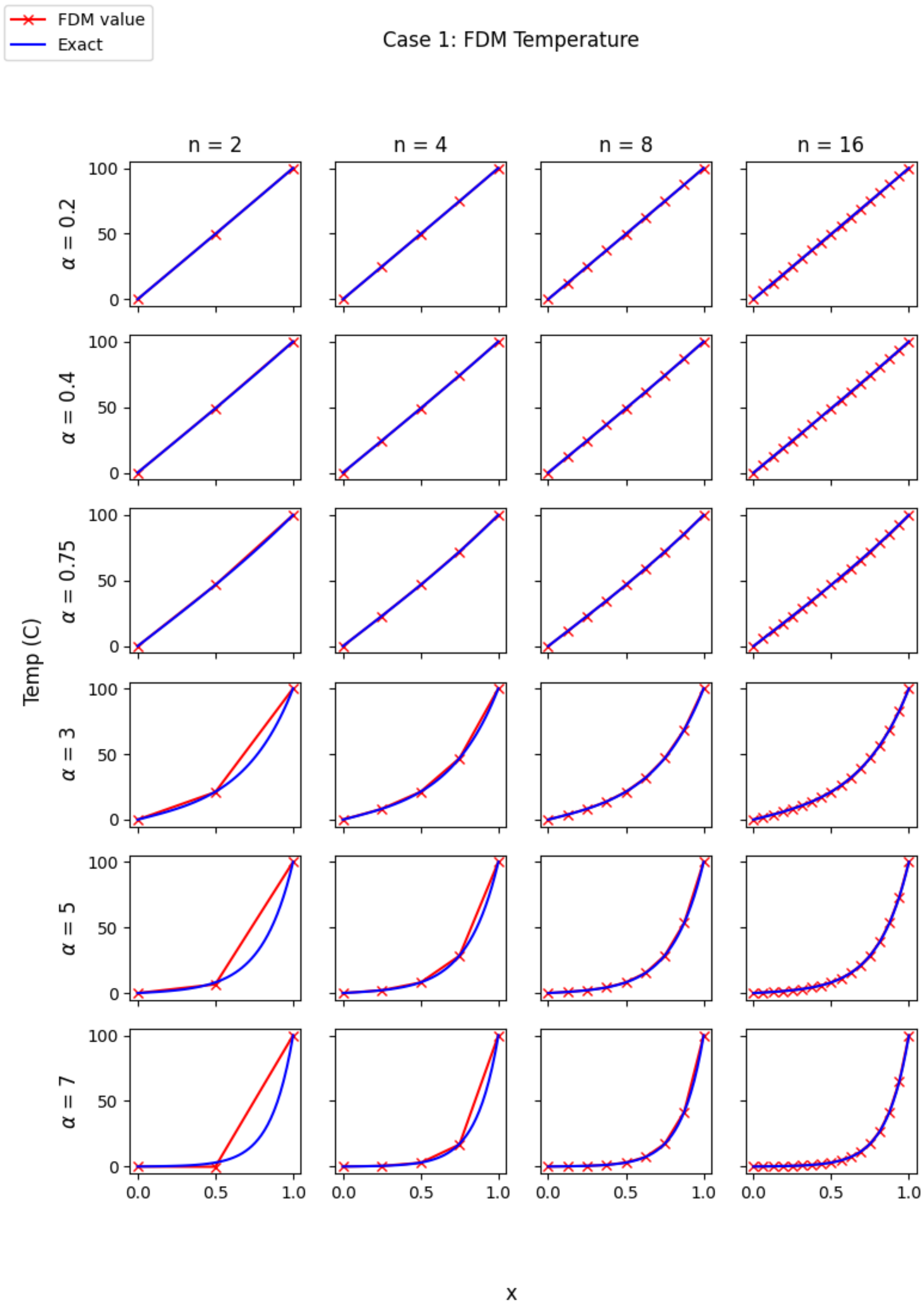

Figure 5: 4th Order: Case 1

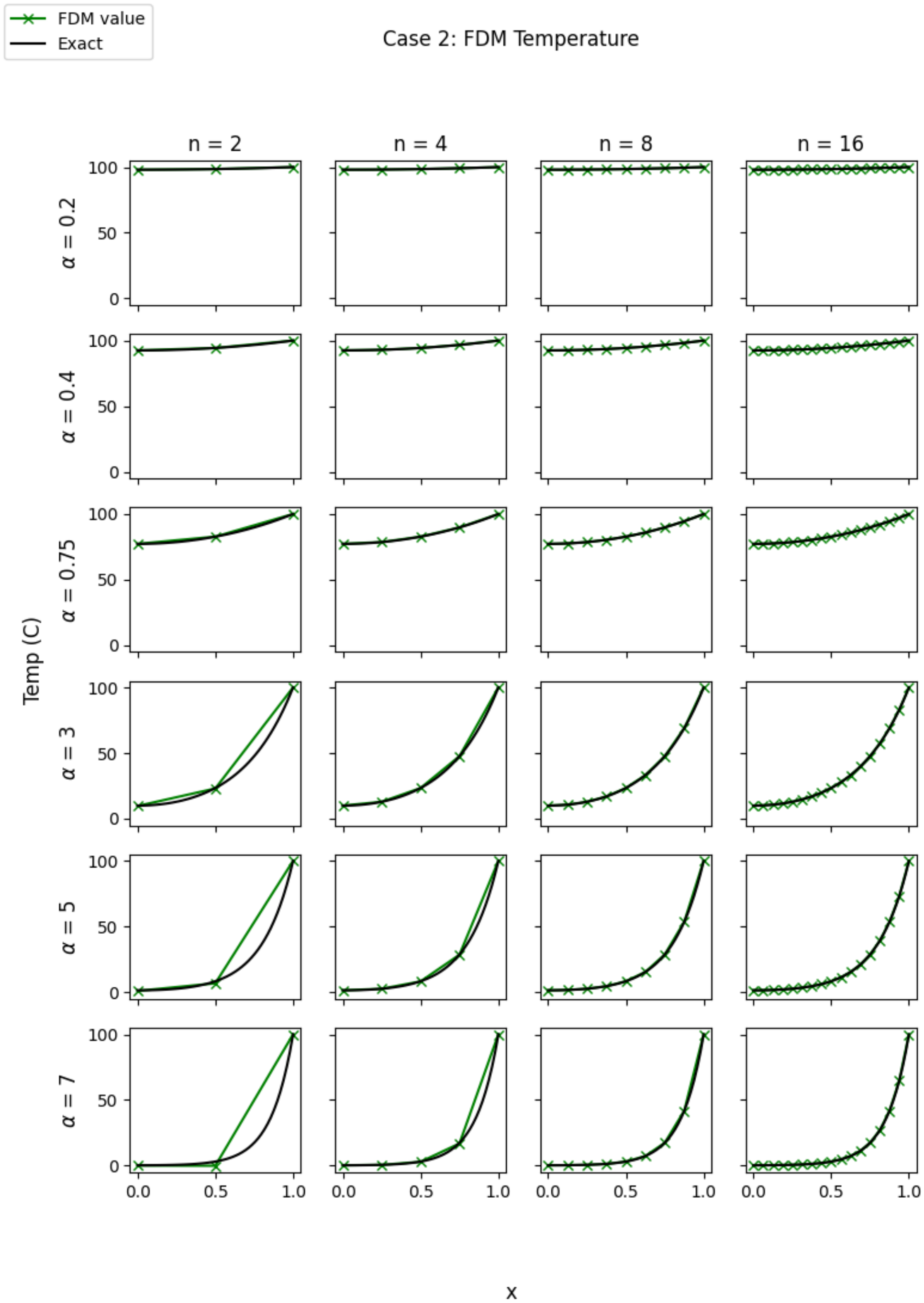

Figure 6: 4th Order: Case 2

5 FDM: 6th Order Scheme

The sixth order scheme is constructed in a similar manner to the fourth order. The only difference will be that the Taylor series is taken out to the sixth term.

$$-U_{i-1} + \kappa U_i - U_{i+1} = 0$$

where

$$\kappa = 2 + \alpha^2 dx^2 + \frac{\alpha^4 dx^4}{12} + \frac{\alpha^6 dx^6}{360}. \tag{22}$$

5.1 Approximation of Heat Loss

To approximate the heat loss at the end of the rod,

$$\dot{Q} = -kaU'(L) \tag{23}$$

where $U'(L)$ is found once again through a Taylor series expansion as:

$$U'(L) = \frac{\left(1 + \frac{dx^2\alpha^2}{2} + \frac{dx^4\alpha^4}{24} + \frac{dx^6\alpha^6}{720}\right)U_n - U_{n-1}}{dx + \frac{dx^3\alpha^2}{6} + \frac{dx^5\alpha^4}{120}} \tag{24}$$

