FEM Demo

Elena Welch

May 11 2022

Problem 1

A beam with a fixed-fixed boundary conditions subjected to distributed load across half the length of the beam. The beam is loaded by its own gravitational load. The beam has a rectangular cross section with a width $b = 1$ and the height $h(x)$ varying linearly between h_0 and h_L.

$$h(x) = h_0(1 - \frac{x}{L}) + h_L(\frac{x}{L}) \tag{1}$$

Consequently, the moment of inertia for the beam is

$$I(x) = \frac{bh(x)^3}{12} \tag{2}$$

The problem is to compute the maximum positive and negative value of the bending moment and shear force, using 3 methods: (1) Simplified FEM, (2) Consistent FEM, and (3) Exact FEM.

The Simplified FEM

Simplified FEM dictates that the moment of inertia and the gravitational load is constant throughout the element. Thus, the equations for the moment of inertia and the gravitational loading are

$$h(\bar{x}) = h(x_{i-1})(1 - \frac{\bar{x}}{\Delta x}) + h(x_i)(\frac{\bar{x}}{\Delta x})$$
$$p(\bar{x}) = -\rho g b h(\bar{x})$$
$$I(\bar{x}) = \frac{bh^3(\bar{x})}{12}$$

and are consistent for the entire element. Therefore, weak solution of equilibrium is

$$\begin{bmatrix} k^i_{0,0} & k^i_{0,1} & k^i_{0,2} & k^i_{0,3} \\ k^i_{1,0} & k^i_{1,1} & k^i_{1,2} & k^i_{1,3} \\ k^i_{2,0} & k^i_{2,1} & k^i_{2,2} & k^i_{2,3} \\ k^i_{3,0} & k^i_{3,1} & k^i_{3,2} & k^i_{3,3} \end{bmatrix} \begin{bmatrix} v_{i-1} \\ v'_{i-1} \\ v_i \\ v'_i \end{bmatrix} = \begin{bmatrix} p^i_0 \\ p^i_1 \\ p^i_2 \\ p^i_3 \end{bmatrix} + \begin{bmatrix} V_{i-1} \\ -M_{i-1} \\ -V_i \\ M_i \end{bmatrix} \tag{3}$$

or, using Gauss-Legendre integration for Hermite shape functions,

$$EI(x_{midpoint})\begin{bmatrix} 12 & 6\Delta x & -12 & 6\Delta x \\ 6\Delta x & 4\Delta x^2 & -6\Delta x & 2\Delta x^2 \\ -12 & -6\Delta x & 12 & -6\Delta x \\ 6\Delta x & 2\Delta x^2 & -6\Delta x & 4\Delta x^2 \end{bmatrix}\begin{bmatrix} v_{i-1} \\ v'_{i-1} \\ v_i \\ v'_i \end{bmatrix} = \begin{bmatrix} \frac{p\Delta x}{2} \\ \frac{p\Delta x^2}{12} \\ \frac{p\Delta x}{2} \\ -\frac{p\Delta x^2}{12} \end{bmatrix} + \begin{bmatrix} V_{i-1} \\ -M_{i-1} \\ -V_i \\ M_i \end{bmatrix}. \tag{4}$$

Results for Simplified FEM

For the simplified FEM, the following results are obtained from the code. I also only display here K and f for $n = 2$, because of the sheer size of the K matrices beyond this mesh.

V(L/2)	M(L/2)	v(L/2)	v'(L/2)
0.119936	0.0707456	-0.0633742	-0.0819197

$$K_{n=2} = \begin{bmatrix} 5.0379 & 1.25948 & -5.0379 & 1.25948 & 0 & 0 \\ 1.25948 & 0.419825 & -1.25948 & 0.209913 & 0 & 0 \\ -5.0379 & -1.25948 & 6.53061 & -0.886297 & -1.49271 & 0.373178 \\ 1.25948 & 0.209913 & -0.886297 & 0.544218 & -0.373178 & 0.0621963 \\ 0 & 0 & -1.49271 & -0.373178 & 1.49271 & -0.373178 \\ 0 & 0 & 0.373178 & 0.0621963 & -0.373178 & 0.124393 \end{bmatrix}$$

$$f_{n=2} = \begin{bmatrix} -0.214286 \\ -0.0178571 \\ -0.357143 \\ 0.00595238 \\ -0.142857 \\ 0.0119048 \end{bmatrix}$$

deltax	Error in Shear	Error in Moments	Error in v	Error in v'
0.5	3.13538	0.558654	0.0777025	0.224265
0.25	1.26615	0.184344	0.0134232	0.0218756
0.125	0.546495	0.0678643	0.00644373	0.0140688
0.0625	0.250793	0.0276536	0.00183303	0.00415946
0.03125	0.119774	0.0122473	0.000472719	0.00108204
0.015625	0.0584627	0.00572234	0.000119104	0.000273182
0.0078125	0.0288461	0.00275282	2.98444e-05	6.84643e-05
0.00390625	0.0142957	0.00134095	7.4784e-06	1.71303e-05
0.00195312	0.00708476	0.000653204	1.87393e-06	4.28612e-06

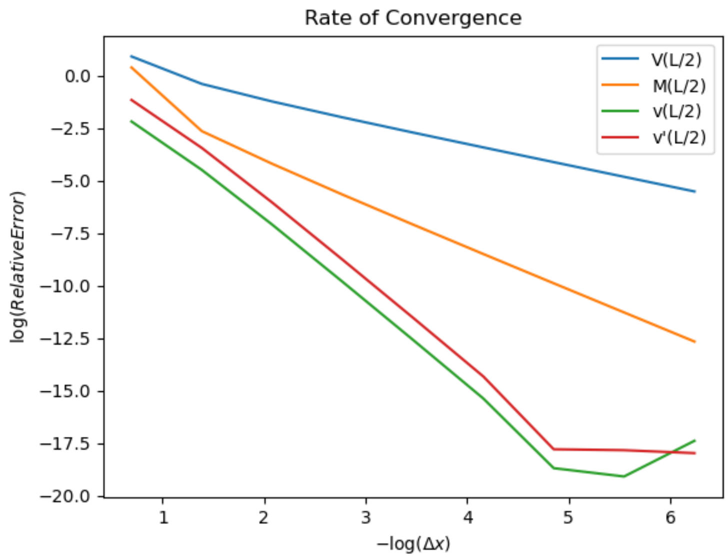


Figure 1: Rate of Convergence, Fixed-Fixed

β_V	β_M	β_v	$\beta_{v'}$
1.30819	1.59955	2.53324	3.35781
1.21217	1.44168	1.05876	0.636827
1.12371	1.29518	1.81367	1.75803
1.06618	1.17501	1.95518	1.94264
1.03473	1.09778	1.98876	1.98582
1.01914	1.05569	1.99669	1.99644
1.01279	1.03765	1.99666	1.9988
1.01279	1.03765	1.99666	1.9988