5.2 Results

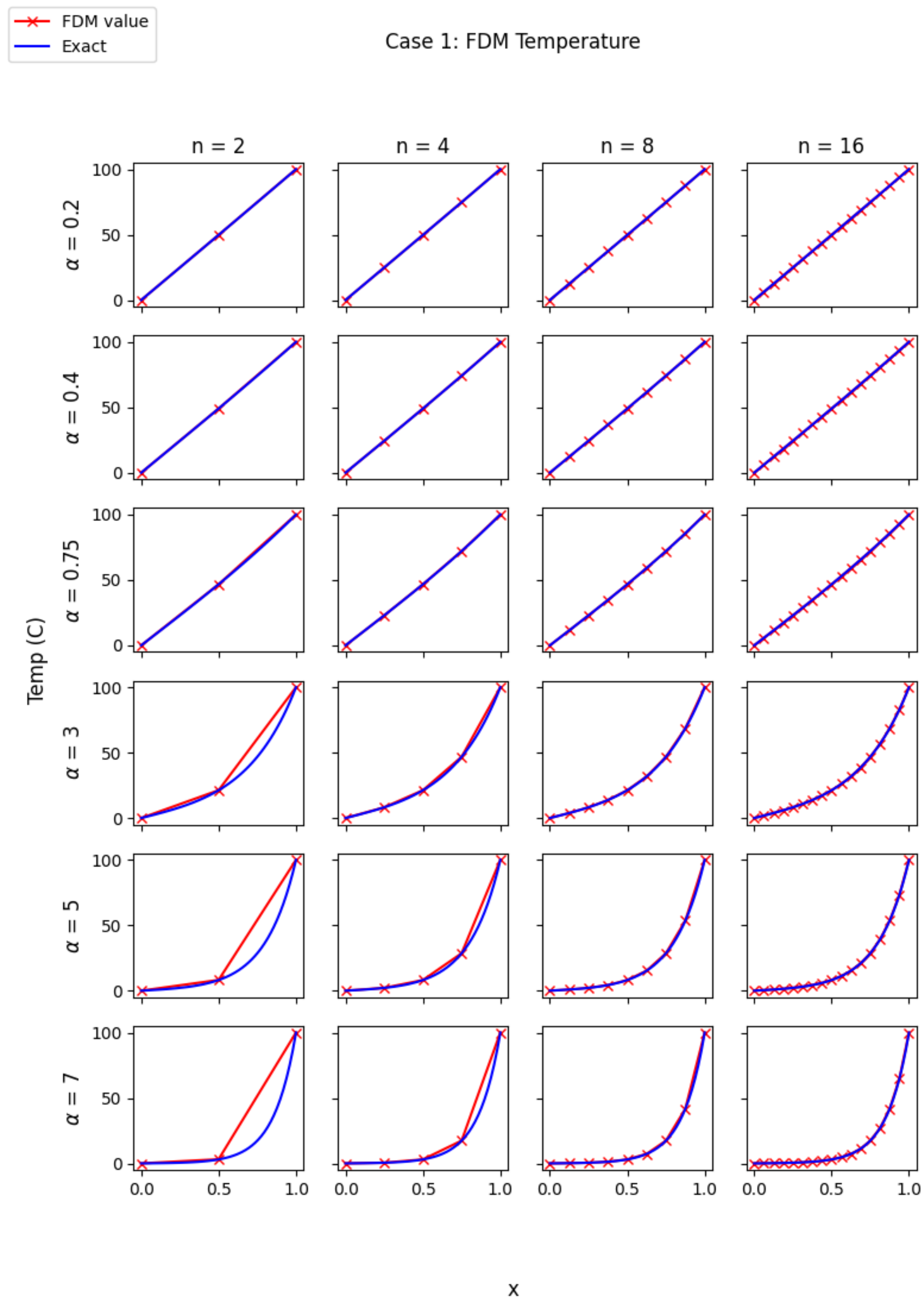

Figure 7: 6th Order: Case 1

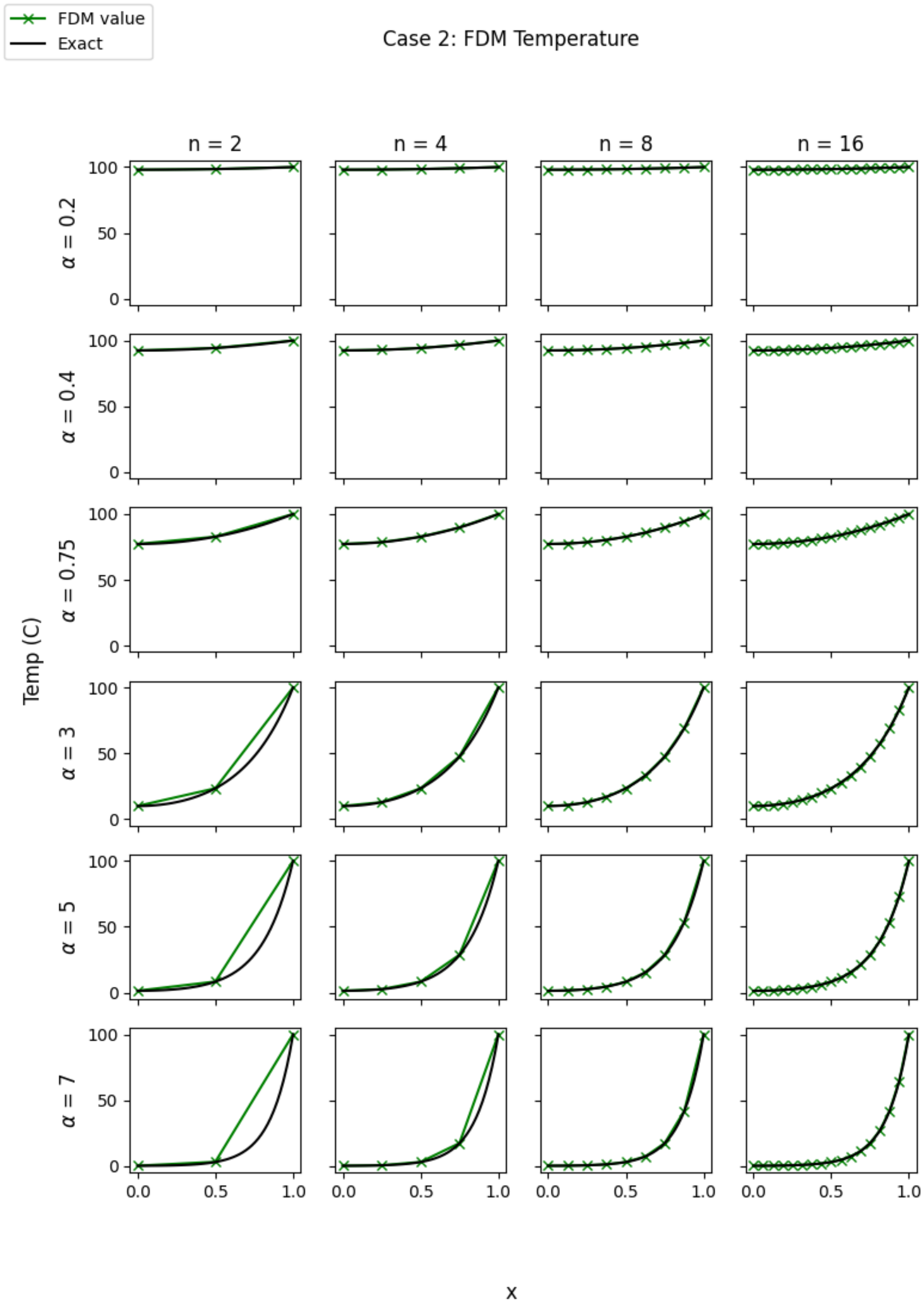

Figure 8: 6th Order: Case 2

6 FDM: 8th Order Scheme

$$-U_{i-1} + \kappa U_i - U_{i+1} = 0$$

where

$$\kappa = 2 + \alpha^2 dx^2 + \frac{\alpha^4 dx^4}{12} + \frac{\alpha^6 dx^6}{360} + \frac{\alpha^8 dx^8}{20160}. \tag{25}$$

6.1 Approximation of Heat Loss

To approximate the heat loss at the end of the rod,

$$\dot{Q} = -kaU'(L) \tag{26}$$

where $U'(L)$ is found once again through a Taylor series expansion as:

$$U'(L) = \frac{\left(1 + \frac{dx^2\alpha^2}{2} + \frac{dx^4\alpha^4}{24} + \frac{dx^6\alpha^6}{720} + \frac{dx^8\alpha^8}{40320}\right) U_n - U_{n-1}}{dx + \frac{dx^3\alpha^2}{6} + \frac{dx^5\alpha^4}{120} + \frac{dx^7\alpha^6}{5040}} \tag{27}$$

6.2 Results

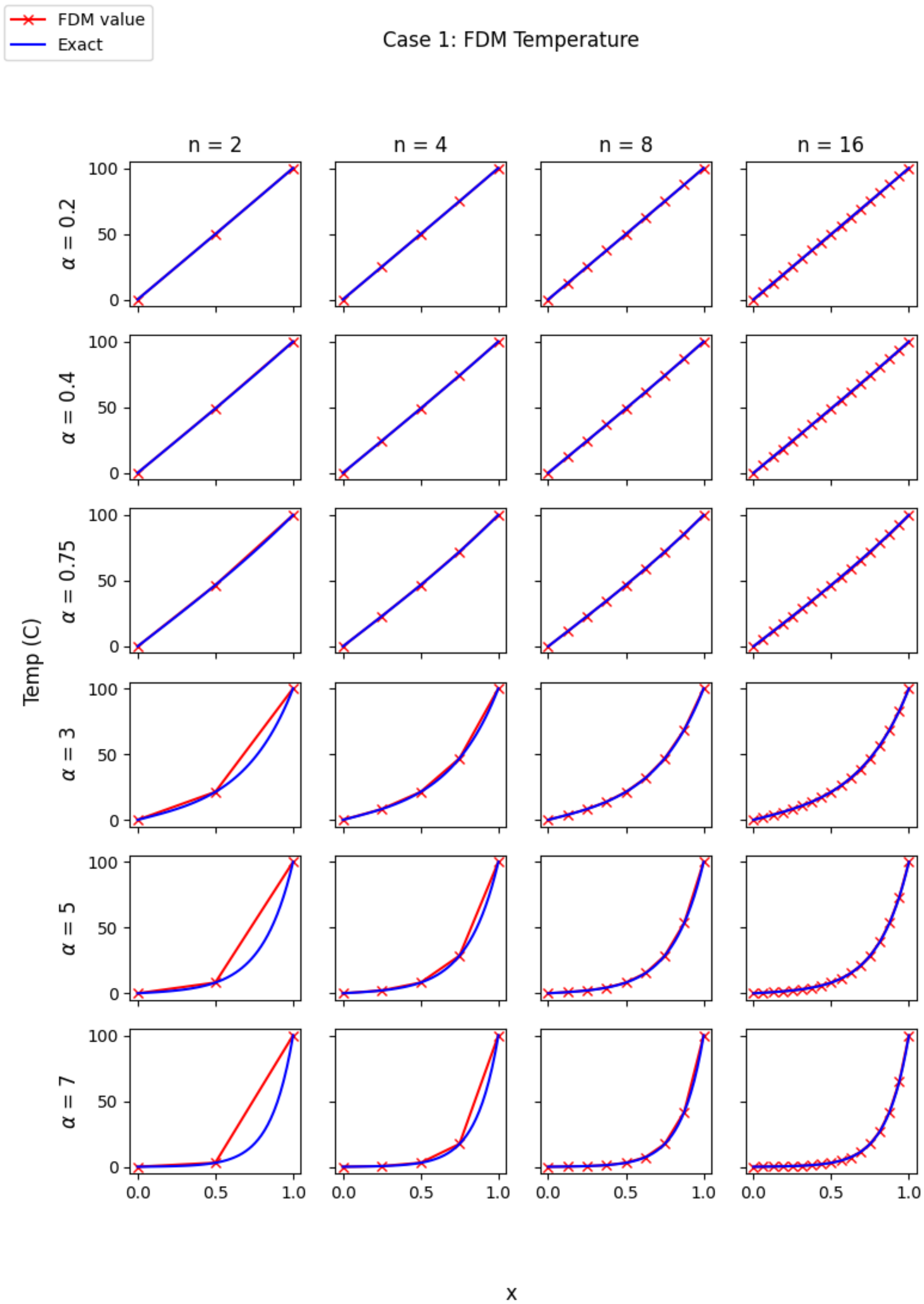

Figure 9: 8th Order: Case 1

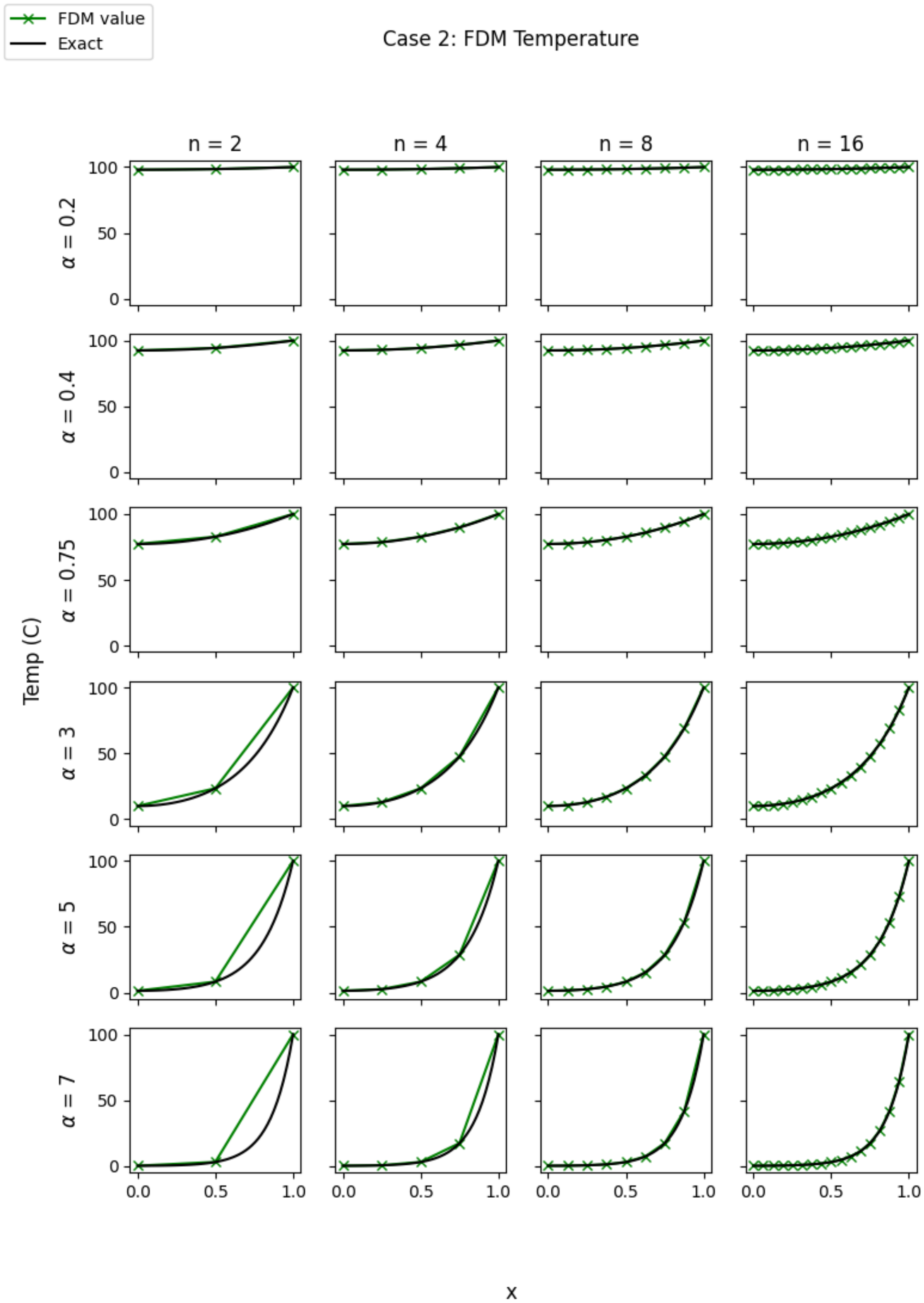

Figure 10: 8th Order: Case 2

7 Rate of Convergence

The rate of convergence can be found in a similar way to Assignment 1, but instead of the exact temperature, we compare to an extrapolated value given by:

$$Q_{extr} = \frac{Q_{dx/2}^2 - Q_{dx} * Q_{dx/4}}{2Q_{dx/2} - Q_{dx} - Q_{dx/4}} \tag{28}$$

where the values of Q are given for different mesh sizes dx, $\frac{dx}{2}$, and $\frac{dx}{4}$. The rate of convergence is then found in a similar manner to the previous assignment.

$$Q_{extr} - Q_{\Delta x} \approx C\left(\Delta x\right)^{\beta}$$
$$Q_{extr} - Q_{\Delta x/2} \approx C\left(\frac{\Delta x}{2}\right)^{\beta}$$

If I divide the first equation by the second,

$$\frac{Q_{extr} - Q_{\Delta x}}{Q_{extr} - Q_{\Delta x/2}} \approx 2^{\beta}$$
$$\beta \approx \frac{\log \frac{Q_{extr} - Q_{\Delta x}}{Q_{extr} - Q_{\Delta x/2}}}{\log 2}$$

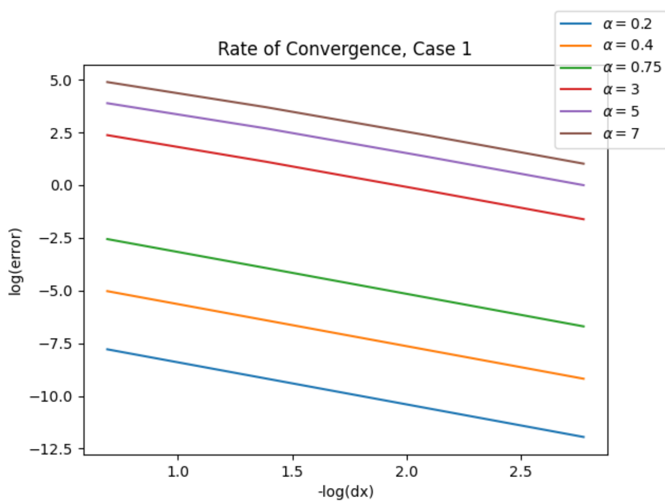

Figure 11: Rate of Convergence, 2nd Order, Case 1

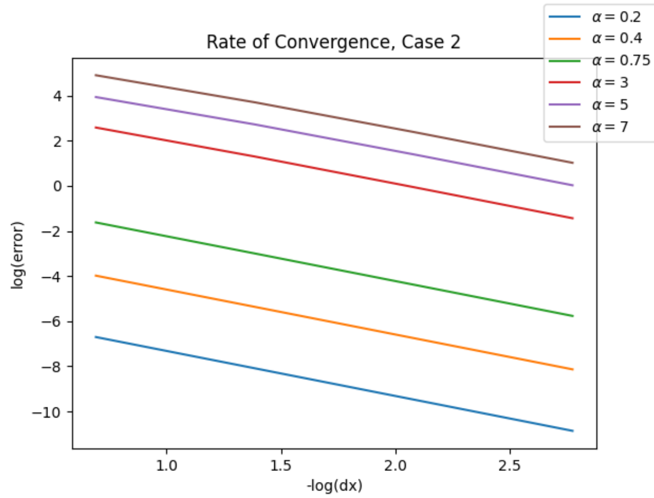

Figure 12: Rate of Convergence, 2nd Order, Case 2

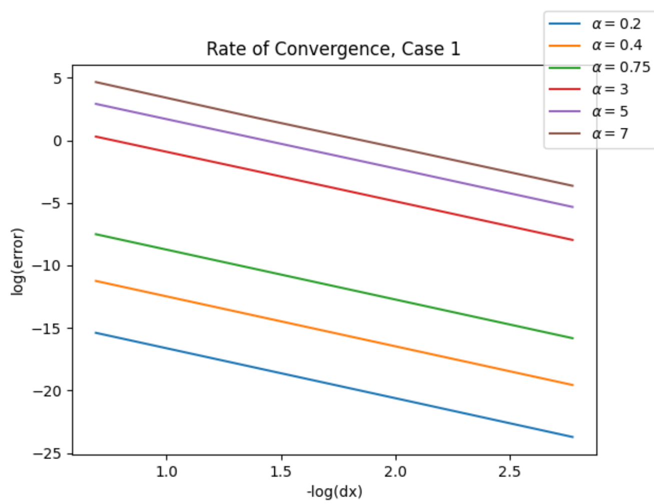

Figure 13: Rate of Convergence, 4th Order, Case 1

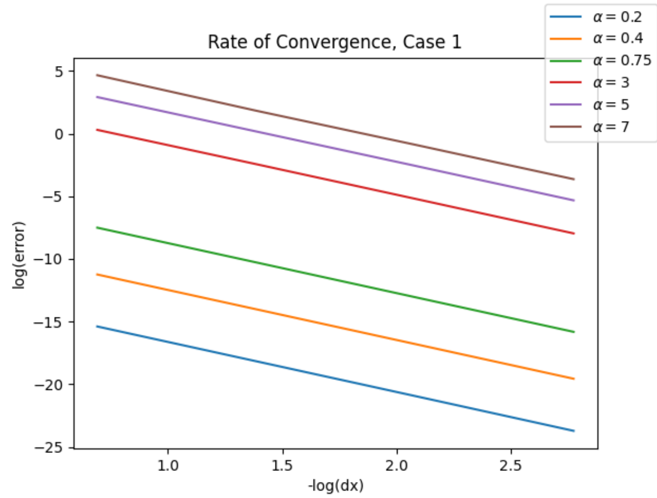

Figure 14: Rate of Convergence, 4th Order, Case 2

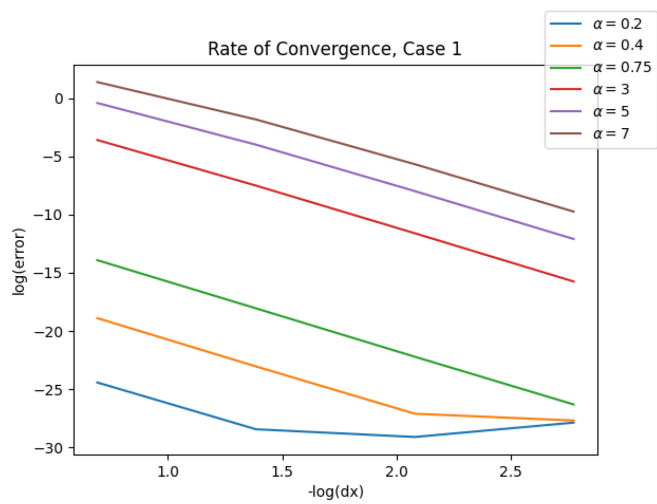

Figure 15: Rate of Convergence, 6th Order, Case 1

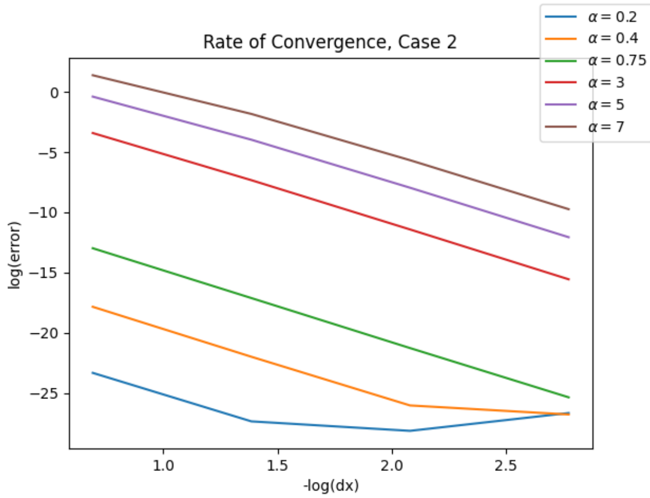

Figure 16: Rate of Convergence, 6th Order, Case 2

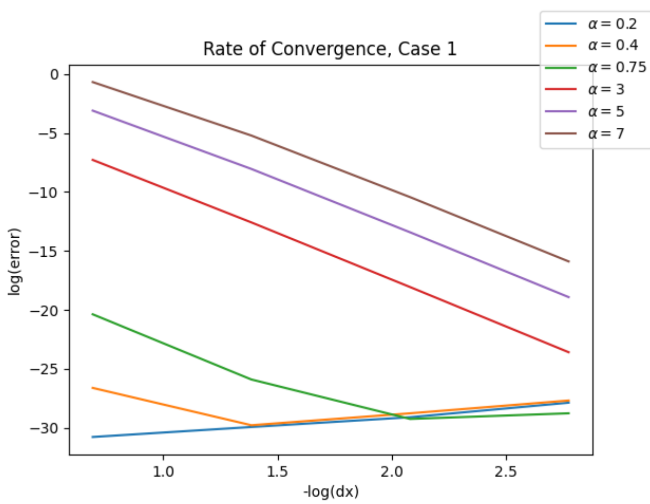

Figure 17: Rate of Convergence, 8th Order, Case 1

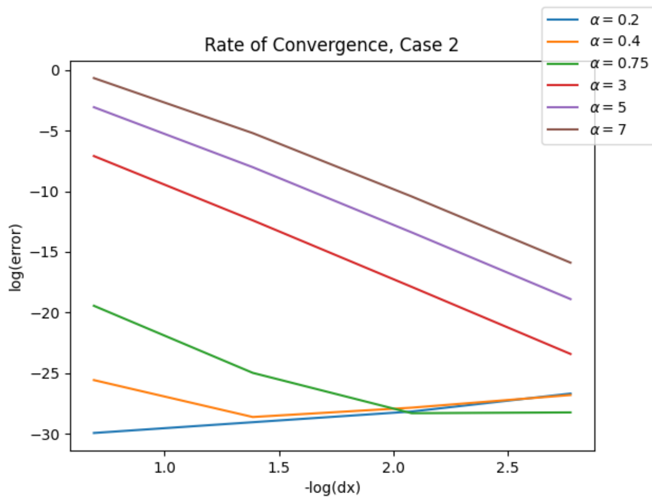

Figure 18: Rate of Convergence, 8th Order, Case 2

8 Conclusion

The rates of convergence look as expected for all except 6th and 8th order case 2. I have been unable to find the reason for these convergences, although they only occur when *alpha* is small. Not included in this report is The values for $U'(L)$, but they should be included in HW 3.

A Tables

A.1 2nd Order

A.1.1 $\alpha = 0.2$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	49.751	49.7512	0.000414732
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	98.0328	98.0344	0.00164263
0.5	98.5234	98.5246	0.00122789
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	24.8445	24.8445	0.000129798
0.5	49.751	49.7511	0.000103786
0.75	74.782	74.7821	6.04917e-05
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	98.0328	98.0332	0.000411063
0.25	98.1554	98.1557	0.000385052
0.5	98.5234	98.5237	0.000307276
0.75	99.1377	99.1379	0.000178507
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	12.4183	12.4184	3.40846e-05
0.25	24.8445	24.8445	3.24575e-05
0.375	37.2861	37.2861	2.97465e-05
0.5	49.751	49.7511	2.5953e-05
0.625	62.2471	62.2471	2.1079e-05
0.75	74.782	74.782	1.51267e-05
0.875	87.3637	87.3637	8.09922e-06
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	98.0328	98.0329	0.000102791
0.125	98.0634	98.0635	0.000101164
0.25	98.1554	98.1555	9.62867e-05
0.375	98.3086	98.3087	8.81713e-05
0.5	98.5234	98.5234	7.68381e-05
0.625	98.7997	98.7997	6.2315e-05
0.75	99.1377	99.1378	4.46377e-05
0.875	99.5378	99.5378	2.38494e-05
1	100	100	0

n = 16

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	6.20869	6.20869	8.62341e-06
0.125	12.4183	12.4183	8.52169e-06
0.1875	18.6299	18.6299	8.35218e-06
0.25	24.8445	24.8445	8.11488e-06
0.3125	31.0628	31.0629	7.80984e-06
0.375	37.2861	37.2861	7.4371e-06
0.4375	43.5152	43.5152	6.99669e-06
0.5	49.751	49.751	6.48867e-06
0.5625	55.9947	55.9947	5.91311e-06
0.625	62.2471	62.2471	5.27007e-06
0.6875	68.5092	68.5092	4.55964e-06
0.75	74.782	74.782	3.78191e-06
0.8125	81.0665	81.0665	2.93697e-06
0.875	87.3637	87.3637	2.02493e-06
0.9375	93.6745	93.6745	1.0459e-06
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	98.0328	98.0328	2.56994e-05
0.0625	98.0405	98.0405	2.55977e-05
0.125	98.0634	98.0635	2.52926e-05
0.1875	98.1017	98.1018	2.47843e-05
0.25	98.1554	98.1554	2.40732e-05
0.3125	98.2243	98.2244	2.31596e-05
0.375	98.3086	98.3087	2.20442e-05
0.4375	98.4083	98.4083	2.07276e-05
0.5	98.5234	98.5234	1.92107e-05
0.5625	98.6538	98.6538	1.74944e-05
0.625	98.7997	98.7997	1.55797e-05
0.6875	98.961	98.961	1.34679e-05
0.75	99.1377	99.1377	1.11601e-05
0.8125	99.33	99.33	8.6579e-06
0.875	99.5378	99.5378	5.96271e-06
0.9375	99.7611	99.7611	3.07617e-06
1	100	100	0

A.1.2 $\alpha = 0.4$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	49.0164	49.0196	0.00654467
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	92.5007	92.5241	0.0252009
0.5	94.3569	94.3745	0.0186551
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	24.3862	24.3867	0.00205737
0.5	49.0164	49.0172	0.00164263
0.75	74.1372	74.1379	0.000955048
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	92.5007	92.5066	0.00632438
0.25	92.9636	92.9691	0.00590963
0.5	94.3569	94.3614	0.00468168
0.75	96.6946	96.6972	0.00268762
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	12.1779	12.1779	0.000540861
0.25	24.3862	24.3863	0.00051485
0.375	36.6555	36.6556	0.000471555
0.5	49.0164	49.0166	0.000411063
0.625	61.4999	61.5001	0.000333493
0.75	74.1372	74.1374	0.000238999
0.875	86.9599	86.96	0.000127763
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	92.5007	92.5022	0.00158261
0.125	92.6164	92.6178	0.0015566
0.25	92.9636	92.965	0.00147883
0.375	93.5433	93.5446	0.00135005
0.5	94.3569	94.358	0.00117155
0.625	95.4065	95.4074	0.000945008
0.75	96.6946	96.6952	0.000672552
0.875	98.2245	98.2248	0.000356637
1	100	100	0

n = 16

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	6.08703	6.08704	0.000136876
0.125	12.1779	12.1779	0.000135249
0.1875	18.2763	18.2763	0.000132538
0.25	24.3862	24.3862	0.000128744
0.3125	30.5113	30.5113	0.00012387
0.375	36.6555	36.6555	0.000117918
0.4375	42.8225	42.8226	0.00011089
0.5	49.0164	49.0165	0.000102791
0.5625	55.2409	55.2409	9.36242e-05
0.625	61.4999	61.5	8.3394e-05
0.6875	67.7974	67.7974	7.21056e-05
0.75	74.1372	74.1372	5.97645e-05
0.8125	80.5234	80.5234	4.63766e-05
0.875	86.9599	86.9599	3.19486e-05
0.9375	93.4507	93.4508	1.64872e-05
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	92.5007	92.5011	0.000395748
0.0625	92.5297	92.53	0.000394121
0.125	92.6164	92.6168	0.000389244
0.1875	92.761	92.7614	0.000381129
0.25	92.9636	92.964	0.000369795
0.3125	93.2243	93.2247	0.000355272
0.375	93.5433	93.5436	0.000337595
0.4375	93.9208	93.9211	0.000316806
0.5	94.3569	94.3572	0.000292957
0.5625	94.8521	94.8523	0.000266103
0.625	95.4065	95.4067	0.000236309
0.6875	96.0205	96.0207	0.000203643
0.75	96.6946	96.6948	0.000168179
0.8125	97.4291	97.4292	0.000129997
0.875	98.2245	98.2246	8.91807e-05
0.9375	99.0813	99.0813	4.58181e-05
1	100	100	0