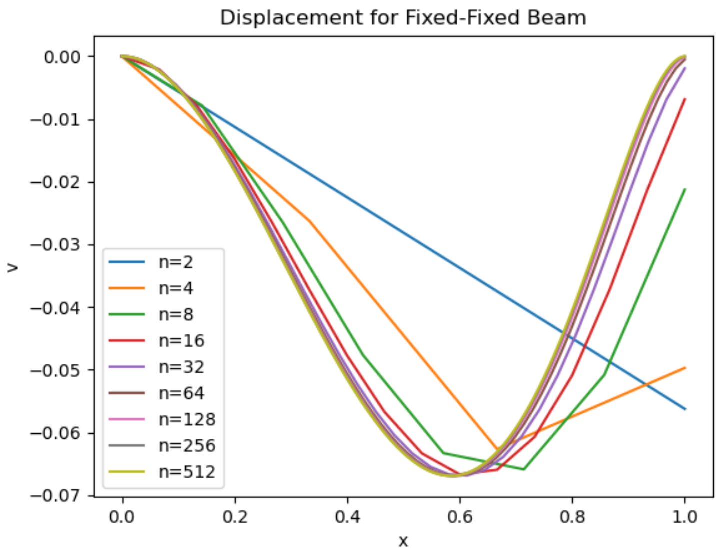


Figure 2: Displacement, Fixed-Fixed

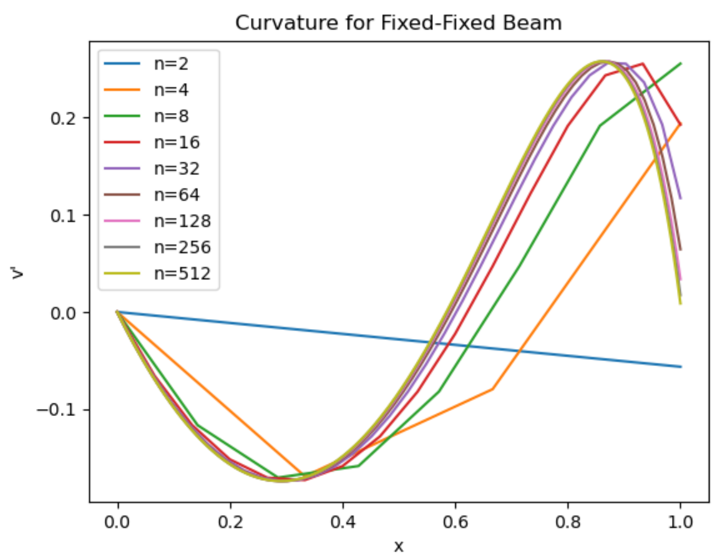


Figure 3: Curvature, Fixed-Fixed

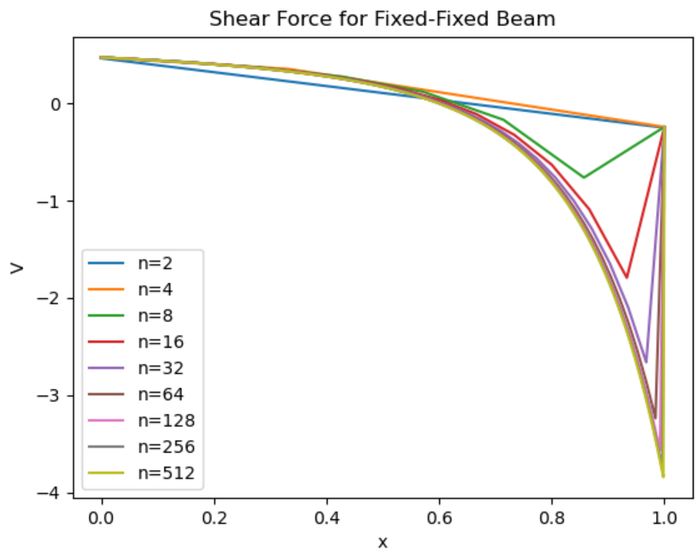


Figure 4: Shear, Fixed-Fixed

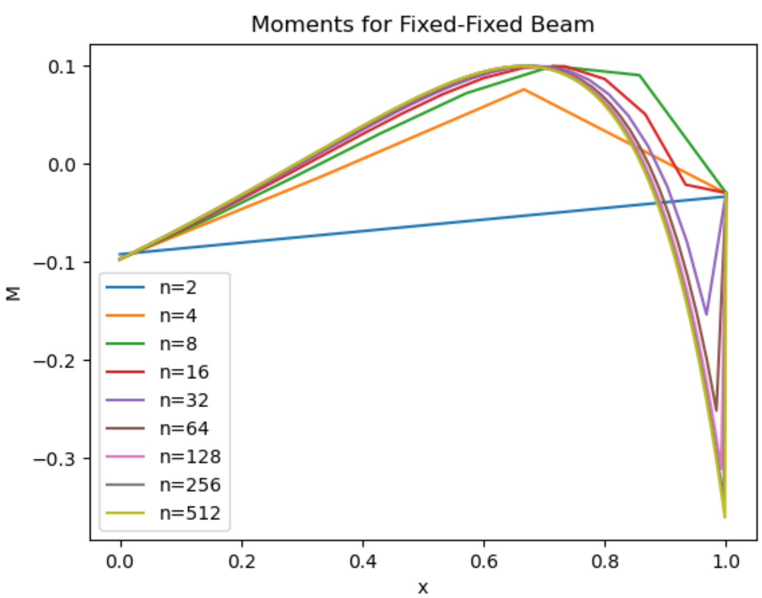


Figure 5: Moments, Fixed-Fixed

Consistent FEM

The consistent FEM dictates that the integral for K_{nm} use the actual functions of the moment of inertia and the area, instead of the modified equations used in the previous assignment. Therefore, we use the exact equations as listed above, 1 and 2. Furthermore,

$$p(x) = -\rho g A(x)$$
$$A(x) = bh(x).$$

Results for Consistent FEM

For the consistent FEM, the following results are obtained from the code. To save space, I only included results from $n = 1 - 16$, but the code goes farther and displays those results as well. I also only display here K and f for $n = 2$, because of the sheer size of the K matrices beyond this mesh.