A.1.3 $\alpha = 0.75$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	46.6792	46.7153	0.0773463
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	77.239	77.4511	0.27461
0.5	82.7338	82.8969	0.197111
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	22.9353	22.9409	0.0246644
0.5	46.6792	46.6884	0.0195966
0.75	72.0691	72.0772	0.0113036
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	77.239	77.2926	0.0694934
0.25	78.6007	78.6513	0.0644233
0.5	82.7338	82.775	0.049887
0.75	89.784	89.8089	0.0276546
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	11.4174	11.4182	0.00650747
0.25	22.9353	22.9367	0.0061869
0.375	34.6548	34.6568	0.0056551
0.5	46.6792	46.6815	0.00491573
0.625	59.1142	59.1165	0.00397379
0.75	72.0691	72.0711	0.00283551
0.875	85.6578	85.6591	0.00150817
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	77.239	77.2524	0.0174269
0.125	77.5786	77.5919	0.0171063
0.25	78.6007	78.6134	0.0161556
0.375	80.314	80.3258	0.014607
0.5	82.7338	82.7441	0.0125106
0.625	85.8812	85.8897	0.00992966
0.75	89.784	89.7903	0.00693542
0.875	94.4765	94.4799	0.00360161
1	100	100	0

n = 16

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	5.70245	5.70254	0.00164834
0.125	11.4174	11.4176	0.00162824
0.1875	17.1575	17.1578	0.00159479
0.25	22.9353	22.9356	0.00154803
0.3125	28.7635	28.7639	0.00148806
0.375	34.6548	34.6553	0.00141497
0.4375	40.6224	40.6229	0.00132889
0.5	46.6792	46.6798	0.00122997
0.5625	52.8386	52.8392	0.00111838
0.625	59.1142	59.1148	0.000994293
0.6875	65.5196	65.5202	0.00085792
0.75	72.0691	72.0696	0.000709482
0.8125	78.7769	78.7773	0.000549214
0.875	85.6578	85.6582	0.000377366
0.9375	92.727	92.7272	0.000194203
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	77.239	77.2423	0.0043601
0.0625	77.3238	77.3272	0.00434
0.125	77.5786	77.582	0.00427989
0.1875	78.0039	78.0072	0.00418028
0.25	78.6007	78.6038	0.00404203
0.3125	79.3701	79.3732	0.00386632
0.375	80.314	80.317	0.00365459
0.4375	81.4344	81.4372	0.00340854
0.5	82.7338	82.7364	0.00313009
0.5625	84.215	84.2173	0.00282129
0.625	85.8812	85.8833	0.00248436
0.6875	87.7362	87.7381	0.00212157
0.75	89.784	89.7856	0.00173522
0.8125	92.0292	92.0304	0.00132764
0.875	94.4765	94.4774	0.000901116
0.9375	97.1315	97.132	0.000457858
1	100	100	0

A.1.4 $\alpha = 3$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	21.2548	23.5294	10.7016
1	100	100	1.42109e-14

Case 2

x	T_exact	T_FDM	Error
0	9.93279	12.4514	25.3561
0.5	23.366	26.4591	13.2378
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	8.20849	8.54597	4.11135
0.5	21.2548	21.8991	3.03111
0.75	46.828	47.5704	1.58535
1	100	100	1.42109e-14

Case 2

x	T_exact	T_FDM	Error
0	9.93279	10.6089	6.80699
0.25	12.8598	13.5927	5.69879
0.5	23.366	24.2223	3.66479
0.75	47.6433	48.477	1.74985
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	3.83166	3.87536	1.14054
0.25	8.20849	8.2957	1.06237
0.375	13.7532	13.8826	0.940723
0.5	21.2548	21.4218	0.785602
0.625	31.7805	31.9734	0.606869
0.75	46.828	47.0212	0.412696
0.875	68.5382	68.6815	0.20907
1	100	100	1.42109e-14

Case 2

x	T_exact	T_FDM	Error
0	9.93279	10.1053	1.73677
0.125	10.6394	10.8158	1.65814
0.25	12.8598	13.0473	1.45816
0.375	16.9099	17.1136	1.20458
0.5	23.366	23.5865	0.943758
0.625	33.1466	33.3763	0.69288
0.75	47.6433	47.8595	0.453852
0.875	68.9188	69.0731	0.223914
1	100	100	0

n = 16

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	1.88264	1.88815	0.292617
0.125	3.83166	3.84268	0.287536
0.1875	5.91578	5.9323	0.279221
0.25	8.20849	8.23048	0.267887
0.3125	10.7906	10.818	0.253808
0.375	13.7532	13.7859	0.237294
0.4375	17.2008	17.2384	0.21867
0.5	21.2548	21.2969	0.198256
0.5625	26.0583	26.1042	0.176353
0.625	31.7805	31.8292	0.153234
0.6875	38.6233	38.6732	0.129139
0.75	46.828	46.8768	0.10427
0.8125	56.6838	56.7284	0.078797
0.875	68.5382	68.5744	0.052858
0.9375	82.8092	82.8312	0.0265633
1	100	100	1.42109e-14

Case 2

x	T_exact	T_FDM	Error
0	9.93279	9.97615	0.4365
0.0625	10.1079	10.1515	0.431411
0.125	10.6394	10.6838	0.416822
0.1875	11.5461	11.5916	0.394508
0.25	12.8598	12.907	0.366752
0.3125	14.627	14.6761	0.335743
0.375	16.9099	16.9612	0.303211
0.4375	19.7891	19.8426	0.270328
0.5	23.366	23.4216	0.237773
0.5625	27.7668	27.8239	0.205873
0.625	33.1466	33.2045	0.174731
0.6875	39.6951	39.7524	0.144321
0.75	47.6433	47.6979	0.11456
0.8125	57.2714	57.3202	0.0853443
0.875	68.9188	68.9577	0.0565714
0.9375	82.9962	83.0195	0.0281491
1	100	100	0

A.1.5 $\alpha = 5$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	8.15356	12.1212	48.6616
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	1.34753	3.02744	124.666
0.5	8.26343	12.4882	51.1257
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	2.15883	2.62549	21.6165
0.5	8.15356	9.35331	14.7144
0.75	28.6359	30.6957	7.19282
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	1.34753	1.78085	32.1565
0.25	2.5447	3.17213	24.6562
0.5	8.26343	9.51987	15.2048
0.75	28.665	30.7424	7.24715
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	0.898199	0.954044	6.21746
0.25	2.15883	2.28076	5.64815
0.375	4.29056	4.4984	4.84417
0.5	8.15356	8.47323	3.92057
0.625	15.3066	15.7579	2.94855
0.75	28.6359	29.198	1.96289
0.875	53.5201	54.0436	0.978188
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	1.34753	1.45683	8.11132
0.125	1.6194	1.74137	7.53186
0.25	2.5447	2.70613	6.34346
0.375	4.49682	4.72797	5.14025
0.5	8.26343	8.59667	4.03265
0.625	15.3644	15.8234	2.98768
0.75	28.665	29.2313	1.97532
0.875	53.5322	54.0575	0.981321
1	100	100	0

n = 16

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	0.428029	0.434917	1.60917
0.125	0.898199	0.912305	1.57056
0.1875	1.4568	1.47879	1.50929
0.25	2.15883	2.18968	1.42918
0.3125	3.0734	3.11441	1.3344
0.375	4.29056	4.34328	1.22884
0.4375	5.93014	5.9963	1.11573
0.5	8.15356	8.2349	0.997575
0.5625	11.1797	11.2777	0.876171
0.625	15.3066	15.4218	0.752755
0.6875	20.9404	21.072	0.628147
0.75	28.6359	28.7799	0.502869
0.8125	39.1507	39.2984	0.377251
0.875	53.5201	53.6547	0.251493
0.9375	73.1587	73.2507	0.125719
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	1.34753	1.37492	2.03267
0.0625	1.41386	1.44205	1.9939
0.125	1.6194	1.65001	1.89064
0.1875	1.98437	2.01911	1.75077
0.25	2.5447	2.58538	1.5985
0.3125	3.35558	3.40413	1.44702
0.375	4.49682	4.55532	1.30094
0.4375	6.08079	6.15136	1.1606
0.5	8.26343	8.34812	1.02487
0.5625	11.2596	11.3601	0.892458
0.625	15.3644	15.4815	0.76229
0.6875	20.9818	21.1148	0.633606
0.75	28.665	28.81	0.505901
0.8125	39.1704	39.3188	0.378852
0.875	53.5322	53.6672	0.252258
0.9375	73.1644	73.2566	0.125999
1	100	100	0

A.1.6 $\alpha = 7$

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	3.01699	7.01754	132.601
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	0.182376	0.994716	445.419
0.5	3.02249	7.08735	134.487
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	0.508906	0.835971	64.2683
0.5	3.01699	4.2321	40.2759
0.75	17.3769	20.5891	18.485
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	0.182376	0.359502	97.121
0.25	0.540598	0.909989	68.3303
0.5	3.02249	4.24732	40.5239
0.75	17.3779	20.5921	18.496
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	0.180736	0.214032	18.422
0.25	0.508906	0.591931	16.3144
0.375	1.25221	1.42303	13.6414
0.5	3.01699	3.34363	10.8268
0.625	7.24283	7.8242	8.0268
0.75	17.3769	18.2952	5.28429
0.875	41.686	42.7734	2.60845
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	0.182376	0.224098	22.877
0.125	0.256762	0.309886	20.6901
0.25	0.540598	0.63293	17.0798
0.375	1.26542	1.44056	13.8407
0.5	3.02249	3.35112	10.873
0.625	7.24512	7.82739	8.03677
0.75	17.3779	18.2965	5.2863
0.875	41.6864	42.7739	2.60879
1	100	100	0

n = 16

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	0.0823596	0.086265	4.74189
0.125	0.180736	0.189042	4.59524
0.1875	0.314263	0.328002	4.37198
0.25	0.508906	0.529744	4.09465
0.3125	0.802521	0.832882	3.78323
0.375	1.25221	1.29544	3.4523
0.4375	1.94543	2.00595	3.11119
0.5	3.01699	3.10042	2.76532
0.5625	4.67529	4.78832	2.41764
0.625	7.24283	7.39274	2.06969
0.6875	11.219	11.4122	1.72221
0.75	17.3769	17.616	1.3756
0.8125	26.9143	27.1916	1.03001
0.875	41.686	41.9718	0.685524
0.9375	64.5648	64.7857	0.342184
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	0.182376	0.192622	5.61791
0.0625	0.20011	0.211056	5.47003
0.125	0.256762	0.269889	5.11243
0.1875	0.363348	0.380379	4.68726
0.25	0.540598	0.563676	4.26912
0.3125	0.822982	0.854865	3.87407
0.375	1.26542	1.30968	3.49773
0.4375	1.95395	2.01518	3.13327
0.5	3.02249	3.10639	2.77583
0.5625	4.67884	4.79218	2.42256
0.625	7.24512	7.39523	2.07195
0.6875	11.2204	11.4138	1.72324
0.75	17.3779	17.617	1.37605
0.8125	26.9149	27.1922	1.0302
0.875	41.6864	41.9722	0.685601
0.9375	64.5649	64.7859	0.342208
1	100	100	0

A.2 4th Order

A.2.1 $\alpha = 0.2$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	49.751	49.751	2.07559e-07
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	98.0328	98.0328	8.22072e-07
0.5	98.5234	98.5234	6.14512e-07
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	24.8445	24.8445	1.62285e-08
0.5	49.751	49.751	1.29763e-08
0.75	74.782	74.782	7.56323e-09
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	98.0328	98.0328	5.13948e-08
0.25	98.1554	98.1554	4.81426e-08
0.5	98.5234	98.5234	3.84184e-08
0.75	99.1377	99.1377	2.23185e-08
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	12.4183	12.4183	1.06495e-09
0.25	24.8445	24.8445	1.0141e-09
0.375	37.2861	37.2861	9.29405e-10
0.5	49.751	49.751	8.10873e-10
0.625	62.2471	62.2471	6.5857e-10
0.75	74.782	74.782	4.72606e-10
0.875	87.3637	87.3637	2.53039e-10
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	98.0328	98.0328	3.21188e-09
0.125	98.0634	98.0634	3.16106e-09
0.25	98.1554	98.1554	3.00866e-09
0.375	98.3086	98.3086	2.75509e-09
0.5	98.5234	98.5234	2.40095e-09
0.625	98.7997	98.7997	1.94717e-09
0.75	99.1377	99.1377	1.3948e-09
0.875	99.5378	99.5378	7.45209e-10
1	100	100	0

n = 16

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	6.20869	6.20869	6.66918e-11
0.125	12.4183	12.4183	6.59142e-11
0.1875	18.6299	18.6299	6.46088e-11
0.25	24.8445	24.8445	6.27476e-11
0.3125	31.0628	31.0628	6.03997e-11
0.375	37.2861	37.2861	5.75316e-11
0.4375	43.5152	43.5152	5.41294e-11
0.5	49.751	49.751	5.02011e-11
0.5625	55.9947	55.9947	4.57328e-11
0.625	62.2471	62.2471	4.07625e-11
0.6875	68.5092	68.5092	3.52838e-11
0.75	74.782	74.782	2.92647e-11
0.8125	81.0665	81.0665	2.27012e-11
0.875	87.3637	87.3637	1.56645e-11
0.9375	93.6745	93.6745	8.07068e-12
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	98.0328	98.0328	1.98566e-10
0.0625	98.0405	98.0405	1.97797e-10
0.125	98.0634	98.0634	1.95432e-10
0.1875	98.1017	98.1017	1.91503e-10
0.25	98.1554	98.1554	1.86041e-10
0.3125	98.2243	98.2243	1.78995e-10
0.375	98.3086	98.3086	1.70385e-10
0.4375	98.4083	98.4083	1.60205e-10
0.5	98.5234	98.5234	1.48479e-10
0.5625	98.6538	98.6538	1.35232e-10
0.625	98.7997	98.7997	1.20433e-10
0.6875	98.961	98.961	1.0411e-10
0.75	99.1377	99.1377	8.62504e-11
0.8125	99.33	99.33	6.68982e-11
0.875	99.5378	99.5378	4.60714e-11
0.9375	99.7611	99.7611	2.3789e-11
1	100	100	0