V(L/2)	M(L/2)	v(L/2)	v'(L/2)
0.168452	0.070744	-0.0633742	-0.0819197

$$K_{n=2} = \begin{bmatrix} 5.2898 & 1.53353 & -5.2898 & 1.11137 & 0 & 0 \\ 1.53353 & 0.544023 & -1.53353 & 0.222741 & 0 & 0 \\ -5.2898 & -1.53353 & 6.95044 & -0.601749 & -1.66064 & 0.3207 \\ 1.11137 & 0.222741 & -0.601749 & 0.517007 & -0.509621 & 0.0707483 \\ 0 & 0 & -1.66064 & -0.509621 & 1.66064 & -0.3207 \\ 0 & 0 & 0.3207 & 0.0707483 & -0.3207 & 0.0896016 \end{bmatrix}$$

$$f_{n=2} = \begin{bmatrix} -0.228571 \\ -0.0184524 \\ -0.357143 \\ 0.0047619 \\ -0.128571 \\ 0.0113095 \end{bmatrix}$$

deltax	Error in Shear	Error in Moments	Error in v	Error in v'
0.5	2.48806	1.47115	0.112326	0.313158
0.25	0.672617	0.0709019	0.0111724	0.0318327
0.125	0.291136	0.0149549	0.000825041	0.00236428
0.0625	0.136742	0.00344784	5.40762e-05	0.000155183
0.03125	0.0663753	0.000829734	3.41636e-06	9.8047e-06
0.015625	0.0326993	0.000203707	2.08663e-07	5.96085e-07
0.0078125	0.0162191	5.05102e-05	7.56138e-09	1.85959e-08
0.00390625	0.0080664	1.26088e-05	5.08455e-09	1.77931e-08
0.00195312	0.00401174	3.14753e-06	2.78404e-08	1.54792e-08

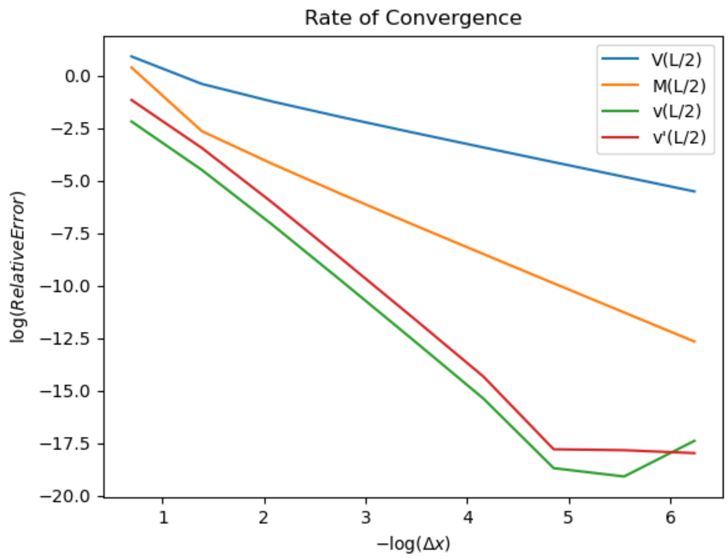


Figure 6: Rate of Convergence

β_V	β_M	β_v	$\beta_{v'}$
1.88716	4.37498	3.32969	3.29831
1.20809	2.2452	3.75933	3.75103
1.09024	2.11686	3.9314	3.92936
1.04274	2.05497	3.98446	3.98436
1.02139	2.02615	4.03321	4.03988
1.01157	2.01185	4.78638	5.00246
1.0077	2.00214	0.572531	0.0636712
1.0077	2.00214	-2.45299	0.200989

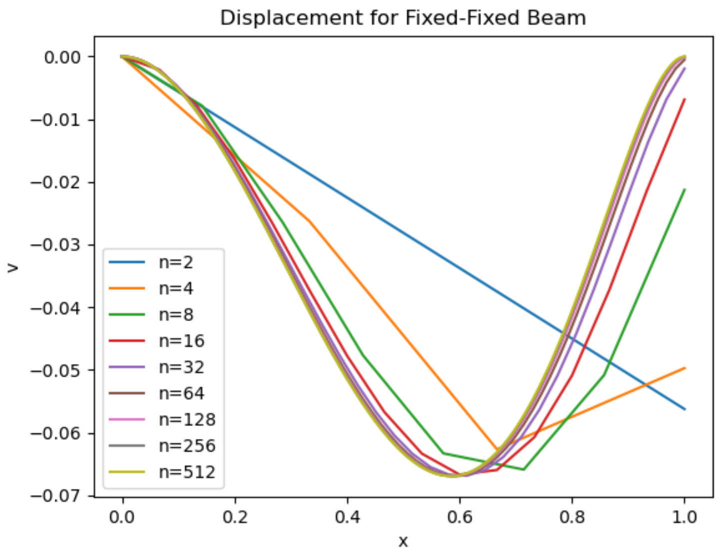


Figure 7: Deflection, Consistent FEM

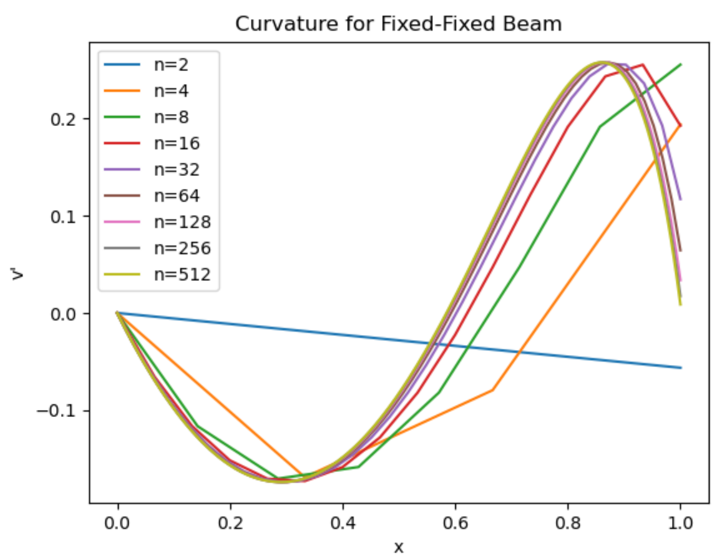


Figure 8: Curvature, Consistent FEM

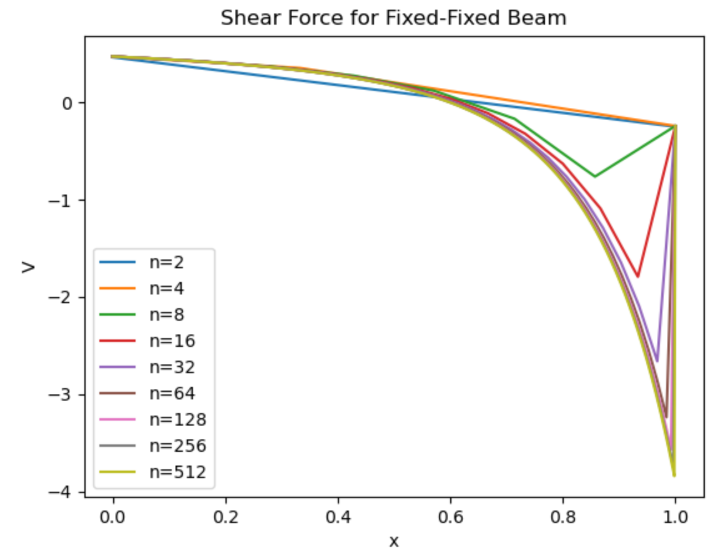


Figure 9: Shear, Consistent FEM

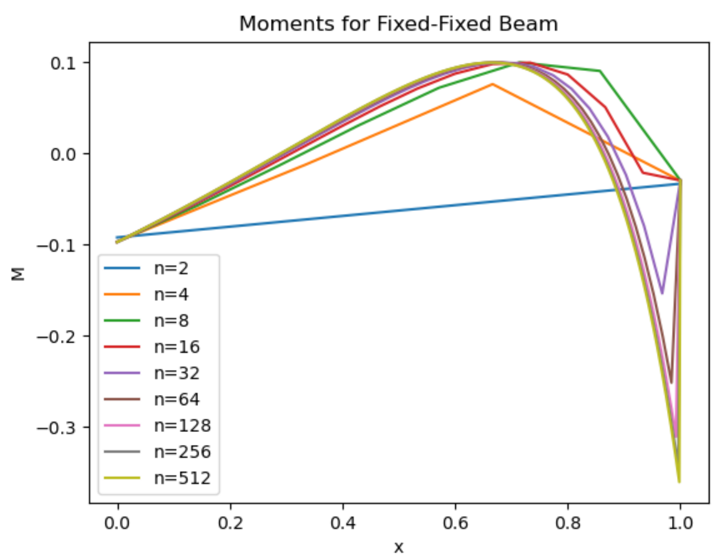


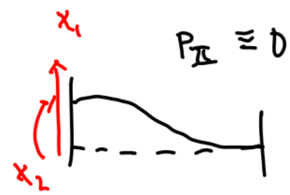
Figure 10: Moments, Consistent FEM

Exact FEM; Force Method

The force method gives the exact K and f matrices. This method consists of subjecting the bar to different loads/displacements and measuring the reactions. With the reactions known, a stiffness matrix is constructed by considering 4 different cases of loading. The f matrix is constructed based on the 5th case, as described below.

Stiffness Matrix

Case 1 describes a loading as seen in the image on the right. x_1 and x_2 are the reactions that will be found for the force method. In case 1, we have the following known conditions on the beam.



$$v_{II}(0) = 1$$
$$v'_{II}(0) = 0$$
$$v_{II}(\Delta x) = 0$$
$$v'_{II}(\Delta x) = 0$$

Using these conditions, it is possible then to calculate the reactions, x_1 and x_2. With the following equation:

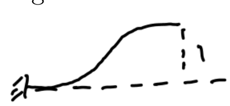
$$\begin{bmatrix} K_{00} & & \dots & \\ \vdots & K_{11} & & \vdots \\ & \dots & K_{22} & \\ \dots & & & K_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x_1 \\ -x_2 \\ -x_1 \\ x_2 + x_1 \Delta x \end{bmatrix}$$

it is possible to find the first column of the stiffness matrix, assuming we set the values of x_1 and x_2.