A.2.2 $\alpha = 0.4$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	49.0164	49.0164	1.31377e-05
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	92.5007	92.5007	5.05802e-05
0.5	94.3569	94.3569	3.74425e-05
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	24.3862	24.3862	1.02963e-06
0.5	49.0164	49.0164	8.22072e-07
0.75	74.1372	74.1372	4.77966e-07
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	92.5007	92.5007	3.16499e-06
0.25	92.9636	92.9636	2.95743e-06
0.5	94.3569	94.3569	2.34292e-06
0.75	96.6946	96.6946	1.345e-06
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	12.1779	12.1779	6.76233e-08
0.25	24.3862	24.3862	6.43711e-08
0.375	36.6555	36.6555	5.8958e-08
0.5	49.0164	49.0164	5.13948e-08
0.625	61.4999	61.4999	4.16963e-08
0.75	74.1372	74.1372	2.98818e-08
0.875	86.9599	86.9599	1.5974e-08
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	92.5007	92.5007	1.9787e-07
0.125	92.6164	92.6164	1.94618e-07
0.25	92.9636	92.9636	1.84894e-07
0.375	93.5433	93.5433	1.68794e-07
0.5	94.3569	94.3569	1.46476e-07
0.625	95.4065	95.4065	1.18152e-07
0.75	96.6946	96.6946	8.40877e-08
0.875	98.2245	98.2245	4.45895e-08
1	100	100	0

n = 16

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	6.08703	6.08703	4.27668e-09
0.125	12.1779	12.1779	4.22583e-09
0.1875	18.2763	18.2763	4.14113e-09
0.25	24.3862	24.3862	4.0226e-09
0.3125	30.5113	30.5113	3.87029e-09
0.375	36.6555	36.6555	3.68433e-09
0.4375	42.8225	42.8225	3.46476e-09
0.5	49.0164	49.0164	3.21172e-09
0.5625	55.2409	55.2409	2.92531e-09
0.625	61.4999	61.4999	2.60566e-09
0.6875	67.7974	67.7974	2.25297e-09
0.75	74.1372	74.1372	1.8674e-09
0.8125	80.5234	80.5234	1.44907e-09
0.875	86.9599	86.9599	9.98258e-10
0.9375	93.4507	93.4507	5.1516e-10
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	92.5007	92.5007	1.23656e-08
0.0625	92.5297	92.5297	1.23148e-08
0.125	92.6164	92.6164	1.21624e-08
0.1875	92.761	92.761	1.19088e-08
0.25	92.9636	92.9636	1.15547e-08
0.3125	93.2243	93.2243	1.11009e-08
0.375	93.5433	93.5433	1.05485e-08
0.4375	93.9208	93.9208	9.89893e-09
0.5	94.3569	94.3569	9.15372e-09
0.5625	94.8521	94.8521	8.31464e-09
0.625	95.4065	95.4065	7.38366e-09
0.6875	96.0205	96.0205	6.36296e-09
0.75	96.6946	96.6946	5.25487e-09
0.8125	97.4291	97.4291	4.06184e-09
0.875	98.2245	98.2245	2.78653e-09
0.9375	99.0813	99.0813	1.43162e-09
1	100	100	0

A.2.3 $\alpha = 0.75$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	46.6792	46.679	0.000550658
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	77.239	77.2375	0.00195194
0.5	82.7338	82.7326	0.00140129
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	22.9353	22.9353	4.34904e-05
0.5	46.6792	46.6792	3.4555e-05
0.75	72.0691	72.069	1.99322e-05
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	77.239	77.2389	0.00012249
0.25	78.6007	78.6006	0.000113554
0.5	82.7338	82.7337	8.79347e-05
0.75	89.784	89.784	4.87484e-05
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	11.4174	11.4174	2.86195e-06
0.25	22.9353	22.9353	2.72097e-06
0.375	34.6548	34.6548	2.48709e-06
0.5	46.6792	46.6792	2.16192e-06
0.625	59.1142	59.1142	1.74767e-06
0.75	72.0691	72.0691	1.24706e-06
0.875	85.6578	85.6578	6.63297e-07
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	77.239	77.239	7.66354e-06
0.125	77.5786	77.5786	7.52256e-06
0.25	78.6007	78.6007	7.1045e-06
0.375	80.314	80.314	6.42351e-06
0.5	82.7338	82.7338	5.50162e-06
0.625	85.8812	85.8812	4.36666e-06
0.75	89.784	89.784	3.04994e-06
0.875	94.4765	94.4765	1.58386e-06
1	100	100	0

n = 16

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	5.70245	5.70245	1.81126e-07
0.125	11.4174	11.4174	1.78918e-07
0.1875	17.1575	17.1575	1.75242e-07
0.25	22.9353	22.9353	1.70104e-07
0.3125	28.7635	28.7635	1.63514e-07
0.375	34.6548	34.6548	1.55483e-07
0.4375	40.6224	40.6224	1.46025e-07
0.5	46.6792	46.6792	1.35155e-07
0.5625	52.8386	52.8386	1.22893e-07
0.625	59.1142	59.1142	1.09257e-07
0.6875	65.5196	65.5196	9.42722e-08
0.75	72.0691	72.0691	7.79612e-08
0.8125	78.7769	78.7769	6.03502e-08
0.875	85.6578	85.6578	4.14668e-08
0.9375	92.727	92.727	2.134e-08
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	77.239	77.239	4.79095e-07
0.0625	77.3238	77.3238	4.76886e-07
0.125	77.5786	77.5786	4.70281e-07
0.1875	78.0039	78.0039	4.59336e-07
0.25	78.6007	78.6007	4.44145e-07
0.3125	79.3701	79.3701	4.24838e-07
0.375	80.314	80.314	4.01573e-07
0.4375	81.4344	81.4344	3.74537e-07
0.5	82.7338	82.7338	3.4394e-07
0.5625	84.215	84.215	3.10009e-07
0.625	85.8812	85.8812	2.72987e-07
0.6875	87.7362	87.7362	2.33122e-07
0.75	89.784	89.784	1.9067e-07
0.8125	92.0292	92.0292	1.45885e-07
0.875	94.4765	94.4765	9.90168e-08
0.9375	97.1315	97.1315	5.03105e-08
1	100	100	0

A.2.4 $\alpha = 3$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	21.2548	20.9677	1.35056
1	100	100	1.42109e-14

Case 2

x	T_exact	T_FDM	Error
0	9.93279	9.64062	2.94154
0.5	23.366	22.9892	1.61275
1	100	100	0

n = 4
Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	8.20849	8.19877	0.118462
0.5	21.2548	21.2362	0.0877048
0.75	46.828	46.8064	0.0461534
1	100	100	1.42109e-14

Case 2

x	T_exact	T_FDM	Error
0	9.93279	9.91365	0.192715
0.25	12.8598	12.839	0.16198
0.5	23.366	23.3414	0.105102
0.75	47.6433	47.6192	0.0506849
1	100	100	0

n = 8
Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	3.83166	3.83135	0.0080657
0.25	8.20849	8.20788	0.00751509
0.375	13.7532	13.7523	0.00665766
0.5	21.2548	21.2536	0.00556326
0.625	31.7805	31.7791	0.00430073
0.75	46.828	46.8266	0.00292711
0.875	68.5382	68.5371	0.0014842
1	100	100	1.42109e-14

Case 2

x	T_exact	T_FDM	Error
0	9.93279	9.93158	0.0122312
0.125	10.6394	10.6382	0.0116806
0.25	12.8598	12.8585	0.0102795
0.375	16.9099	16.9085	0.00850086
0.5	23.366	23.3644	0.00666833
0.625	33.1466	33.145	0.00490192
0.75	47.6433	47.6418	0.00321492
0.875	68.9188	68.9177	0.00158808
1	100	100	0

n = 16
Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	1.88264	1.88263	0.000515098
0.125	3.83166	3.83164	0.000506162
0.1875	5.91578	5.91576	0.00049154
0.25	8.20849	8.20846	0.000471607
0.3125	10.7906	10.7906	0.000446846
0.375	13.7532	13.7532	0.000417799
0.4375	17.2008	17.2007	0.000385035
0.5	21.2548	21.2547	0.000349118
0.5625	26.0583	26.0582	0.000310576
0.625	31.7805	31.7804	0.000269888
0.6875	38.6233	38.6232	0.000227472
0.75	46.828	46.8279	0.000183687
0.8125	56.6838	56.6837	0.000138828
0.875	68.5382	68.5381	9.31383e-05
0.9375	82.8092	82.8091	4.68113e-05
1	100	100	1.42109e-14

Case 2

x	T_exact	T_FDM	Error
0	9.93279	9.93272	0.000767589
0.0625	10.1079	10.1078	0.000758654
0.125	10.6394	10.6393	0.000733034
0.1875	11.5461	11.546	0.000693848
0.25	12.8598	12.8597	0.0006451
0.3125	14.627	14.6269	0.000590631
0.375	16.9099	16.9098	0.000533478
0.4375	19.7891	19.789	0.000475693
0.5	23.366	23.3659	0.000418472
0.5625	27.7668	27.7667	0.000362388
0.625	33.1466	33.1465	0.000307618
0.6875	39.6951	39.695	0.000254121
0.75	47.6433	47.6432	0.00020175
0.8125	57.2714	57.2713	0.000150322
0.875	68.9188	68.9187	9.9658e-05
0.9375	82.9962	82.9961	4.95958e-05
1	100	100	0

A.2.5 $\alpha = 5$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	8.15356	6.6474	18.4725
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	1.34753	0.891638	33.8316
0.5	8.26343	6.70667	18.8392
1	100	100	0

n = 4
Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	2.15883	2.12207	1.70242
0.5	8.15356	8.05626	1.19336
0.75	28.6359	28.4628	0.604712
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	1.34753	1.31514	2.40368
0.25	2.5447	2.4964	1.89826
0.5	8.26343	8.16221	1.22494
0.75	28.665	28.4907	0.608223
1	100	100	0

n = 8
Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	0.898199	0.897109	0.121315
0.25	2.15883	2.15644	0.110464
0.375	4.29056	4.28648	0.0950688
0.5	8.15356	8.14726	0.077265
0.625	15.3066	15.2977	0.0583757
0.75	28.6359	28.6248	0.039048
0.875	53.5201	53.5096	0.0195546
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	1.34753	1.34542	0.156549
0.125	1.6194	1.61704	0.145701
0.25	2.5447	2.54157	0.123364
0.375	4.49682	4.4923	0.100562
0.5	8.26343	8.25688	0.0793449
0.625	15.3644	15.3553	0.0591023
0.75	28.665	28.6538	0.0392792
0.875	53.5322	53.5217	0.0196129
1	100	100	0

n = 16
Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	0.428029	0.427995	0.00785374
0.125	0.898199	0.89813	0.00766651
0.1875	1.4568	1.45669	0.00736928
0.25	2.15883	2.15868	0.00698049
0.3125	3.0734	3.0732	0.00652024
0.375	4.29056	4.2903	0.00600723
0.4375	5.93014	5.92981	0.0054571
0.5	8.15356	8.15316	0.00488186
0.5625	11.1797	11.1793	0.0042902
0.625	15.3066	15.306	0.00368806
0.6875	20.9404	20.9398	0.00307941
0.75	28.6359	28.6352	0.00246676
0.8125	39.1507	39.15	0.0018517
0.875	53.5201	53.5194	0.0012352
0.9375	73.1587	73.1582	0.000617849
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	1.34753	1.34739	0.00989504
0.0625	1.41386	1.41373	0.00970782
0.125	1.6194	1.61925	0.00920903
0.1875	1.98437	1.9842	0.00853303
0.25	2.5447	2.54451	0.00779647
0.3125	3.35558	3.35534	0.00706296
0.375	4.49682	4.49653	0.0063547
0.4375	6.08079	6.08044	0.00567333
0.5	8.26343	8.26302	0.00501342
0.5625	11.2596	11.2592	0.0043687
0.625	15.3644	15.3638	0.00373402
0.6875	20.9818	20.9812	0.00310573
0.75	28.665	28.6643	0.00248138
0.8125	39.1704	39.1696	0.00185942
0.875	53.5322	53.5315	0.00123889
0.9375	73.1644	73.164	0.0006192
1	100	100	0