$$K_{00} = x_1$$
$$K_{10} = -x_2$$
$$K_{20} = -x_1$$
$$K_{30} = x_2 + x_1 \Delta x$$

The subsequent cases will give the other three columns of stiffness matrix K.

Case 2, $v'_{II}(0) = 1$



Case 3, $v_{II}(\Delta x) = 1$

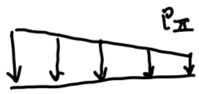
Case 4, $v'_{II}(\Delta x) = 1$

Note that for each case, the reactions are "forced" to be a value we know, and there is no loading on the beam. Therefore, $p_{II} = 0$ for each case.

Load Matrix

To obtain the load matrix f, we will have to apply loading to the beam. The load will be gravitational, the same as described in the consistent FEM.

Case 5 includes just the gravitational load applied. This means



$$v_{II}(0) = 0$$
$$v'_{II}(0) = 0$$
$$v_{II}(\Delta x) = 0$$
$$v'_{II}(\Delta x) = 0$$

This makes the K matrix irrelevant, and the only thing to solve for is the load vector.

The load vector is directly related to the moments and shears of the case 5 loading. From equilibrium, we achieve

$$f = \begin{bmatrix} -x_1^5 \\ -x_2^5 \\ x_1^5 + \frac{2}{3}\Delta x p(x_{i-1}) + \frac{1}{3}\Delta x p(x_i) \\ -x_1^5\Delta x + x_2^5\Delta x^2 \frac{1}{3}p(x_{x-i}) + \frac{1}{6}p(x_i)\Delta x^2 \end{bmatrix}$$

Results

These results can be compared with that of the simplified and consistent FEM, which are listed in their respective sections. Something is wrong with the signs of my K matrix, but I include the results for one mesh regardless.

$$K_{n=2} = \begin{bmatrix} 4.95367 & 1.44482 & -4.95367 & 1.44482 & 0 & 0 \\ -1.44482 & 0.520612 & -1.03201 & 0.201798 & 0 & 0 \\ -4.95367 & -1.44482 & 9.90733 & 0 & -4.95367 & 1.44482 \\ 3.92165 & 0.201798 & -2.88964 & 1.04122 & -1.03201 & 0.201798 \\ 0 & 0 & -4.95367 & -1.44482 & 4.95367 & -1.44482 \\ 0 & 0 & 3.92165 & 0.201798 & -1.44482 & 0.520612 \end{bmatrix}$$

$$f_{n=2} = \begin{bmatrix} -0.266787 \\ -0.0252649 \\ -0.452381 \\ -0.186315 \\ -0.185594 \\ -0.16105 \end{bmatrix}$$

V(L/2)	M(L/2)	v(L/2)	v'(L/2)
-0.214279	-0.0535706	2.20445e-09	-4.06825e-09

Δx	Error in Shear	Error in Moments	Error in v	Error in v'
0.5	1.23967	0.800141	1	1
0.25	0.284818	1.3785	501117	2.11096e+06
0.125	0.0598075	0.710842	234259	1.85593e+06
0.0625	0.0182447	0.365612	76737.2	424397
0.03125	0.00582965	0.183046	21486.4	105015
0.015625	0.00208453	0.0913862	5655.49	26011.5
0.0078125	0.000824085	0.0456296	1449.59	6467.06
0.00390625	0.000346377	0.0227903	367.489	1611.33
0.00195312	0.000145588	0.0113829	93.1631	401.478

β_V	β_M	β_v	β_v'
2.12184	-0.784772	-18.9348	-21.0095
2.25164	0.955496	1.09704	0.185759
1.71285	0.959218	1.61011	2.12865
1.646	0.998107	1.8365	2.01481
1.48369	1.00216	1.9257	2.01338
1.33886	1.002	1.96401	2.00797
1.25045	1.00155	1.97987	2.00486
1.25045	1.00155	1.97987	2.00486