A.2.6 $\alpha = 7$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	3.01699	-0.170648	105.656
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	0.182376	0.000582421	99.6806
0.5	3.02249	-0.170649	105.646
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	0.508906	0.462777	9.0644
0.5	3.01699	2.82844	6.24951
0.75	17.3769	16.8243	3.18006
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	0.182376	0.160258	12.1279
0.25	0.540598	0.489739	9.40781
0.5	3.02249	2.83297	6.27021
0.75	17.3779	16.8251	3.18097
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	0.180736	0.179502	0.682755
0.25	0.508906	0.505802	0.609911
0.375	1.25221	1.24575	0.515978
0.5	3.01699	3.00447	0.41486
0.625	7.24283	7.22026	0.311709
0.75	17.3769	17.3408	0.20799
0.875	41.686	41.6427	0.10406
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	0.182376	0.180863	0.829496
0.125	0.256762	0.254819	0.756759
0.25	0.540598	0.537165	0.634875
0.375	1.26542	1.25881	0.522478
0.5	3.02249	3.0099	0.416363
0.625	7.24512	7.22251	0.312034
0.75	17.3779	17.3417	0.208055
0.875	41.6864	41.643	0.104071
1	100	100	0

n = 16

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	0.0823596	0.0823225	0.0449748
0.125	0.180736	0.180658	0.043611
0.1875	0.314263	0.314132	0.0415324
0.25	0.508906	0.508708	0.038946
0.3125	0.802521	0.802232	0.0360353
0.375	1.25221	1.2518	0.0329342
0.4375	1.94543	1.94485	0.0297283
0.5	3.01699	3.01619	0.0264676
0.5625	4.67529	4.6742	0.0231792
0.625	7.24283	7.24139	0.0198772
0.6875	11.219	11.2171	0.0165686
0.75	17.3769	17.3746	0.0132567
0.8125	26.9143	26.9117	0.0099434
0.875	41.686	41.6833	0.00662929
0.9375	64.5648	64.5626	0.00331477
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	0.182376	0.18228	0.0530248
0.0625	0.20011	0.200007	0.0516611
0.125	0.256762	0.256638	0.0483602
0.1875	0.363348	0.363186	0.0444275
0.25	0.540598	0.540378	0.0405481
0.3125	0.822982	0.822678	0.0368694
0.375	1.26542	1.265	0.0333513
0.4375	1.95395	1.95337	0.0299311
0.5	3.02249	3.02169	0.0265641
0.5625	4.67884	4.67775	0.0232244
0.625	7.24512	7.24368	0.019898
0.6875	11.2204	11.2186	0.016578
0.75	17.3779	17.3756	0.0132609
0.8125	26.9149	26.9122	0.0099452
0.875	41.6864	41.6836	0.00663
0.9375	64.5649	64.5628	0.00331499
1	100	100	0

A.3 6th Order

A.3.1 $\alpha = 0.2$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	49.751	49.751	2.47221e-11
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	98.0328	98.0328	9.78771e-11
0.5	98.5234	98.5234	7.31721e-11
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	24.8445	24.8445	5.57693e-13
0.5	49.751	49.751	4.42741e-13
0.75	74.782	74.782	2.47039e-13
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	98.0328	98.0328	1.72503e-12
0.25	98.1554	98.1554	1.60705e-12
0.5	98.5234	98.5234	1.29815e-12
0.75	99.1377	99.1377	7.45392e-13
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	12.4183	12.4183	3.0039e-13
0.25	24.8445	24.8445	3.00296e-13
0.375	37.2861	37.2861	2.66791e-13
0.5	49.751	49.751	2.28511e-13
0.625	62.2471	62.2471	1.94053e-13
0.75	74.782	74.782	1.33021e-13
0.875	87.3637	87.3637	8.13316e-14
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	98.0328	98.0328	8.11777e-13
0.125	98.0634	98.0634	7.82541e-13
0.25	98.1554	98.1554	7.38374e-13
0.375	98.3086	98.3086	6.64946e-13
0.5	98.5234	98.5234	5.91377e-13
0.625	98.7997	98.7997	4.74656e-13
0.75	99.1377	99.1377	3.58361e-13
0.875	99.5378	99.5378	1.99876e-13
1	100	100	0

n = 16

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	6.20869	6.20869	1.07291e-12
0.125	12.4183	12.4183	1.05852e-12
0.1875	18.6299	18.6299	1.02978e-12
0.25	24.8445	24.8445	1.01529e-12
0.3125	31.0628	31.0628	9.7216e-13
0.375	37.2861	37.2861	9.14712e-13
0.4375	43.5152	43.5152	8.49088e-13
0.5	49.751	49.751	7.85508e-13
0.5625	55.9947	55.9947	7.233e-13
0.625	62.2471	62.2471	6.39233e-13
0.6875	68.5092	68.5092	5.60061e-13
0.75	74.782	74.782	4.75076e-13
0.8125	81.0665	81.0665	4.03187e-13
0.875	87.3637	87.3637	2.76527e-13
0.9375	93.6745	93.6745	1.66875e-13
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	98.0328	98.0328	3.53703e-12
0.0625	98.0405	98.0405	3.50776e-12
0.125	98.0634	98.0634	3.46347e-12
0.1875	98.1017	98.1017	3.40417e-12
0.25	98.1554	98.1554	3.27201e-12
0.3125	98.2243	98.2243	3.12503e-12
0.375	98.3086	98.3086	2.9778e-12
0.4375	98.4083	98.4083	2.8015e-12
0.5	98.5234	98.5234	2.59629e-12
0.5625	98.6538	98.6538	2.33357e-12
0.625	98.7997	98.7997	2.08561e-12
0.6875	98.961	98.961	1.78065e-12
0.75	99.1377	99.1377	1.47645e-12
0.8125	99.33	99.33	1.14454e-12
0.875	99.5378	99.5378	7.99503e-13
0.9375	99.7611	99.7611	3.98857e-13
1	100	100	0

A.3.2 $\alpha = 0.4$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	49.0164	49.0164	6.22706e-09
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	92.5007	92.5007	2.39743e-08
0.5	94.3569	94.3569	1.77472e-08
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	24.3862	24.3862	1.22565e-10
0.5	49.0164	49.0164	9.78481e-11
0.75	74.1372	74.1372	5.68532e-11
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	92.5007	92.5007	3.76823e-10
0.25	92.9636	92.9636	3.52124e-10
0.5	94.3569	94.3569	2.78955e-10
0.75	96.6946	96.6946	1.60149e-10
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	12.1779	12.1779	2.23178e-12
0.25	24.3862	24.3862	2.12701e-12
0.375	36.6555	36.6555	1.93844e-12
0.5	49.0164	49.0164	1.68154e-12
0.625	61.4999	61.4999	1.37487e-12
0.75	74.1372	74.1372	9.58416e-13
0.875	86.9599	86.9599	5.22939e-13
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	92.5007	92.5007	6.62144e-12
0.125	92.6164	92.6164	6.50576e-12
0.25	92.9636	92.9636	6.19102e-12
0.375	93.5433	93.5433	5.63613e-12
0.5	94.3569	94.3569	4.87968e-12
0.625	95.4065	95.4065	3.94719e-12
0.75	96.6946	96.6946	2.79236e-12
0.875	98.2245	98.2245	1.47571e-12
1	100	100	0

n = 16

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	6.08703	6.08703	1.24026e-12
0.125	12.1779	12.1779	1.22529e-12
0.1875	18.2763	18.2763	1.20521e-12
0.25	24.3862	24.3862	1.16548e-12
0.3125	30.5113	30.5113	1.12946e-12
0.375	36.6555	36.6555	1.06614e-12
0.4375	42.8225	42.8225	1.02875e-12
0.5	49.0164	49.0164	9.42241e-13
0.5625	55.2409	55.2409	8.74658e-13
0.625	61.4999	61.4999	7.85642e-13
0.6875	67.7974	67.7974	6.70745e-13
0.75	74.1372	74.1372	5.36713e-13
0.8125	80.5234	80.5234	4.41203e-13
0.875	86.9599	86.9599	2.94153e-13
0.9375	93.4507	93.4507	1.52068e-13
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	92.5007	92.5007	3.10332e-12
0.0625	92.5297	92.5297	3.07163e-12
0.125	92.6164	92.6164	3.03807e-12
0.1875	92.761	92.761	2.97205e-12
0.25	92.9636	92.9636	2.88914e-12
0.3125	93.2243	93.2243	2.77436e-12
0.375	93.5433	93.5433	2.65855e-12
0.4375	93.9208	93.9208	2.49656e-12
0.5	94.3569	94.3569	2.30429e-12
0.5625	94.8521	94.8521	2.11248e-12
0.625	95.4065	95.4065	1.89167e-12
0.6875	96.0205	96.0205	1.62798e-12
0.75	96.6946	96.6946	1.3227e-12
0.8125	97.4291	97.4291	1.03559e-12
0.875	98.2245	98.2245	6.94451e-13
0.9375	99.0813	99.0813	3.58566e-13
1	100	100	0

A.3.3 $\alpha = 0.75$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	46.6792	46.6792	9.06905e-07
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	77.239	77.239	3.21478e-06
0.5	82.7338	82.7338	2.30787e-06
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	22.9353	22.9353	1.81279e-08
0.5	46.6792	46.6792	1.44033e-08
0.75	72.0691	72.0691	8.30824e-09
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	77.239	77.239	5.10567e-08
0.25	78.6007	78.6007	4.73321e-08
0.5	82.7338	82.7338	3.66533e-08
0.75	89.784	89.784	2.03195e-08
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	11.4174	11.4174	2.99046e-10
0.25	22.9353	22.9353	2.84338e-10
0.375	34.6548	34.6548	2.59881e-10
0.5	46.6792	46.6792	2.25892e-10
0.625	59.1142	59.1142	1.82593e-10
0.75	72.0691	72.0691	1.30299e-10
0.875	85.6578	85.6578	6.92975e-11
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	77.239	77.239	8.00852e-10
0.125	77.5786	77.5786	7.86135e-10
0.25	78.6007	78.6007	7.42411e-10
0.375	80.314	80.314	6.71279e-10
0.5	82.7338	82.7338	5.74935e-10
0.625	85.8812	85.8812	4.5632e-10
0.75	89.784	89.784	3.18741e-10
0.875	94.4765	94.4765	1.65549e-10
1	100	100	0

n = 16

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	5.70245	5.70245	5.03085e-12
0.125	11.4174	11.4174	4.94754e-12
0.1875	17.1575	17.1575	4.86602e-12
0.25	22.9353	22.9353	4.7245e-12
0.3125	28.7635	28.7635	4.533e-12
0.375	34.6548	34.6548	4.30572e-12
0.4375	40.6224	40.6224	4.04051e-12
0.5	46.6792	46.6792	3.74457e-12
0.5625	52.8386	52.8386	3.41564e-12
0.625	59.1142	59.1142	3.04102e-12
0.6875	65.5196	65.5196	2.62443e-12
0.75	72.0691	72.0691	2.16902e-12
0.8125	78.7769	78.7769	1.65962e-12
0.875	85.6578	85.6578	1.14473e-12
0.9375	92.727	92.727	5.67043e-13
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	77.239	77.239	1.33941e-11
0.0625	77.3238	77.3238	1.33243e-11
0.125	77.5786	77.5786	1.31706e-11
0.1875	78.0039	78.0039	1.2862e-11
0.25	78.6007	78.6007	1.24208e-11
0.3125	79.3701	79.3701	1.18886e-11
0.375	80.314	80.314	1.12358e-11
0.4375	81.4344	81.4344	1.0453e-11
0.5	82.7338	82.7338	9.58455e-12
0.5625	84.215	84.215	8.63975e-12
0.625	85.8812	85.8812	7.57857e-12
0.6875	87.7362	87.7362	6.4789e-12
0.75	89.784	89.784	5.28649e-12
0.8125	92.0292	92.0292	4.04572e-12
0.875	94.4765	94.4765	2.75263e-12
0.9375	97.1315	97.1315	1.40453e-12
1	100	100	0

A.3.4 $\alpha = 3$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	21.2548	21.2607	0.0277156
1	100	100	1.42109e-14

Case 2

x	T_exact	T_FDM	Error
0	9.93279	9.93885	0.0609488
0.5	23.366	23.3738	0.033224
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	8.20849	8.20856	0.000743033
0.5	21.2548	21.2549	0.000550047
0.75	46.828	46.8281	0.000289403
1	100	100	1.42109e-14

Case 2

x	T_exact	T_FDM	Error
0	9.93279	9.93291	0.00120937
0.25	12.8598	12.86	0.00101638
0.5	23.366	23.3662	0.000659318
0.75	47.6433	47.6435	0.000317863
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	3.83166	3.83166	1.32844e-05
0.25	8.20849	8.2085	1.23775e-05
0.375	13.7532	13.7532	1.09653e-05
0.5	21.2548	21.2548	9.16274e-06
0.625	31.7805	31.7805	7.0833e-06
0.75	46.828	46.828	4.82092e-06
0.875	68.5382	68.5382	2.44445e-06
1	100	100	1.42109e-14