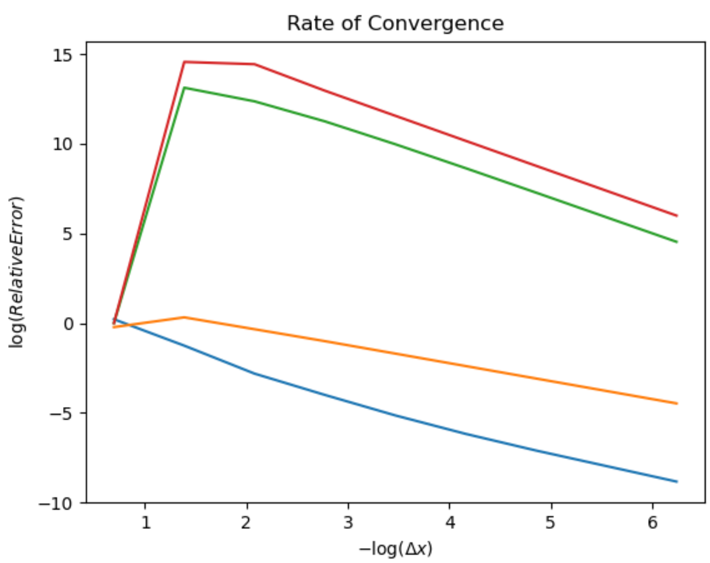


Figure 11: Convergence for Exact Method

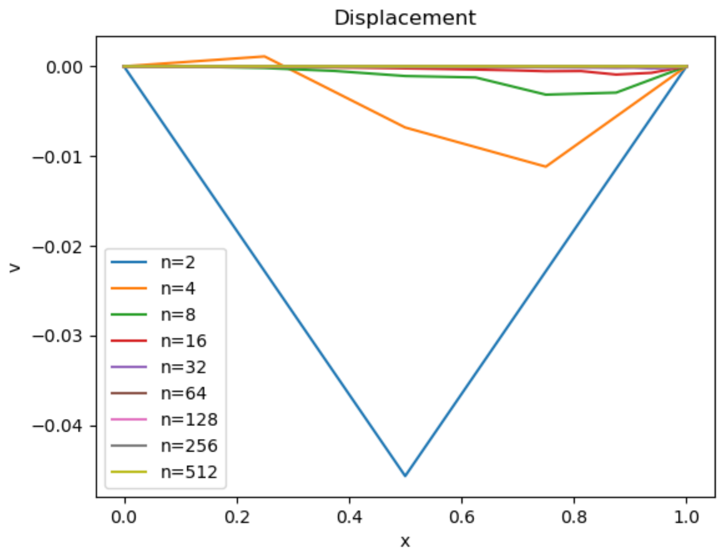


Figure 12: Displacement for Exact

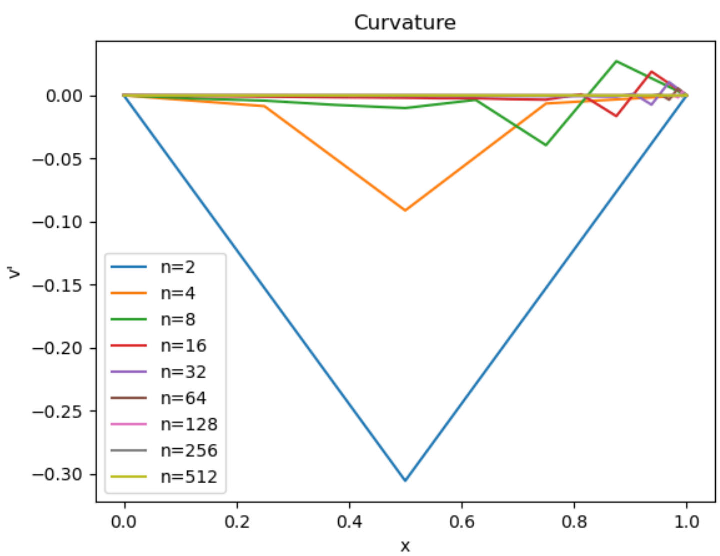


Figure 13: Curvature for Exact

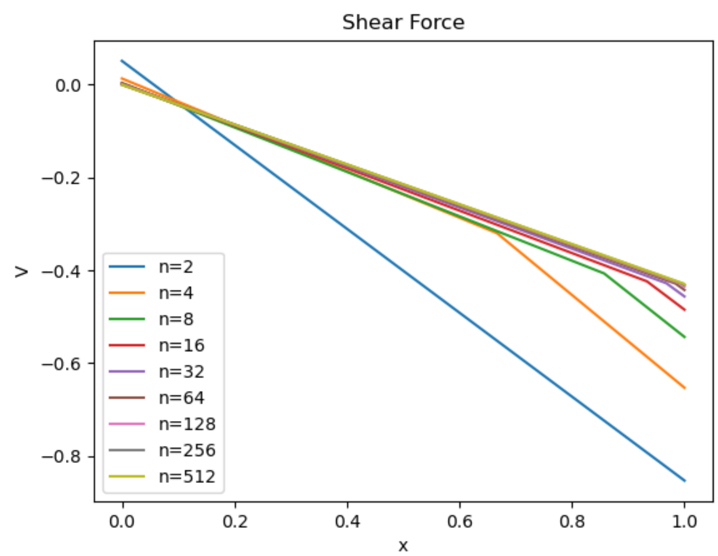


Figure 14: Shear for Exact Method

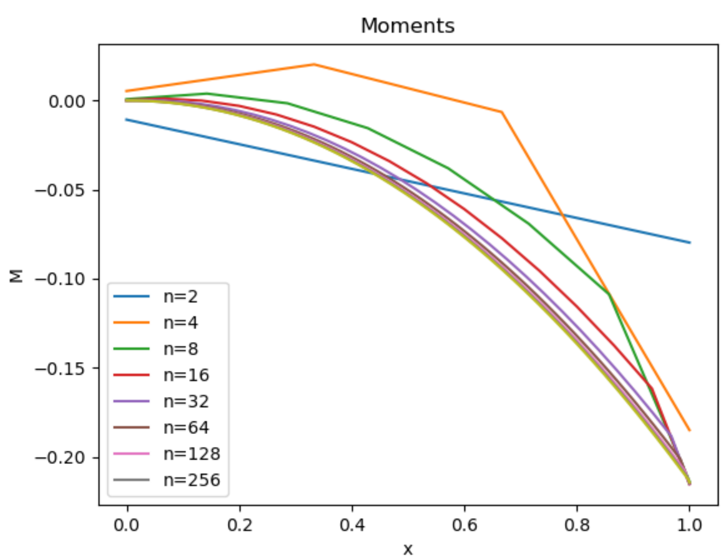


Figure 15: Moments for Exact Method