Case 2

x	T_exact	T_FDM	Error
0	9.93279	9.93279	2.01457e-05
0.125	10.6394	10.6394	1.92388e-05
0.25	12.8598	12.8598	1.69309e-05
0.375	16.9099	16.9099	1.40013e-05
0.5	23.366	23.366	1.0983e-05
0.625	33.1466	33.1466	8.07356e-06
0.75	47.6433	47.6433	5.295e-06
0.875	68.9188	68.9188	2.61556e-06
1	100	100	0

n = 16

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	1.88264	1.88264	2.14706e-07
0.125	3.83166	3.83166	2.10981e-07
0.1875	5.91578	5.91578	2.04886e-07
0.25	8.20849	8.20849	1.96578e-07
0.3125	10.7906	10.7906	1.86257e-07
0.375	13.7532	13.7532	1.74149e-07
0.4375	17.2008	17.2008	1.60493e-07
0.5	21.2548	21.2548	1.45521e-07
0.5625	26.0583	26.0583	1.29456e-07
0.625	31.7805	31.7805	1.12496e-07
0.6875	38.6233	38.6233	9.48161e-08
0.75	46.828	46.828	7.65652e-08
0.8125	56.6838	56.6838	5.7867e-08
0.875	68.5382	68.5382	3.88223e-08
0.9375	82.8092	82.8092	1.95121e-08
1	100	100	1.42109e-14

Case 2

x	T_exact	T_FDM	Error
0	9.93279	9.93279	3.19951e-07
0.0625	10.1079	10.1079	3.16227e-07
0.125	10.6394	10.6394	3.05548e-07
0.1875	11.5461	11.5461	2.89214e-07
0.25	12.8598	12.8598	2.68895e-07
0.3125	14.627	14.627	2.4619e-07
0.375	16.9099	16.9099	2.22367e-07
0.4375	19.7891	19.7891	1.98281e-07
0.5	23.366	23.366	1.7443e-07
0.5625	27.7668	27.7668	1.51053e-07
0.625	33.1466	33.1466	1.28223e-07
0.6875	39.6951	39.6951	1.05924e-07
0.75	47.6433	47.6433	8.40943e-08
0.8125	57.2714	57.2714	6.2658e-08
0.875	68.9188	68.9188	4.15399e-08
0.9375	82.9962	82.9962	2.06727e-08
1	100	100	0

A.3.5 $\alpha = 5$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	8.15356	8.20791	0.6665
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	1.34753	1.3658	1.35571
0.5	8.26343	8.32001	0.684646
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	2.15883	2.1594	0.0264998
0.5	8.15356	8.15507	0.018532
0.75	28.6359	28.6386	0.00936355
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	1.34753	1.34803	0.0375672
0.25	2.5447	2.54546	0.0295985
0.5	8.26343	8.26501	0.0190316
0.75	28.665	28.6677	0.00941907
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	0.898199	0.898204	0.000539309
0.25	2.15883	2.15884	0.000491048
0.375	4.29056	4.29058	0.000422582
0.5	8.15356	8.15359	0.000343415
0.625	15.3066	15.3066	0.000259436
0.75	28.6359	28.636	0.000173522
0.875	53.5201	53.5201	8.68886e-05
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	1.34753	1.34754	0.000696087
0.125	1.6194	1.61941	0.000647826
0.25	2.5447	2.54472	0.000548453
0.375	4.49682	4.49684	0.000447027
0.5	8.26343	8.26346	0.00035267
0.625	15.3644	15.3644	0.000262669
0.75	28.665	28.6651	0.000174551
0.875	53.5322	53.5322	8.71483e-05
1	100	100	0

n = 16

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	0.428029	0.428029	9.02795e-06
0.125	0.898199	0.898199	8.81273e-06
0.1875	1.4568	1.4568	8.47104e-06
0.25	2.15883	2.15883	8.02412e-06
0.3125	3.0734	3.0734	7.49504e-06
0.375	4.29056	4.29056	6.90532e-06
0.4375	5.93014	5.93014	6.27293e-06
0.5	8.15356	8.15356	5.61168e-06
0.5625	11.1797	11.1797	4.93155e-06
0.625	15.3066	15.3066	4.23939e-06
0.6875	20.9404	20.9404	3.53974e-06
0.75	28.6359	28.6359	2.8355e-06
0.8125	39.1507	39.1507	2.12849e-06
0.875	53.5201	53.5201	1.41983e-06
0.9375	73.1587	73.1587	7.102e-07
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	1.34753	1.34753	1.13746e-05
0.0625	1.41386	1.41386	1.11594e-05
0.125	1.6194	1.6194	1.0586e-05
0.1875	1.98437	1.98437	9.80887e-06
0.25	2.5447	2.5447	8.96215e-06
0.3125	3.35558	3.35558	8.11894e-06
0.375	4.49682	4.49682	7.30477e-06
0.4375	6.08079	6.08079	6.5215e-06
0.5	8.26343	8.26343	5.76291e-06
0.5625	11.2596	11.2596	5.02179e-06
0.625	15.3644	15.3644	4.29222e-06
0.6875	20.9818	20.9818	3.57e-06
0.75	28.665	28.665	2.85231e-06
0.8125	39.1704	39.1704	2.13737e-06
0.875	53.5322	53.5322	1.42407e-06
0.9375	73.1644	73.1644	7.11753e-07
1	100	100	0

A.3.6 $\alpha = 7$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	3.01699	3.13858	4.03041
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	0.182376	0.197403	8.23948
0.5	3.02249	3.14478	4.04601
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	0.508906	0.510118	0.238063
0.5	3.01699	3.02187	0.161718
0.75	17.3769	17.391	0.0809614
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	0.182376	0.182968	0.32429
0.25	0.540598	0.541938	0.24788
0.5	3.02249	3.0274	0.162309
0.75	17.3779	17.3919	0.0809871
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	0.180736	0.180747	0.00570646
0.25	0.508906	0.508932	0.00509593
0.375	1.25221	1.25226	0.00430916
0.5	3.01699	3.01709	0.00346295
0.625	7.24283	7.24302	0.00260058
0.75	17.3769	17.3772	0.00173434
0.875	41.686	41.6864	0.000867259
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	0.182376	0.182389	0.00693864
0.125	0.256762	0.256778	0.00632811
0.25	0.540598	0.540626	0.00530563
0.375	1.26542	1.26547	0.00436376
0.5	3.02249	3.02259	0.00347558
0.625	7.24512	7.24531	0.0026033
0.75	17.3779	17.3782	0.00173489
0.875	41.6864	41.6867	0.000867352
1	100	100	0

n = 16

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	0.0823596	0.0823597	0.000100247
0.125	0.180736	0.180737	9.72069e-05
0.1875	0.314263	0.314263	9.2573e-05
0.25	0.508906	0.508907	8.68071e-05
0.3125	0.802521	0.802522	8.03182e-05
0.375	1.25221	1.25221	7.34051e-05
0.4375	1.94543	1.94543	6.62587e-05
0.5	3.01699	3.01699	5.89903e-05
0.5625	4.67529	4.67529	5.16604e-05
0.625	7.24283	7.24284	4.43003e-05
0.6875	11.219	11.219	3.69257e-05
0.75	17.3769	17.3769	2.95443e-05
0.8125	26.9143	26.9144	2.21597e-05
0.875	41.686	41.686	1.47737e-05
0.9375	64.5648	64.5648	7.387e-06
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	0.182376	0.182376	0.000118196
0.0625	0.20011	0.200111	0.000115155
0.125	0.256762	0.256762	0.000107796
0.1875	0.363348	0.363348	9.9028e-05
0.25	0.540598	0.540598	9.03791e-05
0.3125	0.822982	0.822983	8.21779e-05
0.375	1.26542	1.26542	7.43351e-05
0.4375	1.95395	1.95395	6.67108e-05
0.5	3.02249	3.02249	5.92055e-05
0.5625	4.67884	4.67884	5.17611e-05
0.625	7.24512	7.24512	4.43467e-05
0.6875	11.2204	11.2204	3.69468e-05
0.75	17.3779	17.3779	2.95537e-05
0.8125	26.9149	26.9149	2.21638e-05
0.875	41.6864	41.6864	1.47753e-05
0.9375	64.5649	64.5649	7.3875e-06
1	100	100	0

A.4 8th Order

A.4.1 $\alpha = 0.2$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	49.751	49.751	4.28459e-14
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	98.0328	98.0328	1.30464e-13
0.5	98.5234	98.5234	1.00967e-13
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	24.8445	24.8445	1.28698e-13
0.5	49.751	49.751	9.99738e-14
0.75	74.782	74.782	5.70091e-14
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	98.0328	98.0328	3.18912e-13
0.25	98.1554	98.1554	2.89558e-13
0.5	98.5234	98.5234	2.45205e-13
0.75	99.1377	99.1377	1.2901e-13
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	12.4183	12.4183	3.0039e-13
0.25	24.8445	24.8445	3.00296e-13
0.375	37.2861	37.2861	2.66791e-13
0.5	49.751	49.751	2.28511e-13
0.625	62.2471	62.2471	1.94053e-13
0.75	74.782	74.782	1.33021e-13
0.875	87.3637	87.3637	8.13316e-14
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	98.0328	98.0328	8.11777e-13
0.125	98.0634	98.0634	7.82541e-13
0.25	98.1554	98.1554	7.38374e-13
0.375	98.3086	98.3086	6.64946e-13
0.5	98.5234	98.5234	5.91377e-13
0.625	98.7997	98.7997	4.74656e-13
0.75	99.1377	99.1377	3.58361e-13
0.875	99.5378	99.5378	1.99876e-13
1	100	100	0

n = 16

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	6.20869	6.20869	1.07291e-12
0.125	12.4183	12.4183	1.05852e-12
0.1875	18.6299	18.6299	1.02978e-12
0.25	24.8445	24.8445	1.01529e-12
0.3125	31.0628	31.0628	9.7216e-13
0.375	37.2861	37.2861	9.14712e-13
0.4375	43.5152	43.5152	8.49088e-13
0.5	49.751	49.751	7.85508e-13
0.5625	55.9947	55.9947	7.233e-13
0.625	62.2471	62.2471	6.39233e-13
0.6875	68.5092	68.5092	5.60061e-13
0.75	74.782	74.782	4.75076e-13
0.8125	81.0665	81.0665	4.03187e-13
0.875	87.3637	87.3637	2.76527e-13
0.9375	93.6745	93.6745	1.66875e-13
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	98.0328	98.0328	3.53703e-12
0.0625	98.0405	98.0405	3.50776e-12
0.125	98.0634	98.0634	3.46347e-12
0.1875	98.1017	98.1017	3.40417e-12
0.25	98.1554	98.1554	3.27201e-12
0.3125	98.2243	98.2243	3.12503e-12
0.375	98.3086	98.3086	2.9778e-12
0.4375	98.4083	98.4083	2.8015e-12
0.5	98.5234	98.5234	2.59629e-12
0.5625	98.6538	98.6538	2.33357e-12
0.625	98.7997	98.7997	2.08561e-12
0.6875	98.961	98.961	1.78065e-12
0.75	99.1377	99.1377	1.47645e-12
0.8125	99.33	99.33	1.14454e-12
0.875	99.5378	99.5378	7.99503e-13
0.9375	99.7611	99.7611	3.98857e-13
1	100	100	0

A.4.2 $\alpha = 0.4$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	49.0164	49.0164	2.75424e-12
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	92.5007	92.5007	1.06619e-11
0.5	94.3569	94.3569	7.89183e-12
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	24.3862	24.3862	1.60254e-13
0.5	49.0164	49.0164	1.15968e-13
0.75	74.1372	74.1372	3.83366e-14
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	92.5007	92.5007	5.06978e-13
0.25	92.9636	92.9636	4.89167e-13
0.5	94.3569	94.3569	3.76519e-13
0.75	96.6946	96.6946	2.2045e-13
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	12.1779	12.1779	4.23016e-13
0.25	24.3862	24.3862	4.0792e-13
0.375	36.6555	36.6555	3.68303e-13
0.5	49.0164	49.0164	3.18912e-13
0.625	61.4999	61.4999	2.65732e-13
0.75	74.1372	74.1372	1.53347e-13
0.875	86.9599	86.9599	9.80511e-14
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	92.5007	92.5007	1.13686e-12
0.125	92.6164	92.6164	1.1201e-12
0.25	92.9636	92.9636	1.07005e-12
0.375	93.5433	93.5433	9.57079e-13
0.5	94.3569	94.3569	8.1328e-13
0.625	95.4065	95.4065	6.70278e-13
0.75	96.6946	96.6946	4.84989e-13
0.875	98.2245	98.2245	2.60419e-13
1	100	100	0

n = 16

Case 1			
x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	6.08703	6.08703	1.24026e-12
0.125	12.1779	12.1779	1.22529e-12
0.1875	18.2763	18.2763	1.20521e-12
0.25	24.3862	24.3862	1.16548e-12
0.3125	30.5113	30.5113	1.12946e-12
0.375	36.6555	36.6555	1.06614e-12
0.4375	42.8225	42.8225	1.02875e-12
0.5	49.0164	49.0164	9.42241e-13
0.5625	55.2409	55.2409	8.74658e-13
0.625	61.4999	61.4999	7.85642e-13
0.6875	67.7974	67.7974	6.70745e-13
0.75	74.1372	74.1372	5.36713e-13
0.8125	80.5234	80.5234	4.41203e-13
0.875	86.9599	86.9599	2.94153e-13
0.9375	93.4507	93.4507	1.52068e-13
1	100	100	0
Case 2			
x	T_exact	T_FDM	Error
0	92.5007	92.5007	3.10332e-12
0.0625	92.5297	92.5297	3.07163e-12
0.125	92.6164	92.6164	3.03807e-12
0.1875	92.761	92.761	2.97205e-12
0.25	92.9636	92.9636	2.88914e-12
0.3125	93.2243	93.2243	2.77436e-12
0.375	93.5433	93.5433	2.65855e-12
0.4375	93.9208	93.9208	2.49656e-12
0.5	94.3569	94.3569	2.30429e-12
0.5625	94.8521	94.8521	2.11248e-12
0.625	95.4065	95.4065	1.89167e-12
0.6875	96.0205	96.0205	1.62798e-12
0.75	96.6946	96.6946	1.3227e-12
0.8125	97.4291	97.4291	1.03559e-12
0.875	98.2245	98.2245	6.94451e-13
0.9375	99.0813	99.0813	3.58566e-13
1	100	100	0

A.4.3 $\alpha = 0.75$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	46.6792	46.6792	1.41633e-09
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	77.239	77.239	5.02062e-09
0.5	82.7338	82.7338	3.60427e-09
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	22.9353	22.9353	7.07901e-12
0.5	46.6792	46.6792	5.61685e-12
0.75	72.0691	72.0691	3.25353e-12
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	77.239	77.239	1.99256e-11
0.25	78.6007	78.6007	1.84595e-11
0.5	82.7338	82.7338	1.43081e-11
0.75	89.784	89.784	7.92974e-12
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	11.4174	11.4174	2.80049e-13
0.25	22.9353	22.9353	2.47843e-13
0.375	34.6548	34.6548	2.46041e-13
0.5	46.6792	46.6792	1.97884e-13
0.625	59.1142	59.1142	1.44238e-13
0.75	72.0691	72.0691	9.85919e-14
0.875	85.6578	85.6578	6.6361e-14
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	77.239	77.239	7.35942e-13
0.125	77.5786	77.5786	7.14402e-13
0.25	78.6007	78.6007	6.87033e-13
0.375	80.314	80.314	6.016e-13
0.5	82.7338	82.7338	5.15298e-13
0.625	85.8812	85.8812	4.13678e-13
0.75	89.784	89.784	2.84901e-13
0.875	94.4765	94.4765	1.35375e-13
1	100	100	0

n = 16

x	T_exact	T_FDM	Error
Case 1			
0	0	0	nan
0.0625	5.70245	5.70245	4.20536e-13
0.125	11.4174	11.4174	4.35632e-13
0.1875	17.1575	17.1575	4.1413e-13
0.25	22.9353	22.9353	4.02745e-13
0.3125	28.7635	28.7635	4.07599e-13
0.375	34.6548	34.6548	3.89565e-13
0.4375	40.6224	40.6224	3.49828e-13
0.5	46.6792	46.6792	3.19658e-13
0.5625	52.8386	52.8386	2.82396e-13
0.625	59.1142	59.1142	2.64436e-13
0.6875	65.5196	65.5196	2.38584e-13
0.75	72.0691	72.0691	1.97184e-13
0.8125	78.7769	78.7769	1.62354e-13
0.875	85.6578	85.6578	9.95415e-14
0.9375	92.727	92.727	6.13019e-14
1	100	100	0
Case 2			
x	T_exact	T_FDM	Error
0	77.239	77.239	7.17544e-13
0.0625	77.3238	77.3238	7.16756e-13
0.125	77.5786	77.5786	6.96084e-13
0.1875	78.0039	78.0039	6.92289e-13
0.25	78.6007	78.6007	6.87033e-13
0.3125	79.3701	79.3701	6.44563e-13
0.375	80.314	80.314	6.19294e-13
0.4375	81.4344	81.4344	5.93323e-13
0.5	82.7338	82.7338	5.49651e-13
0.5625	84.215	84.215	5.06235e-13
0.625	85.8812	85.8812	4.46772e-13
0.6875	87.7362	87.7362	3.72537e-13
0.75	89.784	89.784	3.00729e-13
0.8125	92.0292	92.0292	2.16184e-13
0.875	94.4765	94.4765	1.35375e-13
0.9375	97.1315	97.1315	5.85221e-14
1	100	100	0

A.4.4 $\alpha = 3$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	21.2548	21.2549	0.00068718
1	100	100	1.42109e-14

Case 2

x	T_exact	T_FDM	Error
0	9.93279	9.93294	0.00151088
0.5	23.366	23.3662	0.000823694
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	8.20849	8.2085	4.63469e-06
0.5	21.2548	21.2548	3.43094e-06
0.75	46.828	46.828	1.80517e-06
1	100	100	1.42109e-14

Case 2

x	T_exact	T_FDM	Error
0	9.93279	9.93279	7.54345e-06
0.25	12.8598	12.8598	6.33969e-06
0.5	23.366	23.366	4.11251e-06
0.75	47.6433	47.6433	1.98268e-06
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	3.83166	3.83166	2.07466e-08
0.25	8.20849	8.20849	1.93303e-08
0.375	13.7532	13.7532	1.71247e-08
0.5	21.2548	21.2548	1.43097e-08
0.625	31.7805	31.7805	1.10622e-08
0.75	46.828	46.828	7.52894e-09
0.875	68.5382	68.5382	3.81754e-09
1	100	100	1.42109e-14

Case 2

x	T_exact	T_FDM	Error
0	9.93279	9.93279	3.1462e-08
0.125	10.6394	10.6394	3.00457e-08
0.25	12.8598	12.8598	2.64415e-08
0.375	16.9099	16.9099	2.18663e-08
0.5	23.366	23.366	1.71524e-08
0.625	33.1466	33.1466	1.26087e-08
0.75	47.6433	47.6433	8.26933e-09
0.875	68.9188	68.9188	4.08479e-09
1	100	100	0

n = 16

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	1.88264	1.88264	8.37397e-11
0.125	3.83166	3.83166	8.22773e-11
0.1875	5.91578	5.91578	7.99179e-11
0.25	8.20849	8.20849	7.66722e-11
0.3125	10.7906	10.7906	7.26469e-11
0.375	13.7532	13.7532	6.79377e-11
0.4375	17.2008	17.2008	6.26034e-11
0.5	21.2548	21.2548	5.67804e-11
0.5625	26.0583	26.0583	5.04994e-11
0.625	31.7805	31.7805	4.38996e-11
0.6875	38.6233	38.6233	3.69958e-11
0.75	46.828	46.828	2.98614e-11
0.8125	56.6838	56.6838	2.25634e-11
0.875	68.5382	68.5382	1.5136e-11
0.9375	82.8092	82.8092	7.60231e-12
1	100	100	1.42109e-14

Case 2

x	T_exact	T_FDM	Error
0	9.93279	9.93279	1.24668e-10
0.0625	10.1079	10.1079	1.23211e-10
0.125	10.6394	10.6394	1.19042e-10
0.1875	11.5461	11.5461	1.12664e-10
0.25	12.8598	12.8598	1.04732e-10
0.3125	14.627	14.627	9.58919e-11
0.375	16.9099	16.9099	8.66227e-11
0.4375	19.7891	19.7891	7.72333e-11
0.5	23.366	23.366	6.79495e-11
0.5625	27.7668	27.7668	5.88307e-11
0.625	33.1466	33.1466	4.99039e-11
0.6875	39.6951	39.6951	4.12237e-11
0.75	47.6433	47.6433	3.27358e-11
0.8125	57.2714	57.2714	2.4379e-11
0.875	68.9188	68.9188	1.61659e-11
0.9375	82.9962	82.9962	8.03036e-12
1	100	100	0

A.4.5 $\alpha = 5$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	8.15356	8.15723	0.0449772
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	1.34753	1.34876	0.0911882
0.5	8.26343	8.26725	0.0461902
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	2.15883	2.15884	0.00045744
0.5	8.15356	8.15359	0.000319911
0.75	28.6359	28.636	0.000161646
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	1.34753	1.34754	0.000648445
0.25	2.5447	2.54472	0.000510916
0.5	8.26343	8.26346	0.000328533
0.75	28.665	28.6651	0.000162605
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	0.898199	0.898199	2.33751e-06
0.25	2.15883	2.15883	2.12834e-06
0.375	4.29056	4.29056	1.83158e-06
0.5	8.15356	8.15356	1.48845e-06
0.625	15.3066	15.3066	1.12447e-06
0.75	28.6359	28.6359	7.52094e-07
0.875	53.5201	53.5201	3.76599e-07
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	1.34753	1.34753	3.01702e-06
0.125	1.6194	1.6194	2.80785e-06
0.25	2.5447	2.5447	2.37714e-06
0.375	4.49682	4.49682	1.93753e-06
0.5	8.26343	8.26343	1.52857e-06
0.625	15.3644	15.3644	1.13848e-06
0.75	28.665	28.665	7.56552e-07
0.875	53.5322	53.5322	3.77725e-07
1	100	100	0

n = 16

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	0.428029	0.428029	9.79261e-09
0.125	0.898199	0.898199	9.55917e-09
0.1875	1.4568	1.4568	9.18853e-09
0.25	2.15883	2.15883	8.70376e-09
0.3125	3.0734	3.0734	8.12987e-09
0.375	4.29056	4.29056	7.49022e-09
0.4375	5.93014	5.93014	6.80426e-09
0.5	8.15356	8.15356	6.087e-09
0.5625	11.1797	11.1797	5.34925e-09
0.625	15.3066	15.3066	4.59846e-09
0.6875	20.9404	20.9404	3.83957e-09
0.75	28.6359	28.6359	3.07567e-09
0.8125	39.1507	39.1507	2.30878e-09
0.875	53.5201	53.5201	1.54008e-09
0.9375	73.1587	73.1587	7.70345e-10
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	1.34753	1.34753	1.2338e-08
0.0625	1.41386	1.41386	1.21046e-08
0.125	1.6194	1.6194	1.14826e-08
0.1875	1.98437	1.98437	1.06397e-08
0.25	2.5447	2.5447	9.72122e-09
0.3125	3.35558	3.35558	8.80659e-09
0.375	4.49682	4.49682	7.92346e-09
0.4375	6.08079	6.08079	7.07383e-09
0.5	8.26343	8.26343	6.25102e-09
0.5625	11.2596	11.2596	5.4471e-09
0.625	15.3644	15.3644	4.65574e-09
0.6875	20.9818	20.9818	3.87235e-09
0.75	28.665	28.665	3.09389e-09
0.8125	39.1704	39.1704	2.31839e-09
0.875	53.5322	53.5322	1.54469e-09
0.9375	73.1644	73.1644	7.72052e-10
1	100	100	0

A.4.6 $\alpha = 7$

n = 2

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.5	3.01699	3.03228	0.506833
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	0.182376	0.184233	1.01811
0.5	3.02249	3.03786	0.508696
1	100	100	0

n = 4

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.25	0.508906	0.508947	0.00800042
0.5	3.01699	3.01715	0.00543668
0.75	17.3769	17.3774	0.00272283
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	0.182376	0.182396	0.0108935
0.25	0.540598	0.540643	0.00832966
0.5	3.02249	3.02265	0.00545651
0.75	17.3779	17.3783	0.00272369
1	100	100	0

n = 8

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.125	0.180736	0.180736	4.84114e-05
0.25	0.508906	0.508906	4.3232e-05
0.375	1.25221	1.25221	3.65575e-05
0.5	3.01699	3.01699	2.93786e-05
0.625	7.24283	7.24284	2.20626e-05
0.75	17.3769	17.3769	1.47138e-05
0.875	41.686	41.686	7.35766e-06
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	0.182376	0.182376	5.88644e-05
0.125	0.256762	0.256762	5.3685e-05
0.25	0.540598	0.540598	4.5011e-05
0.375	1.26542	1.26542	3.70207e-05
0.5	3.02249	3.02249	2.94858e-05
0.625	7.24512	7.24512	2.20857e-05
0.75	17.3779	17.3779	1.47185e-05
0.875	41.6864	41.6864	7.35845e-06
1	100	100	0

n = 16

Case 1

x	T_exact	T_FDM	Error
0	0	0	nan
0.0625	0.0823596	0.0823596	2.13055e-07
0.125	0.180736	0.180736	2.06593e-07
0.1875	0.314263	0.314263	1.96745e-07
0.25	0.508906	0.508906	1.84491e-07
0.3125	0.802521	0.802521	1.707e-07
0.375	1.25221	1.25221	1.56007e-07
0.4375	1.94543	1.94543	1.40819e-07
0.5	3.01699	3.01699	1.25372e-07
0.5625	4.67529	4.67529	1.09794e-07
0.625	7.24283	7.24283	9.41512e-08
0.6875	11.219	11.219	7.8478e-08
0.75	17.3769	17.3769	6.27904e-08
0.8125	26.9143	26.9143	4.7096e-08
0.875	41.686	41.686	3.13985e-08
0.9375	64.5648	64.5648	1.56996e-08
1	100	100	0

Case 2

x	T_exact	T_FDM	Error
0	0.182376	0.182376	2.51201e-07
0.0625	0.20011	0.20011	2.4474e-07
0.125	0.256762	0.256762	2.29098e-07
0.1875	0.363348	0.363348	2.10464e-07
0.25	0.540598	0.540598	1.92082e-07
0.3125	0.822982	0.822982	1.74652e-07
0.375	1.26542	1.26542	1.57984e-07
0.4375	1.95395	1.95395	1.4178e-07
0.5	3.02249	3.02249	1.25829e-07
0.5625	4.67884	4.67884	1.10008e-07
0.625	7.24512	7.24512	9.42499e-08
0.6875	11.2204	11.2204	7.85229e-08
0.75	17.3779	17.3779	6.28104e-08
0.8125	26.9149	26.9149	4.71046e-08
0.875	41.6864	41.6864	3.14019e-08
0.9375	64.5649	64.5649	1.57006e-08
1	100	100	0

Bibliography

[Hof01] Joe D. Hoffman. “Numerical Methods for Engineers and Scientists”. In: Marcel Decker, 2001. Chap. 8